R-equivalence and rational equivalence on varieties over p-adic fields, with special regards to rationally connected varieties

J.-L. Colliot-Thélène
CNRS
Université Paris-Sud

R-equivalence

k a field, X a smooth projective variety over k

Two points A and B in $X(k)$ (the set of k-rational points) are called R-linked if there exists a k-morphism $f : \mathbf{P}^1_k \to X$ such that A and B both belong to $f(\mathbf{P}^1(k))$.

R-equivalence is the equivalence relation spanned by this relation.

Chow group

k a field, X a smooth projective variety over k

The group $Z_0(X)$ of zero-cycles on X is the free abelian group on the closed points $M \in X$ (a point is closed if and only its residue field $k(M)$ is a finite extension of k.)

The *Chow group* $CH_0(X)$ of zero-cycles modulo rational equivalence is the quotient of the group $Z_0(X)$ by the subgroup spanned by elements of the type $p_*(div_C(f))$, where C/k is an irreducible, normal, projective curve over k, $p : C \to X$ is a k-morphism, and f is a rational function on C.

If X/k is proper, then there is a degree map $CH_0(X) \to \mathbf{Z}$, whose kernel is the *reduced Chow* group $A_0(X)$.

What is the structure of the Chow group $CH_0(X)$ over a local field ?

Let k be a p-adic field, and X/k a smooth, projective, absolutely irreducible variety.

Guess : the group $A_0(X)$ admits a filtration whose successive quotients are a finite group, a group isomorphic to a finite sum of copies of $\mathbf{Z}_p$ and a divisible (possibly uniquely divisible) group.

Related questions :

For $n > 0$, is the group ${}_nA_0(X)$ finite?

Is the whole torsion subgroup of $A_0(X)$ finite?

For $n > 0$, is $A_0(X)/n$ finite?

Definition (Kollár, Miyaoka, Mori, 1992)

A smooth, projective, integral variety over a field k of characteristic zero is called *rationally connected* if over a big enough algebraically closed field Ω containing k, there is only one R-equivalence class on the set $X_{\Omega}(\Omega)$.

Examples :

smooth compactifications of connected linear algebraic groups

geometrically unirational varieties

Fano varieties (this is a theorem due to Campana 1992 and to KMM 1992)

A rationally connected surface is just a (geometrically) rational surface.

Assume now that X is a rationally connected variety over the p-adic field k.

Theorem (Kollár 1999). *The set* $X(k)/R$ *is finite.*

Theorem (Kollár/Szabó 2003) *If* X *has good, rationally connected reduction over* $\mathbf{F}$, *then*

(i) $A_0(X) = 0$.

(ii) If the residue field $\mathbf{F}$ *is not too small,* $X(k)/R$ *consists of one class.*

The proof of both theorems uses deformation theory (techniques of Kollár, Miyaoka, Mori).

Two questions

X a rationally connected variety over the p-adic field k

Is the group $A_0(X)$ *finite ?*
Known if X is a surface (via algebraic K-theory).

In the bad reduction case, how can one detect nontrivial elements in $X(k)/R$ *and in* $A_0(X)$ *?*

Surfaces

Theorem. *Let X/k be a smooth, projective geom. irreducible surface over a p-adic field k, with residue class field $\mathbf{F}$.*

(i) For all $n > 0$, the group ${}_nA_0(X)$ is finite.

(ii) For each prime l, the group $A_0(X)\{l\}$ is of cofinite type.

(iii) For n integer prime to p, the quotient $A_0(X)/n$ is finite.

(iv) Suppose that X/k has good reduction $Y/\mathbf{F}$. Then for any l prime, $l \neq p$, the reduction map induces a surjection

$$A_0(X)\{l\} \to A_0(Y)\{l\}.$$

(CT/Sansuc/Soulé 1983, Saito-Sujatha 1993)

Some tools :

Bloch-Ogus theory 1974

Bloch's method (1974) for the study of torsion of codimension 2 Chow groups

the Merkur'ev/Suslin theorem (1982)

finiteness theorems for étale cohomology

hyperplane sections

Theorem. *Let X/k be a smooth, projective, geometrically connected surface over a p-adic field k. Assume $H^2(X, O_X) = 0$.*

Then :

(i) The group $A_0(X)_{tors}$ is finite.

(ii) Under Bloch's conjecture for X over an algebraic closure of k, the group $A_0(X)$ is an extension of a finite abelian group by a finite sum of copies of $\mathbf{Z}_p$.

(iii) Under the same assumption on X, the quotient $A_0(X)/l$ is finite for any prime l and zero for almost all l.

(CT/Raskind 1991, Salberger 1993)

Some tools :

Bloch's Galois cohomological method for computing

$$Ker[CH^2(X) \to CH^2(\overline{X})]$$

Hilbert's theorem 90 for K_2

hyperplane sections

class field theory for curves over a local field (Bloch, Saito)

Suslin's results on torsion in K_2

Roitman's theorem

The good reduction case

Let O be the ring of integers of the p-adic field k. Let $\mathcal{X}/O$ be a smooth, projective relative surface with absolutely irreducible fibres. Let X/k the generic fibre and $Y/\mathbf{F}$ the special fibre.

There is an exact (localization) sequence

$$H^1(X, \mathcal{K}_2) \to Pic(Y) \to$$

$$\to CH^2(\mathcal{X}) \to CH^2(X) \to 0.$$

Let us introduce hypothesis (H) :

(H) *The cokernel of* $H^1(X, \mathcal{K}_2) \to Pic(Y)$ *is a torsion group.*

Since $Pic(Y)$ is finitely generated, the hypothesis amounts to finiteness of this cokernel.

For all we know, this hypothesis could always be satisfied. It has to do with the search for so-called indecomposable elements in $K_1(X)$. Here are cases where the hypothesis is known to hold.

1) $H^2(Y, O_Y) = 0$ (CT/Raskind 1991)

2) X is the product of two elliptic curves with good reduction (Spieß 1999)

3) Some products of two modular curves and related surfaces (Mildenhall, Saito, Langer, Raskind, Otsubo)

Theorem. *Let O be the ring of integers of the p-adic field k. Let $\mathcal{X}/O$ be a smooth, projective relative surface with absolutely irreducible fibres. Let X/k the generic fibre and $Y/\mathbf{F}$ the special fibre.* **Assume** *(H).*

Then :

(i) The prime-to-p part of $A_0(X)_{tors}$ is finite.

(ii) For l prime, $l \neq p$, the specialization map induces an isomorphism of finite groups $A_0(X)\{l\} \simeq A_0(Y)\{l\}$.

(iii) the quotient $A_0(X)/l$ is finite for any prime $l \neq p$ and zero for almost all l.

(iv) $A_0(X)$ is the direct sum of a finite group of order prime to p and a group uniquely divisible by each l prime to p.

(Raskind 1989, CT/Raskind 1991, Spieß 1999)

Some tools : Bloch's method for computing torsion codimension 2 cycles, applied to the integral model $\mathcal{X}$ and compared with the same method for Y. Proper base change in étale cohomology.

Detecting cycles : Pairing with the Brauer group

For X a smooth variety over a field k, the Brauer group $Br(X) = H^2_{\text{ét}}(X, \mathbf{G}_m)$ is a torsion group.

There are natural pairings

$$X(k) \times Br(X) \to Br(k)$$

and

$$Z_0(X) \times Br(X) \to Br(k).$$

For X/k projective, these pairings induce pairings

$$X(k)/R \times Br(X) \to Br(k)$$

$$CH_0(X) \times Br(X) \to Br(k)$$

$$A_0(X) \times Br(X)/Br(k) \to Br(k).$$

For k p-adic, $Br(k) = \mathbf{Q}/\mathbf{Z}$.

Theorem. *Let X/k be a smooth, projective, geometrically connected surface over a p-adic field k. Assume $H^2(X, O_X) = 0$.*

(i) If the Albanese variety of X has good reduction, the pairing

$$A_0(X)_{tors} \times Br(X) \to \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the LHS.

(ii) If moreover the geometric Chow group is representable (Bloch's conjecture) then the pairing

$$A_0(X) \times Br(X) \to \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the LHS.

(Shuji Saito 1992)

The good reduction assumption for the Albanese variety cannot be ignored, as shown by an example of Parimala and Suresh 1995 (conic bundle over a curve with bad reduction).

However in the semistable reduction case, extensions of the above the above theorem are known (K. Sato 1998 ; K. Sato/ S. Saito 2004)

Let $\mathcal{X}/O$ be a smooth, projective relative surface with absolutely irreducible fibres. Let X/k be the generic fibre and $Y/\mathbf{F}$ the special fibre.

Theorem. **Assume** *(H).*

(i) The left kernel of the pairing

$$A_0(X) \times Br(X) \to \mathbf{Q}/\mathbf{Z}$$

consists of elements n-divisible for any integer n prime to p.

(ii) The pairing

$$A_0(X)_{tors}(prime - to - p) \times Br(X) \to \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the LHS.

(Raskind 1989, CT/Raskind 1991, Spieß 1999)

Let us come back to the (possibly) bad reduction case. Let $\mathcal{X}/O$ be a regular, proper flat scheme, with smooth geometrically connected generic fibre X/k. The pairing

$$A_0(X) \times Br(X) \to \mathbf{Q}/\mathbf{Z}$$

is trivial on the subgroup $Br(\mathcal{X}) + Br(k)$ of the group $Br(X)$. Let F_l denote the l-primary part of the quotient $Br(X)/(Br(\mathcal{X}) + Br(k))$.

Theorem (CT/Saito 1996). *For l prime, $l \neq p$, the group F_l is finite and the induced pairing*

$$A_0(X) \times F_l \to \mathbf{Q}/\mathbf{Z}$$

is nondegenerate on the RHS.

Hence for each such l we have a surjective map $A_0(X) \to Hom(F_l, \mathbf{Q}/\mathbf{Z})$. This implies (reduction to case of curves) that the map $A_0(X)\{l\} \to Hom(F_l, \mathbf{Q}/\mathbf{Z})$ is surjective .

p-part (K. Sato/ S. Saito 2004)

Higher dimensional varieties
Examples

Quadric fibrations over a curve

Intersections of two quadrics

Cubic hypersurfaces

Linear algebraic groups

Quadric fibrations over a curve

(Bloch 1981, CT/Sansuc 1981, Salberger 1988, Gros 1987, CT/Skorobogatov 1993, Parimala/Suresh 1995 and 1988)

Let k be a field and $f : X \to C$ a dominant k-morphism of smooth, projective, geom. connected k-varieties, C a curve, and assume that the generic fibre of p is a geometrically irreducible quadric of dimension d over the field $k(C)$. Let

$$CH_0(X/C) = Ker[f_* : CH_0(X) \to CH_0(C)].$$

For $C = \mathbf{P}^1_k$, $CH_0(X/C) = A_0(X)$.

Theorem. *Let k be a p-adic field and let $f : X \to C$ be as above. Then*

(i) The group $CH_0(X/C)$ is finite.

(ii) For $p \neq 2$ and $d \geq 3$, $CH_0(X/C) = 0$.

Tools :

For $d = 1$, X is a surface, the result follows from earlier results.

For $d = 2$, reduction to $d = 1$ by replacing C by a double cover (discriminant of a quadratic form in 4 variables).

For $d \geq 3$, reduction to $d = 2$ (with the same C).

For $p \neq 2$, use of the theorem (Parimala and Suresh 1998) : a quadratic form in $m \geq 11$ variables over $k(C)$ has a nontrivial zero.

Intersections of two quadrics in $\mathbf{P}_k^n$

(Over an algebraic closure, for $n \geq 4$, such a variety is birational to projective space.)

Theorem. *Let k be a p-adic field and let $X \subset \mathbf{P}_k^n$ be a smooth complete intersection of 2 quadrics, of dimension at least 2. Then*

(i) The group $A_0(X)$ is finite.
(ii) For $p \neq 2$ and $n \geq 6$, $A_0(X) = 0$.
(iii) For $n \geq 7$, $A_0(X) = 0$.

Tools :
previous results on quadric fibrations
results on R-equivalence (next slide)

The group $A_0(X)$ may be nonzero for $n = 4$. For $n = 5$, this is an open question (my guess is that it may be nonzero).

Theorem. *For* $X \subset \mathbf{P}^n_k$ *as above and* $n \geq 7$, *the order of* $X(k)/R$ *is at most 1.*
(CT/Sansuc/Swinnerton-Dyer 1987)

The set $X(k)/R$ may consist of more than one element for $n = 4$. For $n = 5, 6$ this is an open question (guess : should get examples with more than one class for $n = 5$).

Smooth cubic hypersurfaces in $\mathbf{P}^n_k$

(Over an algebraic closure, for $n \geq 3$, such a variety is unirational.)

Theorem (Madore 2003). *Let k be a p-adic field and $X \subset \mathbf{P}^n_k$ be a smooth cubic hypersurface. For $n \geq 11$,*

(i) R-equivalence is trivial on $X(k)$: the set $X(k)/R$ consists of one element.

(ii) $A_0(X) = 0$.

Tools :

Intersecting with the tangent hyperplane at a rational point.

Any cubic form in at least 10 variables over a p-adic field has a nontrivial zero (Demjanov, Lewis), and any quadratic form in at least 5 variables has a zero.

For $n = 3$ (case of a surface), there are examples for which R-equivalence is not trivial on $X(k)$ and where $A_0(X) \neq 0$. Nontrivial classes in $A_0(X)$ are detected by the pairing with $Br(X)$.

What happens for $4 \leq n \leq 10$? Here $Br(X) = Br(k)$ is of no help.

Here is one candidate for nontriviality of $A_0(X)$ for $n = 4$ (and $p \neq 3$) :

$$x^3 + y^3 + z^3 + pu^3 + p^2v^3 = 0.$$

One would hope that $J(\mathbf{F}_p)/3$ is a quotient of $A_0(X)$, where J is the jacobian of the curve $x^3 + y^3 + z^3 = 0$ over $\mathbf{F}_p$.

There are similar candidates for $n = 5, 6$.

The idea is to construct a regular proper model over the ring of integers of k and to use intersection theory on this model. This works very well for rational surfaces split over an unramified extension (Dalawat), it works also for some others, such as

$$x^3 + y^3 + z^3 + pt^3 = 0$$

over $\mathbf{Q}_p$, $p \neq 3$. However for rational surfaces, the Brauer group already detects the whole of $A_0(X)$.

For the time being, the only known example of a rationally connected variety X over a p-adic field with a nontrivial zero-cycle in $A_0(X)$ not detected by the Brauer group is an example of Parimala and Suresh 1995. Their X is a quadric bundle of relative dimension 2 over the projective line.

Linear algebraic groups

Theorem. *Let k be a p-adic field, let G be a connected linear algebraic group over k and X a smooth k-compactification of G. Then the prime-to-p part of the torsion group $A_0(X)$ is finite.*

(CT 2004)

Ingredients of the proof

1) The formula $G(k)/R = H^1(k, S)$ for G a semisimple group over a p-adic field. In this formula, which is functorial in k, S is a flasque torus over k associated to G. (P. Gille 1997).

2) The vanishing of $G(k)/R$ when S is split by a cyclic extension (follows from the above formula and a result of Endo and Miyata).

3) For L/k finite field extension of local fields, of degree prime to the degree of the splitting field of S, the restriction map $G(k)/R \to G_L(L)/R$ is a bijection (uses local duality and formula in 1) above).

4) "ramification eats up ramification"

5) Lemma : Let $l \neq p$ be a prime, let k be a p-adic field which contains the l-th roots of 1, let F/k be an extension, and let l^n the highest power of l dividing $[F : k]$. Then there exists a subfield E of F such that $[E : k] = l^n$.