Theorem (Kollár-Szabó 2004)

There exists $\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

if $X/\mathbb{F}$ smooth proj. SRC

$\mathbb{F}$ finite

$\# \mathbb{F} > \Phi(\deg X, \dim X)$

then for any $P, Q \in X(\mathbb{F})$

there exists

$$f : \mathbb{P}^1_{\mathbb{F}} \longrightarrow X \quad \text{very free} \quad \begin{cases} f(0) = P \\ f(\infty) = Q \end{cases}$$

In particular: $\exists \quad \mathbb{P}^1_E \longrightarrow X_E$

over any $\mathbb{E} \supset \mathbb{F}$ $\qquad 0 \longrightarrow P_E$

infinite algebraic $\qquad \infty \longrightarrow Q_E$

(weak form)

Corollary $X/\mathbb{F}$ smooth proj. SRC
then $A_0(X) = 0$

Proof. May assume $X(\mathbb{F}) \ni O$.

Let P = closed point. $\mathbb{F}_1 = \mathbb{F}(P)$

$P \rightsquigarrow P_1 \in X(\mathbb{F}_1)$

thm (week) $\Rightarrow$ exist $\mathbb{F}_2/\mathbb{F}_1$ and $\mathbb{F}_3/\mathbb{F}_1$ coprime degrees $P_1 \sim O$ / $\mathbb{F}_2$
R-equivalent / $\mathbb{F}_3$

$$\begin{array}{ll}
X_{\mathbb{F}_2} & P_1 \underset{rat}{\sim} O \\
\downarrow & \downarrow \\
X_{\mathbb{F}_1} & [\mathbb{F}_2:\mathbb{F}_1](P_1 - O) \underset{rat}{\sim} 0 \\
\downarrow & \downarrow \\
X & [\mathbb{F}_2:\mathbb{F}_1]\left[P - ([\mathbb{F}_1:\mathbb{F}])O\right] \underset{rat}{\sim} 0
\end{array}$$

same $\mathbb{F}_3/\mathbb{F}_1/\mathbb{F}$

$\Rightarrow$ $P - [\mathbb{F}_1:\mathbb{F}]O = 0 \in CH_0(X)$

<u>Compare</u> : Kato-Saito

$X/\mathbb{F}$ smooth proj. abs irreducible

duality of finite groups.

$$A_0(X) \times H^1_{et}(X, \mathbb{Q}/\mathbb{Z})/H^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z})$$

$$\cap$$

$$H^1_{et}(\bar{X}, \mathbb{Q}/\mathbb{Z})^{Gal(\bar{\mathbb{F}}/\mathbb{F})}$$

If $\bar{X}$ SRC $\Longrightarrow$ $\pi_1(\bar{X}) = 0$

(Kollár)

Corollary of main theorem

k p-adic $\qquad k \supset R \twoheadrightarrow \mathbb{F}$

$$\begin{tikzcd} X \arrow[d, dash] & \mathcal{X} \arrow[d] & Y \arrow[d, dash] \\ k & R & \mathbb{F} \end{tikzcd}$$

$Y \to \mathbb{F}$ — smooth proj SRC

$\mathcal{X} \to R$ — smooth proj

$\# \mathbb{F} > \Phi(\deg Y, \dim Y)$

Then $\forall\, P, Q \in X(k)$

$\exists\, f : \mathbb{P}^1_k \longrightarrow X$ very free $\begin{cases} f(0) = P \\ f(\infty) = Q \end{cases}$

<u>Proof</u>: $\quad \underset{/k}{P} \leftarrow \underset{/R}{\tilde{P}} \longrightarrow \underset{/\mathbb{F}}{p} \qquad Q \leftarrow \tilde{Q} \longrightarrow q$

$S = \operatorname{Spec} R$

$$\operatorname{Mor}_S\left(\mathbb{P}^1_S, \mathcal{X}, \begin{matrix} \tilde{0} \to \tilde{P} \\ \tilde{\infty} \to \tilde{Q} \end{matrix}\right)$$

thm $\Rightarrow \exists\, f_0 \quad \mathbb{P}^1_{\mathbb{F}} \to Y \quad \begin{matrix} 0 \mapsto p \\ \infty \mapsto q \end{matrix}$ <u>very free</u>

$\to$ smooth pt of $\operatorname{Mor}_S(\ ) \to S$

S henselian: <u>lift</u>!

Cor. of corollary

k = number field

X/k smooth projective SRC

For almost all places v of k

$$X(k_v)/R = \{*\}$$

[Earlier known case:

smooth model of GL_n/G ← finite group.

Gille, Moret-Bailly]

Corollary

$$\begin{tikzcd} X \arrow[d] & \mathcal{X} \arrow[d] & Y \arrow[d] \\ k & R \arrow[l, phantom, "\supset"] \arrow[r, two heads] & \mathbb{F} \end{tikzcd}$$

smooth proj

Y SRC

Then X/k is SRC

and $A_0(X) = 0$

Pf: similar to that for $A_0(Y) = 0$.

$\longrightarrow$ via very free maps through two points.

Proof of weak form of Theorem

/k **any** field

X/k smooth proj SRC

$p \in X(k)$

FIRST FACT:

Exists V_p/k smooth geom. irreducible

and

$$F_p : V_p \times \mathbb{P}^1 \longrightarrow X \quad \text{(generically) smooth}$$

$$V_p \times 0 \longrightarrow p$$

$$V_p \times \infty \longrightarrow X \quad \text{smooth.}$$

$\forall\, r \in V_p \quad F_r : \mathbb{P}^1_{k(r)} \to X_{k(r)}$ very free.

SECOND FACT (Kollár)

(one Lefschetz type theorem for SRC varieties)

X/k smooth proj SRC

There exists U/k smooth geometrically integral

$$F : U \times \mathbb{P}^1 \longrightarrow X$$

$$\forall u \in U \qquad F_u : \mathbb{P}^1_{k(u)} \to X_{k(u)} \quad \text{very free}$$

such that:

$$\forall\ h : Y \longrightarrow X \,/k$$

generically smooth, with Y/k geometrically integral

$$\exists\ \emptyset \neq U^h \subset U, \quad \forall u \in U^h(\bar{k})$$

the fibre product

$$\begin{tikzcd} \square \arrow[r] \arrow[d] & \bar{Y} \arrow[d] \\ u \times \mathbb{P}^1 \subset \bar{U} \times \mathbb{P}^1_{\bar{k}} \arrow[r, "\bar{F}"] & \bar{X} \end{tikzcd}$$

is irreducible. (and reduced)

PREVIOUS RESULTS

in this direction

k alg closed X/k smooth proj SRC

- $\pi_1(X) = 0$
- If $U \subset X$ open, exists

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\varphi"] & X \\ \cup & \cup \\ \emptyset \neq \varphi^{-1}(U) \arrow[r] & U \end{tikzcd}$$

$$\pi_1(-) \twoheadrightarrow \pi_1(U)$$

i.e. $\begin{matrix} V & \text{connected} \\ \downarrow & \text{finite étale} \\ U & \end{matrix} \Rightarrow \begin{matrix} \varphi^{-1}(V) \text{ connected.} \\ \downarrow \\ V \end{matrix}$

- When k large field, and $X(k) \neq \emptyset$ similar result.

$\rightsquigarrow$ G finite group. is a galois group

$$G = \mathrm{Gal}(K/\mathbb{Q}_p(t))$$

with $\mathbb{Q}_p$ alg closed in K.

(Harbater, CT, Kollár, Moret-Bailly)

Proof, cont'd

X/k smooth, proj. SRC $\qquad P_1, P_2 \in X(k)$

$F_1 : V_1 \times \mathbb{P}^1 \to X \quad \left(\begin{array}{l} V_1 \times 0 \to P_1 \\ V_1 \times \infty \to X \text{ gen smooth} \end{array} \right.$

$F_2 \quad V_2 \times \mathbb{P}^1 \to X \quad \left(\begin{array}{l} V_2 \times 0 \to P_2 \\ V_2 \times \infty \to X \text{ gen. smooth} \end{array} \right.$

$F : U \times \mathbb{P}^1 \to X$ as in the "Lefschetz theorem"

Then:

$$\begin{tikzcd} Z_{1|\mathbb{P}^1} \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_1 \arrow[d, "F_1(\cdot,\infty) \text{ dominant.}"] \\ \mathbb{P}^1_{\bar{u}} \arrow[r] \arrow[d] & U \times \mathbb{P}^1 \arrow[r] \arrow[d] & X \\ \bar{u} \in & U & \end{tikzcd}$$

integral

(shrink U)

$\to$ (fibres of $\begin{array}{c} Z_1 \\ \downarrow \\ U \end{array}$ integral

Same for ($\begin{tikzcd} Z_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_2 \arrow[d, "F_2(\cdot,\infty)"] \\ U \times \mathbb{P}^1 \arrow[r] \arrow[d] & X \\ U & \end{tikzcd}$

- $Z_1 \subset U \times \mathbb{P}^1 \times V_1$

$$(u, t, v_1) \;\Big|\; F(u,t) = F_1(v_1, \infty))$$

if $F(u, \cdot) : \mathbb{P}^1 \longrightarrow X$ embedding (many curves)

t determined by above equation

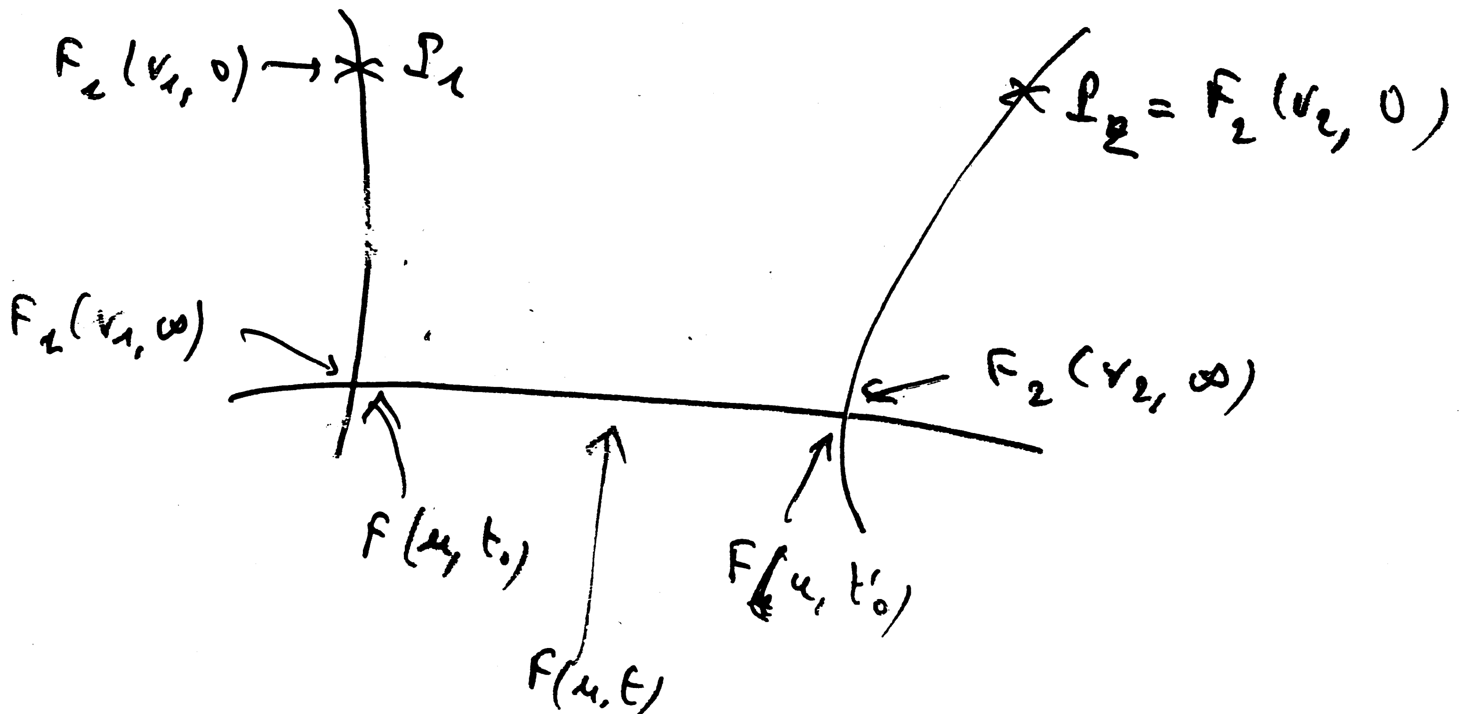
So $Z_1 \subset U \times V_1$

$$(u, v_1) \quad \exists\, t \quad f_u(t) = F_1(v_1, \infty)$$

- Same $Z_2 \subset U \times V_2$

Now

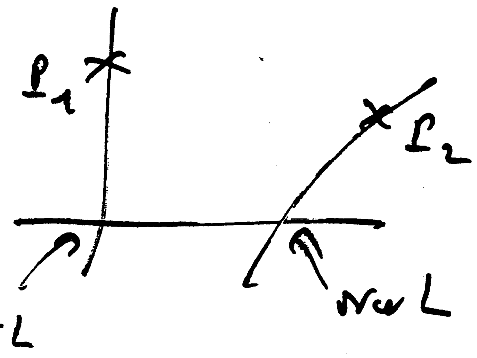
$$Z_1 \times_U Z_2 = \left\{ u, v_1, v_2 \;\middle|\; \begin{array}{l} \exists\, t_0 \;\; F(u, t_0) = F_1(v_1, \infty) \\ \exists\, t'_0 \;\; F(u, t'_0) = F_2(v_2, \infty) \end{array} \right.$$



→ any L-pt of the k-variety

$$W = Z_1 \times_U Z_2$$

defines a triple of very free $\mathbb{P}^1_L \to X_L$ with the property



→ on the other hand $\quad Z_1 \times_U Z_2$ ← fibers geometrically integral
$\downarrow$
$U \quad\quad U/k$ geometrically integral

$\Rightarrow Z_1 \times_U Z_2 = W$ geometrically integral.

If $k = \mathbb{E}$ infinite algebraic ext. of finite field, any open set $\neq \emptyset$ of a geometrically integral variety $/\mathbb{E}$ has a rational point (Lang-Weil)

→ $P_1 \sim P_2$ over $\mathbb{E}$

Can actually smooth the $\not\!\!\times$ to a free $\overset{\times P_1}{\underset{\times P_2}{|}}$

Recent results of Kollár (2004)

R henselian DVR $\qquad K \supset R \twoheadrightarrow \mathbb{k}$ $\leftarrow$ any field

$$\begin{array}{ccc} X & \mathcal{X} & Y \\ | & \downarrow & | \\ K & R & \mathbb{k} \end{array} \quad \text{smooth proj} \qquad Y_{/\mathbb{k}} \ \text{SRC}$$

Then: (a) $X(K)/R \xrightarrow{\text{spec.}} Y(\mathbb{k})/R$ bijection

(b) $CH_0(X) \xrightarrow[\text{bijection}]{\text{spec}} CH_0(Y)$

- onto: obvious
- injectivity: need to pass from R-equivalence (on Y) to very free R-equivalence in order to be able to lift.

<u>Thm</u> (Kollár)

k large field $\quad X/k$ smooth proj SRC

$P_1, \dots, P_n \in X(k)$ all R-equivalent

then $\exists$ $f : \mathbb{P}^1_k \longrightarrow X$ very free

all $P_i \in f(\mathbb{P}^1(k))$

<u>Cor.</u> $U \subset X$ open (k large) $\Rightarrow U(k)/R \hookrightarrow X(k)/R$ <u>injects</u>

If k p-adic $\quad U(k)/R \xrightarrow{\sim} X(k)/R$

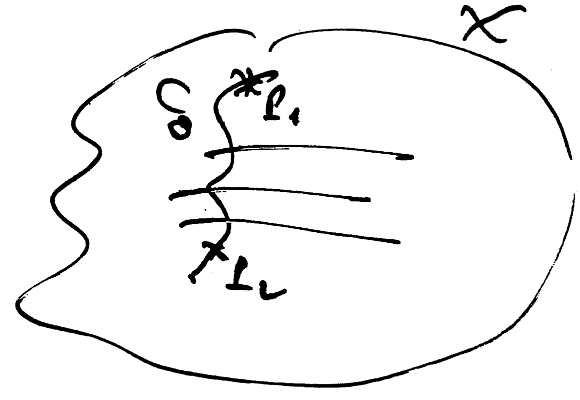
<u>Again</u>: needs to pass from R-equivalence to very free R-equivalence

R-equivalence $\rightsquigarrow$ very free R-equivalence

this is what one does / $k = \bar{k}$ (say char = 0)

for the RCC $\Longrightarrow$ RC theorem.

One starts with a non free



then one adds very free tails

$\rightsquigarrow$ new curve C.

One does not reach $H^1(C, f^*T_X(-2)) = 0$

indeed $f^*T_X(-2) \twoheadrightarrow f_0^*T_X(-2) \to 0$

so $H^1(\quad) \twoheadrightarrow H^1(\quad)$

the deformation is obtained via a dimension argument.

In the process, one looses some teeth

in a non controlled way

$\longrightarrow$ very bad for getting finally / $k \neq \bar{k}$.

New idea in recent paper by Kollár

comes from the proof of the

Graber–Harris–Starr (+ de Jong)

theorem.

$k = \bar{k}$

$$\begin{array}{c} X \\ \downarrow \\ C/\bar{k} \end{array}$$ curve

$X_\eta / k(C)$ SRC

$\Rightarrow \exists$ <u>section</u> $\sigma \, \begin{array}{c} X \\ \downarrow \\ C \end{array}$. (generalizes Max Noether, .., Tsen)

discovered that it is better to

deform curves as points in the Hilbert scheme.

here one replaces C_0 by C

wants

$$H^1(C, \mathcal{O}_C(-\ldots) \otimes \mathcal{N}_C) = 0$$

normal bundle.

Open questions

k p-adic X/k smooth proj. SRC

Is $A_0(X)$ finite ?

$\Leftrightarrow$ $\exists N > 0$ any zero-cycle on X of degree $\geq N$ is rat. equivalent to effective cycle ?

Known for $\dim X = 2$ (1983) K-theory

Some cases in higher dimension.

k number field X/k smooth proj SRC

Is $A_0(X)$ finite ? Known if $\dim X = 2$ (1983)

Is $X(k)/R$ finite ? Unknown even if $\dim X = 2$ (except very special cases) !

$k = \mathbb{Q}(t), \mathbb{R}(t), \mathbb{Q}_p(t), \mathbb{Q}_p((t)), \mathbb{R}((t))$ $X(k)/R$ may be infinite ! (Kollár)