 JANNISEN $\cancel{II}^{o}$ lecture

the reduced motive of (X, p) (depends on p).

Define the <u>homological</u> Tate motive

$$1(1) = \tilde{M}(\mathbb{G}_m)[-1]$$

where the rational point is $1 \in \mathbb{G}_m(k) = k^\times$.

Then define $DM_{gm}(k)$, the triang. category of geometric motives over k, by inverting the Tate motive in $DM^{eff}_{gm}(k)$.

This is a pseudo-abelian triangulated tensor category (one checks that the commutg ~~associa~~tivity constraint on $DM^{eff}_{gm}(k)$ for $1(1) \otimes 1(1) \Rightarrow 1(1) \otimes 1(1)$ is the identity).

Let $X, Y \in P(k)$ be irreducible.

Then one gets a diagram

$$\begin{tikzcd}
 & \operatorname{coker}\beta \arrow[r, "\alpha"] & CH^{\dim Y}(X\times Y) \overset{t}{=} \operatorname{Corr}^0(Y,X) \\
\operatorname{Hom}([X],[Y]) = & c(X,Y) \arrow[r, hook] \arrow[u, two heads] & Z^{\dim Y}(X\times Y) \arrow[u, two heads] \\
 & c(X\times \mathbb{A}^1, Y) \arrow[u, "\beta = s_0^* - s_1^*"] &
\end{tikzcd}$$

Friedlander / Voevodsky: α is an isomorphism

Hence get

$$\begin{tikzcd}
P(k) \arrow[r, "op"] \arrow[d, hook] & C_{rat}(k) \arrow[r, hook] \arrow[d, hook, "``\alpha" op"] & M_{rat}^{eff}(k) \arrow[d, hook, "op"] \\
Sm(k) \arrow[r] & Cor(k) \arrow[r] & H^b(Cor(k))/T \arrow[r] & D_{gm}^{eff}(k)
\end{tikzcd}$$

Fact: "α" is a full embedding

Gysin triangle

$$M(\mathbb{G}_m) \to M(\mathbb{A}^1(0)) \oplus M(\mathbb{A}^1(\infty)) \to M(\mathbb{P}^1_k) \to M(\mathbb{G}_m)[1]$$

gives

$$\tilde{M}(\mathbb{P}^1_k) \cong \tilde{M}(\mathbb{G}_m)[1] \cong 1(1)[2]$$

Hence $L \in CHM^{eff}(k) = M^{eff}_{rat}(k)$ is mapped to $1(1)[2] \in D^{eff}_{gm}(k)$.

easy: $M \mapsto M \otimes L$ is fully faithful

difficult: $N \mapsto N \otimes 1(1)$ is fully faithf.

Result: have fully faithful

$$\begin{tikzcd}
CHM^{eff}(k) = M^{eff}_{rat}(k) \arrow[r, hook] \arrow[d, hook] & M_{rat}(k) = CHM(k) \arrow[d, hook] \\
D^{eff}_{gm}(k) \arrow[r, hook] & D_{gm}(k)
\end{tikzcd}$$

§4 Nisnevich sheaves with transfers and $DM^-(k)$

go on!

$$\begin{tikzcd}
Sm(k) \arrow[r] \arrow[d] & Cor(k) \arrow[r] \arrow[d, hook] & H^b(Cor(k)) \arrow[r] \arrow[d] & DM_{gm}^{eff}(k) \arrow[r] \arrow[d, hook] & DM_{gm}(k) \arrow[d, hook] \\
PST(k) \arrow[r, shift left] & NIST(k) \arrow[l, shift left] \arrow[r] & D^-(NIST(k)) \arrow[r] & DM^-_{eff}(k) \arrow[r] & DM^-(k)
\end{tikzcd}$$

$PST(k)$ = presheaves with transfers over k

= contravariant additive functors from $Cor(k)$ to Ab

Fact: Yoneda embedding

$$\begin{aligned} Cor(k) &\hookrightarrow PST(k) \\ X &\mapsto L(X) = Hom(-,[X]) \end{aligned}$$

$$(\text{so } L(X)(Y) = c(Y,X)) \qquad Hom(-,[X]) = \mathbb{Z}_{tr}(X)$$

has image in

$NIST(k)$ = Nisnevich sheaves with transfers over k

= $\mathcal{F}$ in $PST(k)$ which restricted to $Sm(k)$ are sheaves for the Nisnevich topology*

(Also: $\mathcal{F} \in PST(k) \Rightarrow$ associated Nisnevich sheaf $\tilde{\mathcal{F}}$ has natural structure of transfers, <u>hence is in</u> $NIST(k)$)

$NIST(k)$ is an <u>abelian category</u>; hence can form the bounded above derived category $D^-(NIST(k))$

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* coverings are families $(U_i \to X)_{i \in I}$ of étale morphisms $U_i \to X$ such that for every $x \in X$ $\exists i$ and $y \in U_i$: $y \mapsto x$ and $k(x) \xrightarrow{\sim} k(y)$

Let $\mathcal{A}$ be the smallest triangulated subcategory of $D^-(NST(k))$ containing all complexes

$$L(X \times \mathbb{A}^1_k) \xrightarrow{pr_X} L(X)$$

for $X \in Sm(k)$ and being closed under forming direct sums and direct summands.
$\Rightarrow$ thick subcategory

$$D^-_{eff}(k) := D^-(NST(k)) / \mathcal{A}$$

(Fact: equivalent to subcategory of complexes in $D^-(NST(k))$ with homotopy invariant homology sheaves)

$D_{eff}(k)$: invert $\mathbb{1}(1)$, i.e., $\widetilde{L(\mathbb{P}^1)}$ (reduced part).

MV property: $0 \to L(U \cap V) \to L(U) \oplus L(V) \to L(U \cup V) \to 0$
ex. sequ. of sheaves

$\Rightarrow$ by def. of $\mathcal{T}$ (cf. above) get $D^{eff}_{gm}(k) \to D^-_{eff}(k)$

§5 Around the standard conjectures

Why is $\mathcal{M}_\sim(k)$ rigid?

Let $X \in P(k)$ be irreducible, of dimension d, and $Y \in P(k)$ arbitrary.

Claim: $\underline{\mathrm{Hom}}(h(X), h(Y)) = h(X \times Y)(d)$

$$= h(X) \otimes h(Y)(d)$$

(in particular, $h(X)^\vee = h(X)(d)$)

Proof: If $T \in P(k)$ is irreducible, $f = \dim(T)$

$$\mathrm{Hom}(h(T) \otimes h(X), h(Y)) = \mathrm{Hom}(h(T \times X), h(Y))$$

$$= A_\sim^{f+d}(T \times X \times Y) = \mathrm{Hom}(h(T), h(X \times Y)(d))$$

General formula:

$$\mathrm{Hom}((X,p,i),(Y,q,j)) = q\,\mathrm{Hom}(h(X)(i), h(Y)(j))\,p$$

$$= q\, A_\sim^{\dim(X)+j-i}(X \times Y)\, p \quad .$$

<u>Def.</u> A generalized Weil cohomology theory over k with coefficient field E is a tensor functor

$$H\ \cancel{\Phi} : \mathcal{M}_{rat}(k) \longrightarrow Gr_E = \begin{array}{c}\underline{\text{finite-dim.}}\\ \text{graded } E\\ \text{vector spaces}\end{array}$$

respecting duals, for which $\cancel{\Phi}$

$H(X) := H(h(X))$ is in non-negative degrees for all $X \in P(k)$, and $H(L)$ in degree 2.

$\leadsto$ gives 'classical' notion: $H(X) = \bigoplus_{i \geq 0} H^i(X)$, $H(L) = E[-2] = E$ placed in degree 2, since it is invertible,

$$H(X)^\vee = H(X)(d) \text{ is in degrees } \leq 0$$
$$= H(X)[2d] \qquad d = \dim(X)$$

$\Rightarrow$ $H(X)$ in degrees $0, \dots, 2d$

<u>algebra, graded commutative</u>

$$\begin{array}{ccc} H(X) \otimes H(X) & \xrightarrow{\Delta^*} & H(X) \\ & & \big\downarrow f_* \\ & & E[-2d] \end{array}$$

<u>non-deg. pairing</u>:

$f: X \to \operatorname{Spec} k$

$H(\operatorname{Spec} k) = H(1) = E[0]$

$g: X \to Y$: $\quad g^*: h(Y) \to h(X)$

$$g_*: h(X)(\dim X) \to h(Y)(\dim Y)$$

<u>cycle classes</u>:

$$A^j_{rat}(X) = \operatorname{Hom}(1, h(X)(j))$$
$$\downarrow$$
$$H^{2j}(X) = \operatorname{Hom}(E[0], H(X)[2j])$$

More generally replace Gr_E by rigid abelian $\mathbb{Z}$-graded tensor category

$\rightsquigarrow$ cycle classes $1 \longrightarrow H^{2j}(X)(j)$

Can formulate standard conjectures in this setting

I. $C(X)$ algebraicity of Künneth components

II. $B(X)$ " " $*_L$

III. $I(X)$ positivity of $\langle \cdot, *_L \cdot \rangle$ on <u>algebraic cycles</u>

IV. $D(X)$ $\sim_{num} = \sim_H$

$B + I \Rightarrow D \Rightarrow \mathbf{B} \Rightarrow \mathbf{C}$ if have hard Lefschetz

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Remark: In char $p > 0$, I necessary?

<u>Remarks</u> (a) C holds (for some Weil cohom. H) $\Rightarrow$ can modify commutativity constraints on $\mathcal{M}_{num}(k)_{\mathbb{Q}}$

$\leadsto$ get fibre functor (faithful!)

Deligne's thm. + semi-simplicity (J.)

$$\phi : \dot{\mathcal{M}}_{num} \longrightarrow Vec_E$$

for some field E of char 0.

$\leadsto$ modifying back & using Künneth projectors, get Weil cohomology

$$(\phi \Rightarrow) \quad H' : \mathcal{M}_{num} \longrightarrow Gr_E$$

for which D ($\sim_{num} = \sim_{H'}$) holds.

(b) But it is not clear if B holds: not clear if

$$L^{d-2i} : A^i_{num}(X)_{\mathbb{Q}} \longrightarrow A^{d-i}_{num}(X)_{\mathbb{Q}}$$

is <u>injective</u>.

But if B holds (for original H), it als holds for H' (one has

$$L^{d-j} : h^j(X)(-d+j) \xrightarrow{\sim} h^{2d-j}(X)$$

mod $\sim_H$, hence mod $\sim_{num}$), and also hard Lefschetz holds for H and H'.

(l) If H_1 and H_2 are Weil cohomology theories for which the standard conjectures B and D hold, then they give rise to fibre functors over the coeff. fields E_1, E_2, on $\mathcal{M}_{num}(k)$. Since two such functors are isomorphic in the fppf topology, there is a common field extension E of E_1 and E_2, and a 'comparison isomorphism'

$$H_1^*(X) \otimes_{E_1} E \xrightarrow{\sim} H_2^*(X) \otimes_{E_2} E$$

§6 Different adequate equivalence relations - and finite dimensional motives

$$\sim_{rat} \Rightarrow \sim_{alg} \Rightarrow \sim_{\wedge} \Rightarrow \sim_{hom} \Rightarrow \sim_{num}$$

(smash nilpotence (⊗ ⊗))

$$CH^j(X)_\sim = \{ z \in CH^j(X) \mid z \sim 0 \}$$

$$\leadsto A^j_\sim(X) = CH^j(X) / CH^j(X)_\sim$$

$E = \mathbb{Z}$

also with coefficient in a ring E

$CH^j(X,E)_\sim$, $A^j_\sim(X,E)$

Facts / Conjectures

1) • $A^j_{num}(X)$ is a finitely generated abelian group !

• $A^j_{num}(X) \otimes_{\mathbb{Z}} E \xrightarrow{\sim} A^j_{num}(X,E)$

- $\mathcal{M}_{num}(k)_{\mathbb{Q}}$ is semi-simple abelian (J.)

2) • If E is a field of characteristic 0 and H is a Weil cohomology theory with coefficients in E, then

$$A^j_{hom}(X) \otimes_{\mathbb{Z}} E \twoheadrightarrow A^j_{hom}(X, E)$$

- not known to be injective (except for $E = \mathbb{Q}_\ell$) (this would follow from standard conj. $D(X)$: $\sim_{hom} = \sim_{num}$ for H)

- $A^j_{hom}(X) \otimes \mathbb{Q}$ not known to be finite-dim'l

- But $\{x \in A^j_{hom}(X) \otimes \mathbb{Q} \mid x \sim_{num} 0\} \subseteq A^j_{hom}(X) \otimes \mathbb{Q}$ is a nilpotent ideal if $C(X)$ holds (J.) (in fact, if $S(X)$ holds, cf. below)

3) <u>Def.</u> (Voevodsky) A cycle $z \in CH^j(X)$ is called <u>smash nilpotent</u> ($z \sim_\wedge 0$) if there is some integer $n \geq 0$ such that $\underbrace{z \times \ldots \times z}_{n \text{ times}}$ is $\sim_{rat} 0$ on $\underbrace{X \times \ldots \times X}_{n \text{ times}}$.

- <u>Lemma</u> $\sim_\wedge$ is an adequate equivalence relation
- Obvious: $\sim_\wedge \;\Rightarrow\; \sim_{hom}$ (any Weil coh.)

(In a vector space V, $x \otimes \ldots \otimes x = 0 \Rightarrow x = 0$)

- <u>Conj.</u> (Voevodsky) $\sim_\wedge \; = \; \sim_{num}$

(would imply $\sim_{hom} = \sim_{num}$!) algebraic approach

- <u>Thm</u> (V.) $\sim_{alg} \;\Rightarrow\; \sim_\wedge$ <u>for $\mathbb{Q}$-coeff.</u>

  V. = Voisin + Voevodsky

$z \sim_{alg} 0 \iff z$ comes, via algebraic correspondence, from $z' \in Pic^0(C)$, C curve

$\leadsto$ suffices to show $z' \sim_\wedge 0$; show $z'^{\wedge 2g} = 0$

$g = \text{genus}(C)$

4) $CH^j(X)_{hom} / CH^j(X)_{alg}$ is countable (theory of Hilbert schemes) but can be infinite-dimensional after $\otimes \mathbb{Q}$ (Griffiths, Clemens)

5) $CH^j(X)_{alg}$ is usually huge - already for $j = 1$: $Pic^0(X)$; it is the "continuous part".