**Def.:** (Kimura, O'Sullivan) Let $\mathcal{C}$ be a rigid pseudo-abelian $\mathbb{Q}$-linear $\otimes$-category. An object $M$ in $\mathcal{C}$ is called finite dimensional, if there is a decomposition

$$M = M_+ \oplus M_-,$$

and integers $m, n \geq 0$ such that

$$\Lambda^{m+1} M_+ = 0 \qquad \mathrm{Sym}^{n+1} M_- = 0.$$

In this case

$$\left\{ \begin{matrix} M_+ \\ M_- \end{matrix} \right\} \text{ is called } \left\{ \begin{matrix} \text{evenly} \\ \text{oddly} \end{matrix} \right\} \text{ finite dimensional}$$

and
$$\dim M_+ = \min\{ m \mid \Lambda^{m+1} M_+ = 0 \}$$
$$\dim M_- = \min\{ n \mid \mathrm{Sym}^{n+1} M_- = 0 \}$$
are called the dimensions of $M_+$ and $M_-$, respectively. Call $\dim M = \dim M_+ + \dim M_-$ the dimension of $M$.

This makes sense, because $M_+$ and $M_-$ are well-defined up to (non-canonical) isomorphism, by the following:

<u>Def.</u> (Voevodsky) A morphism $\alpha : M \to N$ in $\mathcal{C}$ is called smash nilpotent, if $0 = \alpha^{\otimes n} : M^{\otimes n} \to N^{\otimes n}$ for some $n \geq 1$.

<u>Lemma</u> (V.) Every smash nilpotent endomorphism $\alpha : M \to M$ is nilpotent. (more precisely: $\alpha^{\otimes n} = 0 \Rightarrow \alpha^n = 0$)

<u>Lemma</u> (Kimura) Every morphism $f : M_+ \to M_-$ or $g : M_- \to M_+$ is smash nilpotent. (precisely: $f^{\otimes(n+1)} = 0$ for $n = \dim M_+ \cdot \dim M_-$)

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$\times$ or $\otimes$-nilpotent

For an algebraic cycle $z \in CH^j(X)_{\mathbb{Q}}$, regarded as a morphism $z : 1 \to h(X)(j)$ one gets the previous notion, because $z^{\otimes n} = z \times \dots \times z$.

In the following, always $\mathbb{Q}$-coefficients.

**Conjecture** (Kimura, O'Sullivan) Every motive $M \in CHM(k) = \mathcal{M}_{rat}(k)$ is finite-dimensional.

**Theorem** (Kimura) If $C$ is a smooth, projective curve, then $h(C) \in CHM(k)$ is finite dimensional.

**Theorem** Let $\mathcal{C}$ be a pseudo-abelian tensor category. For $M_1, M_2$ in $\mathcal{C}$ we have:

(a) $M_1, M_2$ finite dimensional $\Rightarrow$ $M_1 \oplus M_2$ finite-dimensional with $\dim(M_1 \oplus M_2) \leq \dim M_1 + \dim M_2$.

(b) $M_1, M_2$ finite dimensional $\Rightarrow$ $M_1 \otimes M_2$ finite-dimensional with $\dim(M_1 \otimes M_2) \leq \dim M_1 \cdot \dim M_2$.

[
$$\begin{array}{l} \text{even} \\ \text{odd} \end{array} \oplus \begin{array}{l} \text{even} \\ \text{odd} \end{array} = \begin{array}{l} \text{even} \\ \text{odd} \end{array}$$

$$\text{even} \otimes \text{odd} = \text{odd}$$

$$\text{even} \otimes \text{even} = \text{even} = \text{odd} \otimes \text{odd}$$

Important application of finite-dimensionality

<u>Theorem</u> (Kimura) Let $M \in CHM(k)$ be finite dimensional, and let $\alpha : M \longrightarrow M$ be an endomorphism.

(a) There is a polynomial $f(T) \in \mathbb{Q}[T]$ with $f(\alpha) = 0$.

(b) If $\alpha \sim_{num} 0$, then $\alpha$ is nilpotent.

---

refinement:

<u>Theorem</u> Let $M$ and $\alpha : M \to M$ be as above. Let $H$ be any Weil cohomology theory. Assume $M$ is either evenly or oddly finite dimensional.

(a) $\dim M = b := \dim H(M)$.

(b) Let $f(T)$ be the characteristic polynomial of $\alpha$ on $\oplus H^i(M)$. Then $f \in \mathbb{Q}[T]$, and $f(\alpha) = 0$ in $\mathrm{End}_{CHM(k)}(M)$.

(c) If $\alpha \sim_{num} 0$, then $\alpha^{b} = 0$.

<u>Proof</u>: (sketch) (b) $\Rightarrow$ (c) ; (c) $\Rightarrow$ (a):

Let $M_{hom}$ be the motive associated to $M$ by passing to homological equivalence, i.e., the image under the functor $\mathcal{M}_{rat}(k) \to \mathcal{M}_{hom}(k)$.
(c) ~~implies~~ says that $\ker(\mathrm{End}(M) \to \mathrm{End}(M_{hom}))$ is ~~nilpotent~~ a nil ideal. This implies

$$M_{hom} = 0 \Rightarrow M = 0$$

($M$ is not a "phantome motive")*
The same holds for $M^{\otimes n}$, all $n$, and all factors of this. Hence

$$\mathrm{Sym}^n M_{hom} = 0 \iff \mathrm{Sym}^n M = 0,$$

similarly for $\mathrm{Alt}^n$. But obviously,

$$\dim M_{hom} = b,$$

hence $\dim M = b$.

---

* Of course, the same holds for "num" instead of "hom"

(b) : Kimura : $\exists$ correspondence $\Gamma_\alpha$ such that, for every $\beta : M \to M$

$$\Gamma_\alpha\Big(\underbrace{\sum_{\sigma\in S_{n+1}} \mathrm{sig}(\sigma)\,\sigma\circ\beta^{\otimes(n+1)}}_{0 \text{ if } \Lambda^{n+1}M=0}\Big) = \Big(\underbrace{(-1)^n\, n!\,(\beta\alpha)^n + \underbrace{\cdots}_{(*)}}_{g(T) \,=\, \text{polynomial in } \beta\alpha,\ \text{coeff. in } \mathbb{Q},\ \deg n}\Big)\beta$$

(i) $(*)$ has degree $\leq n-1$

(ii) $(*) = 0$ if $\beta\alpha \sim_{num} 0$

Now take $\beta = id$ ! $\Rightarrow$ $g(\alpha) = 0$

( Kimura :: with roles of $\alpha$ and $\beta$ interchanged

(ii) $\Rightarrow$ $\alpha^n = 0$ if $\alpha \sim_{num} 0$

$\underset{\text{as above}}{\Longrightarrow}$ $\dim M = b$ $\Rightarrow$ can take $n = b$

On $H^\cdot(M)$ one has $g(\alpha) = 0$ and $f(\alpha) = 0$

( $f$ = char. polyn. of $\alpha$ on $H^\cdot(M)$ )

Now both polynomials are of degree $b$, so we can conclude $f = g$ if all eigenvalues of $\alpha$ on $H^{\bullet}(M)$ are different. One has to check a universal formula for endomorphisms of vector spaces $\Rightarrow$ done by continuity argument (ugly ...)

---

For a motive $M$, write $\mathcal{J}(M) \subseteq \mathrm{End}(M)$ for the ideal of endomorphisms $\alpha$ with $\alpha \sim_{num} 0$. For $X$ smooth, projective write $\mathcal{J}(X) = \mathcal{J}(h(X))$.

---

<u>Corollary</u>. If $M$ is 1-dimensional (= finite dimensional and of dimension 1), then $\mathcal{J}(M) = 0$.

<u>Corollary</u> Let $M$ be a finite dimensional motive.

(a) $\mathcal{J}(M)$ is a nilpotent ideal.

(b) $M_{num} = 0 \Rightarrow M = 0$.

<u>Proof</u> (a) : If $M$ is evenly or oddly finite-dimensional, we have seen there exists an $n \geq 1$ such that $a^n = 0$ for all $a \in J(M)$. Since $End(M)$ is a $\mathbb{Q}$-algebra, $J(M)$ is nilpotent, by a result of Nagata Higman.* (a) $\Rightarrow$ (b) as before.

---

For a given Weil cohomology theory $H$, let $\pi_+^X$ and $\pi_-^X$ be the projectors onto the even and odd part of the cohomology, respectively, for $X \in P(k)$.

Consider the following "sign conjecture"

$S(X)$ : $\pi_+^X$ and $\pi_-^X$ are algebraic.

* easy : passes to sums

(as elements modulo $\sim_{..}$ )

Moreover consider the conjectures

$N(X)$: $\mathcal{J}(X) \subseteq \mathrm{End}(h(X))$ is a nilpotent ideal

$N'(X)$: $\mathcal{J}(X)$ is a nil ideal

Then we can show the following 'converse' of Kimura's nilpotence theorem:

<u>Proposition</u> For a smooth projective variety $X$ the following conditions are equivalent:

(a) $h(X)$ is finite-dimensional.

(b) $S(X)$ holds, and $N(X^n)$ holds for all $n \geq 1$.

(c) $S(X)$ holds, and $N'(X^n)$ holds for all $n \geq 1$.

<u>Proof</u>: If $h(X) = M_+ \oplus M_-$ with $M_+$ evenly and $M_-$ oddly fin. dim.,

then $H(M_\pm) = H^{\substack{even\\odd}}(X)$ and hence the decomposition corresponds to $\pi_+$ and $\pi_-$ modulo $\sim_H$. Therefore (a) implies $S(X)$, and we have seen that (a) implies $N(X)$. Since $h(X^n) = h(X)^{\otimes n}$ is fin. dim. as well, we also get $N(X^n)$

(b) $\Rightarrow$ (c) is trivial

(c) $\Rightarrow$ (a): $S(X) \Rightarrow \pi_\pm$ are algebraic

$N'(X) \Rightarrow$ they lift to projectors modulo rational equiv., again denoted $\pi_\pm$.

Let $M_\pm = (X, \pi_\pm, 0)$ modulo $\sim_{rat}$.

For $b_\pm := H(M_\pm) = H^{\substack{even\\odd}}(X)$ we have

$\Lambda^{b_+ + 1} M_+ = 0 = \mathrm{Sym}^{b_- + 1} M_-$ modulo $\sim_{hom}$, hence also mod $\sim_{rat}$, because of $N'(X^n)$ for $n = b_+ + 1$ and $n = b_- + 1$.

**Thm.** (Guletskii) If $h(X)$ is Schur finite, ~~and~~ then $N'(X)$ holds.

Since Schur finiteness also extends to tensor products, we get:

**Corollary** The conditions of the previous proposition are further equivalent to:

(d) $S(X)$ holds and $h(X)$ is Schur finite.

(Note: $M$ finite-dimensional $\Rightarrow$ $M$ Schur finite)

---

Schur finiteness is useful, because it behaves well for exact triangles in triangulated tensor categories (Mazza).
Guletskii

Note also :

Voevodsky's nilpotence conjecture $\sim_\wedge = \sim_{num}$ ((for $X$))

$$\Rightarrow \begin{cases} \sim_{hom} = \sim_{num} \Rightarrow C(X) \Rightarrow S(X) \\ N'(X) \end{cases}$$

$\Rightarrow$ finite dimensionality of $h(X)$

---

<u>Thm.</u> (O'Sullivan) Voevodsky's nilpotence conj.

$\Leftrightarrow$ i) finite dimensionality conjecture

ii) Every $\otimes$-functor $CHM(k) \longrightarrow sVec_E$ (char $E=0$) factors through $\sim_{num}$

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(- Murre

# §7 The Bloch-Beilinson(-Murre) filtration

always $CH(\;)$ with $\mathbb{Q}$-coefficients

<u>Conjecture</u> (Bloch-Beilinson) There exists a descending filtration $CH^j(X) = F^0 \supseteq F^1 \supseteq F^2 \supseteq \dots$ on the Chow groups of smooth proj. varieties $X/k$ such that

(a) $F^1 = \{ z \mid z \sim_{hom} 0 \}$

(b) $F^r . F^s \subseteq F^{r+s}$ under intersection

(c) $F^r$ is respected by pull-backs $f^*$ and push-forwards $f_*$ for $f: X \to Y$

(d) $Gr_F^\nu CH^j(X) = F^\nu CH^j(X) / F^{\nu+1} CH^j(X)$ only depends on $h^{2j-\nu}(X)$ mod $\sim_{hom}$.

(e) $F^\nu CH^j(X) = 0$ for $\nu \gg 0$

---

(a) - (c) $\Rightarrow$ alg. correspondences act on $F^\nu$, and on $Gr_F^\nu$ via $Gr_F^0$, hence modulo $\sim_{hom}$!

(d): assume algebraicity of $\pi_i$ mod $\sim_{hom}$.

<u>Suppose</u> there exists an abelian category $MM(k)$ of mixed motives containing $M(k) = M_{hom}(k)$ as a full category and such that $DM_{gm}(k) = D^b(MM(k))$ *

Then one would get such a filtration, because then

$$H^i_M(X, \mathbb{Q}(j)) = \mathrm{Hom}_{D^b(MM(k))}(1, H(X)(j)[i])$$

$$\Big( = \mathrm{Hom}_{DM_{gm}(k)}(M(X), \mathbb{Z}(j)[i]$$

in Voevodsky's homological notation $\Big)$

and the canonical filtration would induce $F^\cdot$ on $H^{2j}_M(X, \mathbb{Q}(j)) = CH^j(X)_{\mathbb{Q}}$ with

---

* It would suffice to have a suitable t-structure on $DM_{gm}(k)$ with heart $MM(k)$

$$\mathrm{Gr}_F^\nu \, CH^j(X) = \mathrm{Ext}^\nu_{MM(\mathbb{Q})}\left(1, h^{2j-\nu}(X)(j)\right)$$

---

$F^r \circ F^s \subseteq F^{r+s}$ under composition, so the axioms (a) and (e) would imply $N(X)$ ($F^1 = \bigoplus_j CH^j(X)_{hom}$ is a nilpotent ideal), and together with $C(X)$ we would get finite-dimensionality of motives.

---

One instance:

<u>Thm.</u> Let $F$ be a finite field, let $C$ be an elliptic or rational curve over $F$, and let $X$ be a <u>product of elliptic curves</u> over $F$. Let $k = F(C)$ (global function field).

Then the Tate conjecture holds for $X_k = X \times_F k$, and the Bloch-Beilinson filtration exists on the products $X_k^n$.

---

Remark: B-B filtr. is unique when it exists.

Tools: • Tate conj.: M. Spieß

• use finite dimensionality for $X \times C$ and $X$ over $F$ + ideas of Geisser and Kahn

• use regulator map

$$H^i_M(X_k, \mathbb{Q}(j)) \otimes \mathbb{Q}_\ell \longrightarrow \underbrace{\tilde{H}^i_{et}(X_k, \mathbb{Q}_\ell(j))}_{\text{modified étale cohomology}}$$

show: iso for all $i \leq 2j$

$\Rightarrow$ $F^0 \supseteq F^1 \supseteq F^2 = 0$ by Hochschild-Serre spectral sequence