Corollary (characterisation of algebraic spaces)
Let $F : (\text{Schemes})^{\circ} \to (\text{Sets})$ be a functor.
Then:
F is an algebraic stack $\Leftrightarrow$ F is an algebraic space

Proof: $\Leftarrow$ trivial
$\Rightarrow$ trivial if F is a Deligne-Mumford stack.

In general, if F is an algebraic stack and a presheaf, the diagonal Δ_F is a monomorphism, hence unramified. By the preceding theorem, F is a D-M stack, so we are done ■

Corollary.
- $\mathcal{M}_{g,n}$ is a D-M. stack $\Leftrightarrow$ $2g-2+n>0$
  $\Leftrightarrow (g,n) \notin \{(0,0),(0,1),(0,2),(1,0)\}$
- $\mathcal{M}_{g,n}$ is an algebraic space $\Leftrightarrow n > 2g+2$. ■

Remark: To prove that $\mathcal{M}_{g,0}$ is a Deligne-Mumford stack for $g \geq 2$, one can use the moduli scheme $\mathcal{M}_{g,0}^{(r)}$ of curves with level-r structure over $\mathbb{Z}[1/r]$ $(r \geq 3)$:

$$U \longmapsto \left\{ \begin{array}{l} \text{curves } p: X \longrightarrow U \ + \\ \text{isomorphism } (\mathbb{Z}/r\mathbb{Z})_U^{2g} \xrightarrow{\sim} R^1 p_* (\mathbb{Z}/r\mathbb{Z}) \end{array} \right\}$$

This approach is less elementary, but gives a bonus: the natural morphism

$$\mathcal{M}_{g,0}^{(r)} \longrightarrow \mathcal{M}_{g,0} | \operatorname{Spec} \mathbb{Z}[1/r]$$

is finite étale.

QUOTIENT STACKS

Assume: S is a scheme,

$X \to S$ an algebraic space

$G \to S$ a sheaf of groups which is a smooth, separated, S-algebraic space of finite type acting on X

We define the quotient stack $\mathcal{M} = [X/G]$, over S, as follows:

(S-scheme U) $\longmapsto$ $\mathcal{M}(U)$ = category of commutative diagrams

$$\begin{tikzcd} \tilde{U} \arrow[rr, "h"] \arrow[dr] & & X_U = X \times_S U \arrow[dl] \\ & U & \end{tikzcd}$$

where $\tilde{U} \to U$ is a G-torsor over U

$h : \tilde{U} \to X_U$ is G-equivariant.

. There is a natural morphism

$$\pi : X \longrightarrow [X/G]$$

$$x \in X(U) \longmapsto \begin{tikzcd} G_U \arrow[rr, "\text{orbit morphism } g \mapsto xg"] \arrow[dr] & & X_U \arrow[dl] \\ & U & \end{tikzcd}$$

Now for any object
$$\begin{tikzcd} \tilde{U} \arrow[r, "h"] \arrow[dr] & X_U \arrow[d] \\ & U \end{tikzcd}$$
defining a point

$\xi : U \to [X/G]$, the diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "h"] \arrow[d] & X \arrow[d, "\pi"] \\ U \arrow[r, "\xi"] & {[X/G]} \end{tikzcd}$$

is Cartesian! In particular, π is surjective and smooth, and is a G-torsor in a natural sense (even if the action of G is not free!)

Special case: $X = S$ with trivial action of G.

The resulting stack is just:

$(S\text{-scheme } U) \mapsto (\text{cat. of } G_U\text{-torsors})$.

This is the classifying stack of G, also denoted by BG.

Note that there is a structural morphism
$$p : BG \to S$$
which is not representable unless G is trivial.

On the other hand, this morphism has a section

$$s : S \longrightarrow BG$$

corresponding to the trivial torsor $G \to S$. This section is representable and is in fact the universal G-torsor over BG.

Remark: we have seen some instances of BG before:

$$\mathrm{BUN}_r \simeq B(GL_r)$$
$$\mathcal{M}_{0,0} \simeq B(PGL_2)$$
$$\mathcal{M}_{0,1} \simeq B\Gamma \quad (\Gamma = \text{affine transformations of } \mathbb{A}^1)$$
$$\mathcal{M}_{0,2} \simeq B\mathbb{G}_m \simeq \mathrm{BUN}_1$$

For $g \geq 2$, we have seen two ways of viewing $\mathcal{M}_{g,0}$ as a quotient:

(1) $\left\{ \begin{array}{l} \text{curves } X \xrightarrow{p} U \\ + \text{basis of } p_* \omega^{\otimes 3} \end{array} \right\} \longrightarrow \mathcal{M}_{g,0} \simeq [\widetilde{M}_{g,0} / GL_{5g-5}]$

<u>scheme</u> $\widetilde{M}_{g,0}$ + action of GL_{5g-5}

(2) $\mathcal{M}^{(n)}_{g,0} \longrightarrow \mathcal{M}_{g,0} | \operatorname{Spec} \mathbb{Z}[1/n] \simeq [\mathcal{M}^{(n)}_{g,0} / GL_{2g}(\mathbb{Z}/n\mathbb{Z})]$

(level-n structures, $n \geq 3$, over $\mathbb{Z}[1/n]$)

The first morphism is a GL-torsor, hence smooth with geometrically connected fibres. Moreover, for any field (or semilocal ring) k,

$\widetilde{M}_{g,0}(k) \longrightarrow \mathcal{M}_{g,0}(k)$ is surjective on objects.

The second morphism is finite étale but

- only over $\mathbb{Z}[1/r]$.
- need a finite extension to lift points of $\mathcal{M}_{g,0}(k)$.
- $\mathcal{M}_{g,0,\mathbb{C}}$ is connected but $\mathcal{M}^{(r)}_{g,0,\mathbb{C}}$ is not.

GROUPOID SPACES

(generalisation of equivalence relations)

Definition A groupoid presheaf on a category C is a set of data:

- two functors $X_1, X_0 : C^\circ \to (\mathrm{Sets})$
  ($X_0 =$ "objects", $X_1 =$ "morphisms")
- morphisms
  $s, t : X_1 \rightrightarrows X_0$ (source + target)
  "identities": $X_0 \to X_1$
  "composition": $X_1 \times_{s, X_0, t} X_1 \to X_1$
  "inverse": $X_1 \to X_1$

which, for every $U \in \mathrm{ob}\, C$, defines a groupoid $X_\bullet(U)$ (whence a fibered groupoid $U \mapsto X_\bullet(U)$)

A groupoid space over $C = (\mathrm{Schemes})$ is a groupoid presheaf $X_\bullet$ where X_1, X_0 are algebraic spaces.

EXAMPLE: if $\mathcal{M}$ is an algebraic stack, X an algebraic space and $P : X \to \mathcal{M}$ a 1-morphism, there is a natural groupoid space

$$X_\bullet = \left(X_1 = X \times_{\mathcal{M}} X \rightrightarrows X_0 = X \right)$$

If P is smooth and surjective, then $\mathcal{M}$ is the étale stack associated to the fibered groupoid $X_\bullet$.

Conversely, given a groupoid space

$$X_\bullet = X_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X_0$$

then the stack associated ($\mathcal{M}$) to $X_\bullet$ is algebraic, provided:

- $p_1, p_2 : X_1 \to X_0$ are smooth
- $(p_1, p_2) : X_1 \to X_0 \times X_0$ is quasicompact and separated.

Moreover, the natural morphism $X_0 \to \mathcal{M}$ is smooth and surjective.

Example: $[X/G]$ can be obtained from:

$$X_0 = X, \qquad X_1 = X \times G$$

$p_1(x, g) = x$

$p_2(x, g) = xg$

composition: $((x_1, g_1), (x_1 g_1, g_2)) \mapsto (x_1, g_1 g_2)$

identities: $x \mapsto (x, e)$

inverse: $(x, g) \mapsto (xg, g^{-1})$

Questions about the definition:

- Why the étale topology? (in particular, the fppf topology is very useful)
- Why ask for a smooth $Y \xrightarrow{P} \mathcal{M}$ (and not just a flat one?)

Theorem (M. Artin)

① Every algebraic stack is a stack for the fppf topology (i.e. we have effective fppf descent)

② Let $\mathcal{M}$ be an fppf stack over (Schemes) such that:

- $\Delta_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$ is representable, quasicompact, separated
- there exists a scheme Y and a 1-morphism

$$P : Y \longrightarrow \mathcal{M}$$

which is faithfully flat, locally of finite presentation.

Then $\mathcal{M}$ is an algebraic stack.

Example: more general quotients:

Let $X_\bullet = X_1 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X_0$ be a groupoid space

with $\begin{cases} p_1, p_2 \text{ flat of finite presentation} \\ (p_1, p_2) : X_1 \longrightarrow X_0 \times X_0 \text{ quasicompact, separated.} \end{cases}$

Then the associated fppf stack is algebraic.

Special case:

$$\begin{tikzcd} G \arrow[dr, "{\text{flat, finite presentation, separated}}"'] & \text{acting on} & X \arrow[dl] \\ & S & \end{tikzcd}$$

then $[X/G]$ is an algebraic stack

(in the "torsor" description, one must take fppf torsors!)

Some examples of "geometry on algebraic stacks":

Let $\mathbb{P}$ be a property of schemes which is local in the étale sense, i.e.:

if $X' \to X$ is étale surjective,

then X has $\mathbb{P}$ $\Leftrightarrow$ X' has $\mathbb{P}$.

Examples:

- locally Noetherian
- —— —— and purely d-dimensional
- reduced
- normal
- regular
- (---)

Then $\mathbb{P}$ carries over to Deligne-Mumford stacks (and algebraic spaces):

if $\mathcal{M}$ is a D-M. stack, choose

$$\Phi : \underset{\text{scheme}}{\underbrace{Y}} \longrightarrow \mathcal{M} \quad \text{étale surjective}$$

and say that $\mathcal{M}$ has $\mathbb{P}$ iff Y does.

This is independent of the choice of Φ:

for another $\Phi' : Y' \to \mathcal{M}$ étale surjective,

consider

$$\begin{tikzcd} & Y \arrow[r, "\Phi"] & \mathcal{M} \\ Y \times_{\mathcal{M}} Y' =: & Z \arrow[u, "g"] \arrow[r, "g'"] \arrow[ur, phantom, "\square"] & Y' \arrow[u, "\Phi'"'] \\ & Z_1 \arrow[u, "\pi"] & \end{tikzcd}$$

π étale surjective (Z is an algebraic space)

Z_1 (scheme)

Then: Y has $\mathcal{P}$, g and π étale surjective

$$\Downarrow$$

$$Z_1 \text{ has } \mathcal{P}$$

$$\Downarrow$$

$$Y' \text{ has } \mathcal{P}.$$

Remark In this situation, Z is in fact a scheme: for Deligne-Mumford stacks $\mathcal{M}$, the diagonal

$$\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$$

is representable in the scheme sense. This boils down to the fact that $\Delta_{\mathcal{M}}$ is separated and quasi-finite (of finite type with finite fibres), hence quasi-affine.

If we try to do the same for Artin stacks, we must restrict to properties which are local in the smooth sense, i.e:

if $X' \to X$ is smooth and surjective, then X has $P \Leftrightarrow X'$ has P.

Examples: all the above, except "purely d-dimensional".

Remark: ~~one~~ can define the dimension of an algebraic stack $\mathcal{M}$ at a point $\mathrm{Spec}(k) \to \mathcal{M}$ (k a field). It may be negative.

For instance if G is an algebraic k-group, the dimension of $BG = [\mathrm{Spec}(k)/G]$ (at the obvious k-point) is $-\dim(G)$.

Definition. If $\mathcal{M}$ is an algebraic stack, a locally closed (resp. open, closed) substack of $\mathcal{M}$ is a 1-morphism $\mathcal{Y} \longrightarrow \mathcal{M}$ which is representable by immersions (resp. --)

A closed substack $\mathcal{Y} \hookrightarrow \mathcal{M}$ has an obvious open complement $\mathcal{V}$ defined by

$$\mathcal{V}(U) = \left\{ x : U \longrightarrow \mathcal{M} \text{ in } \mathcal{M}(U) \;\middle|\; U \times_{x,\mathcal{M}} \mathcal{Y} = \emptyset \right\}.$$

An open substack $\mathcal{V} \hookrightarrow \mathcal{M}$ has a reduced closed complement $\mathcal{Z}$ defined as follows: choose

$$P : Y \longrightarrow \mathcal{M} \text{ smooth surjective,}$$

put $Z =$ reduced closed complement of $Y \times_{\mathcal{M}} \mathcal{V}$ in Y and define

$$\mathcal{Z}(U) = \left\{ x : U \longrightarrow \mathcal{M} \;\middle|\; U \times_{x,\mathcal{M},P} Y \longrightarrow Y \text{ factors through } Z \right.$$

This does not depend on P, because taking the reduced closed complement commutes with smooth base change

Definition. An algebraic stack $\mathcal{M}$ is separated if $\Delta_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is proper.

Proposition (Valuative criterion) $\mathcal{M}$ is separated if and only if:

For every valuation ring Λ, with field of fractions K, and all $x, y \in \mathcal{M}(\Lambda)$,

every isomorphism $x_K \xrightarrow{\sim} y_K$ in $\mathcal{M}(K)$ extends $\underbrace{\text{(uniquely)}}_{\text{automatic since } \Delta_{\mathcal{M}} \text{ is separated.}}$ to $x \xrightarrow{\sim} y$ in $\mathcal{M}(\Lambda)$.

<u>Remarks</u>:
- One has a notion of separated 1-morphism $\mathcal{M} \to \mathcal{N}$ of algebraic stacks.
- If $\mathcal{M}$ is of finite type over a separated Noetherian scheme S, then one can restrict the valuative criterion to discrete valuation rings.
- Many useful stacks (such as BUN_n) are not separated!
- If $\mathcal{M}$ is a Deligne-Mumford stack, then:

$$\mathcal{M} \text{ separated} \iff \Delta_{\mathcal{M}} \text{ finite}$$

PROPER STACKS:

We fix a Noetherian base scheme S.

Definition An algebraic stack $\mathcal{M} \xrightarrow{f} S$ is proper (over S) if:

(1) $\mathcal{M}$ is of finite type, separated over S

(2) For each valuation ring V over S, with fraction field K, and every object $x : \operatorname{Spec} K \to \mathcal{M}$ of $\mathcal{M}(K)$, there is a valuation ring $V' \supset V$ dominating V, with fraction field $K' \supset K$, and an object of $\mathcal{M}(V')$ extending $x_{K'}$.

Example: let G be a finite group, and consider $\mathcal{M} = BG$ (over S) $= [S/G]$ (trivial action):

$$\begin{tikzcd} \mathcal{M} = [S/G] \arrow[d, "f"] \\ S \arrow[u, bend left, "s"'] \end{tikzcd}$$

(trivial torsor) s

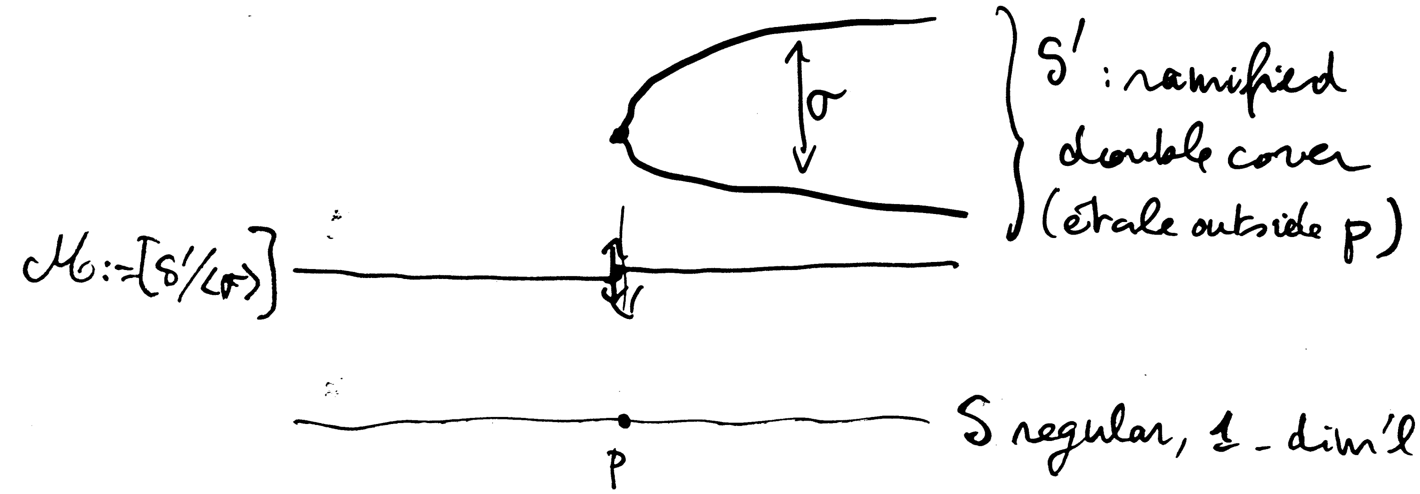
. $\mathcal{M}$ is a separated D.M. stack (the diagonal is finite étale).

. Since $s : S \to \mathcal{M}$ is finite étale surjective, and S is proper over S (!), $\mathcal{M}$ is of finite type, and we expect it to be proper.

Now, if, say, V is a discrete valuation ring, and L a ramified Galois extension of $K = \mathrm{Frac}(V)$, with group G, then $\mathrm{Spec}\, L$ is an object of $\mathcal{M}(K)$ which does not extend to a G-torsor over $\mathrm{Spec}(V)$.

But over $\mathrm{Spec}(L)$ the torsor becomes trivial, hence extends.

Variant:



Then $\mathcal{M} \to S$ is proper, and is an isomorphism over $S - \{p\}$, but is ramified at p.

The section $S - \{p\} \to \mathcal{M}$ does not extend to S (but extends over S').

Remarks.

- Condition (2) of the definition (the valuative criterion) is equivalent to $\mathcal{M} \to S$ being universally closed, in an appropriate sense.
- This notion is hard to use directly. For instance, one would like to restrict to discrete valuation rings, and/or finite extensions K'/K.
- Fortunately, we now have:

Theorem (Gabber-Olsson) Let $S = \operatorname{Spec}(A)$ (A Noetherian). Let $\mathcal{M} \to S$ be a separated algebraic stack, of finite type over S. Then there exists a quasiprojective S-scheme X and an S-morphism

$$p : X \to \mathcal{M}$$

which is proper and surjective.

In particular, $\mathcal{M}$ is proper over S iff X is.

Remarks.

- If $\mathcal{M}$ is a Noetherian Deligne-Mumford stack, there is a scheme X and a morphism $p: X \to \mathcal{M}$ which is finite, surjective, generically étale (no base scheme needed) (Laumon-LMB)
- The theorem implies finiteness of coherent cohomology for proper morphisms of algebraic stacks (other proof by Faltings)

Over a field (under some additional assumptions on $\mathcal{M}$) there is a $p : X \to \mathcal{M}$ finite and flat.

There are other results in the same vein, asserting the existence of nice morphisms from schemes to a given stack.

Here is such a result, of a local nature (and much easier!):

Theorem Let $\mathcal{M}$ be an Artin (resp. Deligne-Mumford) stack, K a field, $x \in \mathcal{M}(K)$. Then there exists a 2-commutative diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \mathrm{Spec}(K) \arrow[ur] \arrow[r, "x"'] & \mathcal{M} \end{tikzcd}$$

where X is a scheme and p is smooth (resp. étale).

What about morphisms from $\mathcal{M}$ to algebraic spaces?

Theorem (Keel-Mori, 1997)

- S: a locally Noetherian scheme
- $\mathcal{M} \to S$: an algebraic stack of finite type over S, such that

$$\Delta_{\mathcal{M}/S} : \mathcal{M} \longrightarrow \mathcal{M} \times_S \mathcal{M} \text{ is finite.}$$

(e.g. $\mathcal{M}$ is a separated Deligne-Mumford S-stack of finite type).

Then $\mathcal{M}$ has a coarse moduli space.

More precisely: there is an S-morphism

$$q : \mathcal{M} \longrightarrow M$$

such that:

① M is a separated algebraic space of finite type $/S$

② for each geometric point ξ of S, the natural map

$$\{\text{isom. classes of } \mathcal{M}(\xi)\} \longrightarrow M(\xi)$$

is bijective

③ q is universal for S-morphisms from $\mathcal{M}$ to algebraic spaces

④ for every flat $M' \xrightarrow{\varphi} M$, the pullback

$$M' \times_M \mathcal{M} \longrightarrow M' \text{ is still universal.}$$