(1) Weibel I

Motivic Cohomology

$$Sm/k \xrightarrow{H^{n,i}} \text{graded abelian groups}$$

$$X \longmapsto H^n(X, \mathbb{Z}(i))$$

Grothendieck
Deligne
Beilinson

- Internal structure
  - universal coefficients $H^n(X, A(i))$
  - projective bundle theorem
  - Steenrod operations $(A = \mathbb{Z}/p)$

- External structure / relations with other math
  - $H^{2n}(X, \mathbb{Z}(n))$ = classical Chowgroup of codimension n cycles / rat. eq.
  - field: $H^n(F, \mathbb{Z}(i)) = \varinjlim_{k(X)=F} H^n(X, \mathbb{Z}(i))$ has $H^n(F, \mathbb{Z}(n)) = K_n^M(F)$ Milnor K-theory
  - $H^n(X, \mathbb{Q}(i)) = H^n(X, \mathbb{Z}(i)) \otimes_{\mathbb{Z}} \mathbb{Q}$ should be the summand $K^{(i)}_{2i-n}(X)$ of $K_{2i-n}(X)_{\mathbb{Q}}$ – Quillen on which Adams ops $\psi^k = k^i$

Problems with $\underline{Cor}$:

(1) not idempotent complete (pseudo-abelian)

eg want $\mathbb{P}^1 = (pt) \oplus \mathbb{L}$

$\uparrow$ Lefschetz motive

(2) want more kernels/cokernels and ~~maybe~~ $\underline{\lim}$

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defn PST is the category of (contra.) (additive) functors $\underline{Cor} \longrightarrow \underline{Ab}$.

[abelian category]

Yoneda $\underline{Corr} \subset PST$

$X \longmapsto \mathbb{Z}_{tr}(X) \quad U \longmapsto Cor(U,X)$

examples: $X = Spec(k)$: $\mathbb{Z}$ "constant" (on irred U)

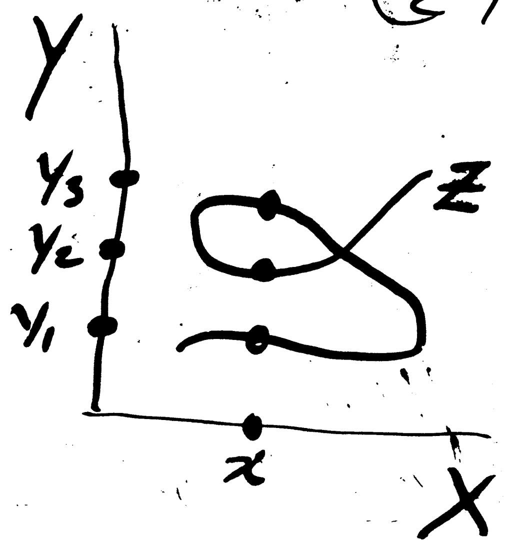
$X = Spec(F)$, F/k Galois : $\mathbb{Z}[G] = \mathbb{Z}_{tr}(F)$

$k \underset{alg}{<} E < X$ $\qquad X \longmapsto \mathbb{Z}[G]^H$.

$$\Gamma \left[\begin{array}{c} \bar{k} \\ \mid H \\ \bar{E} \\ \mid \\ k \end{array} \right.$$

Correspondences as multi-valued functions.

"elementary" $\leftrightarrow$ irred.



Finite (sep) field extn $\quad E \xrightarrow{} \ldots$

Galois

$$\begin{array}{c} E \\ \big|\, G \\ k \end{array}$$

$\mathrm{Cor}_k(E,E) = \mathbb{Z}[G]$

decompose $E \otimes_k E \cong \prod_G E$

$$\begin{tikzcd} & E \arrow[dl, "H_1"'] \arrow[dr, "H_2"] & \\ F_1 \arrow[dr] & & F_2 \arrow[dl] \\ & k & \end{tikzcd}$$

$$\mathrm{Cor}(F_1, F_2) \cong \mathbb{Z}[H_1 \backslash G / H_2]$$

Hecke operations on modular forms

1927 $\quad$ (Shimura $\Gamma = \mathrm{Gal}(\bar{k}/k)$)

Novel structure of a P.S.T., $\mathcal{F}$

1) Zariski – contravariant functor for morphisms of smooth X

2) If $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ is finite $Y \mid \underset{X}{\underline{\;\subseteq\;}}$ Graph Γ

get usual $Y \to X$ in Cor

also a transfer map $X \xrightarrow[\Gamma]{} Y$ (many compatibilities

Eg If $\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ is Galois (G) then $\begin{array}{c} Y \\ \Uparrow \\ X \end{array}$ is ($\times$ degree) $\quad$ $\begin{array}{c} Y \\ \downarrow\uparrow \\ X \end{array}$ is Σg

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Example (Badguy) $X \mapsto \mathcal{O}(X)$ = global functions

$\begin{array}{c} Y \\ \downarrow \\ X \end{array}$ transfer map
$$\begin{array}{ccc} \mathcal{O}(Y) \subset & E = k(Y) \\ \vdots\downarrow & \downarrow \text{trace} \\ \mathcal{O}(X) \subset & F = k(X) \end{array}$$

Good guy $X \mapsto \mathcal{O}^*(X)$ = global units

transfer map
$$\begin{array}{ccc} \mathcal{O}(Y)^* & \subset & E^* \\ \downarrow & & \downarrow \text{Norm.} \\ \mathcal{O}(X)^* & \subset & F^* \end{array}$$

(4)

Classical Chow group $CH^i(X)$ is a PST!

$$\begin{array}{ccc} Y & \xleftarrow{} Z \subseteq X \times Y \\ & \downarrow \\ & X \end{array}$$

$$\begin{array}{l} CH^i(Y) \xrightarrow{p^*} CH^i(X \times Y) \\ \qquad\qquad \downarrow \cup Z \\ \qquad\qquad CH^i(X \times Y)_{\text{finite over } X} \\ \qquad\qquad \downarrow p_* \\ \qquad\qquad CH^i(X) \end{array}$$

Homotopy Invariance

$\mathcal{F}$ is h.invariant if $\mathcal{F}(X) \xrightarrow{\simeq} \mathcal{F}(X\times\mathbb{A}^1)$ for all X.

$$
\begin{tikzcd}
X \arrow[dr, "X\times 0"] & & & & \\
& X\times\mathbb{A}^1 \arrow[r, "\mathrm{proj}"] & X \\
X \arrow[ur, "X\times 1"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \mathcal{F}(X) \\
\mathcal{F}(X) \arrow[r] \arrow[urr, bend left, "="] \arrow[drr, bend right, "="'] & \mathcal{F}(X\times\mathbb{A}^1) \arrow[ur] \arrow[dr] & \\
& & \mathcal{F}(X)
\end{tikzcd}
$$

Naive Approach

$[\mathcal{F}]X$ coker of $\mathcal{F}(X\times\mathbb{A}^1) \xrightarrow{p_1-p_0} \mathcal{F}(X)$

This is h.invariant but has bad homological properties.

Pass to (co)chain complexes. (of PSTs)

Cosimplicial variety $\Delta^\bullet$

$pt \overset{0}{\underset{1}{\rightrightarrows}} \mathbb{A}^1 \equiv \mathbb{A}^2 \equiv \mathbb{A}^3$ $\qquad \mathcal{F}(X\times\Delta^\bullet)$

$\mathcal{F}(X) \leftleftarrows \mathcal{F}(X\times\mathbb{A}^1) \overset{\partial_0}{\Lleftarrow}$

Convert to a complex

$\Sigma \pm \partial_i$