# Generalized Bialgebras and Triples of Operads

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## Classical Bialgebras

$\mathbb{K}$ is a the ground field

*Classical bialgebra:* $(\mathcal{H}, *, \Delta)$, $\mathcal{H} = \mathbb{K}1 \oplus \overline{\mathcal{H}}$

$* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is associative and unital
$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is coassociative and counital
Hopf relation: $\Delta(x * y) = \Delta(x) * \Delta(y)$

*Primitive elements:*
$\mathrm{Prim}\,\mathcal{H} := \{x \in \overline{\mathcal{H}} | \Delta(x) = x \otimes 1 + 1 \otimes x\}$

Observation: $\mathrm{Prim}\,\mathcal{H}$ is a Lie algebra for
$[x, y] := x * y - y * x$

*Connected bialgebra*: $F_0 := \mathbb{K}1$,
$F_r := \{x \in \mathcal{H} | \Delta(x) - x \otimes 1 - 1 \otimes x \in F_{r-1} \otimes F_{r-1}\}$
Condition to be connected: $\mathcal{H} = \bigcup_r F_r$

# Hopf-Borel Theorem

**THM** (Hopf-Borel, 1953) $\mathbb{K}$ = *char 0 field,* $\mathcal{H}$ = *commutative cocommutative bialgebra. TFAE:*
*(a)* $\mathcal{H}$ *is connected*
*(b)* $\mathcal{H} \cong S(V)$, *where* $V = \mathrm{Prim}\,\mathcal{H}$

One of the numerous different proofs involves the Eulerian idempotent

Many applications in algebraic topology and homological algebra (graded version):

$H_*(G, \mathbb{Q}) \cong \Lambda(\pi_*(G) \otimes \mathbb{Q})$, where $G$ = Lie group
$H_*(GL(A), \mathbb{Q}) \cong \Lambda(K_*(A) \otimes \mathbb{Q})$, (Quillen)
$H_*(\mathfrak{g}l(A), \mathbb{Q}) \cong \Lambda(HC_{*-1}(A) \otimes \mathbb{Q})$, (Loday-Quillen-Tsygan)

## PBW and CMM Theorem

**THM** (PBW + CMM) $\mathbb{K} =$ *char 0 field,*
$\mathcal{H} =$ *cocommutative bialgebra. TFAE:*
*(a)* $\mathcal{H}$ *is connected,*
*(b)* $\mathcal{H} \cong U(\mathrm{Prim}\,\mathcal{H})$,
*(c)* $\mathcal{H}$ *is cofree among the connected cocommutative coalgebras.*

$(a) \Rightarrow (b)$ Cartier-Milnor-Moore (CMM) thm
$(b) \Rightarrow (c)$ Poincaré-Birkhoff-Witt (PBW) thm
$(c) \Rightarrow (a)$ is a tautology
$(a) \Rightarrow (c)$ was proved earlier by Leray (1945)

**COR** $T(V) \cong S(Lie(V))$, $\mathrm{Prim}\,T(V) \cong Lie(V)$

**COR** *Structure theorem for cofree cocommutative bialgebras*

QUESTION: Can we remove the hypothesis "cocommutative" ? Several answers. One of them: Etingof-Kazhdan. Another one soon ....

## Unital Infinitesimal Bialgebra

$(\mathcal{H}, \cdot, \Delta) =$ *unital infinitesimal bialgebra* if

$* : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ is associative and unital
$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is coassociative and counital
unital infinitesimal (u. i. ) relation:
$\Delta(x \cdot y) = \Delta(x) \cdot (1 \otimes y) + (x \otimes 1) \cdot \Delta(y) - x \otimes y$

Example: $T(V) = \mathbb{K}1 \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$
$(T(V), \cdot = \text{concatenation}, \Delta = \text{deconcatenation})$

$v_1 \ldots v_p \cdot v_{p+1} \ldots v_n := v_1 \ldots v_n$
$\Delta(v_1 \ldots v_n) := \sum_{p \geq 0} v_1 \ldots v_p \otimes v_{p+1} \ldots v_n$

**THM** (JLL-Ronco, 2003) $\mathcal{H} =$ *u. i. bialgebra*
*TFAE:*
*(a)* $\mathcal{H}$ *is connected*
*(b)* $\mathcal{H} \cong T(V)$*, where* $V = \mathrm{Prim}\,\mathcal{H}$

## Cofree Bialgebra and $B_\infty$-algebra

Let $\mathcal{H}$ be a classical bialgebra, suppose it is cofree:

$\mathcal{H} \cong T^c(V)$ i.e. $\Delta =$ deconcatenation

$$\begin{array}{ccc} T(V)\otimes T(V) & \xrightarrow{\ *\ } & T^c(V) \\ \uparrow & & \downarrow \\ V^{\otimes p}\otimes V^{\otimes q} & \xrightarrow{\ M_{pq}\ } & V \end{array}$$

Associativity of $*$ $\Leftrightarrow$ The $M_{pq}$'s satisfy $\mathcal{R}_{ijk}$

Example: $\mathcal{R}_{111}$:

$$M_{12}(u, vw + wv) + M_{11}(u, M_{11}(v, w)) =$$
$$M_{21}(uv + vu, w) + M_{11}(M_{11}(u, v), w)$$

**DEF** $(R, M_{pq})$ is a $B_\infty$-algebra if the $M_{pq}$'s satisfy the relations $\mathcal{R}_{ijk}$

Claim:
$(R, M_{pq})$ $B_\infty$-alg. $\Leftrightarrow$ $(T^c(R), *, \Delta)$ cofree bialg.

## Cofree Hopf algebra

**DEF** *2-associative algebra* $(A, *, \cdot)$ , operations $*$ and $\cdot$ associative with same unit 1.
Example: a cofree bialgebra $\mathcal{H} = (T^c(V), *, \Delta)$ with concatenation as $\cdot$

**PROP** $\exists\ F$ : $2as-$alg $\to$ $B_\infty-$alg such that Prim $\mathcal{H} \to F(\mathcal{H})$ is a morphism of $B_\infty$-algebras
Example: $M_{11}(u, v) = u * v + u \cdot v + v \cdot u$

**DEF** *2-associative bialgebra* is $(\mathcal{H}, *, \cdot, \Delta)$ s.t.
- $(\mathcal{H}, *, \Delta)$ = classical bialgebra
- $(\mathcal{H}, \cdot, \Delta)$ = unital infinitesimal bialgebra

**THM** (JLL-Ronco) *$\mathcal{H} = 2as$-bialgebra. TFAE:*
*(a) $\mathcal{H}$ is connected,*
*(b) $\mathcal{H} \cong U2$(Prim $\mathcal{H}$),*
*(c) $\mathcal{H}$ is cofree among connected coalgebras.*
$U2 : B_\infty-$alg $\to 2as-$alg left adjoint to $F$

**COR** Structure theorem for cofree Hopf alg.
**COR** Explicitation of free $B_\infty$-algebra (trees)

## Generalized bialgebra, triple of operads

Data $(\mathcal{C}, \between, \mathcal{A} \xrightarrow{F} \mathcal{P})$ abbreviated $(\mathcal{C}, \mathcal{A}, \mathcal{P})$

• $\mathcal{C} =$ operad handling coalgebra structure
• $\mathcal{A} =$ operad handling algebra structure
• $\between$ "spin relations" intertwining operations and cooperations
So $(\mathcal{C}, \between, \mathcal{A})$ determines a notion of bialgebra (prop)
• $\mathcal{P} =$ operad handling algebra structure of the primitive part
• $F : \mathcal{A}{-}alg \to \mathcal{P}{-}alg$ forgetful functor s.t.
Prim $\mathcal{H} \to F(\mathcal{H})$ is a morphism of $\mathcal{P}$-algebras

**DEF** $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is called a *triple of operads* (triplette)

Examples:

| | |
|---|---|
| $(Com, Com, \text{Vect})$ | $\between$ = Hopf relation |
| $(Com, As, Lie)$ | $\between$ = Hopf relation |
| $(As, As, \text{Vect})$ | $\between$ = u.i. relation |
| $(As, 2as, B_\infty)$ | $\between$ = Hopf and u.i. relation |

## Good triples of operads

Let $U : \mathcal{P}$–alg $\to \mathcal{A}$–alg be left adjoint to $F$

**DEF** $(\mathcal{C}, \mathcal{A}, \mathcal{P})$ is called a *good* triple of operads if, for any $(\mathcal{C}, \between, \mathcal{A})$ bialgebra $\mathcal{H}$, TFAE:

(a) $\mathcal{H}$ is connected,
(b) $\mathcal{H} \cong U(\mathrm{Prim}\,\mathcal{H})$,
(c) $\mathcal{H}$ is cofree among connected $\mathcal{C}$-coalgebras.

**COR** $\mathcal{A}(V) \cong \mathcal{C}(\mathcal{P}(V))$ and $\mathrm{Prim}\,\mathcal{A}(V) \cong \mathcal{P}(V)$

All preceding examples are good triples.

**Triples of operads:** $\mathcal{C} \quad \between \quad \mathcal{A} \quad \longrightarrow \quad \mathcal{P}$

| | coalgebra | algebra | primitive |
|---|---|---|---|
| Hopf-Borel | $Com$ | $Com$ | $Vect$ |
| CMM+PBW | $Com$ | $As$ | $Lie$ |
| Ronco | $As$ | $Zinb$ | $Vect$ |
| Ronco | $As$ | $Dend$ | $brace$ |
| JLL-Ronco | $As$ | $Dipt$ | $B_\infty$ |
| JLL-Ronco | $As$ | $As$ | $Vect$ |
| JLL-Ronco | $As$ | $2as$ | $B_\infty$ |
| JLL | $Mag$ | $Mag$ | $Vect$ |
| Holtkamp-JLL | $As$ | $Mag$ | $Mag_{Fine}$ |
| Holtkamp | $Com$ | $Mag$ | ?? |
| Guin-Oudom | $Com$ | $\mathcal{X}$ | $preLie$ |
| Markl-Remm | ?? | $preLie$ | $Lie$ |
| Goncharov | $Com$ | $As \times As$ | ?? |
| JLL | $2as$ | $2as$ | $Vect$ |
| Livernet | $NAperm$ | $preLie$ | $Vect$ |
| Foissy | $Dend$ | $Dend$ | $Vect$ |

| $\mathcal{P}$ | operations | relations |
|---|---|---|
| $As$ | $xy$ | $(xy)z = x(yz)$ |
| $Com$ | $xy = yx$ | $(xy)z = x(yz)$ |
| $Lie$ | $[xy] = -[yx]$ | Jacobi identity |
| $Mag$ | $xy$ | no relation |
| $preLie$ | $xy$ | $(xy)z - x(yz) = (xz)y - x(z$ |
| $Zinb$ | $xy$ | $(xy)z = x(yz) + x(zy)$ |
| 2-$as$ | $x \cdot y,\ x * y$ | both associative |
| $Dend$ | $x \prec y,\ x \succ y$<br>$x * y =$<br>$x \prec y + x \succ y$ | $(x \prec y) \prec z = x \prec (y * z)$<br>$(x \succ y) \prec z = x \succ (y \prec z)$<br>$(x * y) \succ z = x \succ (y \succ z)$ |
| $Dipt$ | $x * y, x \succ y$ | $(x * y) * z = x * (y * z)$<br>$(x * y) \succ z = x \succ (y \succ z)$ |
| $B_\infty$ | $M_{pq}$ | $(\mathcal{R}_{ijk})$ |
| $brace$ | $M_{1q}$ | $(\mathcal{R}_{1jk})$ |
| $NAperm$ | $xy$ | $(xy)z = (xz)y$ |

Thank you for your attention !