Torsion Hermitian Forms

K field, $\operatorname{char} K \neq 2$

(V, h) ε-hermitian form over (A, σ)

A c.s.a $/K$

σ involution on A

$\varepsilon \in K$ $\sigma(\varepsilon)\varepsilon = 1$

V f.g right A-module

$$h(xa, yb) = \sigma(a)\, h(x,y)\, b$$

$$\sigma(h(x,y)) = \varepsilon\, h(x,y)$$

h biadditive

h is torsion (or weakly hyperbolic) if $m \times h$ is hyperbolic for some $m \geq 1$.

Hyperbolic forms and quadratic descent (First situation)

(D, σ) div. alg. $/K$ σ involution

(H) $\lambda, \mu \in D^\times$, $\sigma(\lambda) = -\lambda$, $\sigma(\mu) = -\mu$,
$\lambda\mu = -\mu\lambda$, $K(\lambda)/K$ quadratic ext.

$\tilde{D} = C_D(K(\lambda))$

σ_1, σ_2 involution on D: $\sigma_1 = \sigma|_{\tilde{D}}$, $\sigma_2 = \operatorname{Int}(\mu^{-1}) \circ \sigma_1$

σ_1 of the first kind

σ_2 of the same kind as σ but of different type

$$D = \tilde{D} \oplus \mu \tilde{D}$$

$$\pi_1 : D \longrightarrow \tilde{D} \qquad \pi_i(x_1 + \mu x_2) = x_i$$
$$\pi_2 : D \longrightarrow \tilde{D} \qquad i = 1, 2$$

π_1^{ε}, π_2^{ε} corresponding homomorphisms between Witt groups:

$$\pi_1^{\varepsilon} : W^{\varepsilon}(D, \sigma) \longrightarrow W^{\varepsilon}(\tilde{D}, \sigma_1)$$
$$\pi_2^{\varepsilon} : W^{\varepsilon}(D, \sigma) \longrightarrow W^{-\varepsilon}(\tilde{D}, \sigma_2)$$

Exact sequence (Parimala, Sridharan, Suresh 1995)

$$W^{\varepsilon}(D, \sigma) \xrightarrow{\pi_1^{\varepsilon}} W^{\varepsilon}(\tilde{D}, \sigma_1) \xrightarrow{\rho} W^{-\varepsilon}(D, \sigma) \xrightarrow{\pi_2^{-\varepsilon}} W^{\varepsilon}(\tilde{D}, \sigma_2)$$

π_1^{ε} and π_2^{ε} are not injective in general

$$\ker \pi_1^{\varepsilon} \cap \ker \pi_2^{\varepsilon} = ?$$

<u>Proposition 1</u> h anisotropic, $\pi_1^{\varepsilon} h$ and $\pi_2^{\varepsilon} h$ hyperbolic
$\Rightarrow$ -1 is a similitude factor for h
(i.e. $h \simeq -h$)

proof.

<u>Lemma</u> $\pi_1^{\varepsilon} h$ hyp $\Rightarrow$ $h \simeq \langle \mu c_1, \ldots, \mu c_n \rangle$ $c_i \in \tilde{D}$

<u>Lemma</u> $\pi_2^{\varepsilon} h$ hyp

$$\Downarrow$$

$$\underset{\substack{\wr\wr \\ h}}{\langle \mu c_1, \ldots, \mu c_n \rangle} \simeq \underset{\substack{\wr\wr \\ -h}}{\langle \mu^2 c_1 \mu, \ldots, \mu^2 c_n \mu \rangle}$$

Hyperbolic forms and quadratic descent (Second situation)

$(D, \sigma) \simeq (D_0 \otimes_K L, \sigma_0 \otimes \theta)$

(H') $L|K$ quadratic ext. σ_0, θ involutions $\sigma_0|_K = \theta|_K$

$$\pi_1^\varepsilon : W^\varepsilon(D, \sigma) \longrightarrow W^\varepsilon(\tilde{D}, \sigma_0)$$
$$\pi_2^\varepsilon : W^\varepsilon(D, \sigma) \longrightarrow W^{\pm\varepsilon}(\tilde{D}, \sigma_0)$$

$\pm\varepsilon$ depending on $\theta|_K$

Proposition 2 h anisotropic, $\pi_1^\varepsilon h, \pi_2^\varepsilon h$ hyperbolic $\Longrightarrow$ -1 is a similitude factor for h

In any two cases:

Corollary (1) $\ker \pi_1^\varepsilon \cap \ker \pi_2^\varepsilon \subset$ 2-torsion of the Witt group.

(2) h torsion $\Longleftrightarrow$ $\pi_1^\varepsilon h, \pi_2^\varepsilon h$ torsion

Applications

(I) Pfister local-global principles

q nondeg. quad. form $/K$ TFAE (Pfister 1966)

- The order of $[q]$ in $W(K)$ is a 2-power
- q is torsion
- $\forall P \in X_K$, $\mathrm{Sgn}_P(q) = 0$ X_K = set of orderings of K

hermitian version: (Scharlau 1970)

TFAE
- The order of $[h]$ in $W^\varepsilon(A, \sigma)$ is a 2-power
- h is torsion.

Signature of an involution

(A, σ) c.s.a $/K$ σ inv.

$$\mathrm{Sgn}_P(\sigma) = \begin{cases} \sqrt{\mathrm{Sgn}\, T_\sigma} & \sigma \text{ 1st kind (Lewis-Tignol 1993)} \\ \sqrt{\tfrac{1}{2}\mathrm{Sgn}\, T_\sigma} & \sigma \text{ 2nd kind (Quéguiner 1996)} \end{cases}$$

$P \in X_K$

$$T_\sigma : A \longrightarrow k \quad T_\sigma(x) = \mathrm{Trd}(\sigma(x)x)$$

(Lewis-Unger 2003) TFAE
$$\begin{cases} h \text{ is torsion} \\ \mathrm{sgn}_P(h) = 0 \ \forall P \in X_K \end{cases}$$

→ Alternative proof for both (Scharlau 1970) and (Lewis-Unger 2003) using $\ker \pi_1^\varepsilon \cap \ker \pi_2^\varepsilon \subset$ 2-torsion of the Witt group.

Main ingredient of the proof:

$$\mathrm{sgn}_P(h) = 0 \Rightarrow \mathrm{sgn}_{\tilde{P}}(\pi_1(h)) = \mathrm{sgn}_{\tilde{P}}(\pi_2(h)) = 0$$

$P \subset \tilde{P}$ (Bayer-Fluckiger-Parimala 1998)

(II) Hermitian forms induced by Pfister forms

$(D, \sigma) = \bigotimes (Q_i, \sigma_i)$ $\quad (Q_i, \sigma_i)$ quaternion algebras with involution

k = fixed field of $\sigma|_K$

φ Pfister form $/k$

<u>Proposition</u> (—, and independently by Grenier-Boley)

$\varphi|_{(D,\sigma)}$ isotropic $\Rightarrow m \times \varphi|_{(D,\sigma)}$ hyperbolic

m depends only to $\deg D$

(For $m=1$ $\varphi|_{(D,\sigma)}$ hyp $\Leftrightarrow \varphi|_{(D,\sigma)}$ isotropic (Lewis, Serhir))

<u>proof</u> using prop. 1, 2