On the vanishing of negative K-theory of singular varieties

(talk by Marco Schlichting, 8 Feb. 2005 on joint work with Cortiñas, Häsemeyer, Weibel)

Fundamental theorem (Bass):

$\exists$ exact sequence

$$0 \to K_1(R) \to K_1(R[T]) \oplus K_1(R[T^{-1}]) \to K_1(R[T,T^{-1}]) \to K_0(R) \to 0$$

motivated inductive definition of K_n, $n<0$, as

$$K_n(R) := \operatorname{coker}\left[K_{n+1}(R[T]) \oplus K_{n+1}(R[T^{-1}]) \to K_{n+1}(R[T,T^{-1}]) \right]$$

Some motivation for studying K_n, $n<0$:

- Bass fundamental thm extends to $n \leq 0$ (actually $n \in \mathbb{Z}$)
- Thomason localization: $\exists$ long exact sequence for $U \subset X$ open

$$\text{higher K-theory} \to K_0(X) \to K_0(U) \to \text{lower K-theory}$$

not surjective, in gen.

- certain surgery obstructions

Conjecture (Weibel 1980):

Let R be a noetherian ring of Krull dimension d, then

$K_n(R) = 0, \quad n < -d$, and

R is K_{-d}-regular

Here, R is K_d-regular if $K_d(R) \xrightarrow{\cong} K_d(R[T_1, \dots, T_n]) \ \forall n$

Ex R regular ring $\Rightarrow$ R is K_d-regular $\forall d \in \mathbb{Z}$

Rem Conjecture holds for

- R regular since then $K_n(R) = 0$, $n < 0$
- $\dim R = 0, 1$: classical
- $\dim R \leq 2$, R excellent (Weibel 2001)

Thm 1 (CHSW)

Let X be a scheme essentially of finite type $/_F$, $\operatorname{char} F = 0$,

$d = \dim X$, then

$K_n(X) = 0, \quad n < -d$

X is K_{-d}-regular, and

$K_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$

Rem: Conj. open in char $F \neq 0$, $\dim X \geq 3$

goal of talk: explain proof of thm 1

cdh-topology: is the Grothendieck topology generated by covers given by cartesian squares

$$(*) \qquad \begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \qquad \text{with}$$

a) (Nisnevich squares)

$Y \to X$ open immersion, $X' \to X$ étale,

$(X' - Y')_{red} \xrightarrow{\cong} (X - Y)_{red}$

b) (abstract blow-up squares)

$Y \to X$ closed immersion, $X' \to X$ proper

$(X' - Y')_{red} \xrightarrow{\cong} (X - Y)_{red}$

Dfn A functor $G : Sch_{/F}^{op} \longrightarrow$ Spectra, or chaincomplexes satisfies
- descent for a square $(*)$ if G sends $(*)$ to a homotopy cartesian square
- cdh-descent if it satisfies descent ~~of~~for Nisnevich and abstract blow-up squares

Rem: If G satisfies cdh-descent, then $\exists$ spectral sequence

$$H^p_{cdh}(X, a_{cdh}\pi_{-q}G) \Rightarrow \pi_{-p-q}G(X),$$

a_{cdh} = cdh-sheafification

Non-examples

- K-theory $K: Sch_{/F}^{op} \to$ Spectra
- ~~ne~~ cyclic homology HC
- negative cyclic homology $HN = HC^-$

do not satisfy cdh-descent on $Sch_{/F}$

Examples

a) homotopy K-theory $KH(X) := |n \mapsto K(X \times \Delta^n)|$

Thm (Häsemeyer)

Assume char $F = 0$. Then

KH satisfies cdh-descent on $Sch_{/F}$

Rem: thm, cdh-spectral sequence, $a_{cdh}KH_n \begin{cases} = 0, & n < 0 \\ = \mathbb{Z}, & n = 0 \end{cases}$

$$\Rightarrow \quad KH_n(X) = 0 \qquad n < -d = -\dim X$$

$$KH_{-d}(X) = H^d_{cdh}(X, \mathbb{Z})$$

b/c cdh-cohomological dim of X is $d = \dim X$

b) <u>periodic cyclic homology</u> $HP = HC^{per}$ satisfies

cdh-descent on $Sch_{/F}$, char $F = 0$

c) <u>infinitesimal K-theory</u> $K^{inf}(X)$ is the homotopy fibre of the Chern character $K(X) \xrightarrow{ch} HN(X)$:

$$K^{inf}(X) \longrightarrow K(X) \xrightarrow{ch} HN(X) \quad \text{is a homotopy fibration}$$

<u>Thm 2</u> (CHSW)

K^{inf} satisfies cdh-descent on $Sch_{/F}$ if char $F = 0$.

<u>Proof of thm 1</u>

- For a functor $G: Sch_{/F}^{op} \longrightarrow$ Spectra, or chaincomplexes, write $C^d G(X)$ for the homotopy fibre of

$$G(X) \longrightarrow G(X \times \mathbb{A}^d)$$

- Homotopy fibration $HN \longrightarrow HP \longrightarrow HC[2]$ induces a htpy fibration $C^d HN \longrightarrow C^d HP \longrightarrow C^d HC[2]$

$$\overset{\text{IS}}{\ast}$$

HP is homotopy invariant on $Sch/_F$, $\operatorname{char} F = 0$

$\Rightarrow \quad C^t HP \simeq *, \quad t \geq 1$

$\Rightarrow \quad C^t HC \simeq \Omega\, C^t HN, \quad t \geq 1$

By c),

(**) $\quad C^t HC(X) \longrightarrow C^t K^{inf}(X) \longrightarrow C^t K(X)$ homotopy fibration, $t \geq 1$

- For $X \in Sch/_F$ smooth, $C^t K(X) = 0$, $t \geq 1$, thus:

For $X \in Sch/_F$, X is locally smooth in the cdh-top,

$\Rightarrow \quad C^t K \simeq 0$ in the cdh-topology

(**) $\Rightarrow \quad C^t HC \longrightarrow C^t K^{inf}$ is an equivalence for the cdh-topology,

that is, $\quad a_{cdh} C^t HC_n \xrightarrow{\simeq} a_{cdh} C^t K^{inf}_n, \quad t \geq 1, \; n \in \mathbb{Z}$

Therefore, the cdh-spectral sequence for $C^t K_n^{inf}$ becomes

(***) $\quad H^p_{cdh}(X, a_{cdh} C^t HC_{-q}) \Longrightarrow C^t K^{inf}_{-p-q}(X), \quad t \geq 1$

- $HC_n(X) = 0$, $n < 0$ and X affine

$\rightsquigarrow$ • $a_{zar} C^t HC_n = 0$, $n < 0$ $\rightsquigarrow$ $C^t HC_n(X) = 0$ $n < -\dim X$

$\rightsquigarrow$ $C^t K_n^{inf}(X) \xrightarrow{\simeq} C^t K_n(X)$, $n < -\dim X$, $t \geq 1$

• $a_{cdh} C^t HC_n = 0$, $n < 0$, $t \geq 1$

(***) $\rightsquigarrow$ $C^t K_n^{inf}(X) = C^t K_n(X) = 0$, $n < -\dim X$,

$\rightsquigarrow$ X is K_{-d-1}-regular, $d = \dim X$

The spectral sequence $K_q(X \times \Delta^p) \Rightarrow KH_{p+q}(X)$

then implies $K_n(X) = KH_n(X) = 0$, $n < -\dim X$

- K_{-d} - regularity:

(**) $\rightsquigarrow$ exact sequence

$$\begin{array}{ccccccc} C^t HC_{-d}(X) & \longrightarrow & C^t K_{-d}^{inf}(X) & \longrightarrow & C^t K_{-d}(X) & \longrightarrow & 0 \\ \| & & \| & & & & \\ H^d_{zar}(X, C^t HC_0) & \longrightarrow & H^d_{cdh}(X, C^t HC_0) & & & & \end{array}$$

(by Zariski, cdh-spectral sequence)

surjective by

<u>Thm 3 (CHSW)</u>

$H^d_{zar}(X, \mathcal{O}_X) \longrightarrow H^d_{cdh}(X, \mathcal{O}_X)$ is surjective for $d = \dim X$, $X \in Sch_{/F}$, char $F = 0$