Motivic decompositions of anisotropic projective homogeneous varieties of type F_4

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This is a report on the joint work with S. Nikolenko and N. Semenov.

Let k be a field. For a smooth projective variety X over k consider its Chow motive $\mathcal{M}(X)$. One of the naturally arising problems in the theory of Chow motives is to determine whether the motive of a given variety is decomposable. In what follows we explain the machinery of proving such results for projective homogeneous varieties of type F_4.

The main motivation for our work was the result of N. Karpenko where he gave a shortened construction of a Rost motive for a norm quadric [Ka98]. In our talk we try to give a 'shortened' construction of a generalized Rost motive for a variety that corresponds to a symbol $(3, 3)$.

Before formulating the results we need to introduce the notion of a Chow motive and a projective homogeneous variety of type F_4.

Chow motives. We define the category of Chow motives (over k) following [Ma68]. Fisrt, we define the category of *correspondences* (over k). Its objects are smooth projective varieties over k. For morphisms, called correspondences, we set $Mor(X, Y) := \mathrm{CH}^{\dim X}(X \times Y)$. The pseudo-abelian completion of the category of correspondences is called the category of *Chow motives* and is denoted by $\mathcal{M}_k$. The objects of $\mathcal{M}_k$ are pairs (X, p), where X is a smooth projective variety and p is a projector, that is, $p \circ p = p$. The motive (X, id) will be denoted by $\mathcal{M}(X)$. By the construction $\mathcal{M}_k$ is a tensor additive self-dual category. Moreover, the Chow functor $\mathrm{CH} : \mathcal{V}ar_k \to \mathbb{Z}\text{-}\mathcal{A}b$ (to the category of $\mathbb{Z}$-graded abelian groups) factors through $\mathcal{M}_k$, i.e, one has

$$\begin{array}{ccc} \mathcal{V}ar_k & \xrightarrow{\ \mathrm{CH}\ } & \mathbb{Z}\text{-}\mathcal{A}b \\ & {\scriptstyle\Gamma}\searrow \qquad \nearrow{\scriptstyle R} & \\ & \mathcal{M}_k & \end{array}$$

where $\Gamma : f \mapsto \Gamma_f$ is the graph and $R : (X, p) \mapsto \mathrm{im}(p)$ is the realization functor.

Homogeneous varieties of type F_4**.** It is well known that the classification of algebraic groups of type F_4 is equivalent to the classification of Albert algebras which are 27-dimensional exceptional simple Jordan algebras ([PR94] and [Inv]). All Albert algebras can be obtained from one of the two Tits constructions. We restrict ourselves to those obtained by the first Tits process. Recall that an Albert algebra A obtained by the first Tits construction is produced from a central simple algebra of degree 3. This means that A is either completely split (reduced) or a division algebra. In our talk we restrict ourselves to the latter case, i.e., to the case of an anisotropic group G of type F_4.

Consider the Dynkin diagram corresponding to the group G.

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Let P_i, $i = 1, 4$ be a maximal parabolic subgroup of G that is generated by the first (last) three vertices of the Dynkin diagram. The projective homogeneous varieties $X_1 = G/P_1$ and $X_2 = G/P_4$ will be the main objects of our investigations. Observe that $\dim X_1 = \dim X_2 = 15$ and they are not isomorphic to each other as well as they are not isomorphic to any variety of classical type.

The main result The goal of the present talk is to prove the following

Theorem 1. *Let k be a field of characteristic different from 2 and 3. Let $X = G/P$ be a projective homogeneous variety over k, where G is an anisotropic group of type F_4 obtained by the first Tits process and P its maximal parabolic subgroup corresponding to the first (last) three vertices of the respective Dynkin diagram. Then*

(i) the (integral) Chow motive of X decomposes as

$$\mathcal{M}(X) \cong R \oplus R_{12} \oplus R(3) \oplus R^t \oplus R^t_{12} \oplus R^t(3),$$

where the motive $R = (X, p)$ is the (integral) generalized Rost motive, i.e., over the separable closure k' of k it splits as the direct sum of Lefschets motives $\mathbb{Z} \oplus \mathbb{Z}(4) \oplus \mathbb{Z}(8)$, the motive $R^t = (X, p^t)$ denotes its transpose and the motive R_{12} is isomorphic over k' to the direct sum $R(1) \oplus R(2) \oplus R^t(1) \oplus R^t(2)$

(ii) the Chow motive of X with $\mathbb{Z}/3$-coefficients decomposes as

$$\mathcal{M}(X,\mathbb{Z}/3) \cong \bigoplus_{i=0}^{3}(R(i) \oplus R^t(i)).$$

The following result obtained during the proof of 1 represents independent interest (in view of [Bo03])

Theorem 2. *Under the hypotheses of theorem 1 let $X_1 = G/P_1$ and $X_2 = G/P_4$ be two projective homogeneous varieties corresponding to the maximal parabolic subgroups P_1 and P_4 generated by the first (last) three vertices of the Dynkin diagram respectively. Then the motives of X_1 and X_2 are isomorphic.*

Remark 3. Note that the variety X is a norm variety for the symbol $\{a_1, a_2, a_3\} \in K_3^M(k)/3$ (it follows from the existance of the Rost-Serre invariant $\mathfrak{g}_3$ for an Albert algebra). And the part (ii) of the Theorem follows from Voevodsky's results [Vo03, S. 5].

Remark 4. Observe that the variety X splits completely over its function field (generically split). For anisotropic groups of classical types the examples (see [Ro98, Prop. 19]) of generically split projective homogeneous varieties that have decomposable motives are Pfister quadrics and their neighbours (those are norm varieties for symbols mod 2). For exceptional groups the 'minimal' example was provided by J.-P. Bonnet [Bo03]. He considered the Fano variety G_2/P_2 (an anisotropic group of type G_2 modulo the maximal parabolic subgroup) and proved that the motives of G_2/P_2 and G_2/P_1 are isomorphic. Since the latter variety is a 5-dimensional quadric which has a decomposable motive, so is G_2/P_2. The examples of varieties X_1 and X_2 that we provide can not be reduced to the classical ones.

Remark 5. Our proof works not only for projective homogeneous varieties of type F_4. Applying the similar arguments to Pfister quadrics and their neighbours one obtains the well-known decompositions into Rost motives. For the groups of type G_2 one immediately obtains the motivic decomposition of the Fano variety G_2/P_2 together with the motivic isomorphism found by J.-P. Bonnet.

We start with the following

Definition 6. Let X be a projective variety over a field k of dimension n. Let k' be the separable closure of the field k. Consider the scalar extension $X' = X \times_k k'$. We say a cycle $J \in \mathrm{CH}(X')$ is *rational* if it lies in the image of the pull-back homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(X')$. For instance, there is an obvious rational cycle $\Delta_{X'}$ on $\mathrm{CH}^n(X' \times X')$ that is given by the diagonal class. Clearly, any linear combinations, intersections and correspondence products of rational cycles are rational.

The reduction to a separably closed field. The basic idea of the proof is to produce several rational projectors p_i (idempotents) in the ring of endomorphisms $End_{\mathcal{M}}(X_1') = \mathrm{CH}^{15}(X_1' \times X_1')$ over the separable closure k' of the field k with the property $\sum_i p_i = \Delta_{X_1}$. Then by Rost Nilpotence Theorem we can lift those idempotents to $End_{\mathcal{M}}(X)$ and hence, obtain the desired decomposition. So from this point on we work over the separably closed field $k' = k$.

The properties we use. In order to produce such projectors we use the following properties of the varieties X_1 and X_2:

(i) $Pic(X_i)$ is generated by rational cycles;

(ii) X_1 and X_2 split completely over the function fields of the each other (generically split);

(iii) The Chow group $\mathrm{CH}(X_i)$ has a rational basis element in each codimension $\leq [\dim X_i/2] = 7$.

Observe that the first property holds by the result of Merkurjev and Tignol [MT95] that provides an exact sequence relating $Pic(X)$ and $Pic(X')$ by means of Tits algebras. Since any group of type F_4 is adjoint and simply-connected, these algebras are trivial, and one obtains $Pic(X) = Pic(X')$.

The second property holds since our group is obtained by the first Tits process and, hence, there are only two Tits diagrams (completely split or anisotropic).

The last property holds by Pieri formulas (when computing $\mathrm{CH}(G/P)$) and the fact that any cycle times 3 is rational (our varieties split by a cubic field extension).

Remark 7. Note that the same properties hold for Pfister quadrics and for projective homogeneous varieties of type G_2.

The tools we use To perform all the computations we use the following tools

(a) The generic point diagram

Let X and Y be projective homogeneous varieties over k that split completely over the function fields $k(Y)$ and $k(X)$ respectively. Consider the following pull-back diagram

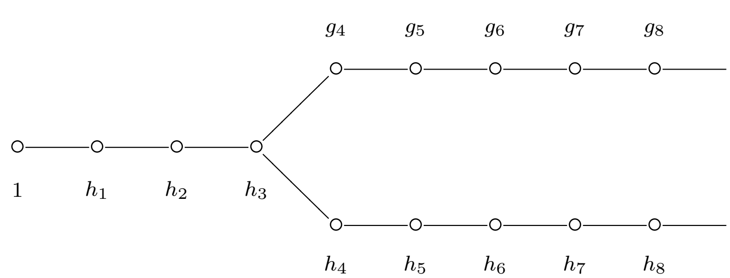
$$\begin{array}{ccc} \mathrm{CH}(X \times Y) & \xrightarrow{g} & \mathrm{CH}(X' \times Y') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathrm{CH}(X_{k(Y)}) & \xrightarrow{=} & \mathrm{CH}(X'_{k'(Y')}) \end{array}$$

where the vertical arrows are surjective. Now take any cycle $\alpha \in \mathrm{CH}^i(X' \times Y')$, $i \leq \dim X$. Let $\beta = (g \circ f)^{-1}(f'(\alpha))$. Then $f'(\beta) = f'(\alpha)$ and β is rational. Hence, $\beta = \alpha + J$, where $J \in ker f'$, and we conclude that $\alpha + J \in \mathrm{CH}^i(X' \times Y')$ is rational.

(b) The Hasse diagram for G/P

(c) The computer program that computes the ring structure of $\mathrm{CH}(G/P)$ for a split group G and a parabolic subgroup P

The proof Consider the part (codimensions $i = 0 \ldots 8$) of the Hasse diagram for the projective homogeneous variety $X = G/P_i$, $i = 1, 4$. Denote by h_i, $i = 0 \ldots 8$, and g_i, $i = 0 \ldots 4$, the respective basis elements of Chow groups $\mathrm{CH}^i(X)$.



By property (i) the generator h_1 is rational. Hence all its powers $h_2 = h_1^2$, $h_3 = h_1^3$ are rational as well. By the generic point diagram (a) and property

(ii) we obtain that the cycle $h_4 \times 1 \pm 1 \times g_4$ is rational. Hence, its square $r = (h_4 \times 1 \pm 1 \times g_4)^2$ is rational as well. Observe that $r \in \mathrm{CH}^8(X)$. Now by property (iii) we have rational cycles r_i in each codimension $i = 0 \dots 7$. Multiplying r by $r_i \times r_{7-i}$ for $i = 4, 5, 6, 7$ we obtain some rational cycles in $\mathrm{CH}^{15}(X \times X)$. It turns out that taking its linear combinations one obtains the desired projectors p_0, p_{12}, p_3.

Applying the similar procedure to the product $X_1 \times X_2$ one obtains the desired motivic isomorphism.

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