ESSENTIAL DIMENSION OF QUADRATIC FORMS WITH TRIVIAL DISCRIMINANT AND CLIFFORD INVARIANT

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ABSTRACT. We conclude the computation of the essential dimension of split spinor groups and give an application in algebraic theory of quadratic forms. We also compute essential dimension of quadratic forms with trivial discriminant and Clifford invariant.

1. INTRODUCTION

Let F be a field and let $\mathcal{F} : \textsf{\textit{Fields}}/F \to \textsf{\textit{Sets}}$ be a functor from the category of field extensions over F to the category of sets. Let $E \in \textsf{\textit{Fields}}/F$ and $K \subset E$ a subfield over F. We say that that K is a *field of definition* of an element $\alpha \in \mathcal{F}(E)$ if α belongs to the image of the map $\mathcal{F}(K) \to \mathcal{F}(E)$. The *essential dimension of* α, denoted $\operatorname{ed}^{\mathcal{F}}(\alpha)$, is the least transcendence degree $\operatorname{tr.deg}_F(K)$ over all fields of definition K of α. The *essential dimension of the functor* $\mathcal{F}$ is

$$\operatorname{ed}(\mathcal{F}) = \sup\{\operatorname{ed}^{\mathcal{F}}(\alpha)\},$$

where the supremum is taken over all fields $E \in \textsf{\textit{Fields}}/F$ and all $\alpha \in \mathcal{F}(E)$ (see [1, Def. 1.2] or [12, Sec.1]). Informally, the essential dimension of $\mathcal{F}$ is the smallest number of algebraically independent parameters required to define $\mathcal{F}$ and may be thought of as a measure of complexity of $\mathcal{F}$.

Let p be a prime integer. The *essential p-dimension of* α, denoted $\operatorname{ed}_p^{\mathcal{F}}(\alpha)$, is defined as the minimum of $\operatorname{ed}^{\mathcal{F}}(\alpha_{E'})$, where E' ranges over all finite field extensions of E of degree prime to p. The *essential p-dimension of* $\mathcal{F}$ is

$$\operatorname{ed}_p(\mathcal{F}) = \sup\{\operatorname{ed}_p^{\mathcal{F}}(\alpha)\},$$

where the supremum ranges over all fields $E \in \textsf{\textit{Fields}}/F$ and all $\alpha \in \mathcal{F}(E)$. By definition, $\operatorname{ed}(\mathcal{F}) \geq \operatorname{ed}_p(\mathcal{F})$ for all p.

For convenience we write $\operatorname{ed}_0(\mathcal{F}) = \operatorname{ed}(\mathcal{F})$, so $\operatorname{ed}_p(\mathcal{F})$ is defined for $p = 0$ and all prime p.

Let G be an algebraic group scheme over F. Write $\mathcal{F}_G$ for the functor taking a field extension E/F to the set $H^1_{et}(E, G)$ of isomorphism classes of principal homogeneous G-spaces (G-torsors) over E. The essential (p-)dimension of $\mathcal{F}_G$

2000 *Mathematics Subject Classification.* 11E04, 11E57, 11E72, 11E81, 14L35, 20G15.

Key words and phrases. Linear algebraic groups, Spinor groups, essential dimension, torsor, non-abelian cohomology, quadratic forms, Witt rings, the fundamental ideal.

The work of the first author has been supported in part by Canada Research Chairs Program and NSERC research grant.

The work of the second author has been supported by the NSF grant DMS #0652316.

is called the *essential (p-)dimension of* G and is denoted by $\mathrm{ed}(G)$ and $\mathrm{ed}_p(G)$ (see [14] and [15]).

In this paper we conclude computation of the essential dimension of the split spinor groups originated in [2] and [7] and continued in [12] (Theorem 2.2). We give an application in algebraic theory of quadratic forms (Theorem 4.2). We also compute essential dimension of quadratic forms with trivial discriminant and Clifford invariant (Theorem 7.1).

2. Essential dimension of $\mathbf{Spin}_n$

Let G be an algebraic group over F and let $C \subset G$ be a normal subgroup over F. For a torsor $E \to \mathrm{Spec}(F)$ of the group $H := G/C$ consider the stack $[E/G]$ (see [16]). Recall that an object of the category $[E/G](K)$ for a field extension K/F is a pair (E', φ), where E' is a G-torsor over K and $\varphi : E'/C \xrightarrow{\sim} E_K$ is an isomorphism of H-torsors over$K$. The essential dimension $\mathrm{ed}[E/G]$ of the stack $[E/G]$ is the essential dimension of the functor $K \mapsto$ set of isomorphism classes of objects in $[E/G](K)$.

The following proposition was proven independently by R. Lötscher in [11, Ex. 3.4]:

Proposition 2.1. *Let C be a normal subgroup of an algebraic group G over F and $H = G/C$. Then*

$$\mathrm{ed}(G) \leq \mathrm{ed}(H) + \max \mathrm{ed}[E/G],$$

where the maximum is taking over all field extensions L/F and all H-torsors E over L.

Proof. Let I' be a G-torsor over a field extension K/F. Then $I := I'/C$ is an H-torsor over K. There is a subextension K_0/F of K/F and an H-torsor E over K_0 such that there is an isomorphism $\varphi : I \xrightarrow{\sim} E_K$ of H-torsors and $\mathrm{tr.deg}(K_0/F) \leq \mathrm{ed}(H)$.

Consider the stack $[E/G]$ over K_0. The pair (I', φ) is an object of $[E/G](K)$. There is a subextension K_1/K_0 of K/K_0 such that (I', φ) is defined over K_1 and $\mathrm{tr.deg}(K_1/K_0) \leq \mathrm{ed}[E/G]$. It follows that I' is defined over the field K_1 with

$$\mathrm{tr.deg}(K_1/F) = \mathrm{tr.deg}(K_0/F) + \mathrm{tr.deg}(K_1/K_0) \leq \mathrm{ed}(H) + \mathrm{ed}[E/G]. \quad \square$$

The following theorem concludes computation of the essential dimension of the spinor groups initiated in [2] and [7] and continued in [12]. We write $\mathbf{Spin}_n$ for the split spinor group of a nondegenerate quadratic form of dimension n and maximal Witt index.

If $\mathrm{char}(F) \neq 2$, then the essential dimension of $\mathbf{Spin}_n$ has the following values for $n \leq 14$ (see [7, §23]:

n	≤ 6	7	8	9	10	11	12	13	14
$\mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n)$	0	4	5	5	4	5	6	6	7

In the following theorem we give the values of $\operatorname{ed}_p(\mathbf{Spin}_n)$ for $n \geq 15$ and $p = 0$ and 2. Note that $\operatorname{ed}_p(\mathbf{Spin}_n) = 0$ if $p \neq 0, 2$ as every $\mathbf{Spin}_n$-torsor over a field is split over an extension of degree a power of 2.

Theorem 2.2. *Let F be a field of characteristic zero. Then for every integer $n \geq 15$ we have:*

$$\operatorname{ed}_2(\mathbf{Spin}_n) = \operatorname{ed}(\mathbf{Spin}_n) = \begin{cases} 2^{(n-1)/2} - \frac{n(n-1)}{2}, & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - \frac{n(n-1)}{2}, & \text{if } n \equiv 2 \pmod 4; \\ 2^{(n-2)/2} + 2^m - \frac{n(n-1)}{2}, & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where 2^m is the largest power of 2 dividing n.

Proof. The case $n \geq 15$ and n is not divisible by 4 has been considered in [2, Th. 3.3].

Now assume that $n > 15$ and n is divisible by 4. The inequality "$\geq$" was obtained in [12, Th. 4.9], so we just need to prove the inequality "$\leq$". The case $n = 16$ was considered in [12, Cor. 4.10]. Assume that $n \geq 20$ and n is divisible by 4.

Consider the following diagram with the exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & \mathbf{Spin}_n \arrow[r] \arrow[d] & \mathbf{Spin}_n^+ \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{O}_n^+ \arrow[r] & \mathbf{PGO}_n^+ \arrow[r] & 1, \end{tikzcd}$$

where $\mathbf{Spin}_n^+$ is the semi-spinor group, $\mathbf{O}_n^+$ is the split special orthogonal group and $\mathbf{PGO}_n^+$ is the split special projective orthogonal group. We see from the diagram that the image of the connecting map

$$\delta_K : H^1_{et}(K, \mathbf{Spin}_n^+) \to H^2_{et}(K, \boldsymbol{\mu}_2) \subset \operatorname{Br}(K)$$

is contained in the image of the other connecting map

$$H^1_{et}(K, \mathbf{PGO}_n^+) \to H^2_{et}(K, \boldsymbol{\mu}_2) \subset \operatorname{Br}(K)$$

for every field extension K/F. The image of the last map consists of the classes $[A]$ of all central simple K-algebras A of degree n admitting orthogonal involutions (see [9, §31]). As $\operatorname{ind}(A)$ is a power of 2 dividing n, we have $\operatorname{ind}(A) \leq 2^m$, where 2^m is the largest power of 2 dividing n.

Let E be a $\mathbf{Spin}_n^+$-torsor over K. We have shown that if $\delta_K([E]) = [A]$ for a central simple K-algebra A, then $\operatorname{ind}(A) \leq 2^m$. It follows from [3, Th. 4.1] that $\operatorname{ed}[E/\mathbf{Spin}_n] = \operatorname{ind}(A) \leq 2^m$.

It is shown in [2, Rem. 3.10] that $\operatorname{ed}(\mathbf{Spin}_n^+) = 2^{(n-2)/2} - \frac{n(n-1)}{2}$ for every integer $n \geq 20$ divisible by 4. Finally, by Proposition 2.1,

$$\operatorname{ed}(\mathbf{Spin}_n) \leq \operatorname{ed}(\mathbf{Spin}_n^+) + 2^m = 2^{(n-2)/2} + 2^m - \frac{n(n-1)}{2}. \qquad \square$$

3. The functors I_n^k

We use the following notation. Let F be a field of characteristic different from 2 and K/F a field extension. We define:

$$I_n^1(K) = \boxed{\begin{array}{c}\text{Set of isomorphism classes of nondegenerate}\\ \text{quadratic forms over } K \text{ of dimension } n\end{array}}$$

We have a natural bijection $I_n^1(K) \simeq H^1_{et}(K, \mathbf{O}_n)$ (see [9, §29.E]).

Recall that the *discriminant* $\operatorname{disc}(q)$ of a form $q \in I_n^1(K)$ is equal to $(-1)^{n(n-1)/2}\det(q) \in K^\times/K^{\times 2}$. Set

$$I_n^2(K) = \{q \in I_n^1(K) \quad \text{such that} \quad \operatorname{disc}(q) = 1\}.$$

We have a natural bijection $I_n^2(K) \simeq H^1_{et}(K, \mathbf{O}_n^+)$ (see [9, §29.E]).

The *Clifford invariant* $c(q)$ of a form $q \in I_n^2(K)$ is the class in the Brauer group $\operatorname{Br}(K)$ of the Clifford algebra of q if n is even and the class of the even Clifford algebra if n is odd [9, §8.B]. Define

$$I_n^3(K) = \{q \in I_n^2(K) \quad \text{such that} \quad c(q) = 0\}.$$

Remark 3.1. Our notation of the functors I_n^k for $k = 1, 2, 3$ is explained by the following property: $I_n^k(K)$ consists of all classes of quadratic forms $q \in W(K)$ of dimension n such that $q \in I(K)^k$ if n is even and $q \perp \langle -1\rangle \in I(K)^k$ if n is odd, where $I(K)$ is the fundamental ideal in the Witt ring $W(K)$ of K.

The functor I_n^3 is related to $\mathbf{Spin}_n$-torsors as follows. The short exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{Spin}_n \to \mathbf{O}_n^+ \to 1$$

yields an exact sequence

$$H^1_{et}(K, \boldsymbol{\mu}_2) \to H^1_{et}(K, \mathbf{Spin}_n) \to H^1_{et}(K, \mathbf{O}_n^+) \xrightarrow{c} H^2_{et}(K, \boldsymbol{\mu}_2), \tag{1}$$

where c is the Clifford invariant. Thus $\operatorname{Ker}(c) = I_n^3(K)$.

The essential dimension of I_n^1 and I_n^2 was computed in [14, Th. 10.3 and 10.4]: we have $\operatorname{ed}(I_n^1) = n$ and $\operatorname{ed}(I_n^2) = n-1$. In Section 7 we compute $\operatorname{ed}(I_n^3)$.

Lemma 3.2. *We have:* $\operatorname{ed}_p(I_n^3) \le \operatorname{ed}_p(\mathbf{Spin}_n) \le \operatorname{ed}_p(I_n^3) + 1$.

Proof. Let K/F be a field extension. The group $H^1_{et}(K, \boldsymbol{\mu}_2) = K^\times/K^{\times 2}$ acts transitively on the fibers of the second map in the sequence (1). It follows that the natural map $\mathbf{Spin}_n$-*Torsors* $\to I_n^3$ is a surjection with $\mathbf{G}_\mathrm{m}$ acting surjectively on the fibers. The statement follows from [1, Prop. 1.13]. □

4. Subforms of forms in I_n^3

In this section we study the following problem in quadratic form theory which will be used in Section 7 in order to compute the essential dimension of I_n^3. Note that the problem is stated entirely in terms of quadratic forms, while in the solution we use the essential dimension. We don't know how to solve the problem by means of quadratic form theory.

Problem 4.1. *Given a field F, determine all integers n such that every form in $I^3_n(K)$ contains a nontrivial subform in $I^2(K)$ for any field extension K/F.*

All forms in $I^3_n(K)$ for $n \le 14$ are classified (see [7, Ex. 17,8, Th.17.13 and Th. 21.3]). Inspection shows that for such n the problem has positive solution.

In the following theorem we show that in the range $n \ge 15$ the problem has negative solution (with possibly two exceptions).

Theorem 4.2. *Let F be a field of characteristic zero, $n \ge 15$ and let b be an even integer with $0 < b < n$. Then there is a field extension K/F and a form in $I^3_n(K)$ that does not contain a subform in $I^2_b(K)$ (with possible exceptions: $(n, b) = (15, 8)$ or $(16, 8)$).*

Let $a := n - b$. Write $H_{a,b}$ for the image of the natural homomorphism

$$\mathbf{Spin}_a \times \mathbf{Spin}_b \to \mathbf{Spin}_n . \tag{2}$$

Note that the kernel of (2) is contained in

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = \operatorname{Ker}\big(\mathbf{Spin}_a \times \mathbf{Spin}_b \to \mathbf{O}^+_a \times \mathbf{O}^+_b\big)$$

and therefore, is the cyclic group of order 2 generated by $(-1, -1)$. Hence we have an exact sequence

$$1 \to \boldsymbol{\mu}_2 \to H_{a,b} \to \mathbf{O}^+_a \times \mathbf{O}^+_b \to 1$$

and therefore a map

$$H^1_{et}(R, H_{a,b}) \to H^1_{et}(R, \mathbf{O}^+_a \times \mathbf{O}^+_b) = H^1_{et}(R, \mathbf{O}^+_a) \times H^1_{et}(R, \mathbf{O}^+_b)$$

for a commutative F-algebra R.

We write $q(\eta) := (q_a, q_b)$ for the image of an element $\eta \in H^1_{et}(R, H_{a,b})$ under this map, where $q_a \in H^1_{et}(R, \mathbf{O}^+_a)$ and $q_b \in H^1_{et}(R, \mathbf{O}^+_b)$.

Consider the commutative diagram with the exact rows

$$\begin{tikzcd}
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] \arrow[d, equal] & H_{a,b} \arrow[r] \arrow[d] & \mathbf{O}^+_a \times \mathbf{O}^+_b \arrow[r] \arrow[d, "\tau"'] & 1 \\
1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{Spin}_n \arrow[r] & \mathbf{O}^+_n \arrow[r] & 1.
\end{tikzcd}$$

The image of an element $\xi \in H^1_{et}(R, \mathbf{Spin}_n)$ in $H^1_{et}(R, \mathbf{O}^+_n)$ will be denoted by $q(\xi)$.

If $\xi \in H^1_{et}(R, \mathbf{Spin}_n)$ is the image of an element $\eta \in H^1_{et}(R, H_{a,b})$, then $q(\xi) = q_a \perp q_b$, the image of $(q_a, q_b) = q(\eta)$ under the map induced by τ. We can reverse this statement as follows:

Lemma 4.3. *Let $\xi \in H^1_{et}(R, \mathbf{Spin}_n)$ with $q(\xi) = q_a \perp q_b$, where $q_a \in H^1_{et}(R, \mathbf{O}^+_a)$ and $q_b \in H^1_{et}(R, \mathbf{O}^+_b)$. Then ξ is the image of an element η under the map $H^1_{et}(R, H_{a,b}) \to H^1_{et}(R, \mathbf{Spin}_n)$ such that $q(\eta) = (q_a, q_b)$.*

Proof. The diagram above yields a commutative diagram with the exact rows:

$$\begin{array}{ccccc} H^1_{et}(R, H_{a,b}) & \longrightarrow & H^1_{et}(R, \mathbf{O}^+_a) \times H^1_{et}(R, \mathbf{O}^+_b) & \xrightarrow{c'} & H^2_{et}(R, \boldsymbol{\mu}_2) \\ \downarrow & & \downarrow & & \| \\ H^1_{et}(R, \mathbf{Spin}_n) & \longrightarrow & H^1_{et}(R, \mathbf{O}^+_n) & \xrightarrow{c} & H^2_{et}(R, \boldsymbol{\mu}_2). \end{array}$$

Moreover, the group $H^1_{et}(R, \boldsymbol{\mu}_2)$ acts transitively on the fibers of the left maps in the two rows. The result follows. □

For non-negative integers a, b and a field extension K/F set

$$I^3_{a,b}(K) := \{(q_a, q_b) \in I^2_a(K) \times I^2_b(K) \quad \text{such that} \quad q_a \perp q_b \in I^3_n(K)\}.$$

Corollary 4.4. *For any $\eta \in H^1_{et}(K, H_{a,b})$ we have $q(\eta) \in I^3_{a,b}(K)$. The morphism of functors $q : H_{a,b}$-Torsors $\to I^3_{a,b}$ is surjective. In particular, $\mathrm{ed}_p(I^3_{a,b}) \le \mathrm{ed}_p(H_{a,b})$ for every $p \ge 0$.*

Proof. Note that the map c' in the proof of Lemma 4.3 when $R = K$ takes a pair (q_a, q_b) to the Clifford invariant of $q_a \perp q_b$ in $\mathrm{Br}(K)$. The pair $(q_a, q_b) \in I^2_a(K) \times I^2_b(K)$ comes from $H^1_{et}(K, H_{a,b})$ if and only if the Clifford invariant of $q_a \perp q_b$ is split, i.e., $q_a \perp q_b \in I^3_n(K)$. □

Lemma 4.5. *For an even a and any b,*

$$\mathrm{ed}_p(I^3_{a,b}) \le \mathrm{ed}_p(I^3_{a-1,b}) + 1$$

for every $p \ge 0$.

Proof. Consider the morphism of functors

$$\alpha : \mathbf{G}_{\mathrm{m}} \times I^3_{a-1,b} \to I^3_{a,b}, \quad (\lambda; f, g) \mapsto (\lambda(f \perp \langle -1 \rangle), g).$$

Every form h in $I^2_a(K)$ can be written in the form $h = \lambda(f \perp \langle -1 \rangle)$ for a value λ of h and a form $f \in I^2_{a-1}(K)$, i.e., α is a surjection, whence the result. □

Write V_n (respectively W_n) for the (semi-)spinor (respectively regular) representation of the group $\mathbf{Spin}_n$. We have

$$\dim(V_n) = \begin{cases} 2^{(n-1)/2}, & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2}, & \text{if } n \text{ is even,} \end{cases}$$

and $\dim(W_n) = n$. We consider the tensor product $V_{a,b} := V_a \otimes V_b$ as the representation of the group $H_{a,b}$. We also view W_a (respectively W_b) as a $H_{a,b}$-representation via the natural homomorphism $H_{a,b} \to \mathbf{O}^+_a$ (respectively $H_{a,b} \to \mathbf{O}^+_b$).

A representation V of an algebraic group H is *generically free* if the stabilizer of a generic vector in V is trivial. In this case by [15],

$$\mathrm{ed}(H) \le \dim(V) - \dim(H).$$

Lemma 4.6. *Let a be odd and b even. Suppose that $V_{a,b}$ is a generically free representation of the image of the homomorphism $H_{a,b} \to \mathbf{GL}(V_{a,b})$. Then $V_{a,b} \oplus W_b$ is a generically free representation of $H_{a,b}$. In particular,*

$$\mathrm{ed}(H_{a,b}) \leq \dim(V_{a,b}) + \dim(W_b) - \dim(H_{a,b}).$$

Proof. Write C_n for the kernel of $\mathbf{Spin}_n \to \mathbf{PGO}_n^+$ and C'_n for the kernel of $\mathbf{Spin}_n \to \mathbf{O}_n^+$ so $C'_n = \{\pm 1\} \subset C_n$. By assumption, the generic stabilizer H of the action of $\mathbf{Spin}_a \times \mathbf{Spin}_b$ on $V_{a,b}$ is contained in the center $C_a \times C_b$. Since $C_b/C'_b = \boldsymbol{\mu}_2$ acts on W_b by multiplication by -1 we have $H \subset C_a \times C'_b \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. Note that $\boldsymbol{\mu}_2 \times 1$ and $1 \times \boldsymbol{\mu}_2$ act by multiplication by -1 on $V_{a,b}$, hence H is generated by $(-1,-1)$. It follows that $H_{a,b} = (\mathbf{Spin}_a \times \mathbf{Spin}_b)/H$ acts generically freely on $V_{a,b} \oplus W_b$. □

Proposition 4.7. *Let* $\mathrm{char}(F) = 0$. *If $n = a + b \geq 15$ with $a \leq b$, then $V_{a,b}$ is a generically free representation of the image of $H_{a,b} \to \mathbf{GL}(V_{a,b})$ if and only if $(a,b) \neq (3,12)$, $(4,11)$, $(4,12)$, $(6,10)$ and $(8,8)$.*

Proof. All the cases of infinite generic stabilizers H are listed in [5, §3] (row 7 of Table 6): H is infinite if and only if $(a,b) = (3,12)$ and $(4,12)$.

If H is finite, by [13, Th. 1] (rows 1, 12 and 13 of Table 1) H is nontrivial if and only if $(a,b) = (4,11)$, $(6,10)$ and $(8,8)$. □

Proof of Theorem 4.2. Note that the case (n,b) with n even implies the case $(n-1,b)$. Indeed suppose that every form in I^3_{n-1} for an even n contains a subform from I^2_b. Take any form $q \in I^3_n(K)$ for a field extension K/F and write $q = \lambda(f \perp \langle -1 \rangle)$ for a $\lambda \in K^\times$ and $f \in I^3_{n-1}(K)$. If f contains a subform $h \in I^2_b(K)$, then q contains λh.

We need to show that the natural morphism of functors $I^3_{a,b} \to I^3_n$ is not surjective. It suffices to prove that $\mathrm{ed}(I^3_{a,b}) < \mathrm{ed}(I^3_n)$. We may assume that n (and hence also a) is even. Moreover, we may assume that $a \leq b$.

Suppose that $n \geq 18$. By Proposition 4.7, Lemma 4.5, Lemma 4.6 and Corollary 4.4,

$$\begin{aligned}
\mathrm{ed}(I^3_{a,b}) &\leq \mathrm{ed}(I^3_{a-1,b}) + 1 \\
&\leq \mathrm{ed}(H_{a-1,b}) + 1 \\
&\leq \dim(V_{a-1,b}) + \dim(W_b) - \dim(H_{a-1,b}) + 1 \\
&= 2^{n/2-2} + b - (a-1)(a-2)/2 - b(b-1)/2 + 1 \\
&= 2^{n/2-2} - (a^2 + b^2 - 3a - 3b)/2 \\
&\leq 2^{n/2-2} - (n^2 - 6n)/4
\end{aligned}$$

as $a^2 + b^2 \geq n^2/2$. The last integer is strictly less than

$$2^{n/2-1} - n(n-1)/2 - 1 \leq \mathrm{ed}(\mathbf{Spin}_n) - 1 \leq \mathrm{ed}(I^3_n)$$

by Theorem 2.2 and Lemma 3.2.

It remains to consider the case $n = 16$. Note that by Theorem 2.2 and Lemma 3.2,

$$\operatorname{ed}(I^3_{16}) \geq \operatorname{ed}(\mathbf{Spin}_{16}) - 1 = 23. \tag{3}$$

We shall prove that $\operatorname{ed}(I^3_{a,b}) < 23$. All possible values of b are 8, 10, 12 and 14.

Case $(n, b) = (16, 10)$: Consider the representation $V := W_6 \oplus V_{6,10} \oplus W_{10}$ of $H_{6,10}$. We claim that V is generically free. The stabilizer in $\mathbf{Spin}_6$ of a point in general position in W_6 is $\mathbf{Spin}_5$. Hence the stabilizer in $H_{6,10}$ of a point in general position in W_6 is $H_{5,10}$. Note that the restriction of $V_{6,10}$ to $H_{5,10}$ is isomorphic to $V_{5,10}$. Finally, the $H_{5,10}$-representation $V_{5,10} \oplus W_{10}$ is generically free by Proposition 4.7.

It follows from (3) and Corollary 4.4 that

$$\operatorname{ed}(I^3_{6,10}) \leq \operatorname{ed}(H_{6,10}) \leq \dim(V) - \dim(H_{6,10}) = 80 - 60 = 20.$$

Case $(n, b) = (16, 12)$: Consider the representation $V := W_3 \oplus W_3 \oplus V_{3,12} \oplus W_{12}$ of $H_{3,12}$. We claim that V is generically free as the representation of $H_{3,12}$. Indeed, the stabilizer in $H_{3,12}$ of a generic vector in W_{12} is $H_{3,11}$. We are reduced to showing that $W_3 \oplus W_3 \oplus V_{3,11}$ is a generically free representation of $H_{3,11}$. By [13, §5, p. 246] the generic stabilizer S of $H_{3,11}$ in $V_{3,11}$ is finite (isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$), and the restriction to S of the natural projection $H_{3,11} \to \mathbf{O}^+_3$ is injective. It remains to notice that the representation $W_3 \oplus W_3$ of $\mathbf{O}^+_3 = \mathbf{PGL}_2$ is generically free.

It follows from Lemmas 4.5 and 4.6 and Corollary 4.4 that

$$\begin{aligned} \operatorname{ed}(I^3_{4,12}) \leq \operatorname{ed}(I^3_{3,12}) + 1 \leq \operatorname{ed}(H_{3,12}) + 1 \leq \dim(V) - \dim(H_{3,12}) + 1 \\ = 82 - 69 + 1 = 14. \end{aligned}$$

Case $(n, b) = (16, 14)$: As every form in I^3_2 is hyperbolic, we have $I^3_{2,14} = I^3_{14}$ and $\operatorname{ed}(I^3_{14}) = 7$ by Theorem 2.2. $\square$

5. Unramified principal homogeneous spaces

Let G be an algebraic group over F and let K/F be a field extension with a discrete valuation v trivial on F. Write O for the valuation ring of v. It is a local F-algebra. We say that a class $\xi \in H^1_{et}(K, G)$ is *unramified* (with respect to v) if ξ belongs to the image of the map $H^1_{et}(O, G) \to H^1_{et}(K, G)$.

Let $\overline{K}$ be the residue field of v. The ring homomorphism $O \to \overline{K}$ yields a map $H^1_{et}(O, G) \to H^1_{et}(\overline{K}, G)$. This map is a bijection if K is complete (see [4, Exp. XXIV. Prop. 8.1]). Hence we have the map

$$H^1_{et}(\overline{K}, G) \xrightarrow{\sim} H^1_{et}(O, G) \to H^1_{et}(K, G). \tag{4}$$

Example 5.1. Let $\operatorname{char}(F) \neq 2$ and $G = \mathbf{O}_n$. Then $H^1_{et}(K, G)$ is the set of isomorphism classes of nondegenerate quadratic forms of dimension n over K. A quadratic form q over a field K with a discrete valuation is unramified if and only if $q \simeq \langle a_1, a_2, \ldots, a_n \rangle$, where a_i are units in the valuation ring O in K. In general, every q can be written $q = q_1 \perp \pi q_2 \perp h$, where π is a prime element, q_1 and q_2 are unramified anisotropic quadratic forms and h is a hyperbolic form. The form q is unramified if and only if $q_2 = 0$. It follows that if two forms q and πq are both unramified, then q is hyperbolic. If K is complete, then the map (4) takes $f = \langle \bar{a}_1, \bar{a}_2, \dots, \bar{a}_n \rangle$ over $\overline{K}$, where a_i are units in O, to $f_K := \langle a_1, a_2, \dots, a_n \rangle$.

6. Essential dimension of PI_n^3

Two quadratic forms f and g over a field K are called *similar* if $f = \lambda g$ for some $\lambda \in K^\times$. If n is even, we write $PI_n^3(K)$ for the set of similarity classes of forms in $I_n^3(K)$. The group $K^\times$ acts transitively on the fibers of the the natural surjective map $I_n^3(K) \to PI_n^3(K)$. Hence

$$\operatorname{ed}_p(PI_n^3) \leq \operatorname{ed}_p(I_n^3) \leq \operatorname{ed}_p(PI_n^3) + 1$$

for any $p \geq 0$ by [1, Prop. 1.13].

Proposition 6.1. *Let* $\operatorname{char}(F) \neq 2$. *For an even* $n \geq 8$, *and* $p = 0$ *or* 2, *we have*

$$\operatorname{ed}_p(PI_n^3) = \operatorname{ed}_p(I_n^3) - 1.$$

Proof. Let K/F be a field extension and let $q \in I_n^3(K)$ be a non-hyperbolic form. Consider the form tq over the field $K((t))$. It suffices to show that

$$\operatorname{ed}_p^{I_n^3}(tq) \geq \operatorname{ed}_p^{PI_n^3}(q) + 1.$$

Let $M/K((t))$ be a finite field extension of degree prime to p (i.e., $M = K((t))$ if $p = 0$ and $[M : K((t))]$ is odd if $p = 2$), let L/F be a subextension of M/F and let $f \in I_n^3(L)$ be such that $\operatorname{tr.deg}(L/F) = \operatorname{ed}_p^{I_n^3}(tq)$ and $tq_M \simeq f_M$.

Let v be the (unique) extension on M of the discrete valuation of $K((t))$ and let w be the restriction of v on L. The residue field $\overline{M}$ is a finite extension of K of degree prime to p. As the form q is not hyperbolic, q_M is not hyperbolic and therefore, the form $tq_M \simeq f_M$ is ramified by Example 5.1. It follows that w is nontrivial, i.e., w is a discrete valuation on L.

Let $\widehat{L}$ be the completion of L. Note that as M is complete, we can identify $\widehat{L}$ with a subfield of M. Write $f_{\widehat{L}} \simeq (f_1)_{\widehat{L}} \perp \pi(f_2)_{\widehat{L}}$, where f_1 and f_2 are quadratic forms over the residue field $\overline{L}$ and $\pi \in L$ is a prime element (see Example 5.1). Note that $f_1, f_2 \in I^2(\overline{L})$ by [6, Lemma 19.4]. If the ramification index e of M/L is even, then π is a unit in the valuation ring O of M modulo squares in $M^\times$, hence f_M is unramified, a contradiction. It follows that e is odd. Writing $\pi = ut^e$ with a unit $u \in O^\times$ we have

$$tq_M \simeq f_M \simeq (f_1)_M \perp \pi(f_2)_M \simeq (f_1)_M \perp ut(f_2)_M,$$

hence $(f_1)_M = 0$ and $q_M = u(f_2)_M$ in $W(M)$. It follows that $(f_1)_{\overline{M}} = 0$ and $q_{\overline{M}} = \bar{u}(f_2)_{\overline{M}}$ in $W(\overline{M})$ and therefore,

$$q_{\overline{M}} = \bar{u}(f_2)_{\overline{M}} = \bar{u}g_{\overline{M}}, \tag{5}$$

where $g := f_1 \perp f_2$ is the form over $\overline{L}$ of dimension n. Note that $f_{\widehat{L}} - g_{\widehat{L}} = \langle \pi, -1\rangle (f_2)_{\widehat{L}} \in I^3(\widehat{L})$, hence $g_{\widehat{L}} \in I^3(\widehat{L})$ and $g \in I^3(\overline{L})$.

It follows from (5) that $q_{\overline{M}}$ is similar to $g_{\overline{M}}$, i.e., the form q is p-defined over $\overline{L}$ for the functor PI^3_n (see [12, §1.1]) and therefore

$$\mathrm{ed}_p^{I^3_n}(tq) = \mathrm{tr.deg}(L/F) \geq \mathrm{tr.deg}(\overline{L}/F) + 1 \geq \mathrm{ed}_p^{PI^3_n}(q) + 1. \qquad \square$$

7. Essential dimension of I^3_n

In this section we compute the essential dimension of I^3_n.

Theorem 7.1. *Let F be a field of characteristic zero. Then for every integer $n \geq 15$ we have:*

$$\mathrm{ed}_2(I^3_n) = \mathrm{ed}(I^3_n) = \begin{cases} 2^{(n-1)/2} - 1 - \frac{n(n-1)}{2}, & \text{if } n \text{ is odd;} \\ 2^{(n-2)/2} - \frac{n(n-1)}{2}, & \text{if } n \equiv 2 \pmod 4; \\ 2^{(n-2)/2} + 2^m - 1 - \frac{n(n-1)}{2}, & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where 2^m is the largest power of 2 dividing n.

If $char(F) \neq 2$, then the essential dimension of I^3_n has the following values for $n \leq 14$:

n	≤ 6	7	8	9	10	11	12	13	14
$\mathrm{ed}_2(I^3_n) = \mathrm{ed}(I^3_n)$	0	3	4	4	4	5	6	6	7

Proof. We will prove the theorem case by case.

7.1. **Case $n \equiv 2$ $(mod\ 4)$ and $n \geq 10$.** The exact sequence

$$1 \to \boldsymbol{\mu}_4 \to \mathbf{Spin}_n \to \mathbf{PGO}^+_n \to 1$$

yields a surjective map $\mathbf{Spin}_n$-*Torsors*$(K) \to PI^3_n(K)$ for any K/F with the group $K^\times$ acting transitively on the fibers of this map. It follows from Theorem 2.2, Proposition 6.1 and Lemma 3.2 that

$$\mathrm{ed}_2(I^3_n) = \mathrm{ed}_2(PI^3_n) + 1 \geq \mathrm{ed}_2(\mathbf{Spin}_n) = \mathrm{ed}(\mathbf{Spin}_n) \geq \mathrm{ed}(I^3_n) \geq \mathrm{ed}_2(I^3_n).$$

Hence $\mathrm{ed}_2(I^3_n) = \mathrm{ed}(I^3_n) = \mathrm{ed}(\mathbf{Spin}_n)$. The latter value is known by Theorem 2.2.

7.2. **Case $n \not\equiv 2$ $(mod\ 4)$ and $n \geq 15$.** Let $n = a + b$ with even $b \neq 2$. Let Z be the trivial group if $b = 0$ and the image of the center C_b of $\mathbf{Spin}_b$ in $H_{a,b}$ if $b \geq 4$. Then Z is central in $H_{a,b}$, hence the group $H^1_{et}(K, Z)$ acts on $H^1_{et}(K, H_{a,b})$.

Lemma 7.2. *Let $\xi, \eta \in H^1_{et}(K, H_{a,b})$ with even $b \neq 2$. Suppose that $q(\xi) = q_a \perp q_b$ and $q(\eta) = q_a \perp \lambda q_b$ with the forms $q_a \in I^2_a(K)$ and $q_b \in I^2_b(K)$ and $\lambda \in K^\times$. Then $\eta = \alpha\xi$ for some $\alpha \in H^1_{et}(K, Z)$.*

Proof. The statement is trivial if $b = 0$, so assume that $b \geq 4$. The restriction of the natural homomorphism $H_{a,b} \to \mathbf{O}_b^+$ to the subgroup Z yields a surjection $\varphi : Z \to \boldsymbol{\mu}_2 = Center(\mathbf{O}_b^+)$. The kernel of φ coincides with the kernel C of the canonical homomorphism $H_{a,b} \to \mathbf{O}_a^+ \times \mathbf{O}_b^+$.

As Z is isomorphic to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ or $\boldsymbol{\mu}_4$, the homomorphism $\varphi^* : H^1_{et}(K, Z) \to H^1_{et}(K, \boldsymbol{\mu}_2) = K^\times/K^{\times 2}$ is surjective. Let $\gamma \in H^1_{et}(K, Z)$ be such that $\varphi^*(\gamma) = \lambda K^{\times 2}$. Then $q(\gamma\xi) = q_a \perp \lambda q_b = q(\eta)$. Then there is $\beta \in H^1_{et}(K, C)$ such that $\eta = \beta(\gamma\xi)$. Hence $\eta = \alpha\xi$, where $\alpha = \beta'\gamma$ with β' the image of β under the map $H^1_{et}(K, C) \to H^1_{et}(K, Z)$ induced by the inclusion of C into Z. $\square$

Let $\xi \in H^1_{et}(K, \mathbf{Spin}_n)$ be such that the form $q = q(\xi) \in I^3_n(K)$ is generic for the functor I^3_n (see [12, §2.2]). In particular, $\mathrm{ed}^{I^3_n}(q) = \mathrm{ed}(I^3_n)$. Note that q is anisotropic.

Identifying $\boldsymbol{\mu}_2$ with the kernel of $\mathbf{Spin}_n \to \mathbf{O}_n^+$, we have an action of $H^1_{et}(E, \boldsymbol{\mu}_2) = E^\times/E^{\times 2}$ on $H^1_{et}(E, \mathbf{Spin}_n)$ where $E = K((t))$. Consider the element $t\xi_E \in H^1_{et}(E, \mathbf{Spin}_n)$ over E. We claim that $t\xi_E$ is ramified. Suppose not, i.e., $t\xi_E$ comes from an element $\rho \in H^1_{et}(O, \mathbf{Spin}_n)$ where $O = K[[t]]$. Let $q' \in H^1_{et}(O, \mathbf{O}_n^+)$ be the image of ρ, viewed as a quadratic form over O. We have

$$q'_E = q(t\xi_E) = q(\xi_E) = q_E,$$

hence $q' = q_O$. Then ρ and ξ_O belong to the same fiber of the map

$$H^1_{et}(O, \mathbf{Spin}_n) \to H^1_{et}(O, \mathbf{O}_n^+).$$

As the group $H^1_{et}(O, \boldsymbol{\mu}_2) = O^\times/O^{\times 2}$ acts transitively on the fiber, there is a unit $u \in O^\times$ satisfying $t\xi_E = u\xi_E$. It follows from [9, Prop. 28.11] that tu^{-1} is in the image spinor norm map

$$\mathbf{O}^+(q_E) \to H^1_{et}(E, \boldsymbol{\mu}_2) = E^\times/E^{\times 2}$$

for the form q_E, hence q is isotropic by [6, Th. 18.3], a contradiction. The claim is proven.

Let L/F be a subextension of E/F and let $\eta \in H^1_{et}(L, \mathbf{Spin}_n)$ be such that $\operatorname{tr.deg}(L/F) = \mathrm{ed}^{\mathbf{Spin}_n}(t\xi)$ and $\eta_E \simeq t\xi_E$. We have $q(\eta)_E = q(t\xi) = q(\xi_E) = q_E$, hence the form $q(\eta)_E$ is anisotropic.

Let v be the restriction on L of the discrete valuation of E. As $t\xi$ is ramified, v is nontrivial, hence v is a discrete valuation. Let $\pi \in L$ be a prime element.

Consider the completion $\widehat{L}$ of L. As E is complete, we can view $\widehat{L}$ as a subfield of E. Write $q(\eta_{\widehat{L}}) = (q_a)_{\widehat{L}} \perp \pi(q_b)_{\widehat{L}}$, where q_a and q_b are anisotropic quadratic forms over the residue field $\overline{L}$ of dimension a and b respectively. As $q(\eta) \in I^3(\widehat{L})$ we have $q_b \in I^2(\overline{L})$ and therefore, b is even and $b \neq 2$. By Lemma 4.3, there is $\eta' \in H^1_{et}(\widehat{L}, H_{a,b})$ that maps to η with $q(\eta') = ((q_a)_{\widehat{L}}, \pi(q_b)_{\widehat{L}})$.

We claim that the ramification index e of the extension $E/\widehat{L}$ is odd. Suppose e is even. Note that $q_a \perp q_b \in I^3_n(\overline{L})$. Lemma 4.3 allows us to choose an unramified element $\nu \in H^1_{et}(\widehat{L}, H_{a,b})$ with $q(\nu) = ((q_a)_{\widehat{L}}, (q_b)_{\widehat{L}})$. By Lemma 7.2, there is $\alpha \in H^1_{et}(\widehat{L}, Z)$ such that $\eta' = \alpha\nu$. If b is divisible by 4, we have

$Z \simeq \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. As e is even, α is unramified over E, hence η'_E is unramified. It follows that $\eta_E \simeq t\xi$ is also unramified, a contradiction.

Suppose that $b \equiv 2 \ (mod\ 4)$. Note that $0 < b < n$ since $n \not\equiv 2 \ (mod\ 4)$. Write $\pi = ut^k$ with a unit $u \in O^\times$ and even k. Then

$$(q_a \perp uq_b)_E \simeq (q_a \perp \pi q_b)_E \simeq q(\eta_E) \simeq q(t\xi_E) = q(\xi_E) = q_E.$$

It follows that $q \simeq (q_a)_K \perp (\bar{u}q_b)_K$, i.e., q contains the subform $(\bar{u}q_b)_K$ in $I^2(K)$ of dimension b. This contradicts Theorem 4.2. The claim is proven.

Thus e is odd. We have

$$(q_a \perp utq_b)_E \simeq (q_a \perp \pi q_b)_E \simeq q(\eta_E) \simeq q(t\xi_E) = q(\xi_E) = q_E.$$

It follows that $(q_b)_K$ is hyperbolic and hence $(q_a \perp q_b)_K = (q_a)_K = q$ in $W(K)$, i.e., $(q_a \perp q_b)_K \simeq q$.

Note that $(q_a)_{\widehat{L}} = (q_a)_{\widehat{L}} + \pi(q_b)_{\widehat{L}} = q(\eta_{\widehat{L}}) \in I^3(\widehat{L})$, hence $q_a \in I^3(\overline{L})$ and $q_a \perp q_b \in I^3_n(\overline{L})$. Therefore, q is defined over $\overline{L}$ for the functor I^3_n, hence

$$\mathrm{ed}^{\mathbf{Spin}_n}(t\xi) = \mathrm{tr.deg}(L/F) \geq \mathrm{tr.deg}(\overline{L}/F) + 1 \geq \mathrm{ed}^{I^3_n}(q) + 1 = \mathrm{ed}(I^3_n) + 1.$$

It follows that $\mathrm{ed}(\mathbf{Spin}_n) \geq \mathrm{ed}(I^3_n) + 1$, hence $\mathrm{ed}(I^3_n) = \mathrm{ed}(\mathbf{Spin}_n) - 1$ by Lemma 3.2. The value of $\mathrm{ed}(\mathbf{Spin}_n)$ is given in Theorem 2.2.

In what follows we use the following observation (see [1]): if a functor $\mathcal{F}$ admits a nontrivial cohomological invariant of degree d with values in $\mathbb{Z}/2\mathbb{Z}$, then $\mathrm{ed}_2(\mathcal{F}) \geq d$.

7.3. **Case** $n = 7$**.** Every form q in $I^3_7(K)$ is the pure subform of a 3-fold Pfister form $\langle\langle a, b, c\rangle\rangle$, hence $\mathrm{ed}(I^3_7) \leq 3$. On the other hand, the Arason invariant $e_3(q \perp \langle -1\rangle) = (a) \cup (b) \cup (c) \in H^3(K, \mathbb{Z}/2\mathbb{Z})$ is nontrivial (see [7, §18.6]), hence $\mathrm{ed}_2(I^3_7) \geq 3$.

7.4. **Case** $n = 8$**.** Every form q in $I^3_8(K)$ is a multiple $e\langle\langle a, b, c\rangle\rangle$ of a 3-fold Pfister form, hence $\mathrm{ed}(I^3_8) \leq 4$. The invariant $a_4(q) = (e) \cup (a) \cup (b) \cup (c) \in H^4(K, \mathbb{Z}/2\mathbb{Z})$ is nontrivial, hence $\mathrm{ed}_2(I^3_8) \geq 4$.

7.5. **Case** $n = 9$ **and** 10**.** Every form q in $I^3_9(K)$ (respectively in $I^3_{10}(K)$) is equal to $f \perp \langle 1\rangle$ (respectively, $f \perp \langle 1, -1\rangle$), where f is a 3-fold Pfister form over K, by [10, XII.2.8]. Hence $I^3_8 \simeq I^3_9 \simeq I^3_{10}$.

7.6. **Case** $n = 11$**.** The degree 5 cohomological invariant a_5 of $\mathbf{Spin}_{11}$ defined in [7, §20.8] factors through a nontrivial invariant of I^3_{11}, hence $\mathrm{ed}_2(I^3_{11}) \geq 5$. On the other hand, $\mathrm{ed}(I^3_{11}) \leq \mathrm{ed}(\mathbf{Spin}_{11}) = 5$.

7.7. **Case** $n = 12$**.** The degree 6 cohomological invariant a_6 of $\mathbf{Spin}_{12}$ defined in [7, §20.13] factors through a nontrivial invariant of I^3_{12}, hence $\mathrm{ed}_2(I^3_{12}) \geq 6$. On the other hand, $\mathrm{ed}(I^3_{12}) \leq \mathrm{ed}(\mathbf{Spin}_{12}) = 6$.

7.8. **Case** $n = 13$ **and** 14**.** By Case 7.1 and Theorem 2.2, $\mathrm{ed}_2(I^3_{14}) = \mathrm{ed}(I^3_{14}) = \mathrm{ed}(\mathbf{Spin}_{14}) = 7$. By Lemma, 4.5, $\mathrm{ed}_2(I^3_{13}) = \mathrm{ed}_2(I^3_{13,0}) \geq \mathrm{ed}_2(I^3_{14,0}) - 1 = 6$. On the other hand, $\mathrm{ed}(I^3_{13}) \leq \mathrm{ed}(\mathbf{Spin}_{13}) = 6$. □

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