# Admissibility of groups over function fields of p-adic curves

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## Abstract

Let $K$ be a field and $G$ a finite group. The question of 'admissibility' of $G$ over $K$ was originally posed by Schacher, who gave partial results in the case $K = \mathbb{Q}$. In this paper, we give necessary conditions for admissibility of a finite group $G$ over function fields of curves over complete discretely valued fields. Using this criterion, we give an example of a finite group which is not admissible over $\mathbb{Q}_p(t)$. We also prove a certain Hasse principle for division algebras over such fields.

## Introduction

Let $K$ be a field and $G$ a finite group. We say that $G$ is *admissible* over $K$ if there exists a division ring $D$ central over $K$ and a maximal subfield $L$ of $D$ which is Galois over $K$ with Galois group $G$. Schacher asked, given a field $K$, which finite groups are admissible over $K$ and proved that if a finite group $G$ is admissible over $\mathbb{Q}$, then every Sylow subgroup of $G$ is metacyclic ([Sc], 4.1). This led to the conjecture that a finite group $G$ is admissible over $\mathbb{Q}$ if and only if every Sylow subgroup of $G$ is meta-cyclic. This conjecture has been proved for all solvable groups ([So]) and for certain non-solvable groups of small order ([CS], [FS], [Fe1], [Fe2], [Fe3]).

Recently Harbater, Hartman and Krashen ([HHK2], 4.5) gave a characterization of admissible groups over function fields of curves over complete discretely valued fields with algebraically closed residue fields. In this paper, we consider the function fields of curves over complete discretely valued fields without any assumptions on the residue fields and prove the following

**Theorem 1.** Let $K$ be a complete discretely valued field with residue field $k$ and $F = K(X)$ be the function field of a curve $X$ over $K$. Let $G$ be a finite group. Suppose that the order of $G$ is coprime to char$(k)$. If $G$ is admissible over $F$ then every Sylow subgroup $P$ of $G$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) $P/P_1$ and $P_2$ are cyclic

(2) $P_1/P_2$ is admissible over a finite extension of the residue field of a discrete valuation of $F$.

A main ingredient for the proof of the above theorem is the following Hasse principle for central simple algebras, which has independent interest.

**Theorem 2.** Let $K$ be a complete discretely valued field with residue field $k$ and $F = K(X)$ be the function field of a curve $X$ over $K$. Let $A$ be a central simple algebra over $F$ of degree $n = \ell^r$ for some prime $\ell$ and $r \geq 1$. Assume that $\ell$ is not equal to char$(k)$ and $K$ contains a primitive $n^{th}$ root of unity. Then index$(A) =$ index$(A \otimes F_v)$ for some discrete valuation $v$ of $F$.

For the proof of the above theorem, we use the patching techniques of ([HHK1]). A similar Hasse principle is proved for quadratic forms over such fields in ([CTPS], 3.1). In ([HHK3], 9.12), it is proved that if a central simple algebra $A$ over $F$ ($F$ as above), is split over $F_\nu$ for all discrete valuations on $F$, then $A$ is split over $F$.

There are some examples of classes of finite groups which are admissible over the rational function fields. However there was no example, in the literature, of a finite group which is not admissible over $\mathbb{Q}_p(t)$. Using Theorem 1, we give an example of a finite group which is not admissible over $\mathbb{Q}_p(t)$. We also prove admissibility of a certain class of groups over $\mathbb{Q}_p(t)$ using patching techniques.

**Theorem 3.** Let K be a $p$-adic field and $F$ the function field of a curve over $F$. Let $G$ be a finite group with order coprime to $char(k)$. If every Sylow subgroup of $G$ is a quotient of $\mathbb{Z}^4$, then $G$ is admissible over $F$.

In [FSS], it was proved that every abelian group on three or less generators is admissible over $\mathbb{Q}(t)$. We conclude by showing that every abelian group of order $n$ with four or less generators is admissible over $\mathbb{Q}(\zeta)(t)$, where $\zeta$ is a primitive $n^{th}$ root of unity.

# 1. Some Preliminaries

In this section we recall a few basic definitions and facts about division algebras and patching techniques ([GS], [HH], [HHK1], [P], [S2], [Sc], [Sch],

[Ser]).

Let $K$ be a field and $Br(K)$ be the Brauer group of central simple algebras over $K$. For an integer $n \geq 2$, let ${}_nBr(K)$ denote the $n$-torsion subgroup of $Br(K)$. If $A$ and $B$ are two central simple algebras over $K$, we write $A \simeq B$ if $A$ and $B$ are isomorphic as $K$-algebras and we write $A = B$ if they represent the same element in $Br(K)$. Let $n$ be an integer coprime to char$(K)$. Suppose $E/K$ is a cyclic extension of degree $n$ and $\sigma$ a generator of $Gal(E/K)$. For $b \in K^*$, let $(E/K, \sigma, b)$ (or simply $(E, \sigma, b)$) be the $K$-algebra generated by $E$ and $y$ with $y^n = b$ and $\lambda y = y\sigma(\lambda)$ for all $\lambda \in E$. Then $(E, \sigma, b)$ is a central simple algebra over $K$ and represents an element in ${}_nBr(K)$. Suppose that $K$ contains a primitive $n^{th}$ root of unity. Then $E = K(\sqrt[n]{a})$ for some $a \in K^*$ and $\sigma(\sqrt[n]{a}) = \zeta\sqrt[n]{a}$ for a primitive $n^{th}$ root of unity $\zeta \in K^*$. The algebra $(E, \sigma, b)$ is generated by $x, y$ with $x^n = a$, $y^n = b$ and $xy = \zeta yx$ and we denote the algebra $(E, \sigma, b)$ by $(a, b)_n$.

Suppose that $\nu$ is a discrete valuation of $K$ with residue field $k$. Let $n$ be a natural number which is coprime to char$(k)$. Then we have a *residue homomorphism* $\partial_\nu : {}_nBr(K) \to H^1(k, \mathbb{Z}/n\mathbb{Z})$, where $H^1(k, \mathbb{Z}/n\mathbb{Z})$ denotes the first Galois cohomology group. Suppose $k$ contains a primitive $n^{th}$ root of unity. By fixing a primitive $n^{th}$ root of unity, we identify $H^1(k, \mathbb{Z}/n\mathbb{Z})$ with $k^*/k^{*n}$. With this identification we have $\partial_\nu((a, b)_n) = \overline{\frac{a^{\nu(b)}}{b^{\nu(a)}}} \in k^*/k^{*n}$, where for any $c \in K^*$ which is a unit at $\nu$, $\overline{c}$ denotes its image in $k^*$. More generally, let $E/K$ be a cyclic unramified inert extension of $K$ and $\sigma \in Gal(E/K)$ be a generator. Let $\pi \in K^*$ be a parameter at $\nu$. Then the residue of $(E/K, \sigma, \pi)$ is $(E_0/k, \sigma_0)$, where $k$ is the residue field of $K$ at $\nu$, $E_0$ is the residue field of $E$ at the unique extension of $\nu$ to $E$ and $\sigma_0$ is the image of $\sigma$.

Let $K$ be a field and $n$ an integer. Then $H^1(K, \mathbb{Z}/n\mathbb{Z})$ classifies the equivalence classes of pairs $(E, \sigma)$, where $E$ is a cyclic Galois field extension of $K$ of degree a divisor of $n$ and $\sigma$ a generator of the Galois group $G(E/K)$ of $E/K$. Let $(E, \sigma)$ be a pair representing an element in $H^1(K, \mathbb{Z}/n\mathbb{Z})$. Let $m \geq 1$ be an integer. We now describe $m(E, \sigma)$ in the group $H^1(K, \mathbb{Z}/n\mathbb{Z})$. Let $d$ be the greatest common divisor of $m$ and $[E : K]$. Let $E(m)$ be the subfield of $E$ fixed by $\sigma^{[E:K]/d}$ and $\sigma(m)$ be the restriction of $\sigma^{m/d}$ to $E(m)$. Then $m(E, \sigma) = (E(m), \sigma^{m/d})$. The order of $(E, \sigma)$ in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is equal to $[E : K]$. Since the identity element of $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is $(K, id)$, we have $m(E, \sigma)$ is trivial in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ if and only if $E(m) = K$ if and only if $[E : K]$ divides $m$.

Let $K$ be a complete discretely valued field with residue field $k$. Let $L/K$ be a finite extension of $K$. Since $K$ is complete, the discrete valuation of $K$ extends uniquely to a discrete valuation of $L$ and $L$ is complete with respect to this discrete valuation. Let $n$ be a natural number which is coprime to the characteristic of $k$. Let $L/K$ be a Galois extension of degree $n$. Let $L_1$ be the maximal unramified extension of $K$. Since $n$ is coprime to char($k$), the residue field $L_0$ of $L$ is same as the residue field of $L_1$. Since $L/K$ is Galois, $L_1/K$ and $L_0/k$ are also Galois and there is a natural isomorphism $Gal(L/K) \to Gal(L_0/k)$. Let $E_0/k$ be a cyclic extension of degree $n$. Then there exists a unique (up to isomorphism) unramified cyclic extension $E$ of $K$ with residue field $E_0$ and a natural isomorphism $Gal(E/K) \to Gal(E_0/k)$. Let $(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$. Then we have a unique $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ with $[E : K] = [E_0 : k]$, $E_0$ as the residue field of $E$ and the image of $\sigma$ equal to $\sigma_0$; $(E, \sigma)$ is called the *lift* of $(E_0, \sigma_0)$.

Let $\mathcal{X}$ be a regular integral scheme with function field $F$. Let $n$ be an integer which is a unit on $\mathcal{X}$. Let $f \in F$ and $P \in \mathcal{X}$ be a point. If $f$ is regular at $P$, then we denote its image in the residue field $\kappa(P)$ at $P$ by $f(P)$. Let $\mathcal{X}^1$ denote the set of codimension one points of $\mathcal{X}$. For each codimension one point $x$ of $\mathcal{X}$, we have discrete valuation $\nu_x$ on $F$. Let $\kappa(x)$ denote the residue field at $x$. Since $n$ is a unit on $\mathcal{X}$, $n$ is coprime to char($\kappa(x)$) and we have the residue homomorphism $\partial_x : {}_nBr(F) \to H^1(\kappa(x), \mathbb{Z}/n\mathbb{Z})$. Let $\alpha \in {}_nBr(F)$. We say that $\alpha$ is *unramified* at $x$ if $\partial_x(\alpha) = 0$. We say that $\alpha$ is *unramified* on $\mathcal{X}$ if it is unramified at every codimension one point of $\mathcal{X}$. Let $A$ be a central simple algebra over $F$. We say tat $A$ is *unramified* if its class in $Br(F)$ is unramified. If $\mathcal{X} = Spec(B)$, then we say that $\alpha$ is unramified on $B$ if it is unramified on $\mathcal{X}$.

Let $F$ be a field and $A$ a central simple algebra over $F$. Then $A \simeq M_m(D)$ for some central division algebra $D$ over $F$. The *degree* of $A$ is defined as $\sqrt{dim_F A}$ and the *index* of $A$ is defined as the degree of $D$. If $L/K$ is a field extension, then index($A \otimes_F L$) divides index($A$). Let $B$ be an integral domain and $F$ its field of fractions. Let $\mathcal{A}$ be an Azumaya algebra over $B$, then we define the *index* of $\mathcal{A}$ to be the index of $\mathcal{A} \otimes_B F$.

Let $B$ be a regular integral domain of dimension at most 2 which is complete with respect to a prime ideal $P$. Assume that $B/P$ is a regular integral domain of dimension at most 1 (for example $P$ is a maximal ideal). Let $\kappa(P)$ be the field of fractions of $B/P$. Let $A$ be a central simple algebra over the field of fractions $F$ of $B$ which is unramified on $B$. Then there exists an Azumaya algebra $\mathcal{A}$ over $B$ such that $\mathcal{A} \otimes_B F \simeq A$ ([CTS], 6.13)). For an ideal $I$ of $B$, we denote the algebra $\mathcal{A} \otimes_B B/I$ by $A(I)$.

**Lemma 1.1** Let $A$, $B$, $\mathcal{A}$ and $P$ be as above. Then index($A$) = index ($\mathcal{A} \otimes_{B/P} \kappa(P)$).

**Proof.** Suppose $A \simeq M_n(D)$ for some division algebra $D$ over $F$. Since $A$ is unramified on $B$, $D$ is also unramified on $B$. Since $B$ is a regular domain of dimension 2, there exists an Azumaya algebra $\mathcal{D}$ over $B$ such that $\mathcal{D} \otimes_B K \simeq D$ ([CTS], 6.13)). Since $Br(B) \to Br(F)$ is injective ([AG], 7.2), $\mathcal{A} = \mathcal{D}$ in $Br(B)$. In particular $\mathcal{A} \otimes_{B/P} \kappa(P) = \mathcal{D} \otimes_{B/P} \kappa(P)$ in $Br(\kappa(P))$. Hence $\mathrm{index}(A) = \mathrm{degree}(D) = \sqrt{\dim_F(D)} = \sqrt{\mathrm{rank}_B(\mathcal{D})} = \sqrt{\mathrm{rank}_{B/P}(\mathcal{D} \otimes_B B/P)} = \sqrt{\dim_{\kappa(P)}(\mathcal{D} \otimes_{B/P} \kappa(P))} \geq \mathrm{index}(\mathcal{A} \otimes_{B/P} \kappa(P))$.

Suppose $\mathcal{A} \otimes_{B/P} \kappa(P) \simeq M_m(D_0)$ for some central division algebra $D_0$ over $\kappa(P)$. Since $B/P$ is regular integral domain of dimension 1, there exists an Azumaya algebra $\mathcal{D}_0$ over $B/P$ such that $\mathcal{D}_0 \otimes_{B/P} \kappa(P) \simeq D_0$. Since $B$ is $P$-adically complete, there exists an Azumaya algebra $\tilde{\mathcal{D}}_0$ over $B$ such that $\tilde{\mathcal{D}}_0 \otimes_B B/P \simeq \mathcal{D}_0$ and $\mathcal{A} = \tilde{\mathcal{D}}_0$ in $Br(B)$ ([C], [KOS]). In particular $A = \tilde{\mathcal{D}}_0 \otimes_B F$ in $Br(F)$. Hence $\mathrm{index}(\mathcal{A} \otimes_{B/P} \kappa(P)) = \mathrm{degree}(D_0) = \sqrt{\dim_{\kappa(P)}(D_0)} = \sqrt{\mathrm{rank}_{B/P}(\mathcal{D}_0)} = \sqrt{\mathrm{rank}_B(\tilde{\mathcal{D}}_0)} = \sqrt{\dim_F(\tilde{\mathcal{D}}_0 \otimes F)} \geq \mathrm{index}(A)$. Thus $\mathrm{index}(A) = \mathrm{index}(\mathcal{A} \otimes_{B/P} \kappa(P))$. $\square$

Let $K$ be a field and $L$ a finite extension of $K$. Then $L$ is called $K$-*adequate* if there is a division ring $D$ central over $K$ containing $L$ as a maximal subfield. A finite group $G$ is called $K$-*admissible* if there is a Galois extension $L$ of $K$ with $G$ as the Galois group of $L$ over $K$, and $L$ is $K$-*adequate*.

A finite group $G$ is called *metacyclic* if $G$ has a normal subgroup $H$ such that $H$ is cyclic and $G/H$ is cyclic.

# 2. A local-global principle for central simple algebras

Let $K$ be a complete discretely valued field with residue field $k$. Let $F$ be the function field of a curve over $K$. Let $n$ be an integer which is coprime to the characteristic of $k$. Assume that $K$ contains a primitive $n^{th}$ root of unity. In this section we prove a certain Hasse principle for central simple algebras over $F$ of index $n$. We begin with the following

**Lemma 2.1.** (cf. [FS], Proposition 1(3) and [JW], 5.15) Let $R$ be a complete discrete valuated ring and $K$ its field of fractions. Let $A$ be a central simple algebra over $K$ of index $n$ which is unramified at $R$. Let $E$ be an unramified cyclic extension of $K$ of degree $m$ and $\sigma$ a generator of the Galois group of $E/K$. Let $\pi$ be a parameter in $R$. Assume that $mn$ is invertible in $R$. Then $\mathrm{index}(A \otimes (E, \sigma, \pi)) = \mathrm{index}(A \otimes E) \cdot [E : K]$.

The following two lemmas (2.2, 2.3) are well known.

**Lemma 2.2.** Let $R$ be a regular ring of dimension 2 with field of fractions $F$. Let $n$ be an integer which is a unit in $R$. Assume that $F$ contains a primitive $n^{\text{th}}$ root of unity. Suppose $A$ is a central simple algebra over $F$ which is unramified on $R$. Let $x \in R$ be a regular prime and $\nu$ be the discrete valuation on $F$ given by $x$. Suppose that $R$ is complete with respect to $(x)$-adic topology. Let $u \in R$ be a unit. Then $\text{index}(A \otimes F_\nu(\sqrt[n]{u})) = \text{index}(A \otimes F(\sqrt[n]{u}))$.

**Proof.** Let $S$ be the integral closure of $R$ in $F(\sqrt[n]{u})$. Since $R$ is a regular ring and $n$, $u$ are units in $R$, $S$ is also a regular ring. Let $x$ be a regular prime in $R$, then $x$ is also a regular prime in $S$. Thus by replacing $R$ by $S$, it is enough to show that $\text{index}(A \otimes F_\nu) = \text{index}(A)$.

Since $R$ is a two-dimensional regular ring, there is an Azumaya algebra $\mathcal{A}$ over $R$ with $\mathcal{A} \otimes F \simeq A$ ([CTS], 6.13)). Since $R$ is complete with respect to $(x)$-adic topology, $\text{index}(A) = \text{index}(\mathcal{A} \otimes \kappa(x))$ (cf. 1.1), where $\kappa(x)$ is the field of fractions of $R/(x)$. Let $R_\nu$ be the ring of integers in $F_\nu$. Since $A$ is unramified on $R$, $A$ is also unramified on $R_\nu$. Since $F_\nu$ is complete, $\text{index}(A \otimes F_\nu) = \text{index}(\mathcal{A} \otimes \kappa(\nu))$ (cf. 1.1). Since $\kappa(\nu) = \kappa(x)$, $\text{index}(A) = \text{index}(A \otimes F_\nu)$. $\square$

**Lemma 2.3.** Let $R$ be a complete regular local ring with field of fractions $F$ and residue field $k$. Let $n$ be an integer which is a unit in $R$. Let $u_1, \cdots, u_r \in R$ be units. Suppose $x \in R$ is a regular prime. Let $\nu$ be the discrete valuation on $F$ given by $x$. Then $[F_\nu(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F_\nu] = [F(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F]$.

**Proof.** Since $F_\nu$ is a complete discretely valued field and $n$, $u_1$, $\cdots$, $u_r$ are units in the ring of integers, $[F_\nu(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F_\nu] = [\kappa(\nu)(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : \kappa(\nu)]$. Since $\kappa(\nu)$ is the field of fractions of the complete local ring $R/(x)$ and the the residue field of $R/(x)$ is $k$, we have $[\kappa(\nu)(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : \kappa(\nu)] = [k(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : k]$. Since $R$ is complete and $u_1, \cdots, u_r$ are units, we also have $[F(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F] = [k(\sqrt[n]{\overline{u}_1}, \cdots, \sqrt[n]{\overline{u}_r}) : k]$. Hence $[F_\nu(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F_\nu] = [F(\sqrt[n]{u_1}, \cdots, \sqrt[n]{u_r}) : F]$. $\square$

**Proposition 2.4.** Let $R$ be a 2-dimensional complete regular local ring with maximal ideal $m = (x, y)$. Let $F$ be the field of fraction of $R$ and $k$ the residue field of $R$. Let $n$ be an integer coprime to $\text{char}(k)$. Assume that $F$ contains a primitive $n^{\text{th}}$ root of unity. Let $A$ be a central simple algebra over $F$ of degree $n$. Suppose that $A$ is unramified on $R$ except possibly at $x$ and

$y$. Then $\mathrm{index}(A) = \mathrm{index}(A \otimes F_\nu)$ for the discrete valuation $\nu$ of $F$ given by the prime ideals either $(x)$ or $(y)$ of $R$.

**Proof.** Suppose that $A$ is unramified on $R$. Let $\nu$ be the discrete valuation of $F$ given by $x$. Since $A$ is unramified on $R$, by (2.2), we have $\mathrm{index}(A \otimes F_\nu) = \mathrm{index}(A)$.

Suppose that $A$ is ramified on $R$ only at the prime ideal $(x)$. Let $\nu$ be the discrete valuation on $F$ given by the prime ideal $(x)$ of $R$. By ([S1]), we have $A = A' \otimes (u, x)$ for some unit $u$ in $R$ and a central simple algebra $A'$ over $F$ which is unramified on $R$. Since $F_\nu$ is complete and $u$ is a unit at $\nu$, there is a unique extension of $\nu$ to $F_\nu(\sqrt[n]{u})$ such that $F_\nu(\sqrt[n]{u})$ is complete with the residue field $\kappa(\nu)(\sqrt[n]{\overline{u}})$, where $\overline{u}$ is the image of $u$ in $\kappa(\nu)$. Since $A'$ is unramified on $R$, we have

$$\begin{aligned}
\mathrm{index}(A \otimes F_\nu) &= \mathrm{index}(A' \otimes (u, x) \otimes F_\nu) \\
&= \mathrm{index}(A' \otimes F_\nu(\sqrt[n]{u})) \cdot [F_\nu(\sqrt[n]{u}) : F_\nu] \quad \text{(by (2.1))} \\
&= \mathrm{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F_\nu(\sqrt[n]{u}) : F_\nu] \quad \text{(by (2.2))} \\
&= \mathrm{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F(\sqrt[n]{u}) : F] \quad \text{(by (2.3))}.
\end{aligned}$$

Since $A = A' \otimes (u, x)$, the index of $A$ divides $\mathrm{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F(\sqrt[n]{u}) : F] = \mathrm{index}(A \otimes F_\nu)$. Thus $\mathrm{index}(A) = \mathrm{index}(A \otimes F_\nu)$.

Assume that $A$ is ramified on $R$ at both the primes $(x)$ and $(y)$. Then by ([S1]), either $A = A' \otimes (u_1, x) \otimes (u_2, y)$ or $A = A' \otimes (uy^r, x)$ where $u_1, u_2, u$ are units in $R$, $r$ coprime with $n$ and $A'$ unramified on $R$.

Suppose that $A = A' \otimes (u_1, x) \otimes (u_2, y)$ for some units $u_1, u_2 \in R$ and $A'$ unramified on $R$. Let $\nu$ be the discrete valuation on $F$ given by $y$. Then by (2.1), we have $\mathrm{index}(A \otimes F_\nu) = \mathrm{index}(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) \cdot [F_\nu(\sqrt[n]{u_2}) : F_\nu]$. By (2.2), we have $[F_\nu(\sqrt[n]{u_2}) : F_\nu] = [F(\sqrt[n]{u_2}) : F]$. We now compute $\mathrm{index}(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2}))$.

Since $A' \otimes (u_1, x)$ is unramified at $\nu$ and $u_2$ is a unit at $\nu$, $\mathrm{index}(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) = \mathrm{index}(A' \otimes (u_1, x) \otimes \kappa(\nu)(\sqrt[n]{\overline{u_2}}))$ (cf. 1.1), where $\overline{u_2}$ is the image of $u_2$ in $\kappa(\nu)$. Since $\kappa(\nu)$ is the field of fractions of $R/(y)$, the image of $x$ in $\kappa(\nu)$ gives a discrete valuation $\mu$ on the residue field $\kappa(\nu)$ with $\kappa(\mu) = k$ and $\kappa(\nu)$ is complete with respect to $\mu$. Since $\overline{u_2}$ is a unit at $\mu$, $\kappa(\nu)(\sqrt[n]{\overline{u_2}})$ is a complete discrete valuated field with $\overline{x}$ as parameter and residue field $k(\sqrt[n]{\overline{u_2}})$. Thus, we have

$$\begin{aligned}
&\mathrm{index}(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) = \mathrm{index}(A' \otimes (u_1, x) \otimes \kappa(\nu)(\sqrt[n]{\overline{u_2}}))\text{(by (1.1))} \\
&= \mathrm{index}(A' \otimes \kappa(\nu)(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}})) \cdot [\kappa(\nu)(\sqrt[n]{\overline{u_2}}, \sqrt[n]{\overline{u_1}}) : \kappa(\nu)(\sqrt[n]{\overline{u_2}})] \quad \text{(by (2.1))} \\
&= \mathrm{index}(A' \otimes F_\nu(\sqrt[n]{u_2}, \sqrt[n]{u_1})) \cdot [F_\nu(\sqrt[n]{u_2}, \sqrt[n]{u_1}) : F_\nu(\sqrt[n]{u_2})] \quad \text{(by (1.1))} \\
&= \mathrm{index}(A' \otimes F(\sqrt[n]{u_2}, \sqrt[n]{u_1})) \cdot [F(\sqrt[n]{u_2}, \sqrt[n]{u_1}) : F(\sqrt[n]{u_2})] \quad \text{(by (2.2), (2.3))}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\operatorname{index}(A \otimes F_\nu) = \operatorname{index}(A' \otimes (u_1, x) \otimes F_\nu(\sqrt[n]{u_2})) \cdot [F_\nu(\sqrt[n]{u_2}) : F_\nu] (by\ (2.1)) \\
&= \operatorname{index}(A' \otimes F(\sqrt[n]{u_2}, \sqrt[n]{u_1})) \cdot [F(\sqrt[n]{u_2}, \sqrt[n]{u_1}) : F(\sqrt[n]{u_2})] \cdot [F(\sqrt[n]{u_2}) : F] \\
&= \operatorname{index}(A' \otimes F(\sqrt[n]{u_2}, \sqrt[n]{u_1})) \cdot [F(\sqrt[n]{u_2}, \sqrt[n]{u_1}) : F].
\end{aligned}$$

On the other hand we have $\operatorname{index}(A) = \operatorname{index}(A' \otimes (u_1, x) \otimes (u_2, y))$ divides $\operatorname{index}(A' \otimes F(\sqrt[n]{u_1}, \sqrt[n]{u_2})) \cdot [F(\sqrt[n]{u_1}, \sqrt[n]{u_2}) : F]$. In particular $\operatorname{index}(A \otimes F_\nu) = \operatorname{index}(A)$.

Assume that $A = A' \otimes (uy^r, x)$ for some unit $u \in R$, $r$ coprime to $n$ and $A'$ unramified on $R$. Let $\nu$ be the discrete valuation on $F$ given by the prime ideal $(x)$ of $R$. By (2.1), we have $\operatorname{index}(A \otimes F_\nu) = \operatorname{index}(A' \otimes F_\nu(\sqrt[n]{uy^r})) \cdot [F_\nu(\sqrt[n]{uy^r}) : F_\nu]$. Since $y$ is a regular prime in $R$ and $r$ is coprime to $n$, it follows that $[F(\sqrt[n]{uy^r}) : F] = [F_\nu(\sqrt[n]{uy^r}) : F_\nu] = n$. Since $F_\nu$ is a complete discrete valuated field and $A'$ is unramified, we have $\operatorname{index}(A' \otimes F_\nu(\sqrt[n]{uy^r})) = \operatorname{index}(A' \otimes \kappa(\nu)(\sqrt[n]{\overline{uy^r}}))$ (cf. 1.1). Since $\kappa(\nu)$ is a complete discrete valuated field with $\overline{y}$ as a parameter and residue field $k$, $\kappa(\nu)(\sqrt[n]{\overline{uy^r}})$ is also a complete discrete valuated field with residue field $k$. Since $A'$ is unramified on $R$, we have $\operatorname{index}(A' \otimes \kappa(\nu)(\sqrt[n]{\overline{uy^r}})) = \operatorname{index}(A' \otimes k)$ (cf. (1.1)). Since $R$ is complete, by (1.1), we have $\operatorname{index}(A') = \operatorname{index}(A' \otimes k) = \operatorname{index}(A' \otimes \kappa(\nu)(\sqrt[n]{\overline{uy^r}})) = \operatorname{index}(A' \otimes F_\nu(\sqrt[n]{uy^r}))$. Therefore, we have

$$\begin{aligned}
\operatorname{index}(A \otimes F_\nu) &= \operatorname{index}(A' \otimes (uy^r, x) \otimes F_\nu) \\
&= \operatorname{index}(A' \otimes F_\nu(\sqrt[n]{uy^r})) \cdot [F_\nu(\sqrt[n]{uy^r}) : F_\nu] \\
&= \operatorname{index}(A') \cdot n.
\end{aligned}$$

Since $A = A' \otimes (uy^r, x)$, $\operatorname{index}(A) \leq \operatorname{index}(A') \cdot n = \operatorname{index}(A \otimes F_\nu)$. Thus $\operatorname{index}(A) = \operatorname{index}(A \otimes F_\nu)$. $\square$

**Proposition 2.5.** Let $R$ be a 2-dimensional regular ring with field of fraction $F$ and $n$ an integer which is a unit in $R$. Assume that $F$ contains a primitive $n^{\text{th}}$ root of unity. Suppose that $R$ is complete with respect to $(s)$-adic topology for some prime element $s \in R$ with $R/(s)$ a Dedekind domain. Let $\nu$ be the discrete valuation on $F$ given by $s$. Let $A$ be a central simple algebra over $F$ of degree $n$ which is unramified on $R$ except at $\nu$. Further assume that the residue of $A$ at $(s)$ is given by a unit $a$ in $R/(s)$. Then $\operatorname{index}(A) = \operatorname{index}(A \otimes F_\nu)$.

**Proof.** Suppose that $A$ is unramified on $R$. Since $R$ is $(s)$-adically complete and $R/(s)$ is a regular domain of dimension 1 with field of fractions $\kappa(\nu)$, $\operatorname{index}(A) = \operatorname{index}(A \otimes_{R/(s)} \kappa(\nu)) = \operatorname{index}(A \otimes F_\nu)$ (cf. (1.1)).

Suppose that $A$ is ramified on $R$. Then by the assumption on $A$, $A$ is ramified on $R$ only at the prime ideal $(s)$ of $R$ and the residue of $A$ at $(s)$ is given by a unit $a$ in $R/(s)$. Let $u \in R$ with image $a \in R/(s)$. Since $R$ is $(s)$-adically complete and the image $a$ of $u$ modulo $(s)$ is a unit, $u$ is a unit in $R$. The cyclic algebra $(u, s)$ is ramified on $R$ only at $(s)$ and the residue of $(u, s)$ at $(s)$ is $(a)$. Since $A$ is ramified only at the prime ideal $(s)$ of $R$, and $(a)$ is the residue of $A$ at $s$, we have $A = A' \otimes (u, s)$ for some central simple algebra $A'$ over $F$ which is unramified on $R$. By (2.1), we have $\text{index}(A \otimes F_\nu) = \text{index}(A' \otimes F_\nu(\sqrt[n]{u})) \cdot [F_\nu(\sqrt[n]{u}) : F_\nu]$. Since $R$ is $(s)$-adically complete and $R/(s)$ is integrally closed domain with $u$ a unit, we have $[F(\sqrt[n]{u}) : F] = [F_\nu(\sqrt[n]{u}) : F_\nu]$. Since $A'$ is unramified on $R$ and $R$ is $(s)$-adically complete, by (2.2), we have $\text{index}(A' \otimes F_\nu(\sqrt[n]{u})) = \text{index}(A' \otimes F(\sqrt[n]{u}))$. Hence we have $\text{index}(A \otimes F_\nu) = \text{index}(A' \otimes F(\sqrt[n]{u})) \cdot [F(\sqrt[n]{u}) : F]$. In particular $\text{index}(A \otimes F_\nu) = \text{index}(A)$. $\square$

**Theorem 2.6.** Let $T$ be a complete discrete valuation ring with fraction field $K$ and residue field $k$. Let $X$ be a regular, projective, geometrically integral curve over $K$ and $F = K(X)$ the function field of $X$. Let $l$ be prime not equal to $\text{char}(k)$ and $A$ a central simple algebra over $F$ of index $n = l^d$ for some $d \geq 1$. Assume that $K$ contains a primitive $n^{th}$ root of unity. Then $\text{index}(A) = \text{index}(A \otimes F_\nu)$ for some discrete valuation $\nu$ of $F$.

**Proof.** Let $A$ be a central simple algebra over $F$. We choose a regular proper model $\mathcal{X}/T$ of $X/K$ such that the support of the ramification divisor $A$ and the components of the special fibre of $\mathcal{X}/T$ are a union of regular curves with normal crossings. Let $Y = \mathcal{X} \times_T k$ denote the special fibre.

For each irreducible curve $C$ in the support of ramification divisor of $A$, let $a_C \in \kappa(C)^*$ be such that the residue of $A$ at $C$ is $(a_C) \in H^1(\kappa(C)^*, \mathbb{Z}/n\mathbb{Z})$.

Let $S$ be a finite set of closed points of the special fibre containing all singular points of $Y$ and all those points of irreducible curves $C$ where $a_C$ is not a unit.

For each $P \in S$, let $F_P$ be the field of fractions of the completion $\hat{R}_P$ of the local ring $R_P$ of $\mathcal{X}$ at $P$. Let $t$ be a uniformizing parameter for $T$. For each irreducible component $U$ of $Y \setminus S$, let $R_U$ be the ring of elements in $F$ which are regular on $U$. It is a regular ring. Let $\hat{R}_U$ be the completion of $R_U$ at $(t)$ and $F_U$ the field of fractions of $\hat{R}_U$. As in ([CTPS], proof of 3.1), we choose $S$ such that for every irreducible component $U$ of $Y \setminus S$, $t = u.s^r$ for some integer $r \geq 1$, a unit $u \in R_U$ and $R_U/s = \hat{R}_U/s$ is a Dedekind domain with field of fractions $k(U)$. In particular, the $t$-adic completion $\hat{R}_U$ coincides with the $s$-adic completion of $R_U$.

Let $\mathcal{U}$ be the set of irreducible components of $X \setminus S$. By ([HHK1], 5.1), we have index$(A)$ = lcm of index$(A \otimes F_\zeta)$, $\zeta \in \mathcal{U} \cup S$. Since index$(A \otimes F_\zeta)$ is a power of the prime $l$ for all $\zeta \in \mathcal{P} \cup P$, we have index$(A) =$ index$(A \otimes F_\zeta)$ for some $\zeta \in \mathcal{U} \cup S$. In particular index$(A)$ equal to either index$(A \otimes F_U)$ for some irreducible components $U$ of $Y \setminus S$ or index$(A \otimes F_P)$ for some $P \in S$.

Suppose that index$(A) =$ index$(A \otimes F_P)$ for some $P \in S$. The local ring $\hat{R}_P$ is a complete regular local ring of dimension 2 with maximal ideal $(x, y)$ such that $A$ is ramified on $\hat{R}_P$ at most at $x$ and $y$. By (2.4), we have index$(A) =$ index$(A \otimes F_{P\nu})$ for the discrete valuation of $F_P$ given by either $(x)$ or $(y)$. Since the restriction $\nu_0$ of $\nu$ to $F$ is non-trivial, $\nu_0$ is a discrete valuation on $F$ and index$(A) =$ index$(A \otimes F_{\nu_0})$.

Suppose that index$(A) =$ index$(A \otimes F_U)$ for some irreducible component $U$ of $X \setminus S$. Then, by the choice of $U$, $A$ is unramified on $\hat{R}_U$ except at $(s)$ and the residue at $(s)$ is given by a unit in $R_U/(s)$. Let $\nu$ be the discrete valuation on $F_U$ given by $(s)$. Since $\hat{R}_U$ is $(s)$-adically complete, by (2.5), index$(A) =$ index$(A \otimes F_{U\nu})$. Since the restriction of $\nu$ to $F$ is also given by the ideal $(s)$ in $R_U$, $\nu$ is non-trivial on $F$. In particular $F_\nu \subset F_{U\nu}$ and index$(A) =$ index$(A \otimes F_\nu)$. $\square$.

**Remark 2.7.** Let $F$ and $A$ be as in the above theorem. Then by (2.6), it follows that there exists a discrete valuation $\nu$ of $F$ such that index$(A) =$ index$(A \otimes F_\nu)$. From the proof of (2.6) it follows that this discrete valuation comes from a codimension one point of a regular proper model of $F$. In particular, the residue field $\kappa(\nu)$ is either a finite extension of $K$ or a function field of a curve over a finite extension of $k$.

# 3. Necessary conditions for Admissibility

In this section, we give a necessary condition for a finite group to be admissible over function fields of curves over complete discretely valued fields.

Let $K$ be a complete discretely valued field with residue field $k$ and $\pi$ a parameter. Let $D$ be a central division algebra over $K$ of degree $n$ and $L$ a maximal subfield of $D$. Suppose that $n$ is coprime to char$(k)$. Let $(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$ be the residue of $D$ and $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ the lift of $(E_0, \sigma_0)$ (cf. §1). We fix an algebraic closure $\overline{K}$ of $K$ and also an algebraic closure $\overline{k}$ of $k$. All the finite extensions of $K$ and $k$ are considered as subfields of $\overline{K}$ and $\overline{k}$ respectively. Let $F = L \cap E$. Let $L_1$ be the maximal unramified extension of $F$ contained in $L$. Since $E/K$ is unramified, $L \cap E = L_1 \cap E = F$.

**Lemma 3.1.** Let $K$, $D$, $L$, $L_1$, $(E_0, \sigma_0)$, $(E, \sigma)$, and $F$ be as above. Then $D \otimes_K F \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified at the discrete valuation of $F$.

**Proof.** Since the residue of $D$ is $(E_0, \sigma_0)$, $D \otimes (E, \sigma, \pi)^{op}$ is unramified at the discrete valuation of $K$. In particular, $D \otimes_K (E, \sigma, \pi) \otimes F$ is unramified at the discrete valuation of $F$. Since $F \subset E$, we have $(E, \sigma, \pi) \otimes_K F = (E/F, \sigma^{[F:K]}, \pi)$ in $Br(F)$. Thus $(D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified at the discrete valuation of $F$. $\square$

**Lemma 3.2.** Let $K$, $D$, $L$, $L_1$, $(E_0, \sigma_0)$, $(E, \sigma)$ and $F$ be as above. Then $[E : F]$ divides $[L : L_1]$.

**Proof.** We have the following commutative diagram (cf., [S1], 10.4)

$$\begin{tikzcd} {}_nBr(K) \arrow[r] \arrow[d, "res"'] & H^1(k, \mathbb{Z}/n\mathbb{Z}) \arrow[d, "e.res"] \\ {}_nBr(L) \arrow[r] & H^1(L_0, \mathbb{Z}/n\mathbb{Z}) \end{tikzcd} ,$$

where $L_0$ is the residue field of $L$ and $e$ the ramification index of $L/K$. From the above commutative diagram, the residue of $D \otimes L$ is the restriction of $e(E_0, \sigma_0)$ to $L_0$. Since $L$ is a maximal subfield of $D$, $D \otimes L$ is a split algebra. In particular, the residue of $D \otimes L$ is trivial. Hence the restriction of $e(E_0, \sigma_0)$ to $L_0$ is trivial and $[E : F]$ divides $e$ (cf. §1). Since $[E : F] = [E_0 : F_0]$ and $e = [L : L_1]$, $[E : F]$ divides $[L : L_1]$. $\square$

**Lemma 3.3.** Let $K$, $D$, $L$, $L_1$, $(E_0, \sigma_0)$, $(E, \sigma)$ and $F$ be as above. Then index$(D \otimes_K E) = [L_1 E : E]$.

**Proof.** Since $(E/F, \sigma^{[F:K]}, \pi)^{op} \otimes E$ is split, we have $D \otimes_K E = D \otimes_K F \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op} \otimes_F E$. By (3.1), $D \otimes_K F \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified at the discrete valuation of $F$. Hence $D \otimes_K E$ is unramified at the discrete valuation of $E$. Since $L$ is a maximal subfield of $D$, $D \otimes_K L$ is split. Since $L/L_1$ is totally ramified, $LE/L_1E$ is totally ramified. In particular, the residue field of $LE$ and $L_1E$ are equal. Since $D \otimes_K E$ is unramified and $(D \otimes_K E) \otimes_E LE$ is plit, $(D \otimes_K E) \otimes_E L_1E$ is split (cf. 1.1). In particular index$(D \otimes_K E) \leq [L_1 E : E]$.

Since $(D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op}$ is unramified and $\pi$ is a parameter in $E$, by (2.1), we have

$$\begin{aligned} &\mathrm{index}(D \otimes_K F) = \mathrm{index}((D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op} \otimes_F (E/F, \sigma^{[F:K]}, \pi)) \\ &= \mathrm{index}((D \otimes_K F) \otimes_F (E/F, \sigma^{[F:K]}, \pi)^{op} \otimes_F E) \cdot [E : F] \\ &= \mathrm{index}(D \otimes_K E) \cdot [E : F] \end{aligned}$$

On the other hand, since $F \subset L$ and $L$ is a maximal subfield of $D$, $\mathrm{index}(D \otimes_K F) = [L : F]$. Hence $\mathrm{index}(D \otimes_K E) \cdot [E : F] = [L : F] = [L : L_1] \cdot [L_1 : F]$. By (3.2), we have $[E : F]$ divides $[L : L_1]$. Hence $\mathrm{index}(D \otimes_K E) \geq [L_1 : F] \geq [L_1E : E]$. Therefore $\mathrm{index}(D \otimes_K E) = [L_1E : E]$. □

**Lemma 3.4.** Let $K$ be a complete discretely valued field with residue field $k$ and $P$ a $p$-group, $p$ a prime. Suppose that $p$ is coprime to $\mathrm{char}(k)$. If $P$ is admissible over $K$, then $P$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) $P/P_1$ and $P_2$ are cyclic

(2) $P_1/P_2$ is admissible over some finite extension of $k$.

**Proof** Suppose that $P$ is admissible over $K$. Then there exists a Galois extension $L/K$ and a division ring $D$ central over $K$ which contains $L$ as maximal subfield such that $P = G(L/K)$. Let $L_0$ be the residue field of $L$. Let $\partial(D)=(E_0, \sigma_0) \in H^1(k, \mathbb{Z}/n\mathbb{Z})$ be the residue of D and $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ be the lift of $(E_0, \sigma_0)$. Let $L_1$ be the maximal unramified extension of $K$ contained in $L$. Since $E$ is unramified extension of $K$, we have $L \cap E = L_1 \cap E$. Let $F = L \cap E$. Since $E/K$ is cyclic, $F/K$ is also cyclic.

Let $P_1$ be the Galois group of $L/F$. Since $F/K$ is cyclic, $P_1$ is a normal subgroup of $P$ and $P/P_1$ is cyclic.

Let $P_2$ be the Galois group of $L/L_1$. Then $P_2$ is a subgroup of $P_1$. Since $L/L_1$ is a totally ramified Galois extension of degree coprime to $\mathrm{char}(k)$, $P_2$ is cyclic ([Se], Cor.2 and Cor.4 of Ch.IV, §2). Since $L_1/F$ is a Galois extension (cf. §1), $P_2$ is a normal subgroup of $P_1$. The residue field $F_0$ of $F$ is the same as the intersection of $L_0$ and $E_0$. We now show that $P_1/P_2$ is admissible over $E_0$.

Let $D'$ be the division algebra with center $E$ which is Brauer equivalent to $D \otimes_K E$. Since $D' \otimes_E L_1E = (D \otimes_K E) \otimes_E L_1E$ and $(D \otimes_K E) \otimes_E L_1E$ is split (cf. proof of (3.3)), $D' \otimes_E L_1E$ is also split. Since, by (3.3) $\mathrm{degree}(D') = [L_1E : E]$, $L_1E$ is a maximal subfield of $D'$. By (3.1), $D' = D \otimes_K E$ is unramified at the discrete valuation of $E$. Let $\overline{D'}$ be the image of $D'$ over the residue field $E_0$ of $E$. Since $E$ is complete, $\overline{D'}$ is central division algebra over $E_0$ and $L_0E_0$ is a maximal subfield of $\overline{D'}$. Hence $Gal(L_0E_0/E_0)$ is admissible over $E_0$. Since the residue field of $L_1E$ is $L_0E_0$ and $L_1E/E$ is an unramified Galois extension, we have $Gal(L_0E_0/E_0) \simeq Gal(L_1E/E)$. Since $L_1/K$ and $E/K$ are Galois and $F = L_1 \cap E$, we have $Gal(L_1E/E) \simeq Gal(L_1/F)$. Since $P_1/P_2 \simeq Gal(L_1/F) \simeq Gal(L_0E_0/E_0)$, $P_1/P_2$ is admissible over $E_0$. $\square$

The above lemma immediately gives the following

**Proposition 3.5.** Let $K$ be a complete discretely valued field with residue filed $k$ and $G$ be a finite group. Suppose that the order of $G$ is coprime to char($k$). If $G$ is admissible over $K$ then every Sylow subgroup $P$ of $G$ has a normal sereis $P \supseteq P_1 \supseteq P_2$ such that

(1) $P/P_1$ and $P_2$ are cyclic

(2) $P_1/P_2$ is admissible over some finite extension of the residue field of $K$.

**Proof.** Let $G$ be an admissible group over $K$. Then there is a field extension $L/K$ and a division algebra $D$ central over $K$ containing $L$ as a maximal subfield with $Gal(L/K) = G$. Let $P$ be a Sylow subgroup of $G$. Let $L^P$ be the fixed of $P$. Then $L^P$ is a complete discretely valued field. Let $D'$ be the commutant of $L^P$ in $D$. Then $D'$ is a central division algebra over $L^P$ and $G(L/L^P) = P$ is admissible over $L^P$. Since $L^P$ is also a complete discrete valued field, the result follows by (3.4). $\square$

**Corollary 3.6.** Let $K$ be a complete discretely valued field with residue filed $k$ either a local field or a global field. Let $G$ be a finite group such that char($k$) coprime to the order of $G$. If $G$ is admissible over $K$ then every Sylow subgroup $P$ of $G$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) $P/P_1$ and $P_2$ are cyclic

(2) $P_1/P_2$ is metacyclic.

**Proof.** Every finite extension of the residue field is either a local field or a global field. The corollary follows from (3.5) and ([Sc]). $\square$

**Theorem 3.7.** Let $K$ be a complete discretely valued field with residue field $k$. Let $F$ be the function field of a curve over $K$. Let $G$ be a finite group with order coprime to char($k$). If $G$ is admissible over $F$ then every Sylow subgroup $P$ of $G$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1) $P/P_1$ and $P_2$ are cyclic

(2) $P_1/P_2$ is admissible over some finite extension of the residue field at a discrete valuation of $F$.

**Proof** First we reduce to a Sylow subgroup as in (3.5). Let $G$ be an admissible group over $F$. Then there is a field extension $L/F$ and a division algebra $D$ central over $F$ containing $L$ as a maximal subfield with $Gal(L/F) = G$. Let $P$ be a Sylow subgroup of $G$. Let $L^P$ be the fixed of $P$. Then $L^P$ is a complete discretely valued field. Let $D'$ be the commutant of $L^P$ in $D$. Then $D'$ is a central division algebra over $L^P$ and $G(L/L^P) = P$ is admissible over $L^P$. Since $L^P$ is a finte extension of $F$, $L^P$ is also a function field of a curve over a finite extension of $K$. Since the degree of $D'$ is a power of prime and coprime to the char($k$), by (2.6), there exists a discrete valuation $\nu$ of $L^P$ such that $D' \otimes_{L^P} L^P_\nu$ is division. Since $L \otimes_{L^P} L^P_\nu$ is a maximal subfield of $D' \otimes_{L^P} L^P_\nu$ and $P = Gal(L \otimes_{L^P} L^P_\nu / L^P_\nu)$, the result follows from (3.5). $\square$

The following is immediate from (3.7) and (3.6).

**Corollary 3.8.** Let $K$ be a local adic field and F the function field of a curve over $K$. Let $n$ be a natural number which is coprime to the characteristic of the residue field of $K$ and $G$ a finite group of order $n$. If $G$ is admissible over $F$ then every Sylow subgroup $P$ of $G$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that
(1) $P/P_1$ and $P_2$ are cyclic
(2) $P_1/P_2$ metacyclic.

The following is proved in ([HHK2], 4.5).

**Corollary 3.9.** Let $K$ be a complete discretely valued field with residue field algebraically closed. Let F be the function field of a curve over $K$. Let $G$ be a finite group of order $n$ such that the characteristic of the residue field of $K$ is coprime to $n$. If $G$ is admissible over $F$ then every Sylow subgroup $P$ of $G$ is metacyclic.

**Proof.** By (3.7), every Sylow subgroup $P$ of $G$ has a normal series $P \supseteq P_1 \supseteq P_2$ such that
(1) $P/P_1$ and $P_2$ are cyclic
(2) $P_1/P_2$ is admissible over some finite extension of the residue field of $K$.

Let $k$ be the residue field of $K$. By (2.7) and the proof of (3.7), the residue field at the discrete valuation given in (2) is either a finite extension $K$ or the function field of a curve over $k$. Since $k$ is algebraically closed, these residue fields have cohomological dimension at most one and there are no non-trivial division algebras over such fields. Hence $P_1 = P_2$. $\square$

We now give an example of a finite group which is not $\mathbb{Q}_p(t)$-admissible.

**Example 3.10.** Let $\ell$ and $p$ be two distinct primes. Let $P = (\mathbb{Z}/l\mathbb{Z})^5$. We claim that this group is not admissible over $\mathbb{Q}_p(t)$. Suppose that $P$ is admissible over $\mathbb{Q}_p(t)$. Then by (3.8), there is a normal series $P \supseteq P_1 \supseteq P_2$ such that

(1)$P/P_1$ and $P_2$ are cyclic

(2)$P_1/P_2$ is metacyclic

Since $P_2$ and $P/P_1$ are cyclic their orders will be at most $l$. This implies that $|P_1/P_2| \geq l^3$. Since every element of $P_1/P_2$ has order at most $l$ and $|P_1/P_2| \geq l^3$, $P_1/P_2$ cannot be metacyclic. $\square$

**Remark 3.11.** Let $F$ be a field and $p$ a prime. Suppose that for every finite extension $E$ of $F$ there is a set $\Omega_E$ of discrete valuations of $E$ such that given a central division algebra $D$ over $E$ of degree a power of $p$, there exists a discrete valuation $\nu \in \Omega_E$ such that $D \otimes E_\nu$ is division. Let $G$ be a finite group. The proof of (3.7) gives us the following: If $G$ is admissible over $F$, then every $p$-Sylow subgroup of $G$ has filtration as in (3.7).

**Remark 3.12.** Let $G$ be a finite group satisfying the conditions of (3.5). Then every homomorphic image of $G$ also satisfy the same conditions. However there is an example of a group $G$ with a homomorphic image $H$ such that $G$ is admissible over the complet discrete valuation field $\mathbb{Q}((t))$ but $H$ is not admissible ([FS]). Hence the conditions given in (3.5) for a group to be admissible are not sufficient.

# 4. A class of Admissible groups over $\mathbb{Q}_p(t)$

Let $K$ be a discretely valued field with residue field $k$. Let $F$ be the function field of a curve over $K$. Let $n$ be an integer which is coprime to the characteristic of $k$. Suppose that $K$ contains a primitive $n^{th}$ root of unity. Then in ([HHK2], 4.4) it is proved that every finite group of order $n$ with every Sylow subgroup product of at most two cyclic groups is admissible over $F$. They used the patching techniques to prove this result. In this section we prove a similar result for groups with every Sylow subgroup is a product of at most 4 cyclic groups, with an additional assumption on the residue field $k$. We begin with the following

**Lemma 4.1.** Let $R$ be a regular local ring of dimension two with residue field $k$ and field of fraction $F$. Let $n_1$ and $n_2$ be natural numbers which are coprime to the char$(k)$. Assume that $F$ contains a primitive $(n_1n_2)^{th}$ root of unity and there is an element in $k^*/k^{*n_2}$ of order $n_2$. Then there is a central division algebra $D$ over $F$ of degree $n_1n_2$.

**Proof.** Let $m$ be the maximal ideal of $R$. Since $R$ is a regular local ring of dimension two, we have $m = (t, s)$. By the assumption on $k$, there is an element $\lambda_0 \in k^*$ such that its order in $k^*/k^{*n_2}$ is $n_2$. Let $\lambda \in R$ which maps to $\lambda_0$. Let $a \in R$ be a unit with $a^{n_1} \neq 1$. Let $\xi_1$ be a primitive $n_1^{th}$ root of unity and $\xi_2$ a primitive $n_2^{th}$ root of unity.

Let

$$D_1 = (\frac{s}{s-t}, \frac{s-t^2}{s-a^{n_1}t^2})_{n_1}$$

and

$$D_2 = (\frac{s}{s-t^2}, \frac{s-\lambda t^2}{s-t^2})_{n_2}.$$

Let $D = D_1 \otimes_F D_2$. Then the degree of $D$ is $n_1n_2$. We now show that $D$ is a division algebra.

Let $S = R[x]/(s - t^2x)$. Then the field of fractions of $S$ is isomorphic to $F$. We have

$$D_1 = (\frac{tx}{tx-1}, \frac{x-1}{x-a^{n_1}})_{n_1}$$

and

$$D_2 = (\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}.$$

The ideal $(t)$ of $S$ is a prime ideal and gives a discrete valuation $\nu$ on $F$. Let $F_\nu$ be the completion of $F$ at $\nu$. To show that $D_1 \otimes_F D_2$ is a division algebra, it is enough to show that $D_1 \otimes_F D_2 \otimes_F F_\nu$ is a division algebra. Since $tx/tx-1$ is a parameter at $\nu$, by (2.1), we have

$$\mathrm{index}(D_1 \otimes_F D_2 \otimes_F F_\nu) = \mathrm{index}(D_2 \otimes_F F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}))\cdot[F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}) : F_\nu].$$

Since $\frac{x-1}{x-a^{n_1}}$ is a unit at $\nu$ and the residue field $\kappa(\nu)$ at $\nu$ is $k(x)$, by the assumption on $a$, we have $[F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}}) : F_\nu] = n_1$. Since $D_2$ is unramified at $\nu$, the index of $D_2 \otimes_F F_\nu(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$ is equal to the index of its image $(\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}$ over the residue field $k(x)(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$. Let $\theta$ be the discrete valuation on $k(x)$ given by $(x)$. Let $\upsilon$ be the extension of $\theta$ to $k(x)(\sqrt[n_1]{\frac{x-1}{x-a^{n_1}}})$.

Then the residue field of $\upsilon$ is $k$. The residue of $(\frac{x}{x-1}, \frac{x-\lambda}{x-1})_{n_2}$ at $\upsilon$ is the class of $\lambda_0$. Since the order of $\lambda_0$ in $k^*/k^{*n_2}$ is $n_2$, the index of $D_2$ is $n_2$. Hence the index of $D_1 \otimes_F D_2 \otimes F_\nu$ is $n_1 n_2$. Since the degree of $D_1 \otimes_F D_2$ is $n_1 n_2$, $D_1 \otimes_F D_2$ is a division algebra. $\square$

**Theorem 4.2.** Let $K$ be a discretely valued field with residue field $k$ and $F$ the function field of a curve over $K$. Let $n$ be an integer which is coprime to the characteristic of $k$. Suppose that $K$ contains a primitive $n^{th}$ root of unity. Assume that for every finite extension $L$ of $k$, there is an element in $L^*/L^{*n}$ of order $n$. If $G$ is a finite group of order $n$ with every Sylow subgroup is a quotient of $\mathbb{Z}^4$, then $G$ is admissible over $F$.

**Proof.** Let $R$ be the ring of integers in $K$. Let $\mathcal{X}$ be a regular proper two dimensional scheme over $R$ with function field $F$ and the reduced special fibre is a union of regular curves with normal crossings. Let $p_1, \cdots, p_r$ be the prime factors of $n$. Let $Q_1, \cdots, Q_r$ be regular closed points on the special fibre of $\mathcal{X}$. Let $R_{Q_i}$ be the regular local ring at $Q_i$, $\hat{R}_{Q_i}$ be the completion of $R_{Q_i}$ at the maximal ideal and $F_{Q_i}$ the field of fractions of $\hat{R}_{Q_i}$. Let $t_i \in R_{Q_i}$ be a prime defining the irreducible component of the special fibre of $\mathcal{X}$ containing $Q_i$. Let $P_i$ be a $p_i$-Sylow subgroup of $G$. By ([HHK2], 4.2), it is enough to show that there exists a central division algebra $D_i$ over $F_{Q_i}$ and maximal subfield $L_i$ of $D_i$ with $Gal(L_i/F_{Q_i}) \simeq P_i$ and $L_i \otimes \hat{F}_{Q_i}$ a split algebra.

For a given $i$, $1 \leq i \leq r$, let $Q = Q_i$, $t = t_i$ and $P = P_i$. Since the residue field $\kappa(Q)$ of $\hat{R}_Q$ is a finite extension of $k$, by the assumption on $k$, there is an element in $\kappa(Q)^*/\kappa(Q)^{*n}$ of order $m$ for any $m$ dividing $n$. Since $P$ is a quotient of $\mathbb{Z}^4$, $P \simeq C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$. Since $\hat{R}_Q$ is regular local ring of dimension 2 and $t$ is a regular prime, we have $m_Q = (t, s)$. Let $\xi_1$ be a primitive ${n_1 n_2}^{th}$ root of unity and $\xi_2$ a primitive ${n_3 n_4}^{th}$ root of unity.

Let

$$D_1 = (\frac{s}{s-t}, \frac{s-t^2}{s-a^{n_1 n_1} t^2})_{n_1 n_2}$$

and

$$D_2 = (\frac{s}{s-t^2}, \frac{s-\lambda t^2}{s-t^2})_{n_3 n_4}.$$

for suitable $a$ and $\lambda$ as in (4.1). Let $D = D_1 \otimes_F D_2$. Then, by (4.1), $D$ is a division algebra over $F_Q$.

In particular $D_1$ and $D_2$ are division algebras over $F_Q$. The cyclic algebra $D_1$ is generated by $x_1$ and $y_1$ with relations

$$x_1^{n_1 n_2} = \frac{s}{s-t}, \ y_1^{n_1 n_2} = \frac{s-t^2}{s-a^{n_1 n_2} t^2} \text{ and } x_1 y_1 = \xi_1 y_1 x_1.$$

Similarly $D_2$ is generated by $x_2$ and $y_2$ with relations

$$x_2^{n_3n_4} = \frac{s}{s-t^2},\ y_2^{n_3n_4} = \frac{s-\lambda t^2}{s-t^2} \text{ and } x_2y_2 = \xi_2 y_2 x_2.$$

Let $L_1$ be the subalgebra of $D_1$ generated by $x_1^{n_1}$ and $y_1^{n_2}$. Let $L_2$ be the subalgebra of $D_2$ generated by $x_2^{n_3}$ and $y_2^{n_4}$. Then $L = L_1 \otimes L_2$ is a maximal subfield of $D_1 \otimes D_2$, $Gal(L/F) = C_{n_1} \times C_{n_2} \times C_{n_3} \times C_{n_4}$ and $L \otimes \hat{F}$ is a split algebra (cf., ([HHK2]) proof of 4.4, ). $\square$

**Corollary 4.3.** Let $K$ be a local field and $k$ its residue field. Let $F$ be the function field of a curve over $K$. Let $n$ be an integer which is coprime to the characteristic of $k$. Suppose that $K$ contains a primitive $n^{th}$ root of unity. If $G$ is a group of order $n$ with every Sylow subgroup a quotient of $\mathbb{Z}^4$, then $G$ is admissible over $F$.

**Proof.** Since $k$ is a finite field, for any finite extension $L$ of $k$ and for any natural number $n$ coprime to the characteristic of $L$, we have $L^*/L^{*n}$ is cyclic group of order $n$. Hence $k$ satisfies the condition of (4.2). $\square$.

Let $K$ be a complete discretely valued field with residue field $k$. Let $G$ be a finite abelian group of order $n$ which is a product of at most four cyclic groups. Suppose that $n$ is coprime to char$(k)$ and the order of $k^*/k^{*n}$ is at least $n$. If $K$ contains a primitive $n^{th}$ root of unity, then by (4.3), $G$ is admissible over $K(t)$. We now prove that a similar result without the completeness assumption on $K$.

We begin with the following

**Lemma 4.4.** Let $R$ be a discrete valuation ring and $\pi \in R$ a parameter. Let $K$ be the field of fractions of $R$ and $k$ the residue filed of $R$. Let $F = K(t)$ be the rational function field in one variable over $K$. Let $n$ be a natural number which is coprime to char$(k)$. Assume that $K$ contains a primitive $n^{th}$ root of unity and there exists a $\lambda_0 \in k^*$ such that $[k(\sqrt[n]{\lambda_0}) : k] = n$. Then $(t, \pi - \lambda t)_n \otimes_F (t+1, \pi)_m$ is division over $F$ for any $\lambda \in R$ which maps to $\lambda_0$.

**Proof** Since $\pi$ is a parameter in $R$, the localisation $R[t]_{(\pi)}$ of $R[t]$ at the prime ideal $(\pi)$ is a discrete valuation ring. Let $\nu$ be the discrete valuation on $F$ given the discrete valuation ring $R[t]_{(\pi)}$ and $F_\nu$ be the completion of $F$ at $\nu$. Since the residue field at $\nu$ is $k(t)$, we have $[F_\nu(\sqrt[m]{t+1}) : F_\nu] = m$. Let $w$ be the extension of $\nu$ to $F_\nu(\sqrt[m]{t+1})$. To show that $(t, \pi - \lambda t)_n \otimes_F (t+1, \pi)_m$ is division, it is enough to show that $(t, \pi - \lambda t)_n \otimes_F (t+1, \pi)_m \otimes_F F_\nu$ is division. Since $F_\nu$ is complete and $[F_\nu(\sqrt[m]{t+1}) : F_\nu] = m$, by (2.1), it is enough to show that $(t, \pi - \lambda t)_n \otimes F_\nu(\sqrt[m]{t+1})$ is division. Since $t$ and $\pi - \lambda t$ are units at $\nu$, the algebra $(t, \pi - \lambda t)_n \otimes F_\nu(\sqrt[m]{t+1})$ is unramified at $w$. Thus it is enough to show that its image $(t, -\lambda_0 t)_n = (t, \lambda_0)_n$ is division over the residue filed $k(t)(\sqrt[m]{t+1})$. Let $\gamma$ be the discrete valuation on $k(t)$ given by $t$ and $\tilde{\gamma}$ be the extension of $\gamma$ to $k(t)(\sqrt[m]{t+1})$. Since $t+1$ is an $m^{th}$ power in the completion of $k(t)$ at $\gamma$, the completion of $k(t)(\sqrt[m]{t+1})$ at $\tilde{\gamma}$ is $k((t))$. It is enough show that $(t, \lambda_0)_n$ is division over $k((t))$. Since $\lambda_0 \in k^*$ is an element of order $n$, $(t, \lambda_0)_n$ is division over $k((t))$. $\square$

**Theorem 4.5.** Let $K$ be a field with a discrete valuation (not necessarily complete) and $k$ its residue field. Let $G = \mathbb{Z}/l_1\mathbb{Z} \times \mathbb{Z}/l_2\mathbb{Z} \times \mathbb{Z}/l_3\mathbb{Z} \times \mathbb{Z}/l_4\mathbb{Z}$. Suppose that the $n = l_1l_2l_3l_4$ is coprime to char$(k)$, $K$ contains a primitive $n^{th}$ root of unity and there is a $\lambda_0 \in k$ such that $[k(\sqrt[l_1l_2]{\lambda_0} : k] = l_1l_2$. Then $G$ is admissible over $K(t)$.

**Proof.** Let $R$ be the ring of integers in $K$ and $\lambda \in R$ mapping to $\lambda_0$ in $k$. Let $\pi \in R$ be a parameter. Let $D_1 = (t, \pi - \lambda t)_{l_1l_2}$ and $D_2 = (t+1, \pi)_{l_3l_4}$. Then, by (4.4), $D_1 \otimes D_2$ is a division algebra over $K(t)$. Let $L_1 = K(t)(y, z)$ where $y^{l_1} = t$, $z^{l_2} = \pi - \lambda t$. Then $L_1$ is a maximal subfield of $D_1$. Let $L_2 = K(t)(r, s)$ where $y^{l_3} = t+1$, $z^{l_4} = \pi$. Then $L_2$ is a maximal subfield of $D_2$. Therefore $L_1 \otimes_{K(t)} L_2$ is a maximal subfield of $D_1 \otimes D_2$. Since $Gal(L_1 \otimes_{K(t)} L_2/K(t)) = G$, $G$ is admissible over $K(t)$. $\square$

**Corollary 4.6.** Let $p$ be a prim and $l_1, l_2, l_3, l_4$ be natural numbers which are coprime to $p$. Let $G = \mathbb{Z}/l_1\mathbb{Z} \times \mathbb{Z}/l_2\mathbb{Z} \times \mathbb{Z}/l_3\mathbb{Z} \times \mathbb{Z}/l_4\mathbb{Z}$. Let $n = l_1l_2l_3l_4$ and $\zeta$ a primitive $n^{th}$ root of unity. Then $G$ is admisible over $\mathbb{Q}(\zeta)(t)$.

**Remark 4.7.** For $p$, $l_1, l_2, l_3, l_4$, $G$ as in (4.6), $G$ is admissible over $\mathbb{Q}_p(t)$ by (4.3). However (4.5) gives an explicite constriction of a division algebra over $\mathbb{Q}_p(t)$ and a maximal subfield with Galois group $G$.

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