WEAKLY COMMENSURABLE S-ARITHMETIC SUBGROUPS IN ALMOST SIMPLE ALGEBRAIC GROUPS OF TYPES B AND C

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To Kevin McCrimmon on the occasion of his retirement

Abstract. Let G_1 and G_2 be absolutely almost simple algebraic groups of types B_ℓ and C_ℓ respectively defined over a number field K. We determine when G_1 and G_2 have the same isomorphism or isogeny classes of maximal K-tori. This leads to the necessary and sufficient conditions for two Zariski-dense S-arithmetic subgroups of G_1 and G_2 to be weakly commensurable.

Contents

1. Introduction and the statement of main results

This paper has two interrelated goals: first, to complete the investigation of weak commensurability of S-arithmetic subgroups of almost simple algebraic groups begun in [PR09], and second, to contribute to the classical problem of characterizing almost simple algebraic groups having the same isomorphism or the same isogeny classes of maximal tori over the field of definition.

Let G_1 and G_2 be two semi-simple algebraic groups over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a (finitely generated) Zariski-dense subgroup for $i = 1, 2$. We recall in §7 below the notion of *weak commensurability* of Γ_1 and Γ_2 introduced in [PR09]. (This notion was inspired by some problems dealing with isospectral and length-commensurable locally symmetric spaces, and we state some geometric consequences of our main results in (7.1) and (7.2).) We further recall that the mere existence of Zariski-dense weakly commensurable subgroups implies that G_1 and G_2 either have the same Killing-Cartan type, or one of them is of type B_ℓ and the other is of type C_ℓ. Moreover, cumulatively the results of [PR09], [PR10]

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and [Gar12] give, by and large, a complete picture of weak commensurability for S-arithmetic subgroups of almost simple algebraic groups having the *same* type.

On the other hand, weak commensurability of S-arithmetic subgroups in the case where G_1 is of type B_ℓ and G_2 is of type C_ℓ has not been investigated so far—it was only pointed out in [PR09] that S-arithmetic subgroups corresponding to the split forms of such groups are indeed weakly commensurable (see also Remark 2.6 below). Our first theorem provides a complete characterization of the situations where S-arithmetic subgroups in the groups of types B and C are weakly commensurable. In its formulation we will employ the description, introduced in [PR09, §1], of S-arithmetic subgroups of $G(F)$, where G is an absolutely almost simple algebraic group over a field F of characteristic zero, in terms of triples $(\mathcal{G}, K, S)$ consisting of a number field $K \subset F$, a finite subset S of places of K, and an F/K-form $\mathcal{G}$ of the adjoint group $\overline{G}$ — we briefly recall this description in §6.

The following definition will enable us to streamline the statements of our results.

Definition 1.1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be absolutely almost simple algebraic groups of types B_ℓ and C_ℓ with $\ell \geqslant 2$, respectively, over a number field K. We say that $\mathcal{G}_1$ and $\mathcal{G}_2$ are *twins* (over K) if for each place v of K, both groups are simultaneously either split or anisotropic over the completion K_v.

Theorem 1.2. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero having Killing-Cartan types B_ℓ and C_ℓ ($\ell \geqslant 3$) respectively, and let Γ_i be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$ for $i = 1, 2$. Then Γ_1 and Γ_2 are weakly commensurable if and only if the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are twins.*

If Zariski-dense $(\mathcal{G}_1, K_1, S_1)$- and $(\mathcal{G}_2, K_2, S_2)$-subgroups are weakly commensurable then necessarily $K_1 = K_2$ and $S_1 = S_2$ by [PR09, Th. 3], so Theorem 1.2 in fact treats the most general situation. Furthermore, for $\ell = 2$ we have $\mathsf{B}_2 = \mathsf{C}_2$, so G_1 and G_2 have the same type; then Γ_1 and Γ_2 are weakly commensurable if and only if $\mathcal{G}_1 \simeq \mathcal{G}_2$ over K by [PR09, Th. 4]. This shows that the assumption $\ell \geqslant 3$ in Theorem 1.2 is essential—the excluded case of $\ell = 2$ is treated in Theorem 1.5 below.

Turning to the second problem, of characterizing almost simple algebraic groups having the same (isomorphic classes of) maximal tori, we would like to point out that, as we will see shortly, one gets more satisfactory results if instead of talking about *isomorphic* groups one talks about *isogenous* ones. We recall that algebraic K-groups H_1 and H_2 are called isogenous if there exists a K-group H with central K-isogenies $\pi_i \colon H \to H_i$, $i = 1, 2$. For semi-simple K-groups G_1 and G_2, this amounts to the fact that the universal covers $\widetilde{G}_1$ and $\widetilde{G}_2$ are K-isomorphic, and for K-tori T_1 and T_2 this simply means that there exists a K-isogeny $T_1 \to T_2$. Furthermore, we say that two semi-simple K-groups G_1 and G_2 have the same isogeny classes of maximal K-tori if every maximal K-torus T_1 of G_1 is K-isogenous to some maximal K-torus T_2 of G_2, and vice versa. Unsurprisingly, K-isogenous groups have the same isogeny classes of maximal tori. Using the results from [PR09] and [Gar12], we prove the following partial converse for almost simple groups over number fields.

Proposition 1.3. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a number field K. Assume that G_1 and G_2 have the same isogeny classes of maximal K-tori. Then at least one of the following holds:*

(1) *G_1 and G_2 are K-isogenous;*
(2) *G_1 and G_2 are of the same Killing-Cartan type, which is one of the following: A_ℓ ($\ell > 1$), $\mathsf{D}_{2\ell+1}$ ($\ell > 1$), or E_6;*
(3) *one of the groups is of type B_ℓ and the other of type C_ℓ for some $\ell \geqslant 3$.*

We will prove the proposition in §8. As Theorem 1.5 below shows, it is possible for two isogenous, but not isomorphic, groups to have the same isomorphism classes of maximal K-tori, so the conclusion in (1) cannot be strengthened even if we assume that G_1 and G_2 have the same maximal tori. On the other hand, for each of the types listed in (2) one can construct non-isomorphic simply connected, hence non-isogenous, groups of this type having the same tori [PR09, §9], so these types are genuine exceptions. In this paper, we will sharpen case (3). Specifically, we prove the following in §6.

Theorem 1.4. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a number field K of types B_ℓ and C_ℓ respectively for some $\ell \geqslant 3$.*

(1) *The groups G_1 and G_2 have the same isogeny classes of maximal K-tori if and only if they are twins.*
(2) *The groups G_1 and G_2 have the same isomorphism classes of maximal K-tori if and only if they are twins, G_1 is adjoint, and G_2 is simply connected.*

We note that one can give examples of groups G_1 and G_2 of types B_ℓ and C_ℓ respectively over the field $\mathbb{R}$ of real numbers, that are neither split nor anisotropic but nevertheless have the same isomorphism classes of maximal $\mathbb{R}$-tori (see Example 3.6). This shows Theorem 1.4, unlike many statements about algebraic groups over number fields, is *not* a global version of the corresponding theorem over local fields. What is crucial for the proof of Theorem 1.4 (and also Theorem 1.2) is that if the real groups G_1 and G_2 are neither split nor anisotropic with G_1 adjoint and G_2 simply connected then they cannot have the same maximal $\mathbb{R}$-tori (see Corollary 3.4).

The special case $\mathsf{B}_2 = \mathsf{C}_2$. Theorem 1.4 completely settles the question of when the groups of types B_ℓ and C_ℓ have isogenous tori for $\ell \geqslant 3$. The case where $\ell = 2$ is special because the root systems B_2 and C_2 are the same.

Let G_1 and G_2 be groups of type $\mathsf{B}_2 = \mathsf{C}_2$. They have the same isogeny classes of maximal tori if and only if they are isogenous by Lemma 8.1 below or [PR09, Th. 7.5(2)]. In particular, when G_1 and G_2 are both adjoint or both simply connected, they have the same isogeny classes of maximal tori if and only if $G_1 \simeq G_2$ if and only if they have the same maximal tori. It remains only to give a condition for G_1 and G_2 to have the same maximal tori when one is adjoint and the other is simply connected, which we now do.

Theorem 1.5. *Let q_1, q_2 be 5-dimensional quadratic forms over a number field K. The groups $G_1 = \mathrm{SO}(q_1)$ and $G_2 = \mathrm{Spin}(q_2)$ have the same isomorphism classes of maximal K-tori if and only if*

(1) *q_1 is similar to q_2; and*
(2) *q_1 and q_2 are either both split or both anisotropic at every completion of K.*

Notation. For a number field K, we let V^K denote the set of all places, and let V^K_∞ (resp., V^K_f) denote the subset of archimedean (resp., nonarchimedean) places. Given a reductive algebraic group G defined over a field K, for any field extension L/K we let $\mathrm{rk}_L\, G$ denote the L-rank of G, i.e., the dimension of a maximal L-split torus.

We write $r\langle a\rangle$ for the symmetric bilinear form $(x, y) \mapsto a \sum_{i=1}^r x_i y_i$ on K^r, and adopt similar notation for quadratic forms and hermitian forms.

In §6, we systematically use the following: for G_1 and G_2 absolutely almost simple groups of types B_ℓ and C_ℓ respectively, we put $G_1^\natural$ for the adjoint group of G_1 ("SO"), and $G_2^\natural$ for the simply connected cover of G_2 ("Sp").

2. Steinberg's theorem for algebras with involution

Our proofs of Theorems 1.2 and 1.4 rely on the well-known fact that groups of classical types can be realized as special unitary groups associated with simple algebras with involutions, so their maximal tori correspond to certain commutative étale subalgebras invariant under the involution. This description enables us to apply the local-global principles for the existence of an embedding of an étale algebra with an involutory automorphism into a simple algebra with an involution [PR10]. To ensure the existence of local embeddings, we will use an analogue for algebras with involution of the theorem, due to Steinberg [Ste65], asserting that if G_0 is a quasi-split simply connected almost simple algebraic group over a field K and G is an inner form of G_0 over K then any maximal K-torus T of G admits a K-defined embedding into G_0. The required analogue roughly states that if (A, τ) is an algebra with involution such that the corresponding group is quasi-split then any commutative étale algebra with involution (E, σ) that can potentially embed in (A, τ) does embed. It can be deduced from the original Steinberg's theorem along the lines of [Gil04, Prop. 3.2(b)], but in fact one can give a simple direct argument. To our knowledge, this has not been recorded in the literature. Further, the argument for type B_n (in Proposition 2.5) extends with minor modifications to other types. So, despite the fact that we will only use this statement for algebras corresponding to groups of type B_n and C_n, we will give the argument for all classical types. We begin by briefly recalling the types of algebras with involution arising in this context, indicating in each case the étale subalgebras that give maximal tori.

Description of tori in terms of étale algebras. Let A be a central simple algebra of dimension n^2 over a field L of characteristic $\neq 2$, and let τ be an involution of A. Set $K = L^\tau$. We recall that τ is said to be of the *first* (resp., *second*) kind if the restriction $\tau|_L$ is trivial (resp., nontrivial). Furthermore, if τ is an involution of the first kind, then it is either *symplectic* (i.e., $\dim_K A^\tau = n(n-1)/2$) or *orthogonal* (i.e., $\dim_K A^\tau = n(n+1)/2$).

We also recall the well-known correspondence between involutions on $A = M_n(L)$ and nondegenerate hermitian or skew-hermitian forms on L^n, cf. [KMRT98]: given such a form f, there exists a unique involution τ_f such that

$$f(ax, y) = f(x, \tau_f(a)y)$$

for all $x, y \in L^n$ and all $a \in A$; then the pair $(M_n(L), \tau_f)$ will be denoted by A_f. Moreover, f is symmetric (resp., skew-symmetric) if and only if τ_f is orthogonal (resp., symplectic). Conversely, for any involution τ there exists a form f on L^n of appropriate type such that $\tau = \tau_f$, and any two such forms are proportional. (Note that for involutions of the second kind one can pick the corresponding form to be either hermitian or skew-hermitian as desired.)

Type $^2\mathsf{A}_\ell$. Let (A, τ) be a central simple L-algebra of dimension n^2 with an involution τ of the second kind. Then $G = \mathrm{SU}(A, \tau)$ is an absolutely almost simple simply connected K-group of type $^2\mathsf{A}_\ell$ with $\ell = n-1$, and conversely any such group corresponds to an algebra with involution (A, τ) of this kind. Any τ-invariant étale commutative subalgebra $E \subset A$ gives a maximal K-torus

$$T = \mathrm{R}_{E/K}(\mathrm{GL}_1) \cap G = \mathrm{SU}(E, \tau|_E)$$

of G, and all maximal K-tori are obtained this way (see, for example, [PR10, Prop. 2.3]). The group G is quasi-split if and only if $A = M_n(L)$ and $\tau = \tau_h$ where h is a nondegenerate hermitian form on L^n of Witt index $[n/2]$.

Type B_ℓ $(\ell \geqslant 2)$. Let $A = M_n(K)$ with $n = 2\ell + 1$, and let τ be an orthogonal involution of A. Then $\tau = \tau_f$ for some nondegenerate symmetric bilinear form f on K^n, and

$$G = \mathrm{SU}(A, \tau) = \mathrm{SO}(f)$$

is an adjoint group of type B_ℓ, and every such group is obtained this way. Furthermore, maximal K-tori T of G bijectively correspond to maximal commutative étale τ-invariant subalgebras E of A (of dimension n) such that $\dim_K E^\tau = \ell + 1$ under the correspondence given by

$$T = \mathrm{R}_{E/K}(\mathrm{GL}_1) \cap G = \mathrm{SU}(E, \tau|_E).$$

Furthermore, any such algebra admits a decomposition

$$(E, \tau) = (E', \tau') \times (K, \mathrm{id}_K) \tag{2.1}$$

where $E' \subset E$ is a τ-invariant subalgebra of dimension 2ℓ. Finally, the group G is quasi-split (in fact, split) if and only if f has Witt index ℓ.

Type C_ℓ $(\ell \geqslant 2)$. Let A be a central simple K-algebra of dimension n^2 with $n = 2\ell$, and let τ be a symplectic involution of A. Then $G = \mathrm{SU}(A, \tau)$ is an absolutely almost simple simply connected group of type C_ℓ, and all such groups are obtained this way. Maximal K-tori of G correspond to maximal commutative étale τ-invariant subalgebras $E \subset A$ (of dimension n) such that $\dim_K E^\tau = \ell$ in the fashion described above. The group G is quasi-split (in fact, split) if and only if $A = M_n(K)$. Then $\tau = \tau_f$ where f is a nondegenerate skew-symmetric form on K^n; there is only one equivalence class of such forms, so in this case $G \simeq \mathrm{Sp}_n$.

Type $^{1,2}\mathsf{D}_\ell$ $(\ell \geqslant 4)$. Let A be a central simple K-algebra K-algebra of dimension n^2 where $n = 2\ell$, and let τ be an orthogonal involution of A. Then $G = \mathrm{SU}(A, \tau)$ is an almost absolutely simple K-group of type $^{1,2}\mathsf{D}_\ell$ which is neither simply connected nor adjoint, and any K-group of this type is K-isogenous to such a group. Maximal K-tori of G correspond to maximal commutative étale τ-invariant subalgebras $E \subset A$ (of dimension n^2) such that $\dim_K E^\tau = \ell$. The group G is quasi-split if and only if $A = M_n(K)$ and $\tau = \tau_f$ where f is a symmetric bilinear form on K^n of Witt index $\ell - 1$ or ℓ.

Summary. Thus, if A is a central simple L-algebra of dimension n^2 (and $L = K$ for all types except ${}^2\mathsf{A}_\ell$) then maximal K-tori of the algebraic K-group $G = \mathrm{SU}(A, \tau)$ correspond in the manner described above to maximal abelian étale τ-invariant subalgebras $E \subset A$ with $\dim_L E = n$ such that for $\sigma = \tau|_E$ we have

$$\dim_K E^\sigma = \begin{cases} n & \text{if } \sigma|_L \neq \mathrm{id}_L \\ \left[\frac{n+1}{2}\right] & \text{if } \sigma|_L = \mathrm{id}_L. \end{cases} \tag{2.2}$$

(Note that the condition is automatically satisfied if $\sigma|_L \neq \mathrm{id}_L$.)

Now, let (E, σ) be an n-dimensional commutative étale L-algebra with an involution satisfying (2.2). Then the question of whether the K-torus $T = \mathrm{SU}(E, \sigma)^\circ$ can be embedded into $G = \mathrm{SU}(A, \tau)$ where A is a central simple L-algebra of dimension n^2 with an involution τ such that $\sigma|_L = \tau|_L$ translates into the question if there is an embedding $(E, \sigma) \hookrightarrow (A, \tau)$ of L-algebras with involution, which we will now investigate in the cases of interest to us. We note that if G is quasi-split then in all cases $A = M_n(L)$. In this case, the universal way to construct an embedding $(E, \sigma) \hookrightarrow (M_n(L), \tau)$ is described in the following well-known statement.

Proposition 2.1. *Let (E, σ) be an n-dimensional commutative étale L-algebra with an involution σ.*

(i) *For any $b \in E^\times$, the map $\phi_b \colon E \times E \to K$ given by*

$$\phi_b(x, y) = \mathrm{tr}_{E/L}(x \cdot b \cdot \sigma(y))$$

is a nondegenerate sesqui-linear form, which is hermitian or skew-hermitian if and only if b is such.

(ii) *Let $b \in E^\times$ be hermitian or skew-hermitian, and let τ_{ϕ_b} be the involution on $A := \mathrm{End}_L(E) \simeq M_n(L)$ corresponding to ϕ_b; then the regular representation of E gives an embedding*

$$(E, \sigma) \hookrightarrow (A, \tau_{\phi_b}) = A_{\phi_b}$$

of algebras with involution.

(iii) *Let τ be an involution on $A = M_n(L)$, and let f be a hermitian or skew-hermitian form on L^n such that $\tau_f = \tau$. Then the following conditions are equivalent:*

  (a) *There exists $b \in E^\times$ of the same type as f such that ϕ_b is equivalent to f.*

  (b) *There exists a form h on $E \simeq L^n$ which is equivalent to f and satisfies*

$$h(ax, y) = h(x, \sigma(a)y) \quad \text{for all} \quad a, x, y \in E. \tag{2.3}$$

  (c) *There exists an embedding $(E, \sigma) \hookrightarrow (A, \tau)$ as L-algebras with involutions.*

Sketch of proof. The nondegeneracy of ϕ_b in (i) follows from the fact that the L-bilinear form on E given by $(x, y) \mapsto \mathrm{tr}_{E/L}(xy)$ is nondegenerate as E/L is étale; other assertions in (i) and (ii) are immediate consequences of the definitions. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) in (iii) are obvious, and the equivalence (a) $\Leftrightarrow$ (c) (which we will not need) is established in [PR10, Prop. 7.1]. $\square$

We also note that in fact any nondegenerate hermitian/skew-hermitian form h on E satisfying (2.3) is of the form ϕ_b for some $b \in E^\times$ of the respective type. Indeed, since the form ϕ_1 is nondegenerate, we can write h in the form $h(x, y) =$

$\mathrm{tr}_{E/L}(x \cdot g(\sigma(y)))$ for some K-linear automorphism g of E. Then (2.3) implies that g is E-linear, and therefore is of the form $g(x) = bx$ for some $b \in E^{\times}$, which will necessarily be of appropriate type.

Example 2.2 (involutions of the first kind)**.** According to Proposition 2.2 in [PR10], if $L = K$ and (E, σ) is a K-algebra with involution of dimension $n = 2\ell$ satisfying (2.2) then

$$(E, \sigma) \simeq (F[\delta]/(\delta^2 - d), \theta)$$

where $F = E^{\sigma}$, $d \in F^{\times}$, and $\theta(\delta) = -\delta$.

For invertible $b \in E^{\sigma}$ and $x_i, y_i \in F$, we have

$$\phi_b(x_1 + y_1\delta, x_2 + y_2\delta) = \mathrm{tr}_{E/K}(bx_1x_2 - bdy_1y_2) = \mathrm{tr}_{F/K}(2b(x_1x_2 - dy_1y_2)),$$

so ϕ_b is the transfer from F to K of the symmetric bilinear form $\langle 2b, -2bd\rangle$. Clearly, if E is $F \times F$, then ϕ_b is hyperbolic.

The example gives the entries in the ϕ_b column of Table 1.

Proposition 2.3 (type C)**.** *Let (E, σ) be an étale K-algebra of dimension $n = 2\ell$ with involution satisfying* (2.2)*. Then for every symplectic involution τ on $M_n(K)$, there is a K-embedding $(E, \sigma) \hookrightarrow (M_n(K), \tau)$.*

Proof. It follows from the structure of (E, σ) in the example that there exists a skew-symmetric *invertible* $b \in E$ (one can take, for example, the element corresponding to δ); then by Proposition 2.1(i), the form ϕ_b is nondegenerate and skew-symmetric. On the other hand, since τ is symplectic, we have $\tau = \tau_f$ for some nondegenerate skew-symmetric form f on K^n. As any two such forms are equivalent, our assertion follows from Proposition 2.1(iii). □

To handle the algebras corresponding to types B and D, we need the following.

Lemma 2.4. *Let (E, σ) be a commutative étale K-algebra with involution of dimension $n = 2\ell$ satisfying* (2.2)*. Then there exists a nondegenerate symmetric bilinear form h on E that satisfies* (2.3) *and has Witt index $\geqslant \ell - 1$.*

Proof. If K is finite then one can take, for example, $h = \phi_1$, so we can assume in the rest of the argument that K is infinite. It follows from the description of E that, for $\overline{K}$ an algebraic closure of K,

$$(E \otimes_K \overline{K}, \sigma \otimes \mathrm{id}_K) \simeq (M, \mu)$$

where $M = \prod_{i=1}^{\ell}(\overline{K} \times \overline{K})$ and μ acts on each copy of $\overline{K} \times \overline{K}$ by switching components. Viewing M as an affine n-space, consider the K-defined subvariety $M_- := \{x \in M \mid \mu(x) = -x\}$. Clearly, M_- is a K-defined vector space, so the K-points $E_- := M_- \cap E$ are Zariski-dense in M_-. On the other hand, let $U \subset M$ be the Zariski-open subvariety of elements with pairwise distinct components; then any $x \in U$ generates M as a $\overline{K}$-algebra. Furthermore, it is easy to see that $U \cap M_- \neq \emptyset$, so $U \cap E_- \neq \emptyset$.

Fix $e \in U \cap E_-$; then $1, e, \ldots, e^{n-1}$ form a K-basis of E. For $x \in E$ we define $c_i(x)$ for $i = 0, \ldots, n-1$ so that $x = \sum_{i=0}^{n-1} c_i(x)e^i$. Set

$$h(x, y) := c_{n-2}(x\sigma(y)).$$

Clearly, h is symmetric bilinear and satisfies (2.3). Let us show that h is nondegenerate. If $x = \sum_{i=0}^{n-1} c_i(x)e^i$ is in the radical of h, then so is $\sigma(x)$, and therefore also $x_+ := \sum_{i=0}^{\ell-1} c_{2i}(x)e^{2i}$ and $x_- := \sum_{i=0}^{\ell-1} c_{2i+1}(x)e^{2i+1}$. From $h(x_+, 1) = 0$, $h(x_+, e^2) = 0$, etc., we successively obtain that $c_{n-2}(x) = 0$, $c_{n-4}(x) = 0$, etc., i.e., $x_+ = 0$. Furthermore, we have $0 = h(x_-, e^{-1}) = -c_{n-1}(x)$. Then from $h(x_-, e) = 0$, $h(x_-, e^3) = 0$, etc., we successively obtain $c_{n-3}(x) = 0$, $c_{n-5}(x) = 0$, etc. Thus, $x_- = 0$, hence $x = 0$, as required. It remains to observe that the subspace spanned by $1, e, \ldots, e^{\ell-2}$ is totally isotropic with respect to h. □

Remark. In an earlier version of this paper, we constructed h in Lemma 2.4 in the form $h = \phi_b$ using some matrix computations. The current proof, which minimizes computations, was inspired by [BG11, §5].

Proposition 2.5 (type B). *Let (E, σ) be an étale K-algebra of dimension $n = 2\ell+1$ with involution satisfying* (2.2). *If τ is an orthogonal involution on $A = M_n(K)$ such that $\tau = \tau_f$ where f is a nondegenerate symmetric bilinear form on K^n of Witt index ℓ then there exists an embedding $(E, \sigma) \hookrightarrow (A, \tau)$ of K-algebras with involution.*

Proof. Pick a decomposition (2.1), and then use Lemma 2.4 to find a form h' on E' with the properties described therein. We can write $h' = h'_1 \perp h'_2$ where h'_1 is a direct sum of $\ell - 1$ hyperbolic planes and h'_2 is a binary form. Choose a 1-dimensional form h'' so that $h'_2 \perp h''$ is isotropic, and consider $h = h' \perp h''$ on $E = E' \times K$. Then h is a nondegenerate symmetric bilinear form on E satisfying (2.3) and having Witt index ℓ. So, h is equivalent to f, hence (E, σ) embeds in (A, τ) by Proposition 2.1(iii). □

Remark 2.6. Let now G_1 be the K-split adjoint group $\mathrm{SO}_{2\ell+1}$ of type B_ℓ and G_2 be the K-split simply connected group $\mathrm{Sp}_{2\ell}$ of type C_ℓ where $\ell \geqslant 2$. It was observed in [PR09], Example 6.7, that G_1 and G_2 have the same isomorphism classes of maximal K-tori over any field K of characteristic $\neq 2$. This was derived from the fact that G_1 and G_2 have isomorphic Weyl groups using the results of [Gil04] or [Rag04]. Now, we are in a position to give a much simpler explanation of this phenomenon. Indeed, $G_1 = \mathrm{SU}(A_1, \tau_1)$ where $A_1 = M_{2\ell+1}(K)$ and τ_1 is an orthogonal involution on A_1 corresponding to a nondegenerate symmetric bilinear form on $K^{2\ell+1}$ of Witt index ℓ, and $G_2 = \mathrm{SU}(A_2, \tau_2)$ where $A_2 = M_{2\ell}(K)$ and τ_2 is a symplectic involution on A_2 corresponding to a nondegenerate skew-symmetric form on $K^{2\ell}$. Any maximal K-torus T_2 of G_2 is of the form $\mathrm{SU}(E_2, \sigma_2)$ where E_2 is a 2ℓ-dimensional commutative τ_2-invariant subalgebra of A_2, $\sigma_2 = \tau_2|_{E_2}$, with (E_2, σ_2) satisfying (2.2). Set $(E_1, \sigma_1) = (E_2, \sigma_2) \times (K, \mathrm{id}_K)$. According to Proposition 2.5, there exists an embedding $(E_1, \sigma_1) \hookrightarrow (A_1, \tau_1)$, which gives rise to a K-isomorphism between T_2 and the maximal K-torus $T_1 = \mathrm{SU}(E_1, \sigma_1)$ of G_1. This, combined with the symmetric argument based on Proposition 2.3, yields the required fact. Then, repeating the argument given in *loc. cit.*, we conclude that if K is a number field then for any finite subset $S \subset V^K$ containing V_∞^K, the S-arithmetic subgroups of G_1 and G_2 are weakly commensurable.

Turning now to type D_ℓ, we first observe that if (E, σ) is a K-algebra with involution of dimension $n = 2\ell$ satisfying (2.2) then the determinant — viewed as an element of $K^\times/K^{\times 2}$ — of the symmetric bilinear form ϕ_b for invertible $b \in E^\sigma$ does not depend on b (cf. [BCKM03, Cor. 4.2]) and will be denoted $d(E, \sigma)$. Now, if τ is an involution on $A = M_n(K)$ that corresponds to a symmetric bilinear form

f on K^n having determinant $d(f)$ then it follows from Proposition 2.1(iii) that an embedding $(E,\sigma) \hookrightarrow (A,\tau)$ can exist only if $d(E,\sigma) = d(f)$ in $K^\times/K^{\times 2}$.

Proposition 2.7. *Let (E,σ) be an étale K-algebra of dimension $n = 2\ell$ with involution satisfying* (2.2)*. If τ is an orthogonal involution on $A = M_n(K)$ such that $\tau = \tau_f$ where f is a nondegenerate symmetric bilinear form on K^n of Witt index $\geqslant \ell - 1$ such that $d(E,\sigma) = d(f)$ (in $K^\times/K^{\times 2}$) then there exists an embedding $(E,\sigma) \hookrightarrow (A,\tau)$ of K-algebras with involution.*

Proof. Let h be the symmetric bilinear form on E constructed in Lemma 2.4. As we observed after Proposition 2.1, h is actually of the form $h = \phi_b$ for some invertible $b \in E^\sigma$, so $d(h) = d(E,\sigma)$. We can write $h = h_1 \perp h_2$ where h_1 is a direct sum of $\ell - 1$ hyperbolic planes and h_2 is a binary form. Similarly, $f = f_1 \perp f_2$ where f_1 is a direct sum of $\ell - 1$ hyperbolic planes and f_2 is binary. Then $d(E,\sigma) = d(f)$ implies that $d(h_2) = d(f_2)$, so h_2 and f_2 are similar. Thus, a suitable multiple of h is equivalent to f, and our claim follows from Proposition 2.1(iii). □

Finally, we will treat algebras corresponding to the groups of type ${}^2\mathsf{A}_\ell$. Here L will be a quadratic extension of K and all involutions will restrict to the nontrivial automorphism of L/K.

Proposition 2.8 (type A)**.** *Let (E,σ) be an étale n-dimensional L-algebra with involution. If τ is a unitary involution on $A = M_n(L)$ such that $\tau = \tau_f$ where f is a hermitian form on L^n having Witt index $m := [n/2]$, then there exists an embedding $(E,\sigma) \hookrightarrow (A,\tau)$ of L-algebras with involution.*

Proof. It is enough to construct a nondegenerate hermitian form on E that satisfies (2.3) and has Witt index m. If K is finite, one can take, for example, $h = \phi_1$, so we can assume that K is infinite. Set $F = E^\sigma$ so that $E = F \otimes_K L$. Since K is infinite, arguing as in the proof of Lemma 2.4, one can find $e \in F$ so that $F = K[e]$. Then any $x \in E$ admits a unique presentation of the form $x = \sum_{i=0}^{n-1} e^i \otimes c_i(x)$ with $c_i(x) \in L$. Define

$$h(x,y) := c_{n-1}(x\sigma(y)).$$

It is easy to see h is a hermitian form satisfying (2.3); let us show that it is nondegenerate. If x is in the radical of h then from $h(x,1) = 0$, $h(x,e) = 0$, etc., we successively obtain that $c_{n-1}(x) = 0$, $c_{n-2}(x) = 0$, etc. Thus, $x = 0$, proving the nondegeneracy of h. Since $2(m-1) < n-1$, the subspace spanned by $1, e, \ldots, e^{m-1}$ is totally isotropic, hence the Witt index of h is m, as required. □

3. Maximal tori in real groups of types B and C

This section is devoted to determining the isomorphism classes of maximal tori in certain linear algebraic groups, primarily of types B and C, over the real numbers. Recall that every torus T over $\mathbb{R}$ is $\mathbb{R}$-isomorphic to the product

$$(\mathrm{GL}_1)^\alpha \times (\mathrm{R}^{(1)}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_1))^\beta \times (\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_1))^\gamma \tag{3.1}$$

for uniquely determined nonnegative integers α, β, γ [Vos98, p. 64], and then the group $T(\mathbb{R})$ is topologically isomorphic to $(\mathbb{R}^\times)^\alpha \times (S^1)^\beta \times (\mathbb{C}^\times)^\gamma$, where S^1 is the group of complex numbers of modulus 1. The fact that T is isomorphic to a maximal $\mathbb{R}$-torus of a given reductive $\mathbb{R}$-group G typically imposes serious restrictions on the numbers α, β and γ. To illustrate this, we first consider the following easy example.

Example 3.1. *Every maximal $\mathbb{R}$-torus in $G = \mathrm{GL}_{n,\mathbb{H}}$, where $\mathbb{H}$ is the algebra of Hamiltonian quaternions, is isomorphic to* $(\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_1))^n$. Indeed, every maximal $\mathbb{R}$-torus in G is of the form $\mathrm{R}_{E/\mathbb{R}}(\mathrm{GL}_1)$ where E is a maximal commutative $2n$-dimensional étale subalgebra of $A = M_n(\mathbb{H})$. Any commutative $2n$-dimensional étale $\mathbb{R}$-algebra E is isomorphic to $\mathbb{R}^\alpha \times \mathbb{C}^\gamma$ with $\alpha + 2\gamma = 2n$. But in order for E to have an $\mathbb{R}$-embedding in A, we must have $\alpha = 0$ and then $\gamma = n$ (cf. [PR10, 2.6]), so our claim follows.

We now recall the standard notations for some classical real algebraic groups. We let $\mathrm{SO}(r, n-r)$ denote the special orthogonal group of the n-dimensional quadratic form $q = r\langle 1\rangle \perp (n-r)\langle -1\rangle$. Similarly, we let $\mathrm{Sp}(r, n-r)$ denote the special unitary group of the n-dimensional hermitian form $h = r\langle 1\rangle \perp (n-r)\langle -1\rangle$ over $\mathbb{H}$ with the standard involution. Every adjoint $\mathbb{R}$-group of type B_ℓ is isomorphic to some $\mathrm{SO}(r, n-r)$ for $n = 2\ell+1$ and some $0 \leqslant r \leqslant n$, and every nonsplit simply connected $\mathbb{R}$-group of type C_ℓ is isomorphic to $\mathrm{Sp}(r, \ell - r)$ some $0 \leqslant r \leqslant \ell$.

Lemma 3.2 (Adjoint B_ℓ over $\mathbb{R}$)**.** *The maximal $\mathbb{R}$-tori in $G = \mathrm{SO}(r, n-r)$, where $n = 2\ell+1$, are of the form* (3.1) *with $\alpha+\beta+2\gamma = \ell$ and $\alpha+2\gamma \leqslant s := \min(r, n-r)$.*

Proof. Let τ be the involution on $A = M_n(K)$ that corresponds to the symmetric bilinear form f associated with the quadratic form $q = r\langle 1\rangle \perp (n-r)\langle -1\rangle$ so that $G = \mathrm{SU}(A, \tau)$. Let T be a maximal $\mathbb{R}$-torus of G written in the form (3.1). Since the rank of G is ℓ, we immediately obtain

$$\dim T = \alpha + \beta + 2\gamma = \ell.$$

Furthermore, we have $T = \mathrm{SU}(E, \sigma)$ where $E \subset A$ is a τ-invariant maximal commutative étale subalgebra, $\sigma = \tau|_E$, and (2.2) holds. There are exactly 4 isomorphism classes of indecomposable étale $\mathbb{R}$-algebras with involution, which are listed in Table 1. Using this information, we can write

$$(E, \sigma) = \mathbb{R}^{\delta_1} \times (\mathbb{R}\times\mathbb{R})^{\delta_2} \times \mathbb{C}^{\delta_3} \times (\mathbb{C}\times\mathbb{C})^{\delta_4}$$

where the involutions on factors are as in the table. Comparing this with the structure of T, we obtain $\delta_2 = \alpha$, $\delta_3 = \beta$, and $\delta_4 = \gamma$. According to Proposition 2.1(iii), there exists $b \in E^\sigma$ such that ϕ_b is equivalent to f. But the Witt index of f is s (which equals the $\mathbb{R}$-rank of G), and the Witt index of ϕ_b is $\geqslant \delta_2 + 2\delta_4$. Thus, $\alpha + 2\gamma \leqslant s$. (We note that $\mathrm{rk}_{\mathbb{R}}\, T = \alpha + \gamma$, immediately yielding the restriction $\alpha + \gamma \leqslant s$. So, the restriction we have actually obtained is stronger than one can a priori expect.)

Conversely, suppose α, β, γ satisfy the two constraints, and assume that $r > n-r$ (otherwise we can replace the quadratic form q defining G with $-q$); in particular, $r > \ell$. Consider the étale $\mathbb{R}$-algebra

$$(E, \sigma) = \mathbb{R} \times (\mathbb{R}\times\mathbb{R})^\alpha \times \mathbb{C}^\beta \times (\mathbb{C}\times\mathbb{C})^\gamma =: (E_1, \sigma_1) \times \cdots \times (E_4, \sigma_4)$$

of dimension $1 + 2\alpha + 2\beta + 4\gamma = 2\ell + 1 = n$ where the involutions on the factors $\mathbb{R}$, $\mathbb{R}\times\mathbb{R}, \ldots$ are as described in Table 1. (Clearly, E satisfies (2.2).) Let us show that there exists $b = (b_1, \ldots, b_4) \in E^\sigma$ such that ϕ_b is equivalent to f. Set $b_2 = ((1,1), \ldots, (1,1))$ and $b_4 = ((1,1), \ldots, (1,1))$. Then the quadratic form associated with the bilinear form $(\phi_{2,4})_{(b_2,b_4)}$ on $E_2 \times E_4$ is equivalent to $(\alpha + 2\gamma)(\langle 1\rangle \perp \langle -1\rangle)$. Since $t := (n-r) - (\alpha + 2\gamma) \geqslant 0$, we can choose $b_1 = \pm 1$ and $b_3 = (\pm 1, \ldots, \pm 1)$ so that the quadratic form associated with $(\phi_{1,3})_{(b_1,b_3)}$ is equivalent to $(2\beta + 1 - t)\langle 1\rangle \perp t\langle -1\rangle$. Then $b = (b_1, \ldots, b_4)$ is as required. By Proposition

E	σ	ϕ_b for $b \in E^\sigma$	$\mathrm{SU}(E,\sigma)$
$\mathbb{R}$	Id	$\langle b\rangle$	$\{1\}$
$\mathbb{R}\times\mathbb{R}$	switch	$\langle 1,-1\rangle$	GL_1
$\mathbb{C}$	conjugation	$\langle b,b\rangle$	$\mathrm{R}^{(1)}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_1)$
$\mathbb{C}\times\mathbb{C}$	switch	$\langle 1,-1\rangle\oplus\langle 1,-1\rangle$	$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_1)$

Table 1. Isomorphism classes of indecomposable étale $\mathbb{R}$-algebras with involution and their associated symmetric bilinear forms and unitary groups.

2.1(iii), there exists an embedding $(E,\sigma)\hookrightarrow(A,\tau)$, and therefore an $\mathbb{R}$-defined embedding $\mathrm{SU}(E,\sigma)\hookrightarrow\mathrm{SU}(A,\tau)=G$. Finally, it follows from our construction and Table 1 that $T=\mathrm{SU}(E,\sigma)$ is a torus having the required structure. □

Lemma 3.3 (Simply connected C_ℓ over $\mathbb{R}$)**.** *The maximal $\mathbb{R}$-tori in the group $G=\mathrm{Sp}(r,\ell-r)$ are of the form* (3.1) *with $\alpha=0$, $\beta+2\gamma=\ell$ and $\gamma\leqslant s:=\min(r,\ell-r)$.*

Proof. Let τ be the involution on $A=M_\ell(\mathbb{H})$ that gives rise to the hermitian form $f=r\langle 1\rangle\perp(\ell-r)\langle -1\rangle$, so that $G=\mathrm{SU}(A,\tau)$. Every maximal $\mathbb{R}$-torus T of G is of the form $T=\mathrm{SU}(E,\sigma)$ for some (2ℓ)-dimensional étale τ-invariant subalgebra E of A, where $\sigma=\tau|_E$ and condition (2.2) holds. As in Example 3.1, $E\simeq\mathbb{C}^\ell$ as $\mathbb{R}$-algebras, and therefore

$$(E,\sigma)=\mathbb{C}^{\delta_1}\times(\mathbb{C}\times\mathbb{C})^{\delta_2}$$

where the involutions on $\mathbb{C}$ and $\mathbb{C}\times\mathbb{C}$ are as in Table 1. Then in (3.1) for $T=\mathrm{SU}(E,\sigma)$ we have $\alpha=0$, $\beta=\delta_1$ and $\gamma=\delta_2$. By dimension count, we get $\beta+2\gamma=\ell$. Furthermore,

$$\gamma=\mathrm{rk}_{\mathbb{R}}\,T\leqslant\mathrm{rk}_{\mathbb{R}}\,G=s.$$

Conversely, suppose that T has parameters α,β and γ satisfying our constraints. Consider

$$(E,\sigma)=\mathbb{C}^{\beta}\times(\mathbb{C}\times\mathbb{C})^{\gamma}$$

with the involutions as above, and assume (as we may) that $\ell-r\leqslant r$. Note that

$$(z,w)\mapsto\begin{pmatrix} z & 0\\ 0 & \bar{w}\end{pmatrix}$$

defines an embedding of algebras with involutions $\mathbb{C}\times\mathbb{C}\hookrightarrow(M_2(\mathbb{H}),\theta)$ where $\theta(x)=J^{-1}\bar{x}^tJ$ with $J=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, where $\bar{x}$ is obtained by applying quaternionic conjugation to all entries. Consider the involution $\hat\theta$ on A given by $\hat\theta(x)=\hat{J}^{-1}\bar{x}^t\hat{J}$ where

$$\hat{J}=\mathrm{diag}(\underbrace{1,\ldots,1}_{r-\gamma},\underbrace{-1,\ldots,-1}_{\beta-(r-\gamma)},\underbrace{J,\ldots,J}_{\gamma}).$$

Then it follows from our construction that there exists an embedding $(E,\sigma)\hookrightarrow(A,\theta)$. Noting that $(A,\tau)\simeq(A,\theta)$, we obtain an embedding $(E,\sigma)\hookrightarrow(A,\tau)$. So, there exists an $\mathbb{R}$-embedding $\mathrm{SU}(E,\sigma)\hookrightarrow\mathrm{SU}(A,\tau)=G$, and it remains to observe that $T=\mathrm{SU}(E,\sigma)$ is a torus having the required structure. □

Alternatively, the results of Lemmas 3.2 and 3.3 can be deduced from the more general classification of maximal $\mathbb{R}$-tori in simple real algebraic groups obtained in [DT94]. For the reader's convenience we have included the direct proofs above, written in the same language as the rest of the paper.

Corollary 3.4. *Let G_1 be an adjoint real group of type B_ℓ, and let G_2 be a simply connected real group of type C_ℓ. The groups G_1 and G_2 have the same isomorphism classes of maximal $\mathbb{R}$-tori if and only if G_1 and G_2 are either both split or both anisotropic.*

Proof. Since every $\mathbb{R}$-anisotropic torus T is of the form $(\mathrm{R}^{(1)}_{\mathbb{C}/\mathbb{R}}(\mathrm{GL}_1))^{\dim T}$, there is nothing to prove if both groups are anisotropic. If both groups are split, our claim follows from Remark 2.6. Clearly, G_1 and G_2 cannot have the same maximal tori if one of the groups is anisotropic and the other is isotropic. So, it remains to consider the case where both groups are isotropic but not split. Then G_1 contains the torus with $\alpha = 1$, $\beta = \ell - 1$, and $\gamma = 0$ by Lemma 3.2, but G_2 does not by Lemma 3.3. □

Remark 3.5. Our argument shows that if G_1 is isotropic and G_2 is not split then G_1 has a maximal $\mathbb{R}$-torus that is not isomorphic to any $\mathbb{R}$-torus of G_2. Moreover, by Lemma 3.2, a maximal $\mathbb{R}$-torus T_1 of G_1 that contains a maximal $\mathbb{R}$-split torus has parameters $\alpha = s$, $\beta = \ell - s$ and $\gamma = 0$, hence does not allow an $\mathbb{R}$-embedding into G_2. In particular, if $G_1 = \mathrm{SO}(n-1, 1)$ and G_2 is not split then every isotropic maximal $\mathbb{R}$-torus of G_1 is not isomorphic to a subtorus of G_2.

Example 3.6 (Absolute rank 3)**.** As an empirical illustration of the landscape over $\mathbb{R}$, we divide the 14 real groups of types B_3 and C_3 into equivalence classes under the relation "have isomorphic collections of maximal tori". For forms of SO_7 or Sp_6, the maximal tori are described by Lemmas 3.2 and 3.3. Also, the four anisotropic (compact) forms obviously make up one equivalence class. For the other groups one can use a computer program such as the Atlas software [AdC09] to find the maximal tori. In summary, the groups $\mathrm{SO}(1, 6)$, $\mathrm{SO}(2, 5)$, and $\mathrm{Spin}(2, 5)$ are each their own equivalence class, and we find the following non-singleton equivalence classes:

$$\{4 \text{ anisotropic forms}\}, \quad \{\mathrm{Sp}_6, \mathrm{SO}(4, 3)\}, \quad \{\mathrm{PSp}_6, \mathrm{Spin}(4, 3)\},$$
$$\text{and} \quad \{\mathrm{Sp}(1, 2), \mathrm{PSp}(1, 2), \mathrm{Spin}(1, 6)\}.$$

In particular, $\mathrm{Spin}(1, 6)$ and $\mathrm{PSp}(1, 2)$ have the same isomorphism classes of maximal tori and yet are neither both split nor both anisotropic. This situation is dual to the one considered and eliminated in Corollary 3.4 (adjoint B_ℓ and simply connected C_ℓ).

For completeness, we mention the (much easier) analogue of Corollary 3.4 for non-archimedean local fields.

Lemma 3.7. *Let G_1 and G_2 be absolutely almost simple groups of type B_ℓ and C_ℓ respectively, with $\ell \geqslant 3$, over K a nonarchimedean local field of characteristic $\neq 2$. The following are equivalent:*

1. *The groups G_1 and G_2 have the same isogeny classes of maximal K-tori.*
2. $\mathrm{rk}_K\, G_1 = \mathrm{rk}_K\, G_2$.
3. *G_1 and G_2 are split.*

Proof. (1) obviously implies (2). Suppose (2) and that G_2 is not split. Then

$$[\ell/2] = \mathrm{rk}_K\, G_2 = \mathrm{rk}_K\, G_1 \geqslant \ell - 1,$$

but this is impossible because $\ell \geqslant 3$, hence (3).

To prove (3) $\Rightarrow$ (1), we may assume that G_1 is split adjoint and G_2 is split simply connected. Combining Propositions 2.3 and 2.5 with (2.1) gives that G_1 and G_2 have the same isogeny classes of maximal tori. ☐

4. Local-global principles for embedding étale algebras with involution

The last ingredient we need to develop before proving Theorem 1.4 in §6 is a result guaranteeing in our situation the validity of the local-global principle for the existence of an embedding of an étale algebra with involution into a simple algebra with involution. This issue was analyzed in [PR10]: although the local-global principle may fail (cf. Example 7.5 in *loc. cit.*), it can be shown to hold under rather general conditions. For our purposes we need the following case.

Let (E, σ) be an étale algebra with involution over a number field K of dimension $n = 2m$ and satisfying (2.2). Then $E = F[x]/(x^2 - d)$ where $F = E^\sigma$ is an m-dimensional étale K-algebra and $d \in F^\times$, with the involution defined by $x \mapsto -x$ as in Example 2.2. We write $F = \prod_{j=1}^r F_j$ where F_j is a field extension of K, and suppose that in terms of this decomposition $d = (d_1, \ldots, d_r)$. Let τ be an orthogonal involution on $A = M_n(K)$.

Proposition 4.1. [PR10, Theorem 7.3] *Assume that for every $v \in V^K$ there exists a K_v-embedding*

$$\iota_v \colon (E \otimes_K K_v, \sigma \otimes \mathrm{id}_{K_v}) \hookrightarrow (A \otimes_K K_v, \tau \otimes \mathrm{id}_{K_v}).$$

If the following condition holds

($\diamond$) *for every finite subset $V \subset V^K$, there exists $v_0 \in V^K \setminus V$ such that for $j = 1, \ldots, r$, if $d_j \notin {F_j^\times}^2$, then $d_j \notin (F_j \otimes_K K_{v_0})^{\times 2}$;*

then there exists an embedding $\iota\colon (E, \sigma) \hookrightarrow (A, \tau)$. Furthermore, ($\diamond$) automatically holds if F is a field.

We will now derive from the proposition the following statement, in which n can be odd or even.

Lemma 4.2. *Let K be a number field, let (E, σ) be an n-dimensional étale algebra with involution satisfying (2.2), and let τ be an orthogonal involution on $A = M_n(K)$. Assume that for every $v \in V^K$ there is an embedding*

$$\iota_v \colon (E \otimes_K K_v, \sigma \otimes \mathrm{id}_{K_v}) \hookrightarrow (A \otimes_K K_v, \tau \otimes \mathrm{id}_{K_v}).$$

Then in each of the following situations

(1) *$n \leqslant 5$, or*
(2) *there is a real $v \in V^K$ such that $(E \otimes_K K_v, \sigma \otimes \mathrm{id}_{K_v})$ is isomorphic to $(\mathbb{C}, ^-)^m$ or $(\mathbb{C}, ^-)^m \times (\mathbb{R}, \mathrm{id}_\mathbb{R})$ depending on whether $n = 2m$ or $n = 2m + 1$,*

there exists an embedding $\iota\colon (E, \sigma) \hookrightarrow (A, \tau)$.

Proof. First, we will reduce the argument to the case of even n, i.e. when E satisfies one of the following conditions:

(1′) $n = 2$ or 4, or

(2′) there is a real $v \in V^K$ such that $(E \otimes_K K_v, \sigma \otimes \mathrm{id}_{K_v})$ is isomorphic to $(\mathbb{C}, ^-)^m$.

Indeed, let $n = 2m+1$ and suppose E satisfies condition (1) or (2) of the lemma. Then by [PR10, Prop. 7.2], $(E, \sigma) = (E', \sigma') \times (K, \mathrm{id}_K)$ and there exists an orthogonal involution τ' on $A' = M_{n-1}(K)$ such that for every $v \in V^K$ there is an embedding

$$\iota'_v \colon (E' \otimes_K K_v, \sigma' \otimes \mathrm{id}_{K_v}) \hookrightarrow (A' \otimes_K K_v, \tau' \otimes \mathrm{id}_{K_v}),$$

and the existence of an embedding $\iota' \colon (E', \sigma') \hookrightarrow (A', \tau')$ is equivalent to the existence of an embedding $\iota \colon (E, \sigma) \hookrightarrow (A, \tau)$. Clearly, E' satisfies the respective condition (1′) or (2′). So, if we assume that the lemma has already been established for E', then the existence of ι follows.

Now, suppose that $\dim_K E = 2m$ and E satisfies (2.2). Write $E = F[x]/(x^2 - d)$ where $F = E^\sigma = \prod_{j=1}^r F_j$, $d = (d_1, \ldots, d_r)$ with $d_j \in F_j^\times$. Assume that there exist K-embeddings $\varphi_j \colon F_j \hookrightarrow \bar{K}$ such that if

$$M = \varphi_1(F_1) \cdots \varphi_r(F_r) \quad \text{and} \quad N = M(\sqrt{\varphi_1(d_1)}, \ldots, \sqrt{\varphi_r(d_r)})$$

then there is $\lambda \in \mathrm{Gal}(N/M)$ with the property

$$\lambda\left(\sqrt{\varphi_j(d_j)}\right) = -\sqrt{\varphi_j(d_j)} \quad \text{whenever } d_j \notin F_j^\times \text{ for } j = 1, \ldots, r. \tag{4.1}$$

Let P be the normal closure of N over K, and let $\mu \in \mathrm{Gal}(P/K)$ be such that $\mu|_N = \lambda$. By Chebotarev's Density Theorem [CF10, Ch. 7, 2.4], for any finite $V \subset V^K$, there exists a nonarchimedean $v_0 \in V^K \setminus V$ that is unramified in P and for which the Frobenius automorphism $\mathrm{Fr}(w_0|v_0)$ is μ for a suitable extension $w_0|v_0$. Then it follows from (4.1) that $d_j \notin (F_{j\,w_0})^{\times 2}$ for any j such that $d_j \notin F_j^{\times 2}$, and therefore condition (◇) holds.

Let now (E, σ) be an étale algebra with involution satisfying (1′) or (2′) for which embeddings ι_v exist for all $v \in V^K$. In order to derive the existence of ι from Proposition 4.1, we need to check (◇), for which it is enough to find an automorphism λ as in the previous paragraph. Suppose that (1′) hods. Then $F = E^\sigma$ has dimension 1 or 2 respectively. Since we don't need to consider the case where F is a field (cf. Proposition 4.1), the only remaining case is where $F = K \times K$. Clearly, $K(\sqrt{d_1}, \sqrt{d_2})$ always has an automorphism λ such that $\lambda(\sqrt{d_j}) = -\sqrt{d_j}$ if $d_j \notin K^{\times 2}$, as required. Finally, suppose that (2′) holds. Then $F \otimes_K K_v \simeq \mathbb{R}^m$, and $d = (\delta_1, \ldots, \delta_m)$ in $\mathbb{R}^m$ with $\delta_i < 0$ for all i. Then for any embeddings $\varphi_j \colon F_j \hookrightarrow \mathbb{C}$ we have $\varphi_j(F_j) \subset \mathbb{R}$ and the restriction λ of complex conjugation satisfies $\lambda(\sqrt{d_j}) = -\sqrt{d_j}$ for all j, concluding the argument. □

Remark. Example 7.5 in [PR10] shows that there exists (E, σ) with E of dimension 6 for which the local-global principle for embeddings fails, so in terms of dimension the condition (1) in Lemma 4.2 is sharp.

For convenience of further reference, we will also quote the local-global principle for embeddings in the case of symplectic involutions.

Lemma 4.3. [PR10, Th. 5.1] *Let A be a central simple K-algebra of dimension n^2 with a symplectic involution τ (then, of course, n is necessarily even), and let*

(E,σ) be an n-dimensional étale K-algbra with involution satisfying (2.2)*. If for every $v \in V^K$ there exists an embedding*

$$\iota_v\colon (E \otimes_K K_v, \sigma \otimes \mathrm{id}_{K_v}) \hookrightarrow (A \otimes_K K_v, \tau \otimes \mathrm{id}_{K_v}),$$

then there exists an embedding $(E,\sigma) \hookrightarrow (A,\tau)$.

5. Function field analogue of Theorem 1.4

We recall the following immediate consequence of the rationality of the variety of maximal tori (see [Har68] or [PR94, Cor. 7.3]) which will be used repeatedly: *Let G be a reductive algebraic group over a number field K; then given any $v \in V^K$ and any maximal K_v-torus $T^{(v)}$ of G there exists a maximal K-torus T of G that is conjugate to $T^{(v)}$ by an element of $G(K_v)$.* In particular, for any $v \in V^K$ there exists a maximal K-torus T of G such that $\mathrm{rk}_{K_v} T = \mathrm{rk}_{K_v} G$. It follows that if G_1 and G_2 are reductive K-groups having the same isogeny classes of maximal K-tori then

$$\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 \quad \text{for all } v \in V^K. \tag{5.1}$$

The remark made in the previous paragraph remains valid for global function fields, which can be used to give the following analogue of Theorem 1.4: Suppose G_1 and G_2 are absolutely almost simple algebraic groups of types B_ℓ and C_ℓ ($\ell \geqslant 3$) over a global field K of characteristic > 2. *The groups G_1 and G_2 have the same isogeny classes of maximal K-tori if and only if they are split.* Indeed, if the two groups have the same isogeny classes of maximal K-tori, then both groups are K_v-split for every v (by (5.1) and Lemma 3.7), hence both groups are K-split (by the Hasse Principle). The converse holds by Remark 2.6.

6. Proof of Theorem 1.4

Throughout this section G_1 and G_2 will denote absolutely almost simple algebraic groups of types B_ℓ and C_ℓ for some $\ell \geqslant 3$ defined over a number field K. In 1.1 we defined what it means for G_1 and G_2 to be *twins*. We now observe that since G_1 and G_2 cannot be K_v-anisotropic for $v \in V_f^K$, they are twins if and only if both of the following conditions hold:

$$\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 = \ell \text{ for all } v \in V_f^K \tag{6.1}$$

$$\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 = 0 \text{ or } \ell \text{ for all } v \in V_\infty^K. \tag{6.2}$$

We also note that if G_1 and G_2 are twins over K then they remain twins over any finite extension L/K. If K has r real places, then (by the Hasse Principle) there are exactly $4 \cdot 2^r$ pairs of K-groups G_1, G_2 that are twins, equivalently, 2^r pairs if one only counts the groups G_1 and G_2 up to isogeny.

Now, let G_1 and G_2 be as above, with G_1 adjoint and G_2 simply connected. Then $G_i = \mathrm{SU}(A_i, \tau_i)$ for $i = 1, 2$ where $A_1 = M_{n_1}(K)$, $n_1 = 2\ell + 1$ and the involution τ_1 is orthogonal, and A_2 is a central simple K-algebra of dimension n_2^2 with $n_2 = 2\ell$ and the involution τ_2 is symplectic. Any maximal K-torus T_i of G_i is of the form $\mathrm{SU}(E_i, \sigma_i)$ where $E_i \subset A_i$ is an n_i-dimensional étale τ_i-invariant K-subalgebra and $\sigma_i = \tau_i|_{E_i}$ so that (2.2) holds. For $i = 1$, we can always write $(E_1, \sigma_1) = (E_1', \sigma_1') \times (K, \mathrm{id}_K)$. For $i = 2$, we set $(E_2^+, \sigma_2^+) = (E_2, \sigma_2) \times (K, \mathrm{id}_K)$.

Proposition 6.1. *Let (A_1, τ_1) and (A_2, τ_2) be algebras with involution as above, and assume that $G_1 = \mathrm{SU}(A_1, \tau_1)$ and $G_2 = \mathrm{SU}(A_2, \tau_2)$ are twins. If (E_1, σ_1) is isomorphic to an n_1-dimensional étale subalgebra of (A_1, τ_1) satisfying* (2.2) *then (E_1', σ_1') is isomorphic to a subalgebra of (A_2, τ_2). Conversely, if (E_2, σ_2) is isomorphic to an n_2-dimensional étale subalgebra of (A_2, τ_2) satisfying* (2.2) *then (E_2^+, σ_2^+) is isomorphic to a subalgebra of (A_1, τ_1). Thus, the correspondences*

$$(E_1, \sigma_1) \mapsto (E_1', \sigma_1') \quad \text{and} \quad (E_2, \sigma_2) \mapsto (E_2^+, \sigma_2^+)$$

implement mutually inverse bijections between the sets of isomorphism classes of n_1- and n_2-dimensional étale subalgebras of (A_1, τ_1) and (A_2, τ_2) that are invariant under the respective involutions and satisfy (2.2).

Proof. If we have $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 = \ell$ for all $v \in V_\infty^K$ then the groups G_1 and G_2 are K-split by (6.1) and the Hasse Principle. Then τ_1 corresponds to a nondegenerate symmetric bilinear form of Witt index ℓ, and $A_2 = M_{n_2}(K)$ with τ_2 corresponding to a nondegenerate skew-symmetric form. In this case, our claim immediately follows from Propositions 2.3 and 2.5, as in Remark 2.6. So, we may assume that there is a real $v_0 \in V_\infty^K$ such that $\mathrm{rk}_{K_{v_0}} G_1 = \mathrm{rk}_{K_{v_0}} G_2 = 0$. Observe that given *any* real $v \in V_\infty^K$ satisfying $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 = 0$, the data in Table 1 shows that that for any n_1-dimensional τ_1-invariant étale subalgebra $E_1 \subset A_1$ satisfying (2.2) and $\sigma_1 = \tau_1|_{E_1}$ we have

$$(E_1 \otimes_K K_v, \sigma_1 \otimes \mathrm{id}_{K_v}) \simeq (\mathbb{C}, {}^-)^\ell \times (\mathbb{R}, \mathrm{id}_\mathbb{R}), \tag{6.3}$$

and for any n_2-dimensional τ_2-invariant étale subalgebra $E_2 \subset A_2$ satisfying (2.2) and $\sigma_2 = \tau_2|_{E_2}$ we have

$$(E_2 \otimes_K K_v, \sigma_2 \otimes \mathrm{id}_{K_v}) \simeq (\mathbb{C}, {}^-)^\ell. \tag{6.4}$$

Let (E_1, σ_1) be as in the statement of the proposition. We first show that for any $v \in V^K$ there is an embedding

$$\iota_v \colon (E_1' \otimes_K K_v, \sigma_1' \otimes \mathrm{id}_{K_v}) \hookrightarrow (A_2 \otimes_K K_v, \tau_2 \otimes \mathrm{id}_{K_v}).$$

If $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 = \ell$, this follows from Proposition 2.3. Otherwise, v is real, and $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 = 0$, so we see from (6.3) that

$$(E_1' \otimes_K K_v, \sigma_1' \otimes \mathrm{id}_{K_v}) \simeq (\mathbb{C}, {}^-)^\ell.$$

Then the existence of ι_v follows from the argument given in the proof of Lemma 3.3. Now, applying Lemma 4.3 we obtain the existence of an embedding $\iota \colon (E_1', \sigma_1') \hookrightarrow (A_2, \tau_2)$, as required.

Conversely, let (E_2, σ_2) be as in the proposition. Then arguing as above (using Proposition 2.5 and the proof of Lemma 3.2) we obtain the existence of local embeddings

$$\iota_v \colon (E_2^+ \otimes_K K_v, \sigma_2^+ \otimes \mathrm{id}_{K_v}) \hookrightarrow (A_1 \otimes_K K_v, \tau_1 \otimes \mathrm{id}_{K_v})$$

for all $v \in V^K$. It follows from (6.4) that

$$(E_2^+ \otimes_K K_{v_0}, \sigma_2^+ \otimes \mathrm{id}_{K_{v_0}}) \simeq (\mathbb{C}, {}^-)^\ell \times (\mathbb{R}, \mathrm{id}_\mathbb{R}).$$

This enables us to use Lemma 4.2 which yields the existence of an embedding $(E_2^+, \sigma_2^+) \hookrightarrow (A_1, \tau_1)$, completing the argument. □

The following consequence of the proposition proves the "if" component in both parts, (1) and (2), of Theorem 1.4.

Corollary 6.2. *Let G_1 and G_2 be absolutely almost simple algebraic groups of types B_ℓ and C_ℓ respectively that are twins. Then*

(i) *G_1 and G_2 have the same isogeny classes of maximal K-tori.*
(ii) *If G_1 is adjoint and G_2 is simply connected then G_1 and G_2 have the same isomorphism classes of maximal K-tori.*

Proof. (ii) easily follows from the proposition, and (i) is an immediate consequence of (ii). □

Remark 6.3. The assumption $\ell \geqslant 3$ was never used in Proposition 6.1 and Corollary 6.2. So, these statements remain valid also for $\ell = 2$, which will be helpful in § 8.

We now turn to the proof of the "only if" direction in both parts of Theorem 1.4 where the assumption $\ell \geqslant 3$ becomes essential and will be kept throughout the rest of the section. This direction requires a bit more work and involves the notion of *generic tori.* To recall the relevant definitions, we let G denote a semi-simple algebraic K-group, and fix a maximal K-torus T of G. Furthermore, we let $\Phi(G,T)$ denote the corresponding root system, and let K_T denote the minimal splitting field of T over K. The natural action of $\mathrm{Gal}(K_T/K)$ on the group of characters $X(T)$ gives rise to an injective group homomorphism

$$\theta_T \colon \mathrm{Gal}(K_T/K) \longrightarrow \mathrm{Aut}(\Phi(G,T)).$$

Then T is called *generic* (over K) if $\theta_T(\mathrm{Gal}(K_T/K))$ contains the Weyl group $W(G,T)$. As the following statement shows, generic tori with prescribed local properties always exist.

Proposition 6.4. [PR09, Corollary 3.2] *Let G be an absolutely almost simple algebraic K-group, and let $V \subset V^K$ be a finite subset. Suppose that for each $v \in V$ we are given a maximal K_v-torus $T^{(v)}$ of G. Then there exists a maximal K-torus T of G which is generic over K and which is conjugate to $T^{(v)}$ by an element of $G(K_v)$ for all $v \in V$.*

We now return to the situation where G_1 and G_2 are absolutely almost simple K-groups of types B_ℓ and C_ℓ ($\ell \geqslant 3$) respectively. We let $G_1^\natural$ denote the adjoint group of G_1, and $G_2^\natural$ the simply connected cover of G_2. Furthermore, given a maximal K-torus T_i of G_i, we let $T_i^\natural$ denote the image of T_i in $G_i^\natural$ if $i = 1$ and the preimage of T_i in $G_i^\natural$ if $i = 2$.

Proposition 6.5. *Let T_i be a generic maximal K-torus of G_i where $i = 1, 2$. If there exists a K-isogeny $\pi \colon T_i \to T_{3-i}$ onto a maximal K-torus of G_{3-i} then there exist a K-isomorphism $T_i^\natural \simeq T_{3-i}^\natural$.*

The proof below is an adaptation of Lemma 4.3 and Remark 4.4 in [PR09].

Proof. We have $K_{T_1} = K_{T_2} =: L$, and let $\mathcal{G} = \mathrm{Gal}(L/K)$. Then θ_{T_j} is an isomorphism of $\mathcal{G}$ on $W_j = W(G_j, T_j)$ for $j = 1, 2$. The isogeny π induces a $\mathcal{G}$-equivariant homomorphism of character groups $\pi^* \colon X(T_{3-i}) \to X(T_i)$. Let $X_j^\natural = X(T_j^\natural)$; we need to prove that there is a $\mathcal{G}$-equivariant *isomorphism* $\psi \colon X_{3-i}^\natural \to X_i^\natural$. (We recall that $X_1^\natural$ is the subgroup of $X(T_1)$ generated by all the roots in $\Phi_1 = \Phi(G_1, T_1)$, and $X_2^\natural$ is generated by the weights of the root system $\Phi_2 = \Phi(G_2, T_2)$.)

To avoid cumbersome notations, we will assume that $i = 1$. (This does not restrict generality as along with π there is always a K-isogeny $\pi'\colon T_{3-i} \to T_i$.) Consider

$$\phi = \pi^* \otimes \mathrm{id}_{\mathbb{R}} \colon V_2 = X(T_2) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow X(T_1) \otimes_{\mathbb{Z}} \mathbb{R} = V_2$$

and $\mu\colon W_2 \to W_1$ defined by $\mu = \theta_{T_1} \circ \theta_{T_2}^{-1}$. Then the fact that π^* is $\mathcal{G}$-equivariant implies that

$$\phi(w \cdot v) = \mu(w) \cdot \phi(v) \quad \text{for all} \quad v \in V_2,\ w \in W_2. \tag{6.5}$$

On the other hand, it follows from the explicit description of the root systems as in [Bou02] that there exists a linear isomorphism $\phi_0\colon V_2 \to V_1$ and a group isomorphism $\mu_0\colon W_2 \to W_1$ such that

$$\phi_0(w \cdot v) = \mu_0(w) \cdot \phi_0(v) \quad \text{for all} \quad v \in V_2,\ w \in W_2, \tag{6.6}$$

ϕ_0 takes the short roots of Φ_2 to the long roots of Φ_1, and $(1/2)\phi_0$ takes the long roots of Φ_2 to the short roots of Φ_1, consequently $\phi_0(X_2^\natural) = X_1^\natural$. (Note that we identify W_j with the Weyl group of the root system Φ_j.)

We claim that there exists a nonzero $\lambda \in \mathbb{R}$ and $z \in W_1$ such that

$$\phi(v) = \lambda \cdot z \cdot \phi_0(v) \quad \text{and} \quad \mu(w) = z \cdot \mu_0(w) \cdot z^{-1} \quad \text{for all } v \in V_2,\ w \in W_2.$$

Indeed, it was shown in [PR09, Lemma 4.3] (using that $\ell \geqslant 3$) that a suitable multiple $\phi' = \lambda^{-1} \cdot \phi$ takes the short roots of Φ_2 to the long roots of Φ_2, and $(1/2)\phi_0$ takes the long roots of Φ_2 to the short roots of Φ_1. Then $z := \phi' \circ \phi_0^{-1}$ is an automorphism of Φ_1, hence can be identified with an element of W_1. This gives the formula for ϕ, and then the formula for μ follows from (6.5) and (6.6).

Put $\psi := \lambda^{-1} \cdot \phi$. Then

$$\psi(X_2^\natural) = z(\phi_0(X_2^\natural)) = X_1^\natural,$$

and ψ is $\mathcal{G}$-equivariant, as required. □

Corollary 6.6. *Let T_i be a generic maximal K-torus of G_i. If there exists $v \in V$ such that $T_i^\natural$ does not allow a K_v-defined embedding into $G_{3-i}^\natural$ then T_i is not K-isogenous to any maximal K-torus T_{3-i} of G_{3-i}. Thus, if G_1 and G_2 have the same isogeny classes of maximal K-tori then $G_1^\natural$ and $G_2^\natural$ have the same isomorphism classes of maximal K_v-tori for all $v \in V$.*

Proof. The first assertion immediately follows from the proposition. To derive the second assertion from the first, we observe that given $v \in V$ and a maximal K_v-torus $\mathcal{T}_i$ of $G_i^\natural$ that does not allow a K_v-embedding into $G_{3-i}^\natural$, we can find a maximal K-torus T_i of G_i such that $T_i^\natural$ is conjugate to $\mathcal{T}_i$ by an element $G_i^\natural(K_v)$. □

Proof of Theorem 1.4, "only if". Assume G_1 and G_2 have the same isogeny classes of maximal K-tori. Then by Corollary 6.6, $G_1^\natural$ and $G_2^\natural$ have the same isomorphism classes of maximal K_v-tori for all v. It follows that G_1 and G_2 are twins (by Corollary 3.4 for v real and Lemma 3.7 for v finite), completing the proof of part (1) of Theorem 1.4.

Now suppose that G_1 and G_2 have the same *isomorphism* classes of maximal K-tori, in particular, there is a K-isomorphism $\pi\colon T_1 \to T_2$ between two generic K-tori. Then as in the proof of Proposition 6.5, π^* induces $\phi\colon V_2 \to V_1$ which necessarily satisfies $\phi(X(T_2)) = X(T_1)$ and $\phi(X(T_2^\natural)) = X(T_1^\natural)$. Since $X(T_1^\natural) \subseteq X(T_1)$ and $X(T_2^\natural) \supseteq X(T_2)$, this is possible only if both inclusions are in fact equalities, i.e., $G_1 = G_1^\natural$ and $G_2 = G_2^\natural$. This completes the proof of part (2) of Theorem 1.4. □

7. Weakly commensurable subgroups and proof of Theorem 1.2

We begin by recalling the notion of weak commensurability of Zariski-dense subgroups introduced in [PR09]. Let G_1 and G_2 be semi-simple algebraic groups over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense subgroup for $i = 1, 2$. Semi-simple elements $\gamma_i \in \Gamma_i$ are *weakly commensurable* if there exist maximal F-tori T_i of G_i such that $\gamma_i \in T_i(F)$ and for some characters $\chi_i \in X(T_i)$ we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

Furthermore, the subgroups Γ_1 and Γ_2 are *weakly commensurable* if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

The focus in [PR09] was on analyzing when two Zariski-dense S-arithmetic subgroups in absolutely almost simple algebraic groups are weakly commensurable. This analysis was based on a description of such S-arithmetic groups in terms of triples, which we will now briefly recall. Let G be a (connected) absolutely almost simple algebraic group defined over a field F of characteristic zero, $\overline{G}$ be its adjoint group, and $\pi\colon G \to \overline{G}$ be the natural isogeny. Suppose we are given the following data:

- a number field K with a fixed embedding $K \hookrightarrow F$;
- a finite set S of valuations of K containing all archimedean valuations; and
- an F/K-form $\mathcal{G}$ of $\overline{G}$ (i.e., a K-defined algebraic group such that there exists an F-defined isomorphism of algebraic groups ${}_F\mathcal{G} \simeq \overline{G}$, where ${}_F\mathcal{G}$ is the group obtained from $\mathcal{G}$ by the extension of scalars F/K).

(Note that it is assumed in addition that S does not contain any nonarchimedean valuations v such that $\mathcal{G}$ is K_v-anisotropic.) We then have an embedding $\iota\colon \mathcal{G}(K) \hookrightarrow \overline{G}(F)$ and a natural S-arithmetic subgroup $\mathcal{G}(\mathcal{O}_K(S))$, where $\mathcal{O}_K(S)$ is the ring of S-integers in K, defined in terms of a fixed K-embedding $\mathcal{G} \hookrightarrow \mathrm{GL}_n$, i.e. $\mathcal{G}(\mathcal{O}_K(S)) = \mathcal{G}(K) \cap \mathrm{GL}_n(\mathcal{O}_K(S))$. A subgroup Γ of $G(F)$ such that $\pi(\Gamma)$ is commensurable with $\iota(\mathcal{G}(\mathcal{O}_K(S)))$ is called $(\mathcal{G}, K, S)$-*arithmetic*. (It should be pointed out that we do not fix an F-defined isomorphism ${}_F\mathcal{G} \simeq \overline{G}$ in this definition, and by varying it we obtain a class of subgroups invariant under F-defined automorphisms of $\mathcal{G}$ in the obvious sense.)

It was shown in [PR09] that if G_i is absolutely almost simple and Γ_i is Zariski-dense and $(\mathcal{G}_i, K_i, S_i)$-arithmetic for $i = 1, 2$ then the weak commensurability of Γ_1 and Γ_2 implies that $K_1 = K_2 =: K$ and $S_1 = S_2 =: S$, and additionally either G_1 and G_2 are of the same type or one of them is of type B_ℓ and the other is of type C_ℓ for some $\ell \geqslant 3$. That paper also contains many precise conditions for two S-arithmetic subgroups to be weakly commensurable in the case where G_1 and G_2 are of the *same* type. The goal of this section is to prove Theorem 1.2 which provides such conditions when one of the groups is of type B_ℓ and the other of type C_ℓ ($\ell \geqslant 3$). In conjunction with the previous results, this completes the investigation of weak commensurability of S-arithmetic subgroups in absolutely almost simple groups over number fields.

Proof of Theorem 1.2. Let G_1 and G_2 be absolutely almost simple algebraic groups of types B_ℓ and C_ℓ ($\ell \geqslant 3$) respectively defined over a number field K, and let Γ_i be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup of G_i.

Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are twins. Then by Theorem 1.4, they have the same isogeny classes of maximal K-tori. This *automatically* implies that Γ_1 and Γ_2 are weakly commensurable. To see this, we basically need to repeat the argument given in [PR09, Example 6.5], which we also give here for the reader's convenience. First, we may assume without any loss of generality that G_1 and G_2 are adjoint (cf. Lemma 2.4 in [PR09]), hence $\Gamma_i \subset \mathcal{G}_i(K)$. Let $\gamma_1 \in \Gamma_1$ be a semi-simple element of infinite order, and let T_1 be a maximal K-torus of $\mathcal{G}_1$ that contains γ_1. Then there exists a K-isogeny $\varphi\colon T_1 \to T_2$ onto a maximal K-torus T_2 of $\mathcal{G}_2$. The subgroup $\varphi(T_1(K) \cap \Gamma_1)$ is an S-arithmetic subgroup of $T_2(K)$, so there exists $n > 0$ such that $\gamma_2 := \varphi(\gamma_1)^n \in \Gamma_2$. Let $\chi_1 \in \varphi^*(X(T_2))$ be a character such that $\chi_1(\gamma_1)$ is not a root of unity, and let $\chi_2 \in X(T_2)$ be such that $\varphi^*(\chi_2) = \chi_1$. Then

$$(n\chi_1)(\gamma_1) = \chi_1(\gamma_1)^n = \chi_2(\gamma_2) \neq 1,$$

which implies that Γ_1 and Γ_2 are weakly commensurable.

Conversely, suppose that Γ_1 and Γ_2 are weakly commensurable. According to [PR09, Theorem 6.2], this in particular implies that

$$\mathrm{rk}_{K_v}\, \mathcal{G}_1 = \mathrm{rk}_{K_v}\, \mathcal{G}_2 \quad \text{for all } v \in V^K.$$

As we have seen in Lemma 3.7, for $v \in V_f^K$ and the groups under consideration, the equality of ranks implies that both groups are actually K_v-split, verifying condition (6.1) in § 6. Assume that condition (6.2) fails for a real $v_0 \in V_\infty^K$. Then by Corollary 3.4, there is an $i \in \{1, 2\}$ and a maximal K_{v_0}-torus $\mathcal{T}_i$ of $\mathcal{G}_i^\natural$ that does not allow a K_{v_0}-embedding into $\mathcal{G}_{3-i}^\natural$; obviously $\mathcal{T}_i$ is K_{v_0}-isotropic. Let $T_i^{(v_0)}$ be a maximal K_{v_0}-torus of $\mathcal{G}_i$ such that $(T_i^\natural)^{(v_0)} = \mathcal{T}_i$. Furthermore, for $v \in S \setminus \{v_0\}$ we let $T_i^{(v)}$ denote a maximal K_v-torus of $\mathcal{G}_i$ such that $\mathrm{rk}_{K_v}\, T_i^{(v)} = \mathrm{rk}_{K_v}\, \mathcal{G}_i$. Using Proposition 6.4, we can find a maximal K-torus T_i of $\mathcal{G}_i$ that is generic and that is conjugate to $T^{(v)}$ by an element of $\mathcal{G}_i(K_v)$ for all $v \in S \cup \{v_0\}$. Then clearly

$$\mathrm{rk}_S\, T_i := \sum_{v \in S} \mathrm{rk}_{K_v}\, T_i > 0$$

as $\mathrm{rk}_S\, \mathcal{G}_i > 0$. By Dirichlet's Theorem [PR94, Theorem 5.12], the group of S-integral points $T_i(\mathcal{O}_K(S))$ has the following structure: $H \times \mathbb{Z}^d$ where $d = \mathrm{rk}_S\, T_i - \mathrm{rk}_K\, T_i$. Since T_i is obviously K-anisotropic, we conclude that there exists $\gamma_i \in T_i(K) \cap \Gamma_i$ of infinite order (as in the previous paragraph, we are assuming that G_1 and G_2 are adjoint, hence $\Gamma_j \subset \mathcal{G}_j(K)$ for $j = 1, 2$). Then γ_i is weakly commensurable to some semi-simple $\gamma_{3-i} \in \Gamma_{3-i}$ of infinite order. Let T_{3-i} be a maximal K-torus of $\mathcal{G}_{3-i}$ containing γ_{3-i}. By the Isogeny Theorem [PR09, Theorem 4.2], the tori T_i and T_{3-i} are K-isogenous. Using Proposition 6.5, we conclude that $T_i^\natural$ and $T_{3-i}^\natural$ are K-isomorphic. This implies that over K_{v_0}, the torus $\mathcal{T}_i \simeq T_i^\natural$ has an embedding into $\mathcal{G}_{3-i}$. A contradiction, proving (6.2), and completing the proof of Theorem 1.2. □

As we already mentioned, the notion of weak commensurability was introduced in order to tackle some differential-geometric problems dealing with length-commensurable and isospectral locally symmetric spaces, and we would like to conclude this section with a sample of geometric consequences of the results of the current paper established in [PR11]. For a Riemannian manifold M, we let $L(M)$ denote the weak length spectrum of M, i.e., the collection of lengths of all closed geodesics in M. Two Riemannian manifolds M_1 and M_2 are called *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

(7.1) *Let M_1 be an arithmetic quotient of the real hyperbolic space $\mathbb{H}^p$ ($p \geqslant 5$), and M_2 be an arithmetic quotient of the quaternionic hyperbolic space $\mathbb{H}^q_{\mathbf{H}}$ ($q \geqslant 2$). Then M_1 and M_2 are not length-commensurable.*

Theorem 1.2 is used to handle the case $p = 2n$ and $q = n - 1$ for $n \geqslant 3$; for other values of p and q, the claim follows from [PR09, Th. 8.15].

Now, let $\mathfrak{X}_1$ be the symmetric space of the real Lie group $\mathcal{G}_1 = \mathrm{SO}(n+1, n)$, and let $\mathfrak{X}_2$ be the symmetric space of the real Lie group $\mathcal{G}_2 = \mathrm{Sp}_{2n}$ where $n \geqslant 3$.

(7.2) *Let M_i be the quotient of $\mathfrak{X}_i$ by a $(\mathcal{G}_i, K)$-arithmetic subgroup of $\mathcal{G}_i$ for $i = 1, 2$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are twins then*

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1) \quad \textit{where} \quad \lambda = \sqrt{\frac{2n+2}{2n-1}}.$$

(We refer to [PR09], § 1, for the notion of arithmeticity and the explanation of other terms used here.) We finally note that even though one can make $\mathfrak{X}_1$ and $\mathfrak{X}_2$ length-commensurable by scaling the metric on one of them, this will never make them isospectral [Yeu11].

8. Proofs of Proposition 1.3 and Theorem 1.5

Proof of Proposition 1.3. We can assume that G_1 and G_2 are connected absolutely almost simple *adjoint* K-groups having the same isogeny classes of maximal K-tori. Assume that provisions (2) and (3) of the proposition do not hold; let us show that (1) must hold. First, by [PR09, Theorem 7.5], G_1 and G_2 have the same Killing-Cartan type. Furthermore, if L_i is the minimal Galois extension of K over which G_i becomes an inner form then $L_1 = L_2$; in other words, G_1 and G_2 are inner twists of the *same* quasi-split K-group. So, the required assertion is a consequence of the following lemma. □

Lemma 8.1. *Let G_1 and G_2 be connected absolutely almost simple* adjoint *K-groups of the same Killing-Cartan type which is different from A_ℓ ($\ell > 1$), $\mathsf{D}_{2\ell+1}$ ($\ell > 1$) or E_6. Assume that G_1 and G_2 are inner twists of the same quasi-split K-group (which holds automatically if G_1 and G_2 are not of type D). If G_1 and G_2 have the same isogeny classes of maximal K-tori then $G_1 \simeq G_2$.*

Proof. First, suppose that the groups are not of type D. As we have seen in § 5, the fact that G_1 and G_2 have the same isogeny classes of maximal K-tori implies that $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2$ for all $v \in V^K$. For groups of one of the types under consideration, this implies that $G_1 \simeq G_2$ over K_v for all $v \in V^K$ and then our assertion follows from the Hasse principle for Galois cohomology of adjoint groups (see [PR09, § 6] for details of the argument).

Now, suppose the groups are of type $\mathsf{D}_{2\ell}$ for some $\ell \geqslant 2$. There exists a maximal K-torus T_1 of G_1 that is generic and such that $\mathrm{rk}_{K_v} T_1 = \mathrm{rk}_{K_v} G_1$ at every place v where at least one of G_1 or G_2 is not quasi-split. (Note that the set of such v's is finite, cf. [PR94, Theorem 6.7].) By hypothesis, T_1 is isogenous to a maximal K-torus T_2 of G_2, which is necessarily also generic. Following Lemma 4.3 and Remark

4.4 in [PR09], one finds a K-isomorphism $T_1 \to T_2$ that extends to a $\bar{K}$-isomorphism $G_1 \to G_2$. Then our assertion follows from Theorem 20 in [Gar12]. □

Proof of Theorem 1.5. The "if" direction is actually contained in Corollary 6.2—see Remark 6.3. For the "only if" direction, we first observe that if G_1 and G_2 have the same isomorphism classes of maximal K-tori then by Lemma 8.1 the groups $\mathrm{SO}(q_1)$ and $\mathrm{SO}(q_2)$ are isomorphic, hence the forms q_1 and q_2 are similar, yielding assertion (1). Thus, we can assume that $G_1 = \mathrm{SO}(q)$ and $G_2 = \mathrm{Spin}(q)$ for a single quadratic form q.

To prove assertion (2), it is enough to show that if $v \in V^K$ is such that the Witt index of q over K_v is 1 then there exists a 2-dimensional K_v-torus T_1 that has a K_v-embedding into G_1 but does not allow a K_v-embedding into G_2. For this we pick a quadratic extension L/K_v and set

$$T_1 = \mathrm{GL}_1 \times \mathrm{R}^{(1)}_{L/K_v}(\mathrm{GL}_1).$$

We can write $q = q' \perp q''$ where q' is a hyperbolic plane. Then $\mathrm{SO}(q') = \mathrm{GL}_1$ and $\mathrm{SO}(q'') = \mathrm{PSL}_{1,D}$ where D is a quaternion division algebra over K_v. Since L embeds in D, the torus $\mathrm{R}^{(1)}_{L/K_v}(\mathrm{GL}_1)$ embeds in $SL_{1,D}$ and then also in $\mathrm{PSL}_{1,D}$. It follows that T_1 embeds in $G_1 = \mathrm{SO}(q)$. On the other hand, let $T_2 \subset G_2$ be a maximal K_v-torus that splits over L. We can identify G_2 with $\mathrm{SU}(A, \tau)$ where $A = M_2(D)$ with D a quaternion division algebra over K and τ is a symplectic involution on A. Let E_2 be the K_v-subalgebra of A generated by $T_2(K_v)$. Then $E_2 \otimes_{K_v} L \simeq L^4$. As in § 3, we conclude that $(E_2, \tau|_{E_2})$ is isomorphic to $(L, \sigma) \times (L, \sigma)$ where σ is the nontrivial automorphism of L, or to $(L \times L, \lambda)$ where λ is the switch involution. Then $T_2 = \mathrm{SU}(E_2, \tau|_{E_2})$ is isomorphic respectively to $\mathrm{R}^{(1)}_{L/K_v}(\mathrm{GL}_1)^2$ or $\mathrm{R}_{L/K_v}(\mathrm{GL}_1)$. Neither such torus can be isomorphic to T_1. □

9. Alternative proofs via Galois cohomology

Although the main body of the paper demonstrates the effectiveness (and in fact the ubiquity) of the technique of étale algebras in dealing with maximal tori of classical groups, it is worth pointing out that some parts of the argument can also be given in the language of Galois cohomology of algebraic groups. In this section, we will illustrate such an exchange by giving a cohomological proof of the "if" direction of Theorem 1.4(2), i.e., of Corollary 6.2(ii).

Our main tool is Proposition 9.1, for which we need some notation. Let G be a connected semi-simple algebraic group over a number field K. Fix a maximal K-torus T of G, and let $N = N_G(T)$ and $W = N/T$ denote respectively its normalizer and the corresponding Weyl group. For any field extension P/K, we let $\theta_P \colon H^1(P, N) \to H^1(P, W)$ denote the map induced by the natural K-morphism $N \to W$, and let

$$\mathcal{C}(P) := \mathrm{Ker}\left(H^1(P, N) \longrightarrow H^1(P, G)\right);$$

its elements are in one-to-one correspondence with the $G(P)$-conjugacy classes of maximal P-tori in G, see for example [PR09, Lemma 9.1] where this correspondence is described explicitly. There is an obvious K-defined map $W \to \mathrm{Aut}\, T$, so for any $\xi \in H^1(K, W)$ one can consider the corresponding twisted K-torus ${}_\xi T$.

Proposition 9.1. *Assume that there exists a subset $V_0 \subset V_\infty^K$ such that G is K_v-anisotropic for all $v \in V_0$ and is K_v-split for all $v \in V^K \setminus V_0$. Then the sequence*

$$\mathcal{C}(K) \xrightarrow{\theta_K} H^1(K, W) \xrightarrow{\prod \rho_v} \prod_{v \in V_0} H^1(K_v, W) \tag{9.1}$$

is exact.

Here ρ_v denotes the natural restriction map $H^1(K, W) \to H^1(K_v, W)$.

Proof. If V_0 is empty then it follows from the Hasse principle for adjoint groups [PR94, Theorem 6.22] that G is K-split. In this case it was shown by Gille [Gil04] and Raghunathan [Rag04] (or earlier by Kottwitz [Kot82]) that $\theta_K(\mathcal{C}(K)) = H^1(K, W)$, and our claim follows. So, we will assume in the rest of the argument that V_0 is not empty.

We first prove that $\rho_v \theta_K = 0$ for all $v \in V_0$. Given $\xi \in \mathcal{C}(K)$, one can pick $g \in G(\bar{K})$ such that $n(\sigma) := g^{-1}\sigma(g)$ belongs to $N(\bar{K})$ for all $\sigma \in \mathrm{Gal}(\bar{K}/K)$, and the cocycle $\sigma \mapsto n(\sigma)$ represents ξ. Then the maximal torus $T' = gTg^{-1}$ is defined over K. Now, let $v \in V_0$. According to our definitions, G is anisotropic over $K_v = \mathbb{R}$, so it follows from the conjugacy of maximal tori in compact Lie groups that T and T' are conjugate by an element of $G(K_v)$. Then the one-to-one correspondence between the elements of $\mathcal{C}(K_v)$ and the $G(K_v)$-conjugacy classes of maximal K_v-tori in G (or a simple direct computation) implies that the image of ξ under the restriction map $\mathcal{C}(K) \to \mathcal{C}(K_v)$ is trivial, and hence the image of $\theta_K(\xi)$ under the restriction map $H^1(K, W) \to H^1(K_v, W)$ is trivial as well.

Now suppose that G is simply connected; we verify that every $\xi \in \cap_{v \in V_0} \ker \rho_v$ is in the image of θ_K. Pick $v \in V_0$. Since ξ lies in the kernel of $H^1(K, W) \to H^1(K_v, W)$, the twisted torus ${}_\xi T$ is K_v-isomorphic to T, hence K_v anisotropic (as G is K_v-anisotropic). Thus,

$$\mathrm{Ker}\left(H^2(K, {}_\xi T) \to \prod_{v \in V^K} H^2(K_v, {}_\xi T)\right) = 0$$

by [PR09, Prop. 6.12]. Invoking now [PR09, Th. 9.2], we see that to prove the inclusion $\xi \in \theta_K(\mathcal{C}(K))$, it is enough to show that $\rho_v(\xi) \in \theta_{K_v}(\mathcal{C}(K_v))$ for all $v \in V^K$. If $v \in V_0$ then by construction $\rho_v(\xi)$ is trivial, and there is nothing to prove. Otherwise, the group G is K_v-split, so by the result of Gille-Kottwitz-Raghunathan we have $\theta_{K_v}(\mathcal{C}(K_v)) = H^1(K_v, W)$, and the inclusion $\rho_v(\xi) \in \theta_{K_v}(\mathcal{C}(K_v))$ is obvious. Since ξ was arbitrary, we have proved that $\cap \ker \rho_v$ contains the image of θ_K.

In case G is not simply connected, we fix a K-defined universal cover $\pi \colon \widetilde{G} \to G$ of G and use the tilde to denote the objects associated with $\widetilde{G}$. Then π yields a K-isomorphism of $\widetilde{W}$ and W and we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{\mathcal{C}}(K) & \xrightarrow{\widetilde{\theta}_K} & H^1(K, \widetilde{W}) & \xrightarrow{\prod \widetilde{\rho}_v} & \prod_{v \in V_0} H^1(K_v, \widetilde{W}) \\
\downarrow & & \| & & \| \\
\mathcal{C}(K) & \xrightarrow{\theta_K} & H^1(K, W) & \xrightarrow{\prod \rho_v} & \prod_{v \in V_0} H^1(K_v, W).
\end{array}$$

The top row is exact by the previous paragraph, hence $\cap \ker \rho_v$ contains the image of θ_K. ☐

We now begin to work our way towards the proof of Theorem 1.4(2)/Corollary 6.2(ii). Let G_1 be adjoint of type B_ℓ and let G_2 be simply connected of type C_ℓ for some $\ell \geqslant 2$. We will use a subscript $i \in \{1, 2\}$ to denote the objects associated with

G_i. In particular, we let T_i denote a maximal torus of G_i, and let $N_i = N_{G_i}(T_i)$ and $W_i = N_i/T_i$ be its normalizer and the Weyl group. Then W_i naturally acts on T_i by conjugation. We say that the morphisms of algebraic groups $\varphi\colon T_1 \to T_2$ and $\psi\colon W_1 \to W_2$ are *compatible* if

$$\varphi(w \cdot t) = \psi(w) \cdot \varphi(t) \quad \text{for all } t \in T_1,\ w \in W_1.$$

Lemma 9.2. *One can pick maximal K-tori T_i of G_i for $i = 1, 2$ so that there exist compatible K-defined isomorphisms*

$$\varphi\colon T_1 \to T_2 \quad \textit{and} \quad \psi\colon W_1 \to W_2.$$

Proof. Imitating the argument given in [PR94, Proposition 6.16], it is easy to see that there exists a quadratic extension L/K that splits *both* G_1 and G_2. Indeed, let V_i be the (finite) set of places $v \in V^K$ such that G_i does not split over K_v, and let $V = V_1 \cup V_2$. Pick a quadratic extension L/K so that the local degree $[L_w : K_v] = 2$ for all $v \in V$ and $w|v$. We claim that L is as required. By the Hasse principle, it is enough to show that both G_1 and G_2 split over L_w for any $w \in V^L$. For a given w, we let $v \in V^K$ be the place that lies below w. If $v \notin V$ then by our construction G_1 and G_2 split already over K_v, and there is nothing to prove. If $v \in V$ then $[L_w : K_v] = 2$, and then the proof of [PR94, Proposition 6.16] that G_1 and G_2 split over L_w, as required.

Now, let $\sigma \in \mathrm{Gal}(L/K)$ be a generator. According to [PR94, Lemma 6.17], for each $i \in \{1, 2\}$, there exists an L-defined Borel subgroup B_i of G_i such that $T_i := B_i \cap B_i^\sigma$ is a maximal K-torus of G_i that splits over L. Considering the action of σ on the root system $\Phi(G_i, T_i)$, we see that it takes the system of positive roots corresponding to B_i into the system of negative roots. For groups of types B_ℓ and C_ℓ, this implies that σ acts on the character group $X(T_i)$ as multiplication by (-1). It easily follows from the description of the corresponding root systems (cf. [Bou02]) that there exist compatible (in the obvious sense) isomorphisms $\varphi^*\colon X(T_2) \to X(T_1)$ (of abelian groups) and $\psi\colon W_1 \to W_2$ (of abstract groups considered as subgroups of $GL(X(T_1))$ and $GL(X(T_2))$). Then φ^* gives rise to an isomorphism $\varphi\colon T_1 \to T_2$ of algebraic groups that is compatible (as defined above) with ψ (which can be considered as a morphism of algebraic groups). It remains to observe that since σ acts on $X(T_1)$ and $X(T_2)$ as multiplication by (-1), both φ and ψ are K-defined (in fact, σ acts on W_1 and W_2 trivially). $\square$

Remark. If both groups G_1 and G_2 are K-split then one can, of course, take for T_1 and T_2 their maximal K-split tori.

For the rest of the paper, we fix compatible K-defined isomorphisms

$$\varphi^0\colon T_1^0 \to T_2^0 \ \text{ and } \ \psi^0\colon W_1^0 \to W_2^0.$$

(Thus, we henceforth slightly change the notations used in Lemma 9.2.) Given *arbitrary* maximal K-tori T_i of G_i for $i = 1, 2$, we pick elements $g_i \in G(\bar{K})$ so that

$$T_i = g_i T_i^0 g_i^{-1},$$

and then for any $\sigma \in \mathrm{Gal}(\bar{K}/K)$, the element $n_i(\sigma) := g_i^{-1}\sigma(g_i)$ belongs to $N_i^0(\bar{K})$. Let $\varphi = \varphi(g_1, g_2)$ be the morphism $T_1 \to T_2$ defined by

$$\varphi(t) = g_2 \varphi^0(g_1^{-1} t g_1) g_2^{-1},$$

and let $\nu_i^0\colon N_i^0 \to W_i^0$ denote the canonical morphism.

Lemma 9.3. *If*

$$\psi^0(\nu_1^0(n_1(\sigma))) = \nu_2^0(n_2(\sigma)) \quad \textit{for all} \quad \sigma \in \mathrm{Gal}(\bar{K}/K) \tag{9.2}$$

then $\varphi = \varphi(g_1, g_2)$ is defined over K.

Proof. We need to show that φ commutes with every $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Since φ^0 is defined over K, for any $t \in T_1(\bar{K})$, we have

$$\begin{aligned}\sigma(\varphi(t)) &= \sigma(g_2)\varphi^0(\sigma(g_1)^{-1}\sigma(t)\sigma(g_1))\sigma(g_2)^{-1} \\ &= g_2 n_2(\sigma)\varphi^0(n_1(\sigma)^{-1}g_1^{-1}\sigma(t)g_1 n_1(\sigma))n_2(\sigma)^{-1}g_2^{-1} \\ &= g_2\,[\,(\nu_2^0(n_2(\sigma)))\cdot\varphi^0((\nu_1^0(n_1(\sigma)))\cdot(g_1^{-1}\sigma(t)g_1))\,]\,g_2^{-1}.\end{aligned}$$

Since φ^0 is compatible with ψ^0, condition (9.2) implies that the latter reduces to

$$g_2\varphi^0(g_1^{-1}\sigma(t)g_1)g_2^{-1} = \varphi(\sigma(t)).$$

It follows that $\sigma(\varphi(t)) = \varphi(\sigma(t))$, i.e. φ commutes with σ, as required. □

Pursuant to the above notations, for an extension P/K and $i = 1, 2$, we set

$$\mathcal{C}_i(P) = \mathrm{Ker}\left(H^1(P, N_i^0) \to H^1(P, G_i)\right),$$

and let $\theta_{iP}\colon H^1(P, N_i^0) \to H^1(P, W_i^0)$ denote the canonical map (induced by ν_i). The isomorphism $H^1(K, W_1^0) \to H^1(K, W_2^0)$ induced by ψ^0 will still be denoted by ψ^0.

Lemma 9.4. *Assume that*

$$\psi^0(\mathcal{C}_1(K)) = \mathcal{C}_2(K). \tag{9.3}$$

Then for $i = 1$ or 2, given any maximal K-torus T_i of G_i and an element $g_i \in G_i(\bar{K})$ such that $T_i = g_i T_i^0 g_i^{-1}$, there exists $g_{3-i} \in G_{3-i}(\bar{K})$ such that the maximal torus

$$T_{3-i} := g_{3-i}T_{3-i}^0 g_{3-i}^{-1}$$

and the isomorphism

$$\varphi(g_1, g_2)\colon T_1 \to T_2$$

are K-defined. Thus, in this case G_1 and G_2 have the same isomorphism classes of maximal K-tori.

Proof. To keep our notations simple, we will give an argument for $i = 1$ (the argument in the case $i = 2$ is totally symmetric). As above, we set $n_1(\sigma) = g_1^{-1}\sigma(g_1) \in N_1^0(\bar{K})$ for $\sigma \in \mathrm{Gal}(\bar{K}/K)$, observing that these elements define a cohomology class $n_1 \in \mathcal{C}_1(K)$. Then (9.3) implies that there exists $h_2 \in G_2(\bar{K})$ such that for the cohomology class $m_2 \in \mathcal{C}_2(K)$ defined by the elements $m_2(\sigma) = h_2^{-1}\sigma(h_2) \in N_2^0(\bar{K})$, we have

$$\psi^0(\theta_{1K}(n_1)) = \theta_{2K}(m_2) \quad \text{in} \quad H^1(K, W_2).$$

Then there exists $w_2 \in W_2(\bar{K})$ such that

$$\psi^0(\nu_1^0(n_1(\sigma))) = w_2^{-1}\nu_2^0(m_2(\sigma))\sigma(w_2) \quad \text{for all} \quad \sigma \in \mathrm{Gal}(\bar{K}/K). \tag{9.4}$$

Picking $z_2 \in N_2^0(\bar{K})$ so that $\nu_2^0(z_2) = w_2$, and setting

$$g_2 = h_2 z_2 \quad \text{and} \quad n_2(\sigma) = g_2^{-1}\sigma(g_2) \in N_2^0(\bar{K}) \quad \text{for } \sigma \in \mathrm{Gal}(\bar{K}/K),$$

we obtain from (9.4) that (9.2) holds. Then g_2 is as required. Indeed, the fact that $n_2(\sigma) \in N_2^0(\bar{K})$ implies that $T_2 = g_2 T_2^0 g_2^{-1}$ is defined over K, and Lemma 9.3 yields that the morphism $\varphi(g_1, g_2)\colon T_1 \to T_2$ is also defined over K. □

Proof of Corollary 6.2(ii). Suppose that G_1 and G_2 are twins, and let V_0 be the set of all archimedean places $v \in V^K$ such that G_1 and G_2 are both K_v-anisotropic. Then for any $v \in V^K \setminus V_0$, both G_1 and G_2 are K_v-split. Then according to Proposition 9.1 we have

$$\theta_{iK}(\mathcal{C}_i(K)) = \ker\left(H^1(K, W_i^0) \to \prod_{v \in V_0} H^1(K_v, W_i^0)\right)$$

for $i = 1, 2$, and as $\psi_0 \colon W_1^0 \to W_2^0$ is an isomorphism, condition (9.3) holds, and the claim follows from Lemma 9.4. □

Remark. It follows from the explicit description of the root systems of types B_ℓ and C_ℓ that the isomorphism φ in Lemma 9.2 can be chosen so that for $t \in T_1(\bar{K})$ there exist $\lambda_1, \ldots, \lambda_\ell \in \bar{K}^\times$ such that the values of the roots $\alpha \in \Phi(G_1, T_1)$ on t are

$$\lambda_i^{\pm 1}, \quad i = 1, \ldots, \ell, \quad \text{and} \quad \lambda_i^{\pm 1} \cdot \lambda_j^{\pm 1}, \quad i, j = 1, \ldots, \ell, \ i \neq j,$$

and the values of the roots $\alpha \in \Phi(G_2, T_2)$ on $\phi(t)$ are

$$\lambda_i^{\pm 2}, \quad i = 1, \ldots, \ell, \quad \text{and} \quad \lambda_i^{\pm 1} \cdot \lambda_j^{\pm 1}, \quad i, j = 1, \ldots, \ell, \ i \neq j.$$

Then any identification of the form $\varphi(g_1, g_2)$ also has this property, which was used in [PR11].

Alternatively, suppose that G_i for $i = 1, 2$ is realized as $\mathrm{SU}(A_i, \tau_i)$ as described in the beginning of §6. Let E_1 be a $(\tau_1 \otimes \mathrm{id}_{\bar{K}})$-invariant maximal commutative étale $\bar{K}$-subalgebra of $A_1 \otimes_K \bar{K}$ satisfying (2.2), and let $\sigma_1 = \tau_1|_{E_1}$. Then in the notations of §6, the algebra (E_1', σ_1') admits a $\bar{K}$-embedding embedding into $(A_2 \otimes_K \bar{K}, \tau_2 \otimes \mathrm{id}_{\bar{K}})$, and we let (E_2, σ_2) the image of this embedding. It is easy to see that if we let T_i denote the maximal torus of G_i defined by (E_i, σ_i) then the isomorphism $T_1 \simeq T_2$ coming from the isomorphism of algebras $(E_1', \sigma_1') \simeq (E_2, \sigma_2)$ is the same as the isomorphism coming from the description of the root systems (cf. the proof of Lemma 9.2); in particular, it is compatible with the natural isomorphism of the Weyl groups. So, the assertion of Lemma 9.2 means that given *any* K-algebras with involution (A_1, τ_1) and (A_2, τ_2) as above, there exists a τ_1-invariant maximal commutative étale K-subalgebra E_1 of A_1 that satisfies (2.2) and is such that for $\sigma_1 = \tau_1|_{E_1}$, the algebra (E_1', σ_1') admits an embedding into (A_2, σ_2). Moreover, by Corollary 6.2(ii), if the corresponding groups G_1 and G_2 are twins then the correspondence $(E_1, \sigma_1) \mapsto (E_1', \sigma_1')$ gives a bijection between the sets of isomorphism classes of maximal commutative étale K-subalgebras of (A_1, τ_1) and (A_2, τ_2) that are invariant under the respective involutions and satisfy (2.2). Thus, we recover Proposition 6.1.

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