MOTIVES AND ORIENTED COHOMOLOGY OF A LINEAR ALGEBRAIC GROUP

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Abstract. For a cellular variety X over a field k of characteristic 0 and an algebraic oriented cohomology theory $\mathtt{h}$ of Levine-Morel we construct a filtration on the cohomology ring $\mathtt{h}(X)$ such that the associated graded ring is isomorphic to the Chow ring of X. Taking X to be the variety of Borel subgroups of a split semisimple linear algebraic group G over k we apply this filtration to relate the oriented cohomology of G to its Chow ring. As an immediate application we compute the algebraic cobordism ring of a group of type G_2 and of some other groups of small ranks, hence, extending several results by Yagita.

Using this filtration we also establish the following comparison result between Chow motives and $\mathtt{h}$-motives of generically cellular varieties: any irreducible Chow-motivic decomposition of a generically split variety Y gives rise to a $\mathtt{h}$-motivic decomposition of Y with the same generating function. Moreover, under some conditions on the coefficient ring of $\mathtt{h}$ the obtained $\mathtt{h}$-motivic decomposition will be irreducible. We also prove that if Chow motives of two twisted forms of Y coincide, then their $\mathtt{h}$-motives coincide as well.

1. Introduction

We work over the base field k of characteristic 0. For an algebraic oriented cohomology theory $\mathtt{h}$ of Levine-Morel [11] and a cellular variety X of dimension N we construct a filtration

$$\mathtt{h}(X) = \mathtt{h}^{(0)}(X) \supseteq \mathtt{h}^{(1)}(X) \supseteq \ldots \supseteq \mathtt{h}^{(N)}(X) \supseteq 0$$

on the cohomology ring such that the associated graded ring

$$Gr^* \, \mathtt{h}(X) = \bigoplus_{i \geq 0} \mathtt{h}^{(i)}(X)/\mathtt{h}^{(i+1)}(X)$$

is isomorphic (as a graded ring) to the Chow ring $\mathrm{CH}^*(X, \Lambda)$ of algebraic cycles modulo rational equivalence relation with coefficients in a ring Λ. We exploit this filtration and isomorphism in two different contexts:

First, we consider the (cellular) variety $X = G/B$ of Borel subgroups of a split semisimple linear algebraic group G over k. By [7, Prop. 5.1] the cohomology ring $\mathtt{h}(G)$ can be identified with a quotient of $\mathtt{h}(G/B)$, so there is an induced filtration on $\mathtt{h}(G)$. One of our key results (Prop. 4.3) shows that $\mathrm{CH}^*(G, \Lambda)$ covers the associated graded ring $Gr^* \, \mathtt{h}(G)$ and describes the kernel of this surjection. As an immediate application for $\mathtt{h} = \Omega$ (the algebraic cobordism of Levine-Morel) we compute the cobordism ring for groups G_2, SO_3, SO_4, $Spin_n$ for $n = 3, 4, 5, 6$ and PGL_n for $n \geq 2$, in terms of generators and relations, hence, extending several previously known results by Yagita [19]; as an application for $\mathtt{h} = K_0$ (the

Grothendieck K_0) we construct certain elements in the difference between topological and the Grothendieck γ-filtration on $K_0(X)$, hence, extending some of the results by Garibaldi-Zainoulline [6].

The second deals with the study of $\mathtt{h}$-motives of generically cellular varieties. The latter is a natural generalization of the notion of the Chow motives to the case of an arbitrary algebraic oriented cohomology theory of Levine-Morel. It was introduced and studied by Nenashev-Zainoulline in [13] and Vishik-Yagita in [17].

Let Λ denote the coefficient ring of $\mathtt{h}$ and let Λ^i denote its i-th graded component. We prove the following theorem which relates $\mathtt{h}$-motives of generically cellular varieties to its Chow motives:

Theorem A. *Let X be a generically cellular variety over k, i.e. cellular over the function field $k(X)$. Assume that the Chow motive of X with coefficients in Λ^0 splits as*

$$M^{\mathrm{CH}}(X,\Lambda^0)=\bigoplus_{i\geq 0}\mathcal{R}(i)^{\oplus c_i},\ c_i\geq 0,$$

for some motive $\mathcal{R}$ which splits as a direct sum of twisted Tate motives $\overline{\mathcal{R}}=\bigoplus_{j\geq 0}\Lambda^0(j)^{\oplus d_j}$ over its splitting field.

Then the $\mathtt{h}$-motive of X (with coefficients in Λ) splits as

$$M^{\mathtt{h}}(X)=\bigoplus_{i\geq 0}\mathcal{R}_{\mathtt{h}}(i)^{\oplus c_i}$$

for some motive $\mathcal{R}_{\mathtt{h}}$ and over the same splitting field $\mathcal{R}_{\mathtt{h}}$ splits as a direct sum of twisted $\mathtt{h}$-Tate motives $\overline{\mathcal{R}_{\mathtt{h}}}=\bigoplus_{j\geq 0}\Lambda(j)^{\oplus d_j}$.

This result can also be derived from the arguments of [17] where it is proved that sets of isomorphism classes of objects of category of Chow motives and Ω-motives coincide. However, our approach gives more explicit correspondence between idempotents defining the (Chow) motive $\mathcal{R}$ and the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$. The latter allows us to prove the following result concerning the indecomposability of the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$:

Theorem B. *Assume that $\Lambda^1=\ldots\Lambda^N=0$, where $N=\dim X$.*

If the Chow motive $\mathcal{R}$ is indecomposable (over Λ^0), then the $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$ is indecomposable (over Λ).

and also the following comparison property:

Theorem C. *Suppose that X,Y are generically cellular and Y is a twisted form of X, i.e. Y becomes isomorphic to X over some splitting field.*

If $M^{\mathrm{CH}}(X,\Lambda^0)\cong M^{\mathrm{CH}}(Y,\Lambda^0)$, then $M^{\mathtt{h}}(X)\cong M^{\mathtt{h}}(Y)$.

The paper is organized as follows: In section 2 we recall concepts of an algebraic oriented cohomology theory $\mathtt{h}$ of Levine-Morel and the corresponding category of $\mathtt{h}$-motives. In section 3 we introduce the filtration on the cohomology ring $\mathtt{h}(X)$ of a cellular variety X which plays a central role in the paper. In section 4 we apply the filtration to obtain several comparison results between $\mathrm{CH}(G)$ and $\mathtt{h}(G)$. In particular, in section 5 we compute the algebraic cobordism Ω for some groups of small ranks and construct explicit elements in the difference of between the topological and the γ-filtration on $K_0(G/B)$. In section 6 we apply the filtration to obtain comparison results between $\mathtt{h}$-motives and Chow-motives of generically split varieties.

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2. Preliminaries

In the present section we recall notions of an algebraic oriented cohomology theory, a formal group law and of a cellular variety. We recall the definition of the category of $\mathtt{h}$-motives with the inverted Tate object.

Oriented cohomology theories. The notion of an algebraic oriented cohomology theory was introduced by Levine-Morel [11] and Panin-Smirnov [14]. Let $\mathbf{Sm}_k$ denote the category of smooth varieties over Spec $k = pt$. An algebraic oriented cohomology theory $\mathtt{h}^*$ is a functor from $\mathbf{Sm}_k^{op}$ to the category of graded rings. We will denote by $f^{\mathtt{h}}\colon \mathtt{h}^*(Y) \to \mathtt{h}^*(X)$ the induced morphism $f\colon X \to Y$ and call it the pullback of f. By definition, the functor $\mathtt{h}^*$ is equipped with the pushforward map $f_{\mathtt{h}}\colon \mathtt{h}^*(X) \to \mathtt{h}^{\dim Y - \dim X + *}(Y)$ for any projective morphism $f\colon X \to Y$. These two structures satisfy the axioms of [11, Def. 1.1.2]. We denote its coefficient ring $\mathtt{h}^*(pt)$ by Λ^*. As for the Chow groups, we will also use the lower grading for $\mathtt{h}$, i.e. $\mathtt{h}_i(X) = \mathtt{h}^{\dim X - i}(X)$ for an irreducible variety X.

Formal group law. For an oriented cohomology theory $\mathtt{h}^*$ there is a notion of the first Chern class of a line bundle. For $X \in \mathbf{Sm}_k$ and a line bundle L over X it is defined as $c_1^{\mathtt{h}}(L) = z^{\mathtt{h}} z_{\mathtt{h}}(1) \in \mathtt{h}^1(X)$ where $z\colon X \to L$ is a zero section. There is a commutative associative 1-dimensional formal group law F over Λ^* such that for any two line bundles L_1 , L_2 over X we have $c_1^{\mathtt{h}}(L_1 \otimes L_2) = F(c_1^{\mathtt{h}}(L_1), c_1^{\mathtt{h}}(L_2))$ [11, Lem. 1.1.3]. We will use the notation $x +_F y$ for $F(x, y)$. For any x we will denote by $-_F x$ the unique element such that $x +_F (-_F x) = 0$. For any $n \in \mathbb{Z}$ we will denote by $n \cdot_F x$ the expression $x +_F \ldots +_F x$ (n times) if n is positive, and $(-_F x) +_F \ldots +_F (-_F x)$ ($-n$ times) if n is negative. By [11] there is a natural transformation that commutes with pushforwards:

$$\nu_X\colon \Omega^*(X) \otimes_{\mathbb{L}^*} \Lambda^* \to \mathtt{h}^*(X),$$

where $\mathbb{L} = \Omega(pt)$ is the Lazard ring and the map $\mathbb{L}^* \to \Lambda^*$ is obtained by specializing the coefficients of the universal formal group law to the coefficients of the corresponding F.

Cellular and generically cellular varieties. A variety $Y \in \mathbf{Sm}_k$ is called cellular if there is a filtration of $Y = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_m \supseteq \emptyset$ such that each $Y_i \setminus Y_{i+1}$ is a disjoint union of affine spaces of the same rank c_i: $Y_i \setminus Y_{i+1} \cong \mathbb{A}_k^{c_i} \coprod \ldots \coprod \mathbb{A}_k^{c_i}$.

We call a variety X generically cellular if $X_{k(X)}$ is a cellular variety over the function field $k(X)$.

2.1. **Example.** Let G be a split semisimple algebraic group, B its Borel subgroup containing a fixed maximal split torus T and W the corresponding Weyl group. For any $w \in W$ let $l(w)$ denote its length. Let $w_0 \in W$ denote the longest element of W and $N = l(w)$. It is well known that the flag variety $X = G/B$ has the cellular structure given by the Schubert cells X_w:

$$X = X_{w_0} \supseteq \bigcup_{l(w)=N-1} X_w \supseteq \bigcup_{l(w)=N-2} X_w \supseteq \ldots \supseteq X_e = pt,$$

where X_w is the closure of BwB/B in X.

2.2. **Example.** Let $\zeta \in Z^1(k,G)$ be a 1-cocycle with values in G. Then the twisted form ${}_\zeta(G/B)$ of $X = G/B$ provides an example of a generically split variety.

$\mathtt{h}$-*motives.* The notion of $\mathtt{h}$-motives for the algebraic oriented cohomology theory $\mathtt{h}$ was studied by Nenashev-Zainoulline in [13], and Vishik-Yagita in [17]. We refer to [17, §2] for definition of the category of effective $\mathtt{h}$-motives. In the present paper we will deal with the category of $\mathtt{h}$-motives $\mathcal{M}_{\mathtt{h}}$ with the inverted Tate object. It is constructed as follows:

Let $\mathbf{SmProj}_k$ denote the category of smooth projective varieties over k. Following [5] we consider the category $Corr_{\mathtt{h}}$ defined as follows: For $X, Y \in \mathbf{SmProj}_k$ with irreducible X and $m \in \mathbb{Z}$ we set

$$Corr_m(X,Y) = \mathtt{h}_{\dim X+m}(X \times Y).$$

Objects of $Corr_{\mathtt{h}}$ are pairs (X,i) with $X \in \mathbf{SmProj}_k$ and $i \in \mathbb{Z}$. For $X \in \mathbf{SmProj}_k$ with irreducible components X_l define the morphisms

$$Hom_{Corr}\big((X,i),(Y,j)\big) = \coprod_l Corr_{i-j}(X_l, Y).$$

For $\alpha \in Hom\big((X,i),(Y,j)\big)$ and $\beta \in Hom\big((Y,j),(Z,k)\big)$ the composition is given by the usual correspondence product: $\alpha \circ \beta = (p_{XZ})_{\mathtt{h}}\big((p_{YZ})^{\mathtt{h}}(\beta) \cdot (p_{XY})^{\mathtt{h}}(\alpha)\big)$, where p_{XY}, p_{YZ}, p_{XZ} denote the projections from $X \times Y \times Z$ onto the corresponding summands.

Taking consecutive additive and idempotent completion of $Corr_{\mathtt{h}}$ we obtain the category $\mathcal{M}_{\mathtt{h}}$ of $\mathtt{h}$-motives with inverted Tate object. Objects of this category are $(\coprod_i (X_i, n_i), p)$ where p is a matrix with entries $p_{i,j} \in Corr_{n_i - n_j}(X_i, X_j)$ such that $p \circ p = p$. Morphisms between objects are given by the set

$$Hom\big((\coprod (X_i,n_i),p),(\coprod (Y_j,m_j),q)\big) = q \circ \bigoplus_{i,j} Corr_{n_i-m_j}(X_i,Y_j) \circ p$$

considered as a subset of $\bigoplus_{i,j} Corr_{n_i-m_j}(X_i,Y_j)$. This is an additive category where each idempotent splits. There is a natural tensor structure inherited from the category $Corr_{\mathtt{h}}$:

$$(\coprod (X_i,n_i),p) \otimes (\coprod (Y_j,m_j),q) = (\coprod_{i,j} (X_i \times Y_j, n_i+n_j), p \times q)$$

where $p \times q$ denotes the projector $p_{(i_1,j_1)(i_2,j_2)} = p_{i_1,i_2} \times q_{j_1,j_2} \colon X_{i_1} \times Y_{j_1} \to X_{i_2} \times Y_{j_2}$.

There is a functor $M^{\mathtt{h}} \colon \mathbf{SmProj}_k \to \mathcal{M}_{\mathtt{h}}$ that maps a variety X to the motive $M^{\mathtt{h}}(X) = ((X,0), id_X)$ and any morphism $f \colon X \to Y$ to the correspondence $(\Gamma_f)_{\mathtt{h}}(1) \in \mathtt{h}_{\dim X}(X \times Y) = Corr_0(X,Y)$, where $\Gamma_f \colon X \to X \times Y$ is the graph inclusion. We will denote by $\Delta \colon X \to X \times X$ the diagonal embedding. Then $\Delta_{\mathtt{h}}(1)$ is the identity in $Corr_0(X,X)$.

Denote $M^{\mathtt{h}}(pt)$ by Λ and $((pt,1), id_{pt})$ by $\Lambda(1)$. We call $\Lambda(1)$ the $\mathtt{h}$-Tate motive. We write $\Lambda(n)$ for $\Lambda(1)^{\otimes n}$ and $M^{\mathtt{h}}(X)(n)$ for $M^{\mathtt{h}}(X) \otimes \Lambda(n)$. The motive $M^{\mathtt{h}}(X)(n)$ is called the n-th twist of the motive $M^{\mathtt{h}}(X)$.

By definition we have

$$\mathtt{h}^i(X) = Hom_{\mathcal{M}_{\mathtt{h}}}(M^{\mathtt{h}}(X), \Lambda(i)) \text{ and } \mathtt{h}_i(X) = Hom_{\mathcal{M}_{\mathtt{h}}}(\Lambda(i), M^{\mathtt{h}}(X)).$$

2.3. **Lemma.** *For $X \in \mathbf{SmProj}_k$ with the structure morphism $\pi\colon X \to pt$ any isomorphism $M^{\mathtt{h}}(X) \cong \bigoplus_i \Lambda(\alpha_i)$ corresponds to a choice of two Λ-basis sets*

$$\{\tau_i \in \mathtt{h}^{\alpha_i}(X)\}_i \text{ and } \{\zeta_i \in \mathtt{h}_{\alpha_i}(X)\}_i$$

such that $\pi_{\mathtt{h}}(\tau_i\zeta_j) = \delta_{i,j}$ in Λ and $\sum_i \zeta_i \otimes \tau_i = \Delta_{\mathtt{h}}(1)$ in $\mathtt{h}(X \times X)$.

Proof. In the direct sum decomposition $M^{\mathtt{h}}(X) \cong \bigoplus_i \Lambda(\alpha_i)$ the i-th projection $p_i\colon M^{\mathtt{h}}(X) \to \Lambda(\alpha_i)$ is defined by an element $\tau_i \in \mathtt{h}^{\alpha_j}(X)$ and the i-th inclusion $\imath_i\colon \Lambda(\alpha_i) \to \mathtt{h}(X)$ is defined by an element $\zeta_i \in \mathtt{h}_{\alpha_i}(X)$. Then by definition of the direct sum we obtain

$$\pi_{\mathtt{h}}(\tau_i\zeta_j) = \delta_{i,j} \text{ and } \sum_i \zeta_i \otimes \tau_i = \Delta_{\mathtt{h}}(1).$$

Let us check that $\{\zeta_i\}_i$ form a basis of $\mathtt{h}(X)$. Indeed, we have $\mathtt{h}(X) = \bigoplus_j \mathtt{h}^j(X) = \bigoplus_j Hom_{\mathcal{M}_{\mathtt{h}}}(M^{\mathtt{h}}(X), \Lambda(j)) \cong \bigoplus_j Hom_{\mathcal{M}_{\mathtt{h}}}(\bigoplus_i \Lambda(\alpha_i), \Lambda(j)) = \bigoplus_i \Lambda_{\alpha_i - *}$ and ζ_i are the images of standard generators. So $\{\zeta_i\}_i$ form a Λ-basis of $\mathtt{h}(X)$. Finally, since $\{\tau_i\}_i$ are dual to $\{\zeta_i\}_i$, $\{\tau_i\}_i$ is also a basis. □

2.4. **Remark.** Observe that any isomorphism $M^{\mathtt{h}}(X) \cong \bigoplus \Lambda(\alpha_i)$ gives rise (canonically) to an isomorphism $\mathtt{h}^*(X) \cong \bigoplus_i \Lambda^{*-\alpha_i}$.

3. Filtration on the cohomology ring

In the present section we construct a filtration on the oriented cohomology $\mathtt{h}(X)$ of a cellular variety X which will play an important role in the sequel.

3.1. **Proposition.** *Assume that X is a cellular variety over k with the cellular decomposition $X = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \emptyset$ where $X_i \setminus X_{i+1} = \coprod_{c_i} \mathbb{A}^{\alpha_i}$. Then*

1. *the $\mathtt{h}$-motive of X splits as $M^{\mathtt{h}}(X) = \bigoplus_i \Lambda(\alpha_i)^{\oplus c_i}$;*
2. *the Künneth formula holds, i.e. the natural map $\mathtt{h}(X) \otimes_\Lambda \mathtt{h}(X) \to \mathtt{h}(X \times X)$ is an isomorphism;*
3. *the specialization maps $\nu_X\colon \Omega(X) \otimes \Lambda \to \mathtt{h}(X)$ and $\nu_{X\times X}\colon \Omega(X\times X)\otimes\Lambda \to \mathtt{h}(X \times X)$ are isomorphisms.*

Proof. By [5, Cor. 66.4] the Chow motive $M^{\mathrm{CH}}(X)$ splits, then [17, Cor. 2.9] implies that the motive $M^\Omega(X)$ splits into a sum of twisted Tate motives $M^\Omega(X) = \bigoplus_{i\in I} \mathbb{L}(\alpha_i)^{\oplus c_i}$. By Lemma 2.3 there are elements $\zeta^\Omega_{i,j} \in \mathtt{h}_{\alpha_i}(X)$ and $\tau^\Omega_{i,j} \in \mathtt{h}^{\alpha_i}(X)$, $j \in \{1..c_i\}$ such that $\pi_\Omega(\zeta^\Omega_{i,j}\tau^\Omega_{i',j'}) = \delta_{(i,j),(i',j')}$ and $\Delta_\Omega(1) = \sum_{i,j} \zeta^\Omega_i \otimes \tau^\Omega_i$. Denote $\zeta^{\mathtt{h}}_{i,j} = \nu(\zeta^\Omega_{i,j} \otimes 1)$ and $\tau^{\mathtt{h}}_{i,j} = \nu(\tau^\Omega_{i,j} \otimes 1)$. Since ν commutes with pullbacks and push-forwards, $\pi_{\mathtt{h}}(\zeta^{\mathtt{h}}_{i,j}\tau^{\mathtt{h}}_{i',j'}) = \delta_{(i,j),(i',j')}$ and $\Delta_{\mathtt{h}}(1) = \sum_{i,j} \zeta^{\mathtt{h}}_{i,j} \otimes \tau^{\mathtt{h}}_{i,j}$. Then by Lemma 2.3 we have $M^{\mathtt{h}}(X) = \bigoplus_i \Lambda(\alpha_i)^{\oplus c_i}$, so (1) holds.

The Künneth map fits into the diagram

$$\begin{tikzcd}
\mathtt{h}(X) \otimes_\Lambda \mathtt{h}(X) \arrow[r] \arrow[d, equal] & \mathtt{h}(X \times X) \arrow[d, equal] \\
(\bigoplus_i \Lambda^{*-\alpha_i}) \otimes_\Lambda (\bigoplus_j \Lambda^{*-\alpha_j}) \arrow[r] & \bigoplus_{i,j} \Lambda^{*-\alpha_i-\alpha_j}
\end{tikzcd}$$

where the bottom arrow is an isomorphism, so the Künneth formula (2) holds.

Note that the natural map ν_X can be factored as follows

$$\nu_X\colon \Omega(X)\otimes\Lambda = \oplus_m Hom_{\mathcal{M}_{\Omega\otimes\Lambda}}(\bigoplus \Lambda(\alpha_i), \Lambda(m)) \to Hom_{\mathcal{M}_{\mathtt{h}}}(\bigoplus \Lambda(\alpha_i), \Lambda(m)) = \mathtt{h}(X).$$

Thus ν_X is an isomorphism. The same reasoning proves the statement for $\nu_{X\times X}$, hence, (3) holds. $\square$

3.2. **Definition.** Let X be a cellular variety. Consider two basis sets $\zeta_i \in \mathtt{h}_{\alpha_i}(X)$ and $\tau_i \in \mathtt{h}^{\alpha_i}(X)$ provided by Proposition 3.1 and Lemma 2.3. We define the filtration $\mathtt{h}^{(l)}(X)$ as the Λ-linear span

$$\mathtt{h}^{(l)}(X) = \bigoplus_{N-\alpha_i \geqslant l} \Lambda\zeta_i = \bigoplus_{\alpha_i \geqslant l} \Lambda\tau_i.$$

We denote $\mathtt{h}^{(l/l+1)}(X) = \mathtt{h}^{(l)}(X)/\,\mathtt{h}^{(l+1)}(X)$ and $Gr^*\,\mathtt{h}(X) = \bigoplus_l \mathtt{h}^{(l/l+1)}(X)$ to be the associated graded group. Lemma 3.4 implies that the latter is a graded ring.

3.3. **Remark.** In the case when the theory $\mathtt{h}$ is generically constant and satisfies the localization property, the filtration introduced above coincides with the topological filtration on $\mathtt{h}(X)$, i.e. with the filtration where the l-th term is generated over Λ by classes $[Z \to X]$ of projective morphisms $Z \to X$ birational on its image and $\dim X - \dim Z \leqslant l$. This fact follows from the generalized degree formula [11, Thm. 4.4.7].

3.4. **Lemma.** $\mathtt{h}^{(l_1)}(X) \cdot \mathtt{h}^{(l_2)}(X) \subseteq \mathtt{h}^{(l_1+l_2)}(X)$.

Proof. We have $\tau_i^\Omega \tau_j^\Omega = \sum_l a_l \zeta_l^\Omega$ in $\Omega(X)$ for some $a_l \in \mathbb{L}$. Then $\alpha_i + \alpha_j = \deg(a_l) + \alpha_l$. Since $\deg(a_l) \leqslant 0$, $\alpha_l \geqslant \alpha_i + \alpha_j \geqslant l_1 + l_2$ for any nontrivial a_l. Since $\zeta_i^{\mathtt{h}} = \nu(\zeta_i^\Omega \otimes 1)$ we have $\zeta_i^{\mathtt{h}}\zeta_j^{\mathtt{h}} = \sum_l (a_l \otimes 1)\zeta_l^{\mathtt{h}}$ with $\alpha_l \geqslant \alpha_i + \alpha_j \geqslant l_1 + l_2$. So $\zeta_i^{\mathtt{h}}\zeta_j^{\mathtt{h}} \in \mathtt{h}^{(l_1+l_2)}(X)$. $\square$

3.5. **Proposition.** *For a cellular X there is a graded ring isomorphism:*

$$\Psi\colon \bigoplus_{i=0}^{N} \mathtt{h}^{(i/i+1)}(X) \to \mathrm{CH}(X, \Lambda).$$

Proof. By Proposition 3.1 it is sufficient to prove the statement for $\mathtt{h} = \Omega$. Observe that $\Omega^{(l/l+1)}(X)$ is a free $\mathbb{L}$-module with the basis $\tau_i^\Omega + \mathtt{h}^{(l+1)}(X)$ with $\alpha_i = l$ and $\mathrm{CH}^i(X, \mathbb{L})$ is a free $\mathbb{L}$-module with basis τ_i^{CH} with $\alpha_i = l$. Thus the $\mathbb{L}$-module homomorphism Ψ_l defined by

$$\Psi_l(\tau_i^\Omega + \mathtt{h}^{(i+1)}(X)) = \tau_i^{\mathrm{CH}}$$

is an isomorphism.

Let us check that $\Psi = \bigoplus \Psi_l$ preserves multiplication. For any i, j we have

$$\tau_i^\Omega \tau_j^\Omega = \sum_m a_m \tau_m^\Omega \qquad (*)$$

for some $a_m \in \mathbb{L}$. Then for any m we have $\deg(a_m) + \alpha_m = \alpha_i + \alpha_j$. Then in $\mathtt{h}^{(\alpha_i+\alpha_j/\alpha_i+\alpha_j+1)}$ we have

$$\tau_i^\Omega \tau_j^\Omega = \sum_{\alpha_m = \alpha_i + \alpha_j} a_m \tau_m^\Omega \text{ modulo } + \mathtt{h}^{(\alpha_i+\alpha_j+1)}(X)$$

Observe that $\mathbb{L}^0 = \mathbb{Z}$ and for all $a_m \in \mathbb{L}$ such that $\deg(a_m) < 0$ we have that $a_m \otimes 1_{\mathbb{Z}} = 0$ in $\mathbb{Z}$. Thus tensoring $(*)$ with $1_{\mathbb{Z}}$ we get

$$\tau_i^{\mathrm{CH}} \tau_j^{\mathrm{CH}} = \sum_{\alpha_m = 0} (a_m \otimes 1)\tau_m^{\mathrm{CH}}.$$

So $\Psi_{\alpha_i+\alpha_j}(\tau_i^\Omega + \mathtt{h}^{(\alpha_i+1)}(X)\cdot\tau_j^\Omega + \mathtt{h}^{(\alpha_j+1)}(X)) = \tau_i^{\mathrm{CH}}\cdot\tau_j^{\mathrm{CH}}$. Hence, Ψ is a graded ring isomorphism. $\square$

3.6. **Lemma.** $\Psi(\zeta_i^{\mathtt{h}} + \mathtt{h}^{(\alpha_i+1)}(X)) = \zeta_i^{\mathrm{CH}}$.

Proof. It is sufficient to show the statement for $\mathtt{h} = \Omega^*$. Consider the expansion $\zeta_i^\Omega = \sum a_j\tau_j^\Omega$ for some $a_j \in \mathbb{L}$ with $\deg a_j + \alpha_j = N - \alpha_i$. Since $\deg a_j \leqslant 0$ we have

$$\zeta_i^\Omega = \sum_{\deg a_j = 0} a_j\tau_j^\Omega \mod \Omega^{(N-\alpha_i+1)}(X).$$

Therefore, $\Psi(\zeta_i^\Omega + \Omega^{(N-|w|+1)}(X)) = \zeta_i^{\mathrm{CH}}$. $\square$

4. Oriented cohomology of a group

In the present section, using the filtration introduced in 3.2 we compute algebraic cobordism for some groups of small ranks and for PGL_n, $n \geq 2$. We also construct nontrivial elements in the difference between the topological and the γ-filtration on $K_0(G/B)$.

In this section we assume that the associated to $\mathtt{h}$ weak Borel-Moore homology theory satisfies the localization property of [11, Definition 4.4.6]. Examples of such theories include $\mathtt{h}(-) = \Omega(-)\otimes\Lambda$, or any oriented cohomology theory in the sense of Panin-Smirnov [14].

Consider the variety $X = G/B$ where G is a split semisimple algebraic group. Let $\pi_{G/B}\colon G \to X$ be the quotient map. According to Example 2.1 X is cellular. For any $w \in W$ we fix a minimal decomposition $w = s_{i_1}\dots s_{i_m}$ into simple reflections. Denote the corresponding multiindex by $I_w = (i_1,\dots i_m)$ and consider the Bott-Samelson variety X_{I_w}/B [4, §11]. Then $p_{I_w}: X_{I_w}/B \to G/B$ is a desingularisation of the Schubert cell X_w. Take

$$\zeta_w = (p_{I_w})_{\mathtt{h}}(1) \in \mathtt{h}_{l(w)}(X) = Hom_{\mathcal{M}_{\mathtt{h}}}(\Lambda(l(w)), M^{\mathtt{h}}(X)$$

to be the embedding in the direct sum decomposition $\bigoplus_{w\in W}\Lambda(l(w)) \cong M^{\mathtt{h}}(G/B)$. So, with this choice of isomorphism $M^{\mathtt{h}}(G/B) \cong \bigoplus_{w\in W}\Lambda(l(w))$ the basis given by Lemma 2.3 coincides with the basis ζ_{I_w} constructed in [4, §13].

Let $\Lambda[[T^*]]_F$ be the formal group algebra introduced by Calmès-Petrov-Zainoulline in [4, §2], where T^* is the character lattice of T and F is the formal group law of the theory $\mathtt{h}$. There is the characteristic map $\mathfrak{c}_F\colon \Lambda[[T^*]]_F \to \mathtt{h}(G/B)$ such that $\mathfrak{c}_F(x_\lambda) = c_1^{\mathtt{h}}(\mathcal{L}(\lambda))$ for a character λ. By [7, Prop. 5.1] there is a short exact sequence

$$0 \to \mathfrak{c}(I_F) \to \mathtt{h}(G/B) \xrightarrow{\pi_{G/B}^{\mathtt{h}}} \mathtt{h}(G) \to 0, \tag{1}$$

where I_F denotes the ideal in $\Lambda[[T^*]]_F$ generated by x_α for $\alpha \in T^*$. By [4, Lem. 4.2] there is a graded algebras isomorphism $\psi\colon \bigoplus_{m=0}^\infty I_F^m/I_F^{m+1} \to S_\Lambda^*(T^*)$ where $S_\Lambda^*(T^*)$ denotes the symmetric algebra over T^*. Let F_a denote the additive formal group law. We will need the following

4.1. **Lemma.** *The following diagram commutes:*

$$\begin{tikzcd}
\bigoplus_{m=0}^{\infty} I_F^m/I_F^{m+1} \arrow[r, "Gr\mathfrak{c}_F"] \arrow[d, "\psi"] & \bigoplus_{m=0}^{\infty} \mathtt{h}^{(m/m+1)}(G/B) \arrow[d, "\Psi"] \\
S^*_\Lambda(T^*) \arrow[r, "\mathfrak{c}_{F_a}"] & \mathrm{CH}(G/B,\Lambda)
\end{tikzcd}$$

Proof. By definition, it is sufficient to prove that $\Psi(pr_1(c_1^{\mathtt{h}}(\mathcal{L}_\lambda))) = c_1^{\mathrm{CH}}(\mathcal{L}_\lambda)$ Consider the expansion $c_1^{\Omega}(\mathcal{L}_\lambda) = \sum r_w \tau_w^{\Omega}$ we have that $r_w \in \mathbb{L}^0 = \mathbb{Z}$ for $|w| = 1$. Then $c_1^{\mathtt{h}}(\mathcal{L}_\lambda) = \sum (r_w \otimes 1_\Lambda)\tau_w^{\mathtt{h}}$ and

$$\Psi(pr_1(c_1^{\mathtt{h}}(\mathcal{L}_\lambda))) = \sum_{|w|=1} r_w \tau_w^{\mathrm{CH}} = c_1^{\Omega}(\mathcal{L}_\lambda) \otimes 1_{\mathbb{Z}} = c_1^{\mathrm{CH}}(\mathcal{L}_\lambda). \qquad \square$$

4.2. **Lemma.** *For the additive group law the induced filtration satisfies*

$$\mathfrak{c}(I_{F_a})\mathrm{CH}(X,\Lambda) \cap \mathrm{CH}^i(X,\Lambda) = \sum \mathfrak{c}(x_\alpha)\mathrm{CH}^{i-1}(X,\Lambda).$$

Proof. Note that $\mathfrak{c}(I_a)\mathrm{CH}(X,\Lambda)$ is generated by the elements $\mathfrak{c}(x_\alpha) \in \mathrm{CH}^1(X,R)$. For any element $z = \sum \mathfrak{c}(x_\alpha)y_\alpha$ of $\mathfrak{c}(I_a)\mathrm{CH}(X,\Lambda)$ we have that z lies in the $\mathrm{CH}^i(X,\Lambda)$ if and only if $y_\alpha \in \mathrm{CH}^{i-1}(X,\Lambda)$. Therefore $\mathfrak{c}(I_a)\mathrm{CH}(X,\Lambda) \cap \mathrm{CH}^i(X,\Lambda) = \sum \mathfrak{c}(x_\alpha)\mathrm{CH}^{i-1}(X,\Lambda)$. $\square$

Let $\mathtt{h}^{(i)}(G)$ denote the image $\pi^{\mathtt{h}}_{G/B}(\mathtt{h}^{(i)}(G/B))$ and let $\mathtt{h}^{(i/i+1)}(G)$ denote the quotient $\mathtt{h}^{(i)}(G)/\,\mathtt{h}^{(i+1)}(G)$.

4.3. **Proposition.** *For every i there is an exact sequence:*

$$0 \to \frac{\mathfrak{c}(I)\,\mathtt{h}^{(i-1)}(X)}{\mathtt{h}^{(i+1)}(X)} \to \frac{(\mathfrak{c}(I)\,\mathtt{h}(X))\cap\mathtt{h}^{(i)}(X)}{\mathtt{h}^{(i+1)}(X)} \to \mathrm{CH}^i(G,\Lambda) \to \mathtt{h}^{(i/i+1)}(G) \to 0.$$

Proof. By [3, Prop. 2, §2.4] we obtain from (1) the short exact sequence:

$$0 \to \frac{(\mathfrak{c}(I)\,\mathtt{h}(X))\cap\mathtt{h}^{(i)}(X)}{\mathtt{h}^{(i+1)}(X)} \to \frac{\mathtt{h}^{(i)}(X)}{\mathtt{h}^{(i+1)}(X)} \to \mathtt{h}^{(i/i+1)}(G) \to 0.$$

By Lemma 4.2 applied to the case of additive formal group law, the above sequence turns into

$$0 \to \sum \mathfrak{c}(x_\alpha)\mathrm{CH}^{i-1}(X,R) \to \mathrm{CH}^i(X,R) \to \mathrm{CH}^i(G,R) \to 0.$$

Observe that for isomorphism $(\Psi^i)^{-1}$ we have

$$(\Psi^i)^{-1}\left(\sum \mathfrak{c}(x_\alpha)\mathrm{CH}^{i-1}(X,\Lambda)\right) = \left(\frac{\mathfrak{c}(I)\,\mathtt{h}^{(i-1)}(X)}{\mathtt{h}^{(i+1)(X)}}\right) \subseteq \left(\frac{\mathfrak{c}(I)\,\mathtt{h}^{(}X)\cap\mathtt{h}^{(i)}(X)}{\mathtt{h}^{(i+1)(X)}}\right) \qquad (*)$$

Then we get the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \sum \mathfrak{c}(x_\alpha)\mathrm{CH}^{i-1}(X,\Lambda) \arrow[r] \arrow[d, hook] & \mathrm{CH}^i(X,\Lambda) \arrow[r] \arrow[d, equal, "(\Psi^i)^{-1}"] & \mathrm{CH}^i(G,\Lambda) \arrow[r] & 0 \\
0 \arrow[r] & \frac{\mathfrak{c}(I)\,\mathtt{h}(X)\cap\mathtt{h}^{(i)}(X)}{\mathtt{h}^{(i+1)(X)}} \arrow[r] & \mathtt{h}^{(i/i+1)}(X) \arrow[r] & \mathtt{h}^{(i/i+1)}(G) \arrow[r] & 0
\end{tikzcd}$$

The latter sequence and $(*)$ gives rise to the exact sequence

$$0 \to \frac{\mathfrak{c}(I)\,\mathtt{h}^{(i-1)}(X)}{\mathtt{h}^{(i+1)}(X)} \to \frac{(\mathfrak{c}(I)\,\mathtt{h}(X))\cap\mathtt{h}^{(i)}(X)}{\mathtt{h}^{(i+1)}(X)} \to \mathrm{CH}^i(G,\Lambda) \to \mathtt{h}^{(i/i+1)}(G) \to 0. \qquad \square$$

4.4. **Corollary.** *Assume that pullbacks* $p^{\mathrm{CH}}_{G/B}(X_{w_i})$ *of Schubert cells generate* $\mathrm{CH}(G)$ *as* $\mathbb{Z}$*-algebra for some elements* $w_1, \dots w_m$ *in* W*. Then the pullbacks of Schubert cells* $p^{\mathtt{h}}_{G/B}(\zeta_{w_i})$ *generate* $\mathtt{h}(G)$ *as* Λ*-algebra.*

Proof. By 4.3 classes of $p^{\mathtt{h}}_{G/B}(\zeta_{w_i})$ generate the associated graded ring $\bigoplus_i \mathtt{h}^{(i/i+1)}(G)$. Then $p^{\mathtt{h}}_{G/B}(\zeta_{w_i})$ generate $\mathtt{h}(G)$, since the filtration is finite. □

The following observations will be useful for computations

4.5. **Lemma.** *If* $\mathrm{CH}^i(G) = 0$ *then* $\frac{\mathfrak{c}(I)\,\mathtt{h}^{(i-1)}(G/B)}{\mathtt{h}^{(i+1)}(G/B)} = \frac{\mathtt{h}^{(i)}(G/B)}{\mathtt{h}^{(i+1)}(G/B)}$

4.6. **Lemma.** *Assume that* $\mathrm{CH}^1(G) = 0$*. Then for* $i \geqslant 2$

$$\mathfrak{c}(I)\,\mathtt{h}(G/B) \cap \mathtt{h}^{(i)}(G/B) = \mathfrak{c}(I)\,\mathtt{h}^{(1)}(G/B) \cap \mathtt{h}^{(i)}(G/B)$$

Proof. Note that ideal $\mathfrak{c}(I)\,\mathtt{h}(G/B)$ is generated by $\mathfrak{c}(x_\alpha)$ where α runs over the basis of the character lattice. □

5. Examples of computations

The results of the previous section allow us to obtain some information concerning the ring $\mathtt{h}(G)$ from $\mathrm{CH}(G)$. Moreover, in some cases it allows us to compute $\mathtt{h}(G)$.

We follow the notation of the previous section. We denote $\bigoplus \mathtt{h}^{(i/i+1)}(G)$ by $Gr^*\,\mathtt{h}(G)$. For $a \in \mathtt{h}^i(G)$ let $\overline{a} \in \mathtt{h}^{(i/i+1)}(G)$ denote its residue class. Let α be the projection $\mathrm{CH}^*(G, \Lambda) \to \oplus Gr^*\,\mathtt{h}(G)$.

Algebraic cobordism of G_2*.* According to [12] we have

$$\mathrm{CH}^*(G_2, \mathbb{Z}) = \mathbb{Z}[x_3]\big/(x_3^2, 2x_3)$$

where $x_3 = \pi^{\mathrm{CH}}(\zeta_{212})$, $\pi\colon G \to G/B$ is the projection and ζ_{212} is the Schubert cell corresponding to the word $w = s_2 s_1 s_2$. Let y_3 denote the pullback $\pi^{\Omega}(\zeta^{\Omega}_{212})$ of the corresponding Schubert cell in the ring $\Omega(G/B)$ (see Theorem 13.12 of [4]). Observe that $\alpha^3(x_3) = \overline{y_3}$. Since $\mathrm{CH}(G_2, \mathbb{L})$ is generated by 1 and x_3, $Gr^*\Omega(G_2)$ is generated by 1 and $\overline{y_3}$. Then by[3, §2.8] $\Omega(G_2)$ is generated by 1 and $y_3 \in \Omega^{(3)}(G_2)$. Since $2x_3 = 0$, then $\overline{2y_3}$ so $2y_3 \in \Omega^{(4)}(G_2)$ which is zero since $\mathrm{CH}^i(G_2) = 0$ for $i \geqslant 4$. Thus $2y_3 = 0$ and $y_3^2 \in \Omega^{(6)}(G_2) = 0$.

Let us now compute $\Omega^{(3)}(G_2)$. Proposition 4.3 gives us the exact sequence

$$0 \to \frac{\mathfrak{c}(I)\Omega^{(2)}(G_2/B)}{\Omega^{(4)}(G_2/B)} \to \frac{\mathfrak{c}(I)\Omega(G_2/B)\cap\Omega^{(3)}(G_2/B)}{\Omega^{(4)}(G_2/B)} \to \mathbb{L}/2 \cdot x_3 \to \Omega^{(3)}(G_2).$$

Note that since $\mathfrak{c}(I)\Omega(G/B)$ is generated by $\mathfrak{c}(x_1)$ and $\mathfrak{c}(x_2)$. By Lemma 4.5 we have

$$\frac{\mathfrak{c}(I)\Omega(G_2/B)}{\Omega^{(4)}(G_2/B)} = \frac{\langle\zeta_{12121}, \zeta_{21212}\rangle\Omega(G_2/B)}{\Omega^{(4)}(G_2/B)}$$

Then $\mathfrak{c}(I) \cap \Omega^{(3)}(G_2/B)/\Omega^{(4)}(G_2/B)$ equals to the set

$$\{x = a\zeta_{12121} + b\zeta_{21212} \mid x \in \Omega^{(3)}\}.$$

It is enough to consider only a, b in $\Omega^{(1)}(G_2/B) \setminus \Omega^{(2)}(G_2/B)$ since for $a, b \in \Omega^{(0)}(G_2/B) \setminus \Omega^{(1)}(G_2/B)$ we have $x \notin \Omega^{(2)}(G_2/B)$. So we consider

$$a = r_1\zeta_{12121} + r_2\zeta_{21212} \text{ and } b = s_1\zeta_{12121} + s_2\zeta_{21212} \text{ for } r_1, r_2, s_1, s_2 \in \mathbb{L}.$$

Using the multiplication table for G_2/B from [4] we obtain that x equals

$$(r_2+s_1+s_2)\zeta_{1212}+(3r_1+r_2+s_1)\zeta_{2121}+(r_2+s_1+3r_1)a_1\zeta_{121}+(r_2+s_1)a_1\zeta_{212}$$

modulo $\Omega^{(4)}(G_2/B)$. Then $x\in\Omega^{(3)}(G_2/B)/\Omega^{(4)}(G_2/B)$ iff $r_2+s_1+s_2=0$ and $3r_1+r_2+s_1=0$. Therefore, $r_2+s_1=-3r_1$ and $s_2=3r_1$. So

$$x+\Omega^{(4)}(G_2/B)=-3r_1a_1\zeta_{212}+\Omega^{(4)}(G_2/B).$$

Hence, the kernel of $\mathbb{L}\cdot 2x_3\to\Omega^{(3)}(G_2)$ is generated by $3a_1x_3$. Then

$$\Omega^{(3)}(G_2)=\mathbb{L}/(2,3a_1)\cdot y_3=\mathbb{L}/(2,a_1)\cdot y_3$$

and we obtain that

$$\Omega(G_2)=\mathbb{L}[y_3]\big/_{(y_3^2,\,2y_3,\,a_1y_3)}. \tag{2}$$

Observe that taking the latter equality modulo 2 we obtain the result established by Yagita in [19].

Algebraic cobordism of groups SO_n, $Spin_m$ for $n=3,4$ and $m=3,4,5,6$. According to [12]

$$\mathrm{CH}(Spin_i)=\mathbb{Z}\text{ for } i=3,4,5,6.$$

Then by Proposition 4.3 we obtain

$$\Omega(Spin_i)=\mathbb{L}\text{ for } i=3,4,5,6. \tag{3}$$

We have $\mathrm{CH}(SO_3)=\mathbb{Z}[x_1]/(2x_1,x_1^2)$ where $x_1=\pi^{\mathrm{CH}}(\zeta_{w_0s_1})$. Since $CH^i(SO_3)=0$ for $i\geqslant 2$, $\Omega^{(2)}(SO_3)=0$. For $i=1$ two left terms of exact sequence of Proposition 4.3 coincide, so there is an isomorphism $\mathrm{CH}^1(SO_3)\to\Omega^{(1)}(SO_3)$. Hence, we obtain

$$\Omega(SO_3)=\mathbb{L}[y_1]\big/_{(2y_1,\,y_1^2)},\quad\text{where } y_1=\pi^{\Omega}(\zeta_{w_0s_1}). \tag{4}$$

Since $\mathrm{CH}(SO_4)=\mathbb{Z}[x_1]/(2x_1,x_1^2)$ the same reasoning proves that

$$\Omega(SO_4)=\mathbb{L}[y_1]\big/_{(2y_1,\,y_1^2)}. \tag{5}$$

Oriented cohomology of PGL_n.

5.1. **Lemma.** *For any oriented cohomology theory $\mathtt{h}$ with the coefficient ring Λ and the formal group law F we have*

$$\mathtt{h}(PGL_n)=\Lambda[x]\big/_{(x^n,\,nx^{n-1},\ldots\binom{n}{d}x^d,\ldots,nx,\ n\cdot_F x)}.$$

Proof. Consider the variety of complete flags $X=SL_n/B$. Let F_i denote the tautological vector bundle of dimension i over X. Let $L_1=F_1$ and $L_i=F_i/F_{i-1}$ for $i=2,\ldots n$. Then, by [9, Thm. 2.6] we have

$$\mathtt{h}(X)\cong\Lambda[x_1,\ldots x_n]/S(x_1,\ldots x_n) \tag{$*$}$$

where $S(x_1,\ldots x_n)$ denotes the ideal generated by positive degree symmetric polynomials in variables $x_1,\ldots x_n$, and the isomorphism sends x_i to the Chern class $c_1^{\mathtt{h}}(L_i)$. The maximal split torus $T\subseteq SL_n$ consists of diagonal matrices with trivial determinant. Let $\chi_i\in\hat{T}$ denote the character that sends the diagonal matrix to its i-th entry. So, the character lattice equals to $M=\mathbb{Z}\chi_1\oplus\ldots\oplus\mathbb{Z}\chi_n/(\chi_1+\ldots+\chi_n)$. Observe that L_i coincides with the line bundle $\mathcal{L}(\chi_i)$, so by definition we have that $x_i=\mathfrak{c}(\chi_i)$, where $\mathfrak{c}\colon\Lambda[[M]]_F\to\mathtt{h}(X)$ is the characteristic map. Note that the roots of $PGL_n=PSL_n/\mu_n$ are equal $n\chi_1,\chi_2-\chi_1,\ldots,\chi_n-\chi_1$.

According to [7, 5.1] we have

$$\mathtt{h}(PGL_n) = \mathtt{h}(X)/(\mathfrak{c}(n\chi_1), \mathfrak{c}(\chi_2 - \chi_1), \dots, \mathfrak{c}(\chi_n - \chi_1)).$$

Then in the quotient we have

$$\overline{\mathfrak{c}(\chi_i)} = \overline{\mathfrak{c}(\chi_1 + \chi_i - \chi_1)} = \overline{\mathfrak{c}(\chi_1) +_F \mathfrak{c}(\chi_i - \chi_1)} = \overline{\mathfrak{c}(\chi_1)}.$$

Taking $x = \overline{\mathfrak{c}(\chi_1)}$ by $(*)$ we get

$$\mathtt{h}(PGL_n) = \Lambda[x]/(S(x, \dots, x), n \cdot_F x).$$

According to [9] $S(x_1, \dots x_n)$ is generated by polynomials $f_n(x_n)$, $f_{n-1}(x_n, x_{n-1})$, …, $f_1(x_n, \dots, x_i)$ where $f_i(x_n, \dots, x_i)$ denotes the sum of all degree i monomials in $x_n, \dots, x_i$. Note that $\binom{n}{d}$ equals to the number of degree d monomials in $n - d + 1$ variables. Then substituting $x_1 = \dots x_n = x$ we obtain that $x^n, nx^{n-1}, \binom{n}{d}x^{n-1}, \dots, nx$ generate the ideal $S(x, \dots, x)$. $\square$

5.2. **Example.** For a prime number p and $0 < d < p$ the coefficient $\binom{p}{d}$ is divisible by p. By [8, Rem. 5.4.8] over $\Lambda/p\Lambda$ we have $p \cdot_F x = p\beta_0(x) + \beta_1(x^p)$. Thus, the ideal $I = (x^p, px^{p-1}, \dots \binom{p}{d}x^d, \dots, px,\ p \cdot_F x)$ is generated by x^p, px. So for any prime p we have

$$\mathtt{h}(PGL_p) = \Lambda[x]\big/(px, x^p).$$

In the case $\mathtt{h} = K_0$ this agrees with [20, 3.6].

Topological and the γ-filtration. Proposition 4.3 allows to estimate the difference between the topological and the Grothendieck γ-filtration on $K_0(G/B)$ for a split linear algebraic group G. Namely, consider two filtrations on $K_0(G/B)$:

γ-filtration: $\gamma^i(G/B) = \langle c_1(\mathcal{L}(\lambda)) \mid \lambda \in T^* \rangle$ [20, Definition 4.2],
topological filtration: $\tau^i(G/B) = \langle [\mathcal{O}_V] \mid \text{codim } (V) \geqslant i \rangle$.

5.3. **Proposition.** *Let G be a split semisimple simply connected linear algebraic group such that $\mathrm{CH}^i(G) = 0$ for $1 \leqslant i \leqslant n - 1$ and $\mathrm{CH}^n(G) \neq 0$. Let ζ_w be a Schubert cell such that $\pi^{\mathrm{CH}}(\zeta)$ is nontrivial in $\mathrm{CH}^n(G)$.*

Then $\gamma^i(G/B) + \tau^{i+1}(G/B) = \tau^i(G/B)$ for $i < n$ and the class of $\zeta_w^{K_0}$ is nontrivial in $\tau^n(G/B)/\gamma^n(G/B)$.

Proof. As shown in [15] $K_0(G) = \mathbb{Z}$ for a simply connected group G. Then characteristic map $\mathfrak{c}$ is surjective [6, §1B]. We have $K_0^{(1)}(G/B) = \tau^1 = \gamma^1$. Note that $K_0^{(i)}(G/B) = \tau^i$. Then $\gamma^1\tau^0 \cap \tau^i = \tau^i$ and the Proposition 4.3 gives us a short exact sequence for all $i \geqslant 1$:

$$0 \to \tfrac{\gamma^1\tau^{i-1}}{\tau^{i+1}} \to \tfrac{\tau^i}{\tau^{i+1}} \to \mathrm{CH}^i(G, \mathbb{Z}[\beta, \beta^{-1}]) \to 0.$$

Then for any $1 \leqslant i < n$ we have $\gamma^1\tau^{i-1}\big/\tau^{i+1} = \tau^i\big/\tau^{i+1}$. By induction we get $\tau^i = \gamma^i + \tau^{i+1}$ for $i < n$ and for $i = n$ we get By induction we get $\tau^i = \gamma^i + \tau^{i+1}$ for $i < n$ and for $i = n$ we get

$$0 \to \tfrac{\gamma^n}{\tau^{n+1}} \to \tfrac{\tau^n}{\tau^{n+1}} \to \mathrm{CH}^n(G, \mathbb{Z}[\beta, \beta^{-1}]) \to 0.$$

So for any nontrivial element of of $\mathrm{CH}^n(G)$ the class of its preimage is nontrivial in τ^n/γ^n. $\square$

6. Applications to $\mathtt{h}$-motivic decompositions

Throughout this section we consider a generically cellular variety X of dimension N and an oriented cohomology theory $\mathtt{h}^*$ that is generically constant and is associated with weak Borel-Moore homology $\mathtt{h}_*$ which satisfies the localization property. These assumptions imply that the generalized degree formula of Levine-Morel [11, Theorem 4.4.7] holds. The aim of this section is to prove theorems A, B and C of the introduction which provide a comparison between the Chow motive $M(X)$ and the $\mathtt{h}$-motive $M^{\mathtt{h}}(X)$ of X.

Let L be the splitting field of X and $\overline{X} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$. Let p denote the projection $p\colon \overline{X} \times \overline{X} \to X \times X$. Since $\overline{X}$ is cellular, we may consider a filtration on $\mathtt{h}(\overline{X})$ introduced in 3.2. It gives rise to a filtration on $\mathtt{h}(\overline{X} \times \overline{X}) = \mathtt{h}(\overline{X}) \otimes_\Lambda \mathtt{h}(\overline{X})$. Namely, we set

$$\mathtt{h}^{(l)}(\overline{X} \times \overline{X}) = \sum_{i+j=l} \mathtt{h}^{(i)}(\overline{X}) \otimes_\Lambda \mathtt{h}^{(j)}(\overline{X}).$$

On $\mathtt{h}(X \times X)$ we consider the induced filtration

$$\mathtt{h}^{(l)}(X \times X) = (p^{\mathtt{h}})^{-1}(\mathtt{h}^{(l)}(\overline{X} \times \overline{X})).$$

Denote the quotient $\mathtt{h}^{(l)}(\overline{X} \times \overline{X})/\,\mathtt{h}^{(l+1)}(\overline{X} \times \overline{X})$ by $\mathtt{h}^{(l/l+1)}(\overline{X} \times \overline{X})$ and denote by $pr_l\colon\, \mathtt{h}^{(l)}(\overline{X} \times \overline{X}) \to \mathtt{h}^{(l/l+1)}(\overline{X} \times \overline{X})$ the usual projection. Denote

$$\mathtt{h}^{(i)}_{2N-i}(X \times X) = \mathtt{h}^{(i)}(X \times X) \cap \mathtt{h}_{2N-i}(X \times X) \text{ and}$$
$$\mathtt{h}^{(i)}_{2N-i}(\overline{X} \times \overline{X}) = \mathtt{h}^{(i)}(\overline{X} \times \overline{X}) \cap \mathtt{h}_{2N-i}(\overline{X} \times \overline{X}).$$

6.1. **Lemma.** *There is a graded ring isomorphism*

$$\Phi\colon \bigoplus_{i=0}^{2N} \mathtt{h}^{(i/i+1)}(\overline{X} \times \overline{X}) \to \mathrm{CH}^*(\overline{X} \times \overline{X}, \Lambda).$$

Proof. Since

$$\bigoplus_{i=0}^{2N} \mathtt{h}^{(i/i+1)}(\overline{X} \times \overline{X}) = \bigoplus_{i=0}^{N} \mathtt{h}^{(i/i+1)}(\overline{X}) \otimes_\Lambda \bigoplus_{i=0}^{N} \mathtt{h}^{(i/i+1)}(\overline{X})$$

and $\mathrm{CH}(\overline{X} \times \overline{X}, \Lambda) = \mathrm{CH}(\overline{X}, \Lambda) \otimes_\Lambda \mathrm{CH}(\overline{X}, \Lambda)$ take $\Phi = \Psi \otimes \Psi$, where Ψ is defined in 3.5. $\square$

6.2. **Remark.** The restriction of Φ_i gives an isomorphism $\Phi_i\colon\, \mathtt{h}^{(i/i+1)}_{2N-i}(\overline{X} \times \overline{X}) \to \mathrm{CH}^i(\overline{X} \times \overline{X}, \Lambda^0)$.

The following lemma provides an $\mathtt{h}$-version of the Rost Nilpotence Theorem:

6.3. **Lemma.** *The kernel of the pullback map $p^{\mathtt{h}}\colon End(M^{\mathtt{h}}(X)) \to End(M^{\mathtt{h}}(\overline{X}))$ consists of nilpotents.*

Proof. Consider a diagram

$$\begin{tikzcd} End(M^{\Omega}(X)) \arrow[r, "p^{\Omega}"] \arrow[d] & End(M^{\Omega}(\overline{X})) \arrow[d] \\ End(M^{\mathrm{CH}}(X)) \arrow[r] & End(M^{\mathrm{CH}}(\overline{X})) \end{tikzcd}$$

where vertical arrows are ring homomorphisms that arise from the canonical map $\Omega(-) \to \mathrm{CH}(-)$. By [17, Prop. 2.7] they are surjective with kernels consisting of nilpotents. The kernel of the bottom arrow consists of nilpotents by [18, Prop 3.1]. Then the kernel of the upper arrow consists of nilpotents as well.

Tensoring the upper arrow with Λ we obtain $\ker(p^\Omega)\otimes\Lambda \to \Omega_N(X\times X)\otimes\Lambda \xrightarrow{p^\Omega\otimes id} \Omega(\overline{X}\times\overline{X})\otimes\Lambda$, so $\ker(p^\Omega)\otimes\Lambda$ covers the kernel of $p^\Omega\otimes id$, thus $\ker(p^\Omega\otimes id)$ consists of nilpotents. Now the specialization maps fit into the commutative diagram.

$$\begin{tikzcd}
\Omega_N(X\times X)\otimes_{\mathbb{L}}\Lambda \arrow[r, "p^\Omega\otimes id"] \arrow[d, "\nu_{X\times X}"] & \Omega(\overline{X}\times\overline{X})\otimes_{\mathbb{L}}\Lambda \arrow[d, "\cong"] \\
\mathtt{h}(X\times X) \arrow[r, "p^{\mathtt{h}}"] & \mathtt{h}(\overline{X}\times\overline{X})
\end{tikzcd}$$

where the right arrow is an isomorphism by 2.4 and 3.1, and the map $\nu_{X\times X}$ is surjective. So the kernel of the bottom map consists of nilpotents. ☐

6.4. **Lemma.** *We have* $\mathtt{h}^{(N+i)}(\overline{X}\times\overline{X})\circ\mathtt{h}^{(N+j)}(\overline{X}\times\overline{X})\subseteq\mathtt{h}^{(N+i+j)}(\overline{X}\times\overline{X})$.

Proof. Consider a generator $\zeta_m\otimes\tau_n\in\mathtt{h}^{(N+i)}(\overline{X}\times\overline{X})$ where $N-\alpha_m+\alpha_n\geqslant N+i$ and $\zeta_{m'}\otimes\tau_{n'}\in\mathtt{h}^{(N+j)}(\overline{X}\times\overline{X})$ where $N-\alpha_{m'}+\alpha_{n'}\geqslant N+j$. The composition

$$(\zeta_m\otimes\tau_n)\circ(\zeta_{m'}\otimes\tau_{n'})=\deg(\tau_n\zeta_{m'})(\zeta_m\otimes\tau_{n'})=\delta_{n,m'}\cdot(\zeta_m\otimes\tau_{n'})$$

is nonzero iff $n=m'$. In this case $N-m+n'=(N-m+n)+(N-m'+n')-N\geqslant N+i+j$. Thus $\zeta_m\otimes\tau_{n''}$ lies in $\mathtt{h}^{(N+i+j)}(\overline{X}\times\overline{X})$. ☐

6.5. **Remark.** Indeed, the lemma implies that $\mathtt{h}^{(N)}(\overline{X}\times\overline{X})$ is a ring with respect to the composition product, and $\mathtt{h}^{(N+1)}(\overline{X}\times\overline{X})$ is its two-sided ideal. Since the composition of homogeneous elements is homogeneous, $\mathtt{h}^{(N)}_N(\overline{X}\times\overline{X})$ is also a ring with respect to the composition.

6.6. **Lemma.** *The isomorphism* $\Phi_N\colon \mathtt{h}^{(N/N+1)}(\overline{X}\times\overline{X})\to\mathrm{CH}^N(\overline{X}\times\overline{X},\Lambda)$ *is a ring homomorphism with respect to the composition product.*

Proof. This immediately follows from the fact that Φ maps residue classes of $\zeta_w^{\mathtt{h}}\otimes\tau_v^{\mathtt{h}}$ to $\zeta_w^{\mathrm{CH}}\otimes\tau_v^{\mathrm{CH}}$. ☐

6.7. **Lemma.** *Let Y be a twisted form of X, i.e. $Y_L\cong X_L=\overline{X}$. For every codimension m consider the diagram, where $p\colon \overline{X}\times\overline{X}\to X\times Y$ denotes the projection.*

$$\begin{tikzcd}
\mathtt{h}^{(m)}_{2N-m}(X\times Y) \arrow[r, "pr_m\circ p^{\mathtt{h}}"] & \mathtt{h}^{(m/m+1)}_{2N-m}(\overline{X}\times\overline{X}) \\
\mathrm{CH}^m(X\times Y,\Lambda^0) \arrow[r, "p^{\mathrm{CH}}"] & \mathrm{CH}^m(\overline{X}\times\overline{X},\Lambda^0) \arrow[u, "\Phi^m"']
\end{tikzcd}$$

Then $\mathrm{im}(\Phi^m\circ p^{\mathrm{CH}})\subseteq\mathrm{im}\,pr_m\circ p^{\mathtt{h}}$.

Proof. Note that $\mathrm{CH}^m(X\times Y,\Lambda^0)$ is generated over Λ^0 by classes $i_{\mathrm{CH}}(1)$ where $i\colon\widetilde{Z}\to Z\hookrightarrow X\times Y$, Z is a closed integral subscheme of codimension m, $\widetilde{Z}\in\mathbf{Sm}_k$

and $\widetilde{Z} \to Z$ is projective birational. Consider the Cartesian diagram

$$\begin{tikzcd} \widetilde{Z} \arrow[d, "i"] & \widetilde{Z}_L \arrow[l, "q"'] \arrow[d, "j"] \\ X \times_k Y & \overline{X} \times_L \overline{X} \arrow[l, "p"'] \end{tikzcd}$$

Since this diagram is transverse, then

$$j_{\mathtt{h}} \circ q^{\mathtt{h}} = p^{\mathtt{h}} \circ i_{\mathtt{h}} \text{ and } j_{\mathrm{CH}} \circ q^{\mathrm{CH}} = p^{\mathrm{CH}} \circ i_{\mathrm{CH}}.$$

By lemma 6.8 we have $\Phi^m \circ j_{\mathrm{CH}}(1) = pr_m(j_{\mathtt{h}}(1))$. Then $\Phi^m \circ p^{\mathrm{CH}}(i_{\mathrm{CH}}(1)) = \Phi^m \circ j_{\mathrm{CH}}(1) = pr_m(j_{\mathtt{h}}(1)) = pr_m(p^{\mathtt{h}} \circ i_{\mathtt{h}}(1)) \in \operatorname{im} pr_m \circ p^{\mathtt{h}}_m$. □

6.8. **Lemma.** *Consider a morphism $j\colon \widetilde{Z} \to \overline{X} \times_L \overline{X}$, where $\widetilde{Z}$ is a smooth irreducible scheme and j is projective of relative dimension $-m$. It induces two pushforward maps $j_{\mathtt{h}}\colon \mathtt{h}(\widetilde{Z}) \to \mathtt{h}(\overline{X} \times \overline{X})$ and $j_{\mathrm{CH}}\colon \mathrm{CH}(\widetilde{Z}, \Lambda) \to \mathrm{CH}(\overline{X} \times \overline{X}, \Lambda)$.*

Then $j_{\mathtt{h}}(1) \in \mathtt{h}^{(m)}_{2N-m}(\overline{X} \times \overline{X})$ and $\Phi^m(j_{\mathrm{CH}}(1)) = pr_m(j_{\mathtt{h}}(1))$.

Proof. Observe that

$$j_{\mathtt{h}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_\Lambda \text{ and } j_{\mathrm{CH}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_{\mathbb{Z}}.$$

Expanding in the basis we obtain

$$j_\Omega(1) = \sum_{i_1,i_2} r_{i_1,i_2} \tau^\Omega_{i_1} \otimes \tau^\Omega_{i_2} \text{ for some } r_{i_1,i_2} \in \mathbb{L}. \qquad (*)$$

Since $j_\Omega(1)$ is homogeneous of degree m, we have

$$r_{i_1,i_2} \in \mathbb{L}^{m-\alpha_{i_1}-\alpha_{i_2}}. \qquad (**)$$

Then for every nonzero r_{i_1,i_2} we have $\alpha_{i_1}+\alpha_{i_2} \geqslant m$. So each $\tau^\Omega_{i_1}\otimes\tau^\Omega_{i_2} \in \Omega^{(m)}(\overline{X}\times\overline{X})$ and, thus, $j_\Omega(1) \in \Omega^{(m)}_{2N-m}(\overline{X} \times \overline{X})$. Taking $(*)$ modulo $\Omega^{(m+1)}(\overline{X} \times \overline{X})$ we obtain

$$j_\Omega(1) + \Omega^{(m+1)}(\overline{X} \times \overline{X}) = \sum_{\alpha_{i_1}+\alpha_{i_2}=m} r_{i_1,i_2} \tau^\Omega_{i_1} \otimes \tau^\Omega_{i_2} + \Omega^{(m+1)}(\overline{X} \times \overline{X}).$$

If $\alpha_{i_1} + \alpha_{i_2} = m$ then $r_{i_1,i_2} \in \mathbb{L}^0 = \mathbb{Z}$ by $(**)$. Thus taking $j_{\mathtt{h}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_\Lambda$ and $j_{\mathrm{CH}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_{\mathbb{Z}}$ we get

$$pr_m(j_{\mathtt{h}}(1)) = pr_m(\sum_{\alpha_{i_1}+\alpha_{i_2}=m} r_{i_1,i_2} \tau^{\mathtt{h}}_{i_1} \otimes \tau^{\mathtt{h}}_{i_2})$$

and

$$j_{\mathrm{CH}}(1) = j_\Omega(1) \otimes_{\mathbb{L}} 1_{\mathbb{Z}} \otimes_{\mathbb{Z}} 1_\Lambda = \sum_{\alpha_{i_1}+\alpha_{i_2}=m} r_{i_1,i_2} \tau^{\mathrm{CH}}_{i_1} \otimes \tau^{\mathrm{CH}}_{i_2}.$$

Then $\Phi^m(j_{\mathrm{CH}}(1)) = pr_m(j_{\mathtt{h}}(1))$, since $\Phi^m(\tau^{\mathrm{CH}}_{i_1} \otimes \tau^{\mathrm{CH}}_{i_2}) = pr_m(\tau^{\mathtt{h}}_{i_1} \otimes \tau^{\mathtt{h}}_{i_2})$. □

6.9. **Lemma.** *The kernel of the composition homomorphism*

$$pr_N \circ p^{\mathtt{h}}\colon \mathtt{h}^{(N)}_N(X \times X) \to \mathtt{h}^{(N)}_N(\overline{X} \times \overline{X}) \to \mathtt{h}^{(N/N+1)}_N(\overline{X} \times \overline{X})$$

consists of nilpotents.

Proof. This follows from Rost nilpotence and the fact that $\mathtt{h}^{(N+1)}(\overline{X} \times \overline{X})$ is nilpotent by Lemma 6.4. □

6.10. **Lemma.** *Let $\mathcal{C}$ be an additive category, $A, B \in Ob(\mathcal{C})$. Let $f \in Hom_{\mathcal{C}}(A, B)$ and $g \in Hom_{\mathcal{C}}(B, A)$such that $f \circ g - id_B$ is nilpotent in the ring $End_{\mathcal{C}}(B)$ and $g \circ f - id_A$ is nilpotent in the ring $End_{\mathcal{C}}(A)$. Then A is isomorphic to B.*

Proof. Denote $\alpha = id_A - gf$ and $\beta = id_B - fg$. Take natural n such that $\alpha^{n+1} = 0$ and $\beta^{n+1} = 0$. Then $gf = id_A - \alpha$ is invertible and $(gf)^{-1} = id_A + \alpha + \ldots + \alpha^n$. Analogously $(fg)^{-1} = id_B + \beta + \ldots + \beta^n$. So we have

$$gf(id_A + (id_A - gf) + \ldots + (id_A - gf)^n) = id_A$$

Since $(id_A - gf)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (gf)^i$, we have

$$g \sum_{m=0}^{n} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} (fg)^i f \right) = id_A \tag{$*$}$$

and

$$fg(id_B + (id_B - fg) + \ldots + (id_B - fg)^n) = id_B$$

implies

$$\sum_{m=0}^{n} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} f(gf)^i \right) g = id_B. \tag{$**$}$$

Then take

$$f_1 = \sum_{m=0}^{n} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} (fg)^i f \right) = \sum_{m=0}^{n} \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} f(gf)^i \right).$$

Then $(*)$ implies $gf_1 = id_A$ and $(**)$ implies $f_1 g = id_B$. So f_1 and g establish inverse isomorphisms between A and B. □

6.11. **Corollary.** *Suppose p_1 and p_2 are two idempotents in $End(M^{\mathtt{h}}(\overline{X}))$ such that $p_1 - p_2$ is nilpotent. Then the motives $(\overline{X}, p_1)$ and $(\overline{X}, p_2)$ are isomorphic.*

Proof. Take

$f = p_2 \circ p_1 \in Hom_{\mathcal{M}_{\mathtt{h}}}((\overline{X}, p_1), (\overline{X}, p_2))$ and $g = p_1 \circ p_2 \in Hom_{\mathcal{M}_{\mathtt{h}}}((\overline{X}, p_2), (\overline{X}, p_1))$.

Let us check that $f \circ g - id_{(X,p_2)} = p_2 p_1 p_2 - p_2 = p_2(p_1 - p_2)p_2$ is nilpotent.

It is sufficient to check that $(p_2(p_1 - p_2)p_2)^m = p_2(p_1 - p_2)^m p_2$ for any m. Note that if $x \in \ker p_2 \cap \operatorname{im} p_1$ then $(p_1 - p_2)(x) = x$. Since $p_1 - p_2$ is nilpotent, $x = 0$. Thus, $\ker p_2 \cap \operatorname{im} p_1 = 0$. Since p_2 is idempotent, $\operatorname{im} p_2 \cap \ker p_2 = 0$. Then endomorphism $p_1 - p_2$ of $M(\overline{X}) = \ker p_2 \oplus \operatorname{im} p_2$ can be represented as the matrix

$$p_1 - p_2 = \begin{pmatrix} E_1 & E_2 \\ 0 & 0 \end{pmatrix}$$

where E_1 is a homomorphism from $\operatorname{im} p_2$ to $\operatorname{im} p_2$ and E_2 is a homomorphism from $\ker p_2$ to $\operatorname{im} p_2$. We have

$$p_2(p_1 - p_2)^m p_2 = p_2 \circ \begin{pmatrix} E_1^m & E_1^{m-1} E_2 \\ 0 & 0 \end{pmatrix} \circ p_2 = \begin{pmatrix} E_1^m & 0 \\ 0 & 0 \end{pmatrix} = (p_2(p_1 - p_2)p_2)^m.$$

Then $f \circ g - id_{(X,p_2)} = p_2(p_1 - p_2)p_2$ is nilpotent. Symmetrically, $g \circ f - id_{(X,p_1)}$ is nilpotent. So (X, p_1) and (X, p_2) are isomorphic by Lemma 6.10. □

We are now ready to prove theorems A, B and C of the introduction:

Theorem A. *Suppose X is generically cellular. Assume that there is a decomposition of Chow motive with coefficients in Λ^0*

$$M^{\mathrm{CH}}(X,\Lambda^0)=\bigoplus_{i=0}^{n}\mathcal{R}(\alpha_i) \qquad (*)$$

such that over the splitting field L the motive $\mathcal{R}$ equals to the sum of twisted Tate motives: $\overline{\mathcal{R}}=\bigoplus_{j=0}^{m}\Lambda^0(\beta_j)$.

Then there is a $\mathtt{h}$-motive $\mathcal{R}_{\mathtt{h}}$ such that

$$M^{\mathtt{h}}(X)=\bigoplus_{i=0}^{n}\mathcal{R}_{\mathtt{h}}(\alpha_i)$$

such that over the splitting field $\mathcal{R}_{\mathtt{h}}$ splits into the $\mathtt{h}$-Tate motives $\overline{\mathcal{R}_{\mathtt{h}}}=\bigoplus_{j=0}^{m}\Lambda(\beta_j)$.

Proof. We may assume that $\alpha_0=0$ in $(*)$. Then each summand $\mathcal{R}(\alpha_i)$ equals to (X,p_i) for some idempotent p_i and there are mutually inverse isomorphisms ϕ_i and ψ_i of degree α_i between (X,p_0) and (X,p_i). So we have

- idempotents $p_i\in\mathrm{CH}^N(X\times X)$, $\sum p_i=\Delta^X_{\mathtt{h}}(1)$
- isomorphisms $\phi_i\in p_0\circ\mathrm{CH}^{N+\alpha_i}(X\times X)\circ p_i$ and $\psi_i\in p_i\circ\mathrm{CH}^{N-\alpha_i}(X\times X)\circ p_0$
- such that $\phi_i\circ\psi_i=p_0$ and $\psi_i\circ\phi_i=p_i$

Consider the diagram of Lemma 6.7

$$\begin{tikzcd}
\mathtt{h}^{(m)}_{2N-m}(X\times Y) \arrow[r, "pr_m\circ p^{\mathtt{h}}"] & \mathtt{h}^{(m/m+1)}_{2N-m}(\overline{X}\times\overline{X}) . \\
\mathrm{CH}^m(X\times Y,\Lambda^0) \arrow[r, "p^{\mathrm{CH}}"] & \mathrm{CH}^m(\overline{X}\times\overline{X},\Lambda^0) \arrow[u, "\Phi^m"']
\end{tikzcd}$$

By 6.7 the elements $\Phi^N\circ p^{\mathrm{CH}}(p_i)$ and $\Phi^{N+\alpha_i}\circ p^{\mathrm{CH}}(\phi_i)$ and $\Phi^{N-\alpha_i}\circ p^{\mathrm{CH}}(\psi_i)$ lie in $\operatorname{im} pr_N\circ p^{\mathtt{h}}$, $\operatorname{im} pr_{N-\alpha_i}\circ p^{\mathtt{h}}$ and $\operatorname{im} pr_{N+\alpha_i}\circ p^{\mathtt{h}}$ respectively.

By Lemma 6.9 the kernel of $pr_N\circ p^{\mathtt{h}}\colon \mathtt{h}^{(N)}_N(X\times X)\to\mathtt{h}^{(N/N+1)}_N(\overline{X}\times\overline{X})$ is nilpotent. Then by [1, Prop. 27.4] there is a decomposition r_i such that $pr_N\circ p^{\mathtt{h}}(r_i)=p_i$.

Let us construct the isomorphisms between r_i and r_0. Let ϕ'_i and ψ'_i be some preimages of $\Phi^{N+i}\circ p^{\mathrm{CH}}(\phi_i)$ and $\Phi^{N-i}\circ p^{\mathrm{CH}}(\psi_i)$. Then [16, Lem. 2.5] implies that there are elements $\phi''_i\in r_0\,\mathtt{h}^{(N)}_N(X\times X)r_i$ and $\psi''_i\in r_i\,\mathtt{h}^{(N)}_N(X\times X)r_0$, such that $\phi_{i,j}\psi_{i,j}=r_{0,1}$ and $\psi_{i,j}\phi_{i,j}=r_{i,j}$. So the $\mathtt{h}$-motives (X,r_i) and $(X,r_0)(\alpha_i)$ are isomorphic. Taking $\mathcal{R}_{\mathtt{h}}=(X,r_0)$ we have

$$M^{\mathtt{h}}(X)=\bigoplus_{i=0}^{n}(X,r_i)=\bigoplus_{i=0}^{n}(X,r_0)(\alpha_i)=\bigoplus_{i=0}^{n}\mathcal{R}_{\mathtt{h}}(\alpha_i).$$

Over the splitting field the motive $\overline{\mathcal{R}_{\mathtt{h}}}$ becomes isomorphic to $(\overline{X},p^{\mathtt{h}}(r_0))$ and $pr_N\circ p^{\mathtt{h}}(r_0)=\Phi^N(p^{\mathrm{CH}}(p_0))$. Since the Chow motive $(\overline{X},p^{\mathrm{CH}}(p_{0,1}))$ splits into $\bigoplus_j\Lambda^0(\beta_j)$ we have $p^{\mathrm{CH}}(p_{0,1})=\sum_j f_j\otimes g_j$ with $f_j\in\mathrm{CH}^{\alpha_j}(\overline{X})$, $g_j\in\mathrm{CH}_{\alpha_j}(\overline{X})$ and $\pi_{\mathrm{CH}}(f_jg_l)=\delta_{j,l}$. Take φ_j and γ_j to be the liftings of f_j and g_j in $\mathtt{h}^{(\alpha_j)}_{N-\alpha_j}(\overline{X})$ and $\mathtt{h}^{(N-\alpha_j)}_j(\overline{X})$ respectively.

Note that $\varphi_j\gamma_l+\mathtt{h}^{N+1}(\overline{X})=\Psi^N(f_jg_l)$. Since $\mathtt{h}^{(N+1)}(\overline{X})=0$, we have $\pi_{\mathtt{h}}(\varphi_j\gamma_l)=\pi_{\mathrm{CH}}(f_jg_l)=\delta_{j,l}$. Then the element $\sum\varphi_j\otimes\gamma_j$ is an idempotent in $Corr_0(\overline{X}\times\overline{X})$.

Since

$$pr_N(p^{\mathtt{h}}(r_0)) = \Phi^N(p^{\mathrm{CH}}(p_0)) = pr_N(\sum_j \varphi_j \otimes \gamma_j),$$

$p^{\mathtt{h}}(r_0) - \sum \varphi_j \otimes \gamma_j$ lies in $\mathtt{h}^{N+1}(\overline{X} \times \overline{X})$, so is nilpotent. Then by Corollary 6.11 we obtain

$$\overline{\mathcal{R}_{\mathtt{h}}} = (\overline{X}, p^{\mathtt{h}}(r_0)) \cong (\overline{X}, \sum_j \varphi_j \otimes \gamma_j) = \bigoplus \Lambda(\beta_j). \qquad \square$$

6.12. **Lemma.** *Assume that* $\Lambda^1 = \ldots \Lambda^N = 0$. *Then* $\mathtt{h}_N(\overline{X} \times \overline{X}) \subseteq \mathtt{h}^{(N)}(\overline{X} \times \overline{X})$ *and in the diagram of Lemma 6.7*

$$\begin{tikzcd}
\mathtt{h}_N^{(N)}(X \times Y) \arrow[r, "pr_N \circ p^{\mathtt{h}}"] & \mathtt{h}_N^{(N/N+1)}(\overline{X} \times \overline{X}) \\
\mathrm{CH}^N(X \times Y, \Lambda^0) \arrow[r, "p^{\mathrm{CH}}"] & \mathrm{CH}^N \overline{X} \times \overline{X}, \Lambda^0) \arrow[u, "\Phi^N"']
\end{tikzcd}$$

the inverse inclusion holds: $\operatorname{im} pr_N \circ p^{\mathtt{h}} \subseteq \operatorname{im} \Phi^N \circ p^{\mathrm{CH}}$.

Proof. By the degree formula [11, Thm 4.4.7] $\mathtt{h}(X \times X)$ is generated as Λ-module by pushforwards $i_{\mathtt{h}}(1)$, where $i\colon Z \to X \times X$ is projective, $Z \in Sm_k$ and $i\colon Z \to i(Z)$ is birational. Following [11] we will denote such classes by $[Z \to X \times X]_{\mathtt{h}}$. Then $\mathtt{h}_N(X \times X)$ is additively generated by elements $\lambda[Z \to X \times X]_{\mathtt{h}}$, where λ is homogeneous such that $\deg \lambda + \operatorname{codim} Z = N$. Since $\Lambda^1 = \ldots \Lambda^N = 0$, we have $\operatorname{codim} Z \geqslant N$. Then in $\Omega(\overline{X} \times \overline{X})$ we have

$$[Z_L \to \overline{X} \times \overline{X}]_\Omega = \sum \omega_{i,j} \zeta_i \otimes \tau_j \text{ for some } \omega_{i,j} \in \mathbb{L}.$$

Since all elements of the Lazard ring have negative degrees and $[Z_L \to \overline{X} \times \overline{X}]_\Omega$ has degree N, each $\zeta_i \otimes \tau_j$ in the expansion is contained in $\Omega^{(n)}(\overline{X} \times \overline{X})$. Then

$$[Z_L \to \overline{X} \times \overline{X}]_{\mathtt{h}} = \nu_{\overline{X} \times \overline{X}}[Z_L \to \overline{X} \times \overline{X}]_\Omega \in \mathtt{h}^{(N)}(\overline{X} \times \overline{X}) \text{ and}$$

$[Z \to X \times X]_{\mathtt{h}} \in \mathtt{h}^{(N)}(X \times X)$. By the same reasons $[Y \to X \times X]$ belongs to $\mathtt{h}^{(N+1)}(X \times X)$ if $\operatorname{codim} Y > N$. Then $\operatorname{im} pr_N \circ p^{\mathtt{h}}$ is generated over Λ^0 by classes of $[Z_L \to \overline{X} \times \overline{X}]_{\mathtt{h}}$, where $Z \to X \times X$ has codimension N.

By Lemma 6.8 for any $Z \to X \times X$ of codimension N we have

$$pr_N \circ p^{\mathtt{h}}([Z \to X \times X]_{\mathtt{h}}) = \Phi^N \circ p^{\mathrm{CH}}([Z \to X \times X]).$$

Then $\operatorname{im} pr_N \circ p^{\mathtt{h}} \subseteq \Phi^N \circ p^{\mathrm{CH}}$ and the theorem is proven. $\square$

Theorem B. *Let* $\mathtt{h}$ *be oriented cohomology theory with coefficient ring* Λ. *Assume that the Chow motive* $\mathcal{R}$ *is indecomposable over* Λ^0 *and* $\Lambda^1 = \ldots = \Lambda^N = 0$. *Then the* $\mathtt{h}$*-motive* $\mathcal{R}_{\mathtt{h}}$ *from theorem A is indecomposable.*

Proof. By definition, $\mathcal{R}_{\mathtt{h}} = (X, r_0)$ where r_0 is an idempotent in $\mathtt{h}_N^{(N)}(X \times X)$. If $\mathcal{R}_{\mathtt{h}}$ is decomposable, then $r_0 = r_1 + r_2$ for some idempotents in $r_1, r_2 \in \mathtt{h}_N(X \times X)$ Then by Lemma 6.12 $r_1, r_2 \in \mathtt{h}_N^{(N)}(X \times X)$ and $p_1 = (\Phi^N)^{-1} \circ pr_N \circ p^{\mathtt{h}}(r_1)$ and $p_2 = (\Phi^N)^{-1} \circ pr_N \circ p^{\mathtt{h}}(r_2)$ are rational idempotents and $p^{\mathrm{CH}}(p_0) = p_1 + p_2$. These idempotents are nontrivial, since $\ker(\Phi^N)^{-1} \circ pr_N \circ p^{\mathtt{h}}$ is nilpotent. Hence, the Chow motive $\mathcal{R} = (X, p_0)$ is decomposable, a contradiction. $\square$

6.13. **Example.** If $\mathtt{h} = \Omega$ or connective K-theory, all the elements in the coefficient ring have negative degree. Then Theorems A and B prove that $\mathtt{h}$-motivic irreducible decomposition coincides with integral Chow-motivic decomposition. This gives another proof of the result by Vishik-Yagita [17, Cor. 2.8].

6.14. **Example.** Take $\mathtt{h}$ to be Morava K-theory $\mathtt{h} = K(n)^*$. The coefficient ring is $\mathbb{F}_p[v_n, v_n^{-1}]$, where $\deg(v_n) = -2(p^n - 1)$. In the case $n > \log_p(\frac{N}{2} + 1)$ Theorems A and B prove that $M^{K(n)}(X)$ has the same irreducible decomposition as Chow motive modulo p.

Theorem C. *Suppose that X, Y are generically cellular and Y is a twisted form of X, i.e. $\overline{Y} \cong \overline{X}$.*

If $M^{\mathrm{CH}}(X, \Lambda^0) \cong M^{\mathrm{CH}}(Y, \Lambda^0)$, then $M^{\mathtt{h}}(X) \cong M^{\mathtt{h}}(Y)$.

Proof. Let $f \in \mathrm{CH}^N(X \times Y)$ and $g \in \mathrm{CH}^N(Y \times X)$ be correspondences, that give mutually inverse isomorphisms between $M^{\mathrm{CH}}(X)$ and $M^{\mathrm{CH}}(Y)$. Consider the diagram

$$\begin{tikzcd}
\mathtt{h}_N^{(N)}(X \times Y) \arrow[r, "pr_N \circ p^{\mathtt{h}}"] & \mathtt{h}_N^{(N/N+1)}(\overline{X} \times \overline{X}) \\
\mathrm{CH}^N(X \times Y, \Lambda^0) \arrow[r, "p^{\mathrm{CH}}"] & \mathrm{CH}^N(\overline{X} \times \overline{X}, \Lambda^0) \arrow[u, "\Phi^N"']
\end{tikzcd}$$

Then by Lemma 6.7 we can find $f_1 \in \mathtt{h}_N^{(N)}(X \times Y)$ and $g_1 \in \mathtt{h}_N^{(N)}(Y \times X)$ such that $pr_N \circ p^{\mathtt{h}}(f_1) = \Phi^N(f)$ and $pr_N \circ p^{\mathtt{h}}(g_1) = \Phi^N(g)$. Then $g_1 \circ f_1 - \Delta_X$ lies in the kernel of the map

$$\mathtt{h}_N^{(N)}(X \times X) \xrightarrow{pr_N \circ p^{\mathtt{h}}} \mathtt{h}_N^{(N/N+1)}(\overline{X} \times \overline{X})$$

which consists of nilpotents by Lemma 6.9. So $g_1 \circ f_1 - \Delta_X$ is nilpotent. By the same reasons $f_1 \circ g_1 - \Delta_Y$ is nilpotent. Then $M^{\mathtt{h}}(X)$ and $M^{\mathtt{h}}(Y)$ are isomorphic by Lemma 6.10 and the theorem is proven. □

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