INCOMPRESSIBILITY OF PRODUCTS OF WEIL TRANSFERS OF GENERALIZED SEVERI-BRAUER VARIETIES

NIKITA A. KARPENKO

Abstract. We generalize the result of [11] on incompressibility of Galois Weil transfer of generalized Severi-Brauer varieties, to direct products of varieties of such type; as shown in [11], this is needed to compute essential dimension of representations of finite groups. We also provide a generalization to non-Galois (separable) Weil transfer.

Contents

1. Introduction

Let F be a field and let L/F be a finite Galois field extension. A central division L-algebra D is *balanced*, if for every $g \in \mathrm{Gal}(L/F)$, the conjugate L-algebra $g(D)$, obtained from D by the base change $g : L \to L$, is Brauer-equivalent to a tensor power of D.

For a balanced D as above, we refer as $\mathcal{R}\mathrm{SB}$-*variety* to the Weil transfer $\mathcal{R}_{L/F}$ of a generalized Severi-Brauer variety $\mathrm{SB}(j, D)$ of D ($j = 1, \dots, \deg D$). Let p be a prime integer. An $\mathcal{R}\mathrm{SB}$-variety is a $p\mathcal{R}\mathrm{SB}$-*variety*, if the integers $[L : F]$, $\deg D$, and j are p-powers.

The following problem has been raised in [11] for needs of computing essential p-dimension of a representation of a finite group: compute canonical p-dimension of an arbitrary finite direct product of $\mathcal{R}\mathrm{SB}$-varieties. This problem has been solved in [11] in

Date: 26 June 2014.

Key words and phrases. Central simple algebras, algebraic groups, projective homogeneous varieties, Severi-Brauer varieties, Weil transfer, Chow groups and motives, canonical dimension and incompressibility. *Mathematical Subject Classification (2010):* 20G15; 14C25.

The author acknowledges a partial support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005; his work has been also supported by the start-up grant of the University of Alberta and a Discovery Grant from the National Science and Engineering Board of Canada.

the case of a single $\mathcal{R}$SB-variety. The core result producing the solution was the following statement on p-incompressibility of $p\mathcal{R}$SB-varieties (we refer to [11] for an introduction and further references on p-incompressibility, canonical p-dimension, Weil transfer, etc.):

Theorem 1.1 ([11, Theorem 11.2]). *Any $p\mathcal{R}$SB-variety is p-incompressible.*

In the present paper, we generalize Theorem 1.1 in two directions. First, we replace Galois field extensions in the definition of $p\mathcal{R}$SB-varieties, by separable ones. Although this generalization is not motivated anymore by study of representations, it is a natural step to do. The statement remains the same (see Theorem 2.3), as does the main outline of the proof. The main change is in the definition of a balanced algebra over a separable extension. All this is done in §2.

As a second (and principal) generalization, we establish a criterion of p-incompressibility of a product of $p\mathcal{R}$SB-varieties, see Corollary 3.6. The basic result here is a p-incompressibility criterion for a product of a $p\mathcal{R}$SB-variety by an arbitrary projective homogeneous (under an action of a semisimple affine algebraic group) variety given in Theorem 3.1.

As a particular case of Corollary 3.6, we recover in §4a (with a new proof and a simplified statement) an old result [10, Theorem 2.1] on p-incompressibility of products of Severi-Brauer varieties. But now we can also determine canonical p-dimension of any p-primary Weil transfer of a product of this type, see §4b. In particular, we show that a p-primary Weil transfer of a p-incompressible product of this type is also p-incompressible. Moreover, the latter statement also holds for generalized Severi-Brauer varieties in place of the usual ones, see §4c.

In the last section (§5), we drop the balance assumption to do a complete analysis of a quadratic field extension. Note however, that the balanced case, motivated by the representations, also looks interesting (not only for a quadratic L/F) from the following viewpoint: for a $p\mathcal{R}$SB-variety Y, the integer $\operatorname{cd}_p(Y) = \dim Y$ turns out to be much bigger than $\operatorname{cd}_p(Y_L) = (\dim Y)/[L:F]$ so that the role of the field extension and the Weil transfer shows up as crucial. In the imbalanced case however, the variety Y_L may, for instance, happen to be p-incompressible (c.f. Example 5.3), trivially implying p-incompressibility of Y itself.

Most of our terminology and notation being introduced "on the move", we only mention here that smooth projective varieties X and Y over F are called *equivalent* if there exist rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$. Equivalent varieties have the same canonical (p-)dimension, [11, Lemma 3.3(a)].

Acknowledgements. I am grateful to Alexander Merkurjev and Zinovy Reichstein for useful comments on early versions of the preprint. Its current version has been prepared during my stay at the Universität Duisburg-Essen (research group of Marc Levine), under ideal work conditions.

2. Non-Galois extensions

In this section we generalize Theorem 1.1 to the case of an arbitrary finite separable (not necessarily Galois) field extension L/F of degree a power of p.

Here is the "Galois to separable" generalization of [11, Lemma 11.1] (with practically the same proof). Here and everywhere else in the paper, the motive we are talking about are Chow motives with coefficients in $\mathbb{F}_p$ (the finite field of p elements), see e.g. [2, §64]. Weil transfer of motives has been introduced in [5]. *Corestriction* of motives is from [7, §3].

Lemma 2.1. *Let L/F be an arbitrary finite separable field extension and let E/F be a finite Galois field extension containing L. For and any $m \geq 1$, let $M_1, \dots, M_m$ be m motives over L. Then the motive $\mathcal{R}_{L/F}(M_1 \oplus \dots \oplus M_m)$ decomposes into a direct sum*

$$\mathcal{R}_{L/F}(M_1 \oplus \dots \oplus M_m) \simeq \mathcal{R}_{L/F}M_1 \oplus \dots \oplus \mathcal{R}_{L/F}M_m \oplus N,$$

where N is a direct sum of corestrictions to F of motives over fields K with $F \subsetneq K \subset E$. □

Let L/F be a finite separable field extension. We generalize the notion of a *balanced* central simple L-algebra D (which we already have in the Galois case, see §1). A central simple L-algebra D is *balanced*, if for a Galois field extension E/F containing L and for any F-imbedding of L into E with an image L_0, the LL_0-algebra obtained from D by the base change $L \to L_0 \subset LL_0$ is Brauer-equivalent to a tensor power of D_{LL_0}. This definition does not depend on the choice of E; E can be taken to be a Galois closure of L/F.

From now on, we extend the notion of a $p\mathcal{R}\mathrm{SB}$-variety, introduced in §1, by allowing separable (not necessarily Galois) Weil transfer:

Definition 2.2. Let p be a prime number. An F-variety Y is a $p\mathcal{R}\mathrm{SB}$-*variety*, if there exist a finite p-primary separable field extension L/F and a balanced division L-algebra D of degree p^n such that $X \simeq \mathcal{R}_{L/F}\,\mathrm{SB}(p^i, D)$ for some $i = 0, \dots, n$.

Theorem 2.3. *For any prime number p, any $p\mathcal{R}\mathrm{SB}$-variety is p-incompressible.*

Proof. Let p be a prime number, L/F a finite separable field extension of degree a power of p, D a balanced central division L-algebra of degree p^n for some $n \geq 0$, and X the generalized Severi-Brauer variety $\mathrm{SB}(p^i, D)$ of D with some $i = 0, 1, \dots, n$. Let us prove that the variety $\mathcal{R}_{L/F}X$, given by the Weil transfer of X, is p-incompressible.

Let E be a finite Galois field extension of F containing L (for instance, E can be a Galois closure of L/F). The question on canonical p-dimension of $\mathcal{R}_{L/F}X$ easily reduces to the case where $[E : F]$ is also a power of p. Indeed, let G be the Galois group of E/F and let H be its subgroup corresponding to L. Let G' be a Sylow p-subgroup of G containing a Sylow p-subgroup H' of H. We get the following diagram of subgroups and the corresponding diagram of subfields, where the sign $p^?$ marks p-primary while the sign $p\!\!\not|$ marks p-coprime indexes/degrees:

Since the degree $[F' : F]$ is not divisible by p, canonical p-dimension of $\mathcal{R}_{L/F}X$ does not change under the base change F'/F. Moreover, $(\mathcal{R}_{L/F}X)_{F'} \simeq \mathcal{R}_{L'/F'}(X_{L'})$ because

$L' \simeq L \otimes_F F'$. Finally, observe that the index of a p-primary central simple L-algebra is not changed under the base change L'/L. In particular, $D_{L'}$ is still a division algebra. Moreover, this central division L'-algebra is still balanced (now with respect to the subfield $F' \subset L'$).

For the rest of the proof we will assume that the degree $[E : F]$ is a p-power. We follow the proof of Theorem 1.1, given in [11, Theorem 11.2], and only indicate the changes.

The first place where a change is needed is the place where we compute the index of D over the field $L' = F' \otimes_F L$, where F' is the function field of the variety $\mathcal{R}_{L/F}\,\mathrm{SB}(p^{n-1}, D)$. In the case of Galois L/F, the Weil transfer disappears because $\mathcal{R}_{L/F}\,\mathrm{SB}(p^{n-1}, D)_L$ is the product of the generalized Severi-Brauer varieties given by the conjugate algebras. This happens because of the isomoprhism

$$L \otimes_F L \simeq L \times \cdots \times L$$

of L-algebras, where L acts on $L \otimes L$ on the left. Note that this isomorphism is also L-linear for the right action of L on the tensor product if one lets L act on the factors of $L \times \cdots \times L$ by means of the F-automorphisms of L.

Under the assumptions of Theorem 2.3 however, $L \otimes_F L$ is identified with $L_1 \times \cdots \times L_r$, where each L_i is LL_0 for an appropriate (depending on i) choice of L_0 as in the definition of balanced algebra. Since $[L_i : L] > 1$ for at least one value of i, we cannot avoid the Weil transfer here and may want to use the index reduction formulas for the Weil transfer of generalized Severi-Brauer varieties from [15] (see also [14]). But this is not really necessary. Indeed, since D is balanced, for any i, the L_i-algebra in question is Brauer-equivalent to a tensor power of D_{L_i}, its index will divide p^{n-1} if we extend L to the function field of $\mathrm{SB}(p^{n-1}, D)$. The Weil transfer becomes rational over this extension of L so that the index of D won't be affected by passing to the function field of the Weil transfer.

We have explained why the index of $D_{L'}$ is indeed p^{n-1}. The final adjustment we need to make, in order to adopt the proof of Theorem 1.1 to the setting of Theorem 2.3, is in the choice of the degree p Galois field subextension $\tilde{L}/F$ of L/F. Recall that the Galois group G of E/F is a p-group. Let H be its subgroup corresponding to the field L. By [3, Theorem 4.2.1], H is contained in a normal subgroup $\tilde{H}$ of G such that $[G : \tilde{H}] = p$. We take for $\tilde{L}$ the field corresponding to $\tilde{H}$ and we have to compute canonical p-dimension of the variety $\mathcal{R}_{L/F}(X)_{\tilde{L}}$.

The variety $\mathcal{R}_{L/F}(X)_{\tilde{L}}$ is isomorphic to the product $\prod_{\tilde{g} \in \tilde{G}} \tilde{g}(\mathcal{R}_{L/\tilde{L}} X)$, where $\tilde{G} = G/\tilde{H}$ is the Galois group of $\tilde{L}/F$. Since D is balanced, this product is equivalent to its factor $\mathcal{R}_{L/\tilde{L}} X$ given by $\tilde{g} = 1$. In particular, the canonical p-dimension of $\mathcal{R}_{L/F}(X)_{\tilde{L}}$ is given by the dimension of $\mathcal{R}_{L/\tilde{L}} X$. After this is established, the remainder of the proof of Theorem 1.1 goes through unchanged. □

3. Incompressibility of products

Here is our basic result on p-incompressibility of products:

Theorem 3.1. *Let Y be a $p\mathcal{R}\mathrm{SB}$-variety (for a given prime number p) and let Z be a projective homogeneous F-variety. The product $Y \times Z$ is p-incompressible provided that the varieties $Y_{F(Z)}$ and $Z_{F(Y)}$ are so.*

Proof. Let $Y := \mathcal{R}_{L/F}\,\mathrm{SB}(p^i, D)$, where L/F is a finite separable field extension of degree p^r with $r \geq 0$, D is a balanced central division L-algebra of degree p^n with $n \geq 0$, and $i = 0, \dots, n$. Let Z be a projective homogeneous F-variety such that the varieties $Y_{F(Z)}$ and $Z_{F(Y)}$ are p-incompressible. We are going to prove that the product $Y \times Z$ is p-incompressible.

By Theorem 2.3, the assumption requiring that $Y_{F(Z)}$ is p-incompressible is equivalent to the assumption that $D_{L(Z)}$ is a division algebra.

Since canonical p-dimension is not changed under base field extensions of degree prime to p (see [12, Proposition 1.5(2)]), we may assume that there exists a finite p-primary Galois field extension E/F containing L and such that Z_E is of inner type. This assumption allows us to apply results of [7] and [8, §6].

For any $j = 0, \dots, n$, we set $X^j := \mathrm{SB}(p^j, D)$ and $Y^j := \mathcal{R}_{L/F}X^j$. We also set $X := X^i$ so that $Y = \mathcal{R}_{L/F}X$.

We induct on n. For $n = 0$ the statement is trivial. From now on, we assume that $n > 0$.

For $i = n$ the statement is trivial. From now on, we assume that $i < n$.

Let F' be the function field of the variety Y^{n-1}. Let $L' := F' \otimes_F L$. Since D is balanced, the index of the central simple L'-algebra $D_{L'} := D \otimes_L L' = D \otimes_F F'$ is p^{n-1} so that there exists a central division L'-algebra D' such that the algebra of $(p \times p)$-matrices over D' is isomorphic to $D_{L'}$. We set $X'^j := \mathrm{SB}(p^j, D')$, $Y'^j := \mathcal{R}_{L'/F'}X'^j$, $X' := X'^i$, and $Y' := Y'^i$. By [4] (see also [1, Theorem 7.5]) and [9, Theorem 3.8], the motive of the variety $X_{L'}$ decomposes into a direct sum

$$M(X_{L'}) \simeq M(X') \oplus M(X')(p^{i+n-1}) \oplus \\ M(X')(2p^{i+n-1}) \oplus \cdots \oplus M(X')((p-1)p^{i+n-1}) \oplus N,$$

where N is a direct sum of some shifts of the upper motives $U(X'^j)$ of some varieties X'^j with $j < i$. Therefore, by Lemma 2.1 and [5, Theorem 5.4] (which we use just to determine the shifts), the motive of the variety $Y_{F'} = \mathcal{R}_{L'/F'}(X_{L'})$ decomposes into a direct sum

$$M(Y_{F'}) \simeq M(Y') \oplus M(Y')(p^{r+i+n-1}) \oplus \\ M(Y')(2p^{r+i+n-1}) \oplus \cdots \oplus M(Y')((p-1)p^{r+i+n-1}) \oplus N \oplus N',$$

where now N is a direct sum of shifts of $U(Y'^j)$ with $j < i$, and N' is a direct sum of corestrictions of motives over fields K with $F \subsetneq K \subset E$. It follows that

$$M(Y \times Z)_{F'} \simeq M(Y' \times Z_{F'}) \oplus M(Y' \times Z_{F'})(p^{r+i+n-1}) \oplus \\ M(Y' \times Z_{F'})(2p^{r+i+n-1}) \oplus \cdots \oplus M(Y' \times Z_{F'})((p-1)p^{r+i+n-1}) \oplus N \oplus N',$$

with N' of the same shape as before and with N being a direct sum of shifts of

$$U(Y'^j) \otimes M(Z), \;\; j < i.$$

We claim that the variety $Y' \times Z_{F'}$ is p-incompressible by the induction hypothesis. To check the claim, we check that the varieties $Z_{F'(Y')}$ and $Y'_{F'(Z)}$ are p-incompressible. To check that $Z_{F'(Y')}$ is p-incompressible, we check that Z over a lager field $F'(Y' \times Y_{F'})$ is so. The field $F'(Y' \times Y_{F'}) = F'(Y)(Y')$ is purely transcendental over $F(Y)$ and $Z_{F(Y)}$ is p-incompressible (this is the place in the proof of Theorem 3.1, where the assumption on $Z_{F(Y)}$ is used). Therefore Z over $F'(Y' \times Y_{F'})$ is p-incompressible. (To see that canonical p-dimension of projective homogeneous varieties does not change under a purely transcendental base field extension, one may use the characterization [8, Corollary 6.2] of the canonical p-dimension in terms of algebraic cycles together with the fact that a purely transcendental base field extension does not affect the Chow groups.)

To check that $Y'_{F'(Z)}$ is p-incompressible, we check that $D'_{L'(Z)}$ is a division algebra. Since $L' = L(Y^{n-1})$, we have $L'(Z) = L(Y^{n-1} \times Z) = L(Z)(Y^{n-1})$. Since $D_{L(Z)}$ is a balanced (over $F(Z)$) division algebra, $D_{L(Z)(Y^{n-1})}$ has index p^{n-1}. Therefore $D'_{L'(Z)}$ is a division algebra.

The claim being proved, it follows that

$$M(Y \times Z)_{F'} \simeq U(Y' \times Z_{F'}) \oplus U(Y' \times Z_{F'})(p^{r+i+n-1}) \oplus U(Y' \times Z_{F'})(2p^{r+i+n-1}) \oplus \cdots \oplus U(Y' \times Z_{F'})((p-1)p^{r+i+n-1}) \oplus N, \tag{3.2}$$

with N having the property that no summand of its complete decomposition is isomorphic to a shift of $U(Y' \times Z_{F'})$.

We want to show that the variety $Y \times Z$ is p-incompressible. Let l be the number of those summands in the complete decomposition of the motive $U(Y \times Z)_{F'}$, which are isomorphic to a shift of $U(Y' \times Z_{F'})$. We have $1 \leq l \leq p$ and it suffices to show that $l = p$. Indeed, $l = p$ implies that the complete decomposition of $U(Y \times Z)_{F'}$ contains the summand $U(Y' \times Z_{F'})$ of (3.2) with the maximal shift $(p-1)p^{r+i+n-1}$. Therefore

$$\operatorname{cd}_p(Y \times Z) = \dim U(Y \times Z) = \dim U(Y \times Z)_{F'} \geq \dim U(Y' \times Z_{F'}) + (p-1)p^{r+i+n-1} = \dim(Y \times Z).$$

(We refer to [6, Theorem 5.1] for the relation between canonical p-dimension and dimension of the upper motive of a projective homogeneous variety, used here.)

Our next claim is: l divides p (therefore $l = 1$ or $l = p$, and we will only need to show that $l \neq 1$). To prove the claim, we consider the complete motivic decomposition of $Y \times Z$. It contains several shifts of $U(Y \times Z)$ (it contains one non-shifted $U(Y \times Z)$ and - as we hope - no other shifts of this motive, but we do not know whether the hope comes true by now). Let N be any of the remaining (indecomposable) summands. We affirm that no summand of the complete decomposition of $N_{F'}$ is isomorphic to a shift of $U(Y' \times Z_{F'})$. Clearly, this affirmation implies the claim that l divides p.

To prove the affirmation, let us note that N can be of two alternative types. The first type is given by corestriction to F of a motive over a field K with $F \subsetneq K \subset E$. For such N, any indecomposable summand of $N_{F'}$ is a corestriction to F' of a motive over $K' := K \otimes_F F'$ (see [7, Proposition 3.1]) which is never isomorphic to a shift of the upper motive of a projective homogeneous F'-variety.

The second type of N is a shift of $U(T)$, where T is a projective homogeneous variety with $\deg \operatorname{Ch}_0(T_{F(Y \times Z)}) = 0$ (and with $\deg \operatorname{Ch}_0(Y \times Z)_{F(T)} = \mathbb{F}_p$). Over F' we still have $\deg \operatorname{Ch}_0(T_{F'(Y \times Z)}) = 0$ (and $\deg \operatorname{Ch}_0(Y \times Z)_{F'(T)} = \mathbb{F}_p$) because the field extension

$$F'(Y \times Z)/F(Y \times Z)$$

is purely transcendental. Any indecomposable summand of $N_{F'}$, which is not corestriction from some $F \subsetneq K \subset E$, is a shift of $U(S)$, where S is a projective homogeneous F'-variety with $\deg \mathrm{Ch}_0(T_{F'(S)}) = \mathbb{F}_p$. It follows that $\deg \mathrm{Ch}_0(S_{F'(Y\times Z)}) = 0$ (otherwise we would be in contradiction with $\deg \mathrm{Ch}_0(T_{F'(Y\times Z)}) = 0$) implying that no shift of $U(S)$ (and consequently no shift of $U(S)$) is isomorphic to $U(Y' \times Z_{F'})$, see [9, Corollary 2.15] for the criterion of isomorphism for upper motives.

We proved the affirmation and the claim. This means that we only need to show that $l \neq 1$ to finish the proof of Theorem 3.1. So, we assume that $l = 1$ and we look for a contradiction. By [8, Proposition 2.4], the complete decomposition of $U(Y\times Z)_{F'}$ contains as a summand the motive $U(Y' \times Z_{F'})$ shifted by the difference

$$\dim U(Y \times Z) - \dim U(Y' \times Z_{F'}).$$

Therefore, $l = 1$ implies that the above difference is 0, and we come to

$$\mathrm{cd}_p(Y \times Z) = \dim U(Y \times Z) = \dim U(Y' \times Z_{F'}) = d,$$

where $d := \dim(Y' \times Z_{F'}) = \dim Y' + \dim Z$.

By [8, Proposition 6.1], there exist $\alpha \in \mathrm{Ch}^d(Y \times Z)_{F(Y\times Z)}$ and $\beta \in \mathrm{Ch}_d(Y \times Z)$ with $\deg(\alpha \cdot \beta) \neq 0 \in \mathbb{F}_p$. In the last formula, we consider both cycles over a common field extension of their fields of definition, before we multiply them. We use this convention below (in similar formulas on degree of products) as well.

Since $\mathrm{cd}_p(Y_{F'}) = \dim Y' =: d'$, we can find $\alpha' \in \mathrm{Ch}^{d'}(Y_{F'(Y)})$ and $\beta' \in \mathrm{Ch}_{d'}(Y_{F'})$ with $\deg(\alpha' \cdot \beta') \neq 0$. Using these α' and β' and a rational point $\mathbf{pt} \in Z_{F(Z)}$, we get the cycles

$$\alpha' \times [\mathbf{pt}] \in \mathrm{Ch}^d(Y \times Z)_{F'(Y\times Z)} \quad \text{and} \quad \beta' \times [Z] \in \mathrm{Ch}_d(Y \times Z)_{F'},$$

having the same property as α and β:

$$\deg\Big((\alpha' \times [\mathbf{pt}]) \cdot (\beta' \times [Z])\Big) \neq 0.$$

It follows by [8, Lemma 6.5] that one can "mix up" the old cycles with the new ones and get the relation

$$\deg\Big((\alpha' \times [\mathbf{pt}]) \cdot \beta\Big) \neq 0.$$

Since $\alpha' \times [\mathbf{pt}] = (\alpha' \times [Z]) \cdot ([Y] \times [\mathbf{pt}])$, the last degree relation can be rewritten as $\deg(\alpha' \cdot \beta'') \neq 0$, where $\beta'' \in \mathrm{Ch}_{d'}(Y_{F(Z)})$ is the push-forward of the product $([Y]\times[\mathbf{pt}])\cdot\beta$ along the projection $(Y\times Z)_{F(Z)} \to Y_{F(Z)}$. Since the field extension $F'(Y)/F(Y)$ is purely transcendental, there exists $\alpha'' \in \mathrm{Ch}^{d'}(Y_{F(Y)})$ mapped to α' under the change of field homomorphism. Changing notation, we write α'' for the image of α'' in $\mathrm{Ch}^{d'}(Y_{F(Z)(Y)})$.

The cycles $\alpha'' \in \mathrm{Ch}^{d'}(Y_{F(Z)(Y)})$ and $\beta'' \in \mathrm{Ch}_{d'}(Y_{F(Z)})$ thus constructed have the property $\deg(\alpha'' \cdot \beta'') \neq 0$. It follows by [8, Corollary 6.2] that $\mathrm{cd}_p(Y_{F(Z)}) \leq d'$. Since

$$d' = \dim Y' = p^i(p^{n-1} - p^i) < p^i(p^n - p^i) = \dim Y, \tag{3.3}$$

the relation $\mathrm{cd}_p(Y_{F(Z)}) \leq d'$ obtained contradicts the assumption on p-incompressibility of the variety $Y_{F(Z)}$. □

The necessary condition for p-incompressibility of a product of projective homogeneous varieties, showing up in the following lemma, turns out to be sufficient in the case of the special varieties we are interested in (see Corollary 3.5):

Lemma 3.4. *For any projective homogeneous F-varieties X and Y,*

$$\operatorname{cd}_p(X \times Y) \le \operatorname{cd}_p(X) + \operatorname{cd}_p(Y_{F(X)}).$$

In particular, a necessary condition for p-incompressibility of $X \times Y$ is p-incompressibility of $X_{F(Y)}$ and $Y_{F(X)}$.

Proof. Again, we are using the characterization of canonical p-dimension of projective homogeneous varieties given in [8, Corollary 6.2]. Since canonical p-dimension is not changed under base field extensions of degree prime to p (see [12, Proposition 1.5(2)]), we may assume that the condition of [8, Corollary 6.2] on the projective homogeneous variety is satisfied for X and Y: both of them become of inner type over a finite p-primary extension of F. We set $x := \operatorname{cd}_p(X)$ and $y := \operatorname{cd}_p(Y_{F(X)})$. We find $\alpha_X \in \operatorname{Ch}^x(X_{F(X)})$ and $\beta_X \in \operatorname{Ch}_x(X)$ with $\deg(\alpha_X \cdot \beta_X) \ne 0$. Similarly, we find $\alpha_Y \in \operatorname{Ch}^y(Y_{F(X)(Y)})$ and $\beta_Y \in \operatorname{Ch}_y(Y_{F(X)})$ with $\deg(\alpha_Y \cdot \beta_Y) \ne 0$. Let $\beta'_Y \in \operatorname{Ch}_{\dim X+y}(X \times Y)$ be an element mapped to β_Y under the surjection

$$\operatorname{Ch}_{\dim X+y}(X \times Y) \twoheadrightarrow \operatorname{Ch}_y(Y_{F(X)})$$

given by the pull-back along the morphism $Y_{F(X)} \to X \times Y$ induced by the generic point of X. We set

$$\alpha := \alpha_X \times \alpha_Y \in \operatorname{Ch}^{x+y}(X \times Y)_{F(X\times Y)} \quad \text{and} \quad \beta := (\beta_X \times [Y]) \cdot \beta'_Y \in \operatorname{Ch}_{x+y}(X \times Y).$$

We have the relation $\deg(\alpha \cdot \beta) \ne 0$ showing that $\operatorname{cd}_p(X \times Y) \le x + y$. □

Corollary 3.5. *For two products X and Y of $p\mathcal{R}\mathbb{S}\mathbb{B}$-varieties over F, their product $X \times Y$ is p-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible.*

Proof. The "only if" part being served by Lemma 3.4, we only prove the "if" part. We write X as product $X_1 \times \cdots \times X_r$ of $p\mathcal{R}\mathbb{S}\mathbb{B}$-varieties and induct on r. The case of $r = 1$ follows from Theorem 3.1. For $r > 1$, set $X' := X_2 \times \cdots \times X_r$. To show that $X \times Y = X_1 \times (X' \times Y)$ is p-incompressible, by Theorem 3.1, it suffices to check that $(X_1)_{F(X'\times Y)}$ and $(X' \times Y)_{F(X_1)}$ are p-incompressible. The assumption on p-incompressibility of $X_{F(Y)}$ implies (by Lemma 3.4) p-incompressibility of $(X_1)_{F(X'\times Y)}$. And the variety $(X' \times Y)_{F(X_1)}$ is p-incompressible by induction hypothesis. □

Here is the most convenient statement to check p-incompressibility of a general product of $p\mathcal{R}\mathbb{S}\mathbb{B}$-varieties. Basically, it reduces the problem to application of index reduction formulas.

Corollary 3.6. *Product $X_1 \times \cdots \times X_r$ of $p\mathcal{R}\mathbb{S}\mathbb{B}$-varieties over F is p-incompressible if and only if $(X_i)_{F(X_1\times\cdots\times X_{i-1}\times X_{i+1}\times\cdots\times X_r)}$ for every $i = 1, \dots, r$, is p-incompressible.* □

4. Particular products

4a. **Products of Severi-Brauer varieties.** Applying Corollary 3.6 to a product of Severi-Brauer varieties (Weil transfer do not show up here), we get a version of [10, Theorem 2.1]. Note that the proof is different from the original one: it does not involve K-theory.

Corollary 4.1. *Let $D_1, \dots, D_r$ be p-primary central division F-algebras. The product of their Severi-Brauer varieties is p-incompressible if and only if each D_i remains division algebra over the function field of the product of the Severi-Brauer varieties of the remaining algebras.* □

Remark 4.2. By the index reduction formula for Severi-Brauer varieties, the condition on D_i means that the index of every product of D_i by tensor powers of the remaining algebras is $\geq \operatorname{ind} D_i$. With this, one may see that $D_1, \dots, D_r$ satisfy the condition of Corollary 4.1 if and only if their nonzero Brauer classes (put in some/any order increasing the degrees) form a minimal basis in the sense of [10, Remark 2.9].

Remark 4.3. In general, the property of being p-incompressible for a projective homogeneous variety X is weaker than the property of having indecomposable motive. However, for a generically split X (i.e., for X such that the motive of $X_{F(X)}$ is split, i.e., is a direct sum of shifts of the motive of a point), the above two properties are equivalent. Indeed, $M(X)$ for a generically split X is a direct sum of shifts of $U(X)$ so that $M(X) = U(X)$ is and only if $\dim U(X) = \dim X$.

A product of Severi-Brauer varieties is a generically split projective homogeneous variety. Therefore the motive of a p-incompressible product of Severi-Brauer varieties is indecomposable.

There are numerous simplification in the proof of Theorem 3.1 when we adopt it to products of Severi-Brauer varieties. It might be therefore helpful for the reader to look at the simplified proof below before going through the actual proof of Theorem 3.1.

Proposition 4.4. *Let X be the product of Severi-Brauer varieties $X_1, \dots, X_r$ over F such that for every $i = 1, \dots, r$, the variety $(X_i)_{F(X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_r)}$ is p-incompressible. Then the motive of X is p-incompressible.*

Proof. Using induction on r, we are reduced to prove the following statement (which looks more like the statement of Theorem 3.1): *the product $Y \times Z$ of a Severi-Brauer variety Y by a product of Severi-Brauer varieties Z is p-incompressible provided that the varieties $Y_{F(Z)}$ and $Z_{F(Y)}$ are so.*

Actually, instead of being a product of Severi-Brauer varieties, Z can be any generically split projective homogeneous variety: we will only use this property of Z in the proof.

Now we go along the lines of the proof of Theorem 3.1, removing everything superfluous.

The variety Y is the Severi-Brauer variety of a p-primary central division algebra D, say, $\deg D = p^n$. Note that D remains division over $F(Z)$.

We induct on n. For $n = 0$ the statement we are proving is trivial. We assume that $n \geq 1$ below.

Let F' be the function field of the variety $\mathrm{SB}(p^{n-1}, D)$. The index of the central simple F'-algebra $D_{F'}$ is p^{n-1} so that there exists a central division F'-algebra D' such that the algebra of $(p \times p)$-matrices over D' is isomorphic to $D_{F'}$. We set $Y' := \mathrm{SB}(D')$. The motive of the variety $Y_{F'}$ decomposes into the direct sum of p summands

$$M(Y_{F'}) \;\simeq\; M(Y') \oplus M(Y')(p^{n-1}) \oplus M(Y')(2p^{n-1}) \oplus \cdots \oplus M(Y')((p-1)p^{n-1}).$$

It follows that

$$(4.5)\quad M(Y\times Z)_{F'}\simeq M(Y'\times Z_{F'})\oplus M(Y'\times Z_{F'})(p^{n-1})\oplus \\ M(Y'\times Z_{F'})(2p^{n-1})\oplus\cdots\oplus M(Y'\times Z_{F'})((p-1)p^{n-1}).$$

We claim: the decomposition (4.5) is complete, i.e., the motive of the variety $Y'\times Z_{F'}$ is indecomposable (see Remark 4.3), and that – by the induction hypothesis. To check the claim, we check that the varieties $Z_{F'(Y')}$ and $Y'_{F'(Z)}$ are p-incompressible. This is done precisely as in the proof of Theorem 3.1.

Now we know that (4.5) is the complete decomposition and we want to show that the motive of $Y\times Z$ is indecomposable. In other words, we want to show that

$$U(Y\times Z)=M(Y\times Z),$$

or – equivalently – that $U(Y\times Z)_{F'}$ contains all the p indecomposable summands of the decomposition (4.5).

Let l be the number of summand in the complete decomposition of the motive $U(Y\times Z)_{F'}$ (all of them are automatically isomorphic to a shift of $M(Z_{F'})$). We have $1\le l\le p$ and all we want to show is $l=p$.

It is now very easy to see that l divides p. Indeed, since the variety $Y\times Z$ is generically split, every summand of its complete motivic decomposition is a shift of $U(Y\times Z)$. Therefore l divides p, as claimed, and we only need to show that $l\ne 1$.

So, we assume that $l=1$ and we look for a contradiction. Clearly, $l=1$ implies that

$$\operatorname{cd}_p(Y\times Z)=\dim U(Y\times Z)=\dim M(Y'\times Z_{F'})=d,$$

where $d:=\dim Y'+\dim Z=p^{n-1}-1+\dim Z$. By [8, Proposition 6.1], there exist $\alpha\in\operatorname{Ch}^d(Y\times Z)_{F(Y\times Z)}$ and $\beta\in\operatorname{Ch}_d(Y\times Z)$ with $\deg(\alpha\cdot\beta)\ne 0\in\mathbb{F}_p$. Starting from this point, the proof of Proposition 4.4 ends precisely as the proof of Theorem 3.1. Note that in (3.3) we will have $i=0$ here. □

4b. **Weil transfer of products of Severi-Brauer varieties.**

Corollary 4.6. *Let L/F be a p-primary separable field extension and let X be a product over L of Severi-Brauer varieties of some balanced p-primary division L-algebras. Then $\operatorname{cd}_p(\mathcal{R}_{L/F}X)=[L:F]\cdot\operatorname{cd}_p(X)$. In particular, $\mathcal{R}_{L/F}X$ is p-incompressible provided that X is so.*

Proof. Taking a minimal basis of the subgroup in $\operatorname{Br}(L)$ generated by the algebras (note that any L-algebra representing an element of this subgroup is balanced), consider the product X' of their Severi-Brauer varieties. The variety X' is p-incompressible and equivalent to X. Moreover, its Weil transfer $\mathcal{R}_{L/F}X'$ is equivalent to $\mathcal{R}_{L/F}X$. Therefore we reduced the proof of the first statement of Corollary 4.6 to the proof of the second one. The second statement follows directly from Corollary 3.6. □

4c. **Weil transfer of products of generalized Severi-Brauer varieties.** The second conclusion of Corollary 4.6 also holds for generalized Severi-Brauer varieties. Again, the result is an immediate consequence of Corollary 3.6:

Corollary 4.7. *Let L/F be a p-primary separable field extension and let X be a product over L of some generalized Severi-Brauer varieties of some balanced p-primary division L-algebras. Then $\mathcal{R}_{L/F}X$ is p-incompressible if (and only if) X is so.* □

5. Quadratic extensions

Let L/F be a separable quadratic field extension and let D be a central division L-algebra of a 2-primary index. In this section, we determine canonical 2-dimension of the variety $\mathcal{R}_{L/F}\operatorname{SB}(D)$ without imposing any restrictions on the conjugate algebra $g(D)$, where g is the non-trivial element of $G := \operatorname{Gal}(L/F)$. We will provide two different recipes; which of them has to be applied depends on a property of the group A generated by the Brauer classes of D and $g(D)$. Since the case of cyclic A has been already treated, we assume that A is not cyclic. Therefore $A/2A \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ with g acting by exchanging the summands.

Let $\alpha \in A \setminus 2A$ be an element of the smallest index. Let D_α be a central division L-algebra representing α. If the image of α in $A/2A$ is not invariant under the action of G, then the images of α and $g(\alpha)$ form a minimal basis of $A/2A$ in the sense of [10, Remark 2.9]. It follows that the variety $(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))_L \simeq \operatorname{SB}(D_\alpha)\times\operatorname{SB}(g(D_\alpha))$ is 2-incompressible. Therefore the variety $\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha)$ is 2-incompressible as well. Since α and $g(\alpha)$ generate A, the varieties $\mathcal{R}_{L/F}\operatorname{SB}(D)$ and $\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha)$ are equivalent. So, $\operatorname{cd}_2\mathcal{R}_{L/F}\operatorname{SB}(D) = \operatorname{cd}_2\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha)$ and we get that

$$\operatorname{cd}_2\mathcal{R}_{L/F}\operatorname{SB}(D) = \dim\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha) = 2(\operatorname{ind}\alpha - 1). \tag{5.1}$$

Now we assume that the image of α in $A/2A$ is invariant under G. This means that this image is equal to the element $(1,1) \in \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Let $\beta \in A$ be an element of the smallest index with the property that its image in $A/2A$ is outside of the subgroup generated by the image of α. We claim that in this case

$$\operatorname{cd}_2\mathcal{R}_{L/F}\operatorname{SB}(D) = \dim\mathcal{R}_{L/F}\operatorname{SB}(D_\beta) = 2(\operatorname{ind}\beta - 1). \tag{5.2}$$

Since the image of β in $A/2A$ is $(1,0)$ or $(0,1)$, it is not invariant under G. Moreover, A is generated by β and $g(\beta)$. It follows that the variety $\mathcal{R}_{L/F}\operatorname{SB}(D)$ is equivalent to $\mathcal{R}_{L/F}\operatorname{SB}(D_\beta)$, where D_β is a central division L-algebra representing β. To prove the claim, it suffices to prove that the variety $\mathcal{R}_{L/F}\operatorname{SB}(D_\beta)$ is 2-incompressible. It is so because the $F(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))$-variety

$$\mathcal{R}_{L/F}\operatorname{SB}(D_\beta)_{F(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))} \simeq \mathcal{R}_{L(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))/F(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))}\operatorname{SB}((D_\beta)_{L(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))})$$

is 2-incompressible. Indeed, $g(D_\beta)_{L(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))}$ is Brauer-equivalent to a tensor power of $(D_\beta)_{L(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))}$ so that Theorem 1.1 applies. On the other hand, $(D_\beta)_{L(\mathcal{R}_{L/F}\operatorname{SB}(D_\alpha))}$ is still a division algebra by the Schofield–van den Bergh index reduction formula [16] (see also [13]), because

$$\operatorname{ind}(D_\beta\otimes D_\alpha^{\otimes i}) \geq \operatorname{ind} D_\beta$$

for any i by the minimality of $\operatorname{ind}\beta = \operatorname{ind} D_\beta$.

We finish this section by examples where the above recipes apply.

Example 5.3. Let l be a field of characteristic $\neq 2$, let L be the rational function field over l in variables x, y, x', y', let g be the l-automorphism of L exchanging x with x' and y

with y', and let F be the subfield of L consisting of the elements fixed by g. The variety $\mathcal{R}_{L/F}\operatorname{SB}(D)$, where D is the quaternion L-algebra (x, y), is 2-incompressible by formula (5.1) with $D_\alpha = D$. Note that in this example the algebras D and $g(D)$ are, informally speaking, "completely independent".

In the next example, D and $g(D)$ will be "partially dependent". To get it, we replace F and L by the rational function fields

$$F(s_1, s_2, t_1, t_2) \quad \text{and} \quad L(s_1, s_2, t_1, t_2)$$

in some variables s_1, s_2, t_1, t_2. For $D := C \otimes (x, y)$ with $C := (s_1, t_1) \otimes (s_2, t_2)$, the variety $\mathcal{R}_{L/F}\operatorname{SB}(D)$ is 2-incompressible by formula (5.2) with $D_\alpha = (x, y) \otimes (x', y')$ and $D_\beta = D$. Note that the non-zero elements of the group A in this situation are the Brauer classes of the index 8 conjugate algebras $C \otimes (x, y)$ and $C \otimes (x', y')$ and the index 4 invariant algebra $(x, y) \otimes (x', y')$; besides, $2A = 0$.

References

[1] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[2] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[3] Hall, Jr., M. *The theory of groups*. The Macmillan Co., New York, N.Y., 1959.

[4] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[5] Karpenko, N. A. Weil transfer of algebraic cycles. *Indag. Math. (N.S.) 11*, 1 (2000), 73–86.

[6] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[7] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.

[8] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.

[9] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[10] Karpenko, N. A., and Merkurjev, A. S. Essential dimension of finite p-groups. *Invent. Math. 172*, 3 (2008), 491–508.

[11] Karpenko, N. A., and Reichstein, Z. A numerical invariant for linear representations of finite groups. Linear Algebraic Groups and Related Structures (preprint server) 534 (2014, May 15, revised: 2014, June 18), 24 pages.

[12] Merkurjev, A. S. Essential dimension. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.

[13] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.

[14] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. II. *K-Theory 14*, 2 (1998), 101–196.

[15] Saltman, D. J. The Schur index and Moody's theorem. *K-Theory 7*, 4 (1993), 309–332.

[16] Schofield, A., and Van den Bergh, M. The index of a Brauer class on a Brauer-Severi variety. *Trans. Amer. Math. Soc. 333*, 2 (1992), 729–739.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko