INCOMPRESSIBILITY OF PRODUCTS

NIKITA A. KARPENKO

ABSTRACT. We show that the conjectural criterion of p-incompressibility for products of projective homogeneous varieties in terms of the factors, previously known in a few special cases only, holds in general. We identify the properties of projective homogeneous varieties actually needed for the proof to go through. For instance, generically split (non-homogeneous) varieties also satisfy these properties.

Let F be a field. A smooth complete irreducible F-variety X is *incompressible*, if every rational self-map $X \dashrightarrow X$ is dominant. This means that $\operatorname{cdim} X = \dim X$, where the *canonical dimension* $\operatorname{cdim} X$ is defined as the minimum of $\dim Y$ for Y running over closed irreducible subvarieties of X admitting a rational map $X \dashrightarrow Y$.

For the whole exposition, let p be a fixed prime number. *Canonical p-dimension* $\operatorname{cdim}_p X$ is defined as the minimum of $\dim Y$ for Y running over closed irreducible subvarieties of X admitting a degree 0 correspondence $X \overset{p'}{\rightsquigarrow} Y$ of p-prime multiplicity. The variety X is *p-incompressible*, if every degree 0 self-correspondence $X \overset{p'}{\rightsquigarrow} X$ of p-prime multiplicity is dominant, i.e., if $\operatorname{cdim}_p X = \dim X$. The closure of the graph of a rational map is a degree 0 correspondence of multiplicity 1; therefore a p-incompressible (for at least one p) variety is incompressible.

Studying canonical p-dimension, instead of the integral Chow group CH, it is more appropriate to use the Chow group Ch with coefficients in $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Multiplicities of correspondences as well as degrees of 0-cycles take then values in $\mathbb{F}_p$. We also consider the Chow motives with coefficients in $\mathbb{F}_p$, see [2, Chapter XII].

Now we are going to introduce a class of varieties, called *nice* here, for which we can prove that the following criterion holds (see Theorem 9): the product $X \times Y$ of F-varieties X and Y is p-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible.

A smooth complete variety is *split*, if its motive decomposes into a finite direct sum of Tate motives. By *Tate motive*, we mean an arbitrary shift of the motive of the point $\operatorname{Spec} F$. For instance, an (absolutely) cellular variety is split, [2, Corollary 66.4].

A smooth complete variety X is *nice*, if it has the following three properties:

(i) The variety X is *geometrically split*, that is, there exists a field extension L/F such that the L-variety X_L is split.

Date: January 31, 2015.

Key words and phrases. Algebraic groups; projective homogeneous varieties; Severi-Brauer varieties; Chow groups and motives; canonical dimension and incompressibility. *Mathematical Subject Classification (2010):* 20G15; 14C25.

This work has been supported by a Start-Up Grant of the University of Alberta and a Discovery Grant from the National Science and Engineering Board of Canada.

(ii) The variety X is *A-trivial* (cf. [11, Definition 2.3]), that is, for any field extension L/F with $X(L) \neq \emptyset$, the degree homomorphism $\deg : \mathrm{Ch}_0(X_L) \to \mathbb{F}_p$ is an isomorphism.

(iii) For any field extension L/F, one has $\mathrm{cdim}_p X \geq d$, where d is the minimal integer such that there exist an element $a \in \mathrm{Ch}_d X_L$ and an element $b \in \mathrm{Ch}^d(X_{L(X)})$ with $\deg(a_{L(X)} \cdot b) = 1$ (see Remarks 3 and 4).

Remark 1. The definition of "nice" depends on the prime p. We should probably better say "p-nice", but we keep saying "nice" for short. The same applies to "split" and "A-trivial". On the other hand, we do not abbreviate "p-incompressible".

Remark 2. A nice variety remains nice under any base field extension. On the other hand, it is not clear if the product of two nice varieties is necessarily nice.

Remark 3. Property (iii), referring to the function field of X_L, is well-defined because any A-trivial variety is geometrically integral, see [11, Remark 2.4]. In particular, any nice variety is geometrically integral.

Remark 4. The opposite to the inequality in (iii) always holds (cf. [9, Proof of Theorem 5.8, part "$\leq$"]). Indeed, take the minimal d such that there exist $a \in \mathrm{Ch}_d X$, and $b \in \mathrm{Ch}^d(X_{F(X)})$ with $\deg(a_{F(X)} \cdot b) = 1$. We may assume that $a = [Y]$ and $b = [Z]$ for closed subvarieties $Y \subset X$ and $Z \subset X_{F(X)}$. Since the product $[Y_{F(X)}] \cdot [Z] \in \mathrm{Ch}(X_{F(X)})$, which is a 0-cycle class of degree 1, can be represented by a 0-cycle with support on the intersection $Y_{F(X)} \cap Z$ (see [3, §8.1]), the variety $Y_{F(X)}$ has a 0-cycle of degree 1, that is, there exists a degree 0 correspondence $X \rightsquigarrow Y$ of multiplicity 1 (see [2, Page 328] concerning the relation between correspondences and 0-cycles). Therefore $\mathrm{cdim}_p X \leq \dim Y = d$.

Here is our basic example of nice varieties:

Example 5. Any projective homogeneous (under an action of a semi-simple affine algebraic group) variety over a p-special field is nice: see [13] for (i), [11, Example 2.5] for (ii), and [6, Proposition 6.1] for (iii). A field F is *p-special*, if it has no finite extension fields of degree prime to p. The condition that F is p-special is only needed for (iii).

A smooth complete geometrically irreducible F-variety is *generically split*, if for any field extension L/F with $X(L) \neq \emptyset$, the L-variety X_L is split.

Example 6. Any generically split variety is nice. Indeed, (i) holds for $L = F(X)$, (ii) holds by [9, discussion after Remark 5.6], and (iii) holds by [9, Theorem 5.8 with Remark 5.6].

The direct product of two projective homogeneous varieties is also projective homogeneous and therefore – over a p-special field – nice. Similarly, the direct product of two generically split varieties is generically split (and nice). The mixed product (over a p-special field) turns out to be nice as well:

Example 7. Over a p-special field, the direct product X of a projective homogeneous variety by a generically split one is nice. Indeed, X is, clearly, geometrically split and A-trivial. Property (iii) can be obtained for X in the same way as it is obtained for a projective homogeneous variety in [6, Proposition 6.1]. The upper motive $U(X)$, used in the proof of [6, Proposition 6.1], is defined for X in [7]; [5, Theorem 5.1 and Proposition

5.2], also used in the proof of [6, Proposition 6.1], can be proved for X by almost literal repetition of their proofs; the same is valid for [7, Theorem 1.1], used in the proof of [5, Proposition 5.2].

The following well-known criterion of p-incompressibility for projective homogeneous varieties actually holds for arbitrary A-trivial varieties:

Lemma 8. *An A-trivial variety X is p-incompressible if and only if* $\operatorname{mult}\rho = \operatorname{mult}\rho^t$ *for any degree* 0 *correspondence $\rho : X \rightsquigarrow X$, where ρ^t is the transpose of ρ. In particular, this criterion holds for any nice variety X.*

Proof. We almost repeat the proof of [5, Lemma 2.7].

If X is p-compressible, there exists a correspondence $\alpha : X \rightsquigarrow Y$ of degree 0 and multiplicity 1 to a proper closed subvariety $Y \subset X$. Considering α as a correspondence $X \rightsquigarrow X$, we have $\operatorname{mult}\alpha = 1$ and $\operatorname{mult}\alpha^t = 0$. Therefore the "only if" part of Lemma 8 holds for arbitrary smooth complete irreducible varieties X, not only for A-trivial ones.

The other way round, suppose that we are given a degree 0 correspondence $\alpha : X \rightsquigarrow X$ with $\operatorname{mult}\alpha \neq \operatorname{mult}\alpha^t$. Adding a multiple of the diagonal class and multiplying by an element of $\mathbb{F}_p$, we may achieve that $\operatorname{mult}\alpha = 1$ and $\operatorname{mult}\alpha^t = 0$. In this case the pull-back of α with respect to the morphism $X_{F(X)} \to X \times X$ induced by the generic point of the second factor of the product $X \times X$, is a 0-cycle class of degree 0. Since X is A-trivial, the degree homomorphism $\operatorname{Ch}_0(X_{F(X)}) \to \mathbb{F}_p$ is an isomorphism. Therefore the pull-back of α is 0. By the continuity property of Chow groups [2, Proposition 52.9], there exists a non-empty open subset $U \subset X$ such that the pull-back of α to $X \times U$ is already 0. By the localization sequence [2, Proposition 57.9], it follows that α is the push-forward of some degree 0 correspondence $\beta : X \rightsquigarrow Y \in \operatorname{Ch}_{\dim X}(X \times Y)$, where Y is the proper closed subset $Y := X \setminus U$ of X. Since $\operatorname{mult}\beta = \operatorname{mult}\alpha = 1$, the variety X is p-compressible. □

The main result of this note is the "$\geq$" part of equality (10) in the following theorem:

Theorem 9. *Let X and Y be nice F-varieties such that the product $X \times Y$ is also nice. The variety $X \times Y$ is p-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible. Moreover*

$$\operatorname{cdim}_p(X \times Y) = \operatorname{cdim}_p X_{F(Y)} + \operatorname{cdim}_p Y_{F(X)} \tag{10}$$

provided that at least one of the three varieties $X_{F(Y)}$, $Y_{F(X)}$, $X \times Y$ is p-incompressible.

Corollary 11. *The product $X \times Y$ of projective homogeneous F-varieties X and Y is p-incompressible if and only if the varieties $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible. Moreover,* (10) *holds provided that at least one of the varieties $X_{F(Y)}$, $Y_{F(X)}$, $X \times Y$ is p-incompressible.*

Proof. Since canonical p-dimension of a variety does not change under any base field extension of degree prime to p (see [16, Proposition 1.5]), we may assume that F is p-special. By Example 5, X, Y, and $X \times Y$ are nice in this case so that Theorem 9 applies. □

Partial cases of Corollary 11, dealing with some special types of projective homogeneous varieties, have been recently proved in [8] and [4]. For an older result in this direction see Example 13 below.

The p-incompressibility criterion, given in Theorem 9 for nice products of two nice varieties, immediately generalizes to finite products of arbitrary length:

Corollary 12. *For $n \geq 1$, let $X_1, \dots, X_n$ be F-varieties such that every sub-product of the product $X := X_1 \times \dots \times X_n$ is nice. Then X is p-incompressible if and only if for every $i = 1, \dots, n$ the variety $(X_i)_{F(X_1 \times \dots \times X_{i-1} \times X_{i+1} \times X_n)}$ is p-incompressible. The criterion also holds if for any $i = 1, \dots, n$ the variety X_i is projective homogeneous or generically split.*

Proof. Assuming that the statement holds for some $n \geq 1$, we prove it for $n+1$. Set $X := X_1 \times \dots \times X_n$ and $Y := X_{n+1}$. If $X \times Y = X_1 \times \dots \times X_{n+1}$ is p-incompressible, $X_{F(Y)}$ and $Y_{F(X)}$ are p-incompressible, and it follows by induction hypothesis that the variety $(X_i)_{F(X_1 \times \dots \times X_{i-1} \times X_{i+1} \times X_{n+1})}$ is p-incompressible for any $i = 1, \dots, n+1$.

The other way round, if $(X_i)_{F(X_1 \times \dots \times X_{i-1} \times X_{i+1} \times X_{n+1})}$ is p-incompressible for any i, then, in particular, $Y_{F(X)}$ is p-incompressible and – by induction hypothesis – $X_{F(Y)}$ is p-incompressible. It follows that $X \times Y$ is p-incompressible. The first statement is proved.

Since any finite direct product of projective homogeneous or generically split varieties over a p-special field is nice (see Example 7), the second statement follows. □

Example 13. For purpose of computing the essential dimension of finite groups, Corollary 12 for Severi-Brauer varieties $X_1, \dots, X_n$ has been obtained in [10]. A second and simpler proof has been given in [8]. The third proof, given here (see Proof of Theorem 9), is particularly simple. The result has numerous further applications, see, e.g., [14, 15].

Example 14. For purpose of computing the essential dimension of representations of finite groups, introduced in [12], Corollary 12 for Weil transfers of generalized Severi-Brauer varieties has been obtained in [8] under assumption that the corresponding central simple algebras are *balanced*. Corollary 12 shows that this assumption is superfluous. Another area of applications for this result is provided in [1].

Proof of Theorem 9. We start by introducing some notation and by making some preliminary observations.

We fix a field extension $\bar{F}/F$ splitting both X and Y. For any finite direct product T of copies of X and Y, we write $\bar{T}$ for $T_{\bar{F}}$. We work with the Chow group $\operatorname{Ch}\bar{T}$ with coefficients in $\mathbb{F}_p$. Note that for any field extension $E/\bar{F}$, the change of field homomorphism $\operatorname{Ch}\bar{T} = \operatorname{Ch}T_{\bar{F}} \to \operatorname{Ch}T_E$ is an isomorphism, so that we may identify $\operatorname{Ch}T_E$ with $\operatorname{Ch}\bar{T}$. For a geometrically integral F-variety S (e.g., $S = X$, $S = Y$, or $S = X \times Y$), an element $c \in \operatorname{Ch}\bar{T} = \operatorname{Ch}T_{\bar{F}} = \operatorname{Ch}T_{\bar{F}(S)}$ is $F(S)$-*rational*, if it lies in the image of the change of field homomorphism $\operatorname{Ch}T_{F(S)} \to \operatorname{Ch}T_{\bar{F}(S)}$.

Since the varieties $\bar{X}$ and $\bar{Y}$ are split, any correspondence $\lambda : \bar{T} \rightsquigarrow \bar{T}'$, where T and T' are finite direct products of copies of X and Y, decomposes in a finite sum of external products $c \times c'$ with homogeneous $c \in \operatorname{Ch}\bar{T}$ and $c' \in \operatorname{Ch}\bar{T}'$. This makes it easy to perform computations with correspondences. For instance, the composition of composable correspondences $(e' \times e'') \circ (c \times c')$ is equal to

$$(e' \times e'') \circ (c \times c') = \deg(c' \cdot e') \cdot (c \times e''). \tag{15}$$

For $e \in \operatorname{Ch}\bar{T}$, $(c \times c')_*(e) = \deg(c \cdot e) \cdot c'$, where $(c \times c')_* : \operatorname{Ch}\bar{T} \to \operatorname{Ch}\bar{T}'$ is the induced by $c \times c'$ homomorphism, see [2, §62]. If $b \in \operatorname{Ch}T'_{\bar{F}(T)} = \operatorname{Ch}\bar{T}'$ is the image of λ under the pull-back with respect to the morphism $\bar{T}'_{\bar{F}(T)} \to \bar{T} \times \bar{T}'$, given by the generic point of T, then λ decomposes as

$$\lambda = [\bar{T}] \times b + \dots, \tag{16}$$

where $\dots$ stands for a sum of $c \times c'$ with $\operatorname{codim} c > 0$ (and $\dim c' > \dim b$). If the correspondence λ has degree 0, then it decomposes as

$$\lambda = (\operatorname{mult} \lambda) \cdot ([\bar{T}] \times [\mathbf{pt}']) + \dots,$$

where $\mathbf{pt}'$ is a rational point on $\bar{T}'$ and where $\dots$ stands for a sum of $c \times c'$ with $\operatorname{codim} c = \dim c' > 0$; if moreover $\dim T = \dim T'$, then

$$\lambda = (\operatorname{mult} \lambda) \cdot ([\bar{T}] \times [\mathbf{pt}']) + (\operatorname{mult} \lambda^t) \cdot ([\mathbf{pt}] \times [\bar{T}']) + \dots, \tag{17}$$

where $\mathbf{pt}$ is a rational point on $\bar{T}$ and where $\dots$ stands for a sum of $c \times c'$ with $\operatorname{codim} c = \dim c' > 0$ and $\dim c = \operatorname{codim} c' > 0$.

In order to prove Theorem 9 in whole, we only need to prove equality (10). We start the prove of its (more difficult) "$\geq$" part now. If the variety $X \times Y$ is p-incompressible, the "$\geq$" part is however trivial. We therefore assume that the $F(X)$-variety $Y_{F(X)}$ is p-incompressible, that is, $\operatorname{cdim}_p Y_{F(X)} = \dim Y$.

Let d be an integer such that there exist F-rational $a \in \operatorname{Ch}_d(\bar{X} \times \bar{Y})$ and $F(X \times Y)$-rational $b \in \operatorname{Ch}^d(\bar{X} \times \bar{Y})$ with $\deg(a \cdot b) = 1$. Since the product $X \times Y$ is nice, we have $\operatorname{cdim}_p(X \times Y) \geq d$. Our aim is to show that $d \geq \operatorname{cdim}_p X_{F(Y)} + \dim Y$.

Let $\alpha \in \operatorname{Ch}(\bar{X} \times \bar{Y} \times \bar{X} \times \bar{Y})$ be the push-forward of a under the diagonal morphism of $\bar{X} \times \bar{Y}$. The element α is F-rational. Note that $\alpha = (a \times [\bar{X}] \times [\bar{Y}]) \cdot \Delta$, where $\Delta \in \operatorname{Ch}(\bar{X} \times \bar{Y} \times \bar{X} \times \bar{Y})$ is the diagonal class.

Let β be a homogeneous F-rational preimage of b under the flat pull-back

$$\operatorname{Ch}\big((\bar{X} \times \bar{Y}) \times (\bar{X} \times \bar{Y})\big) \to \operatorname{Ch}(\bar{X} \times \bar{Y})_{\bar{F}(X \times Y)},$$

along the morphism induced by the generic point of the first factor of the product $(\bar{X} \times \bar{Y}) \times (\bar{X} \times \bar{Y})$. For existence of β, see [2, Corollary 57.11].

Let $\delta \in \operatorname{Ch}(\bar{Y} \times \bar{X} \times \bar{Y})$ be the image of the diagonal class of Y under the push-forward with respect to the closed imbedding $\bar{Y} \times \bar{Y} \hookrightarrow \bar{Y} \times \bar{X} \times \bar{Y}$ induced by a closed rational point $\mathbf{pt}_{\bar{X}}$ on $\bar{X}$. Since the element $[\mathbf{pt}_{\bar{X}}] \in \operatorname{Ch} \bar{X}$ is $F(X)$-rational, the element δ is also $F(X)$-rational.

Finally, let $\gamma \in \operatorname{Ch}(\bar{X} \times \bar{Y} \times \bar{Y})$ be the class of the graph of the projection $\bar{X} \times \bar{Y} \to \bar{Y}$. The element γ is F-rational.

We consider the elements $\alpha, \beta, \gamma, \delta$ as correspondences and take their composition ρ in the following order:

$$\rho : \bar{Y} \overset{\delta}{\rightsquigarrow} \bar{X} \times \bar{Y} \overset{\beta}{\rightsquigarrow} \bar{X} \times \bar{Y} \overset{\alpha}{\rightsquigarrow} \bar{X} \times \bar{Y} \overset{\gamma}{\rightsquigarrow} \bar{Y}.$$

The correspondence $\rho : \bar{Y} \rightsquigarrow \bar{Y}$ is $F(X)$-rational.

Let $\mathbf{pt}_{\bar{Y}}$ be a rational point on $\bar{Y}$. Since the variety Y is A-trivial, the class $[\mathbf{pt}_{\bar{Y}}]$ does not depend on the choice of $\mathbf{pt}_{\bar{Y}}$. A direct computation shows that

$$\rho_*([\mathbf{pt}_{\bar{Y}}]) = [\mathbf{pt}_{\bar{Y}}],$$

where $\rho_* : \operatorname{Ch}\bar{Y} \to \operatorname{Ch}\bar{Y}$ is the homomorphism induced by ρ. Indeed,

$$[\mathbf{pt}_{\bar{Y}}] \overset{\delta_*}{\mapsto} [\mathbf{pt}_{\bar{X}}] \times [\mathbf{pt}_{\bar{Y}}] \overset{\beta_*}{\mapsto} b \overset{\alpha_*}{\mapsto} [\mathbf{pt}_{\bar{X}}] \times [\mathbf{pt}_{\bar{Y}}] \overset{\gamma_*}{\mapsto} [\mathbf{pt}_{\bar{Y}}],$$

where the image under β_* is computed via the formulae (16) and (15).

The general formula $\rho_*([\mathbf{pt}_{\bar{Y}}]) = (\operatorname{mult}\rho)[\mathbf{pt}_{\bar{Y}}]$ implies that $\operatorname{mult}\rho = 1$. Since the A-trivial $F(X)$-variety $Y_{F(X)}$ is p-incompressible while ρ is $F(X)$-rational, it follows by Lemma 8 that $\operatorname{mult}\rho^t = 1$. The general formula $\rho_*([\bar{Y}]) = (\operatorname{mult}\rho^t)[\bar{Y}]$ shows now that $\rho_*([\bar{Y}]) = [\bar{Y}]$. We therefore have

$$[\bar{Y}] \overset{\delta_*}{\mapsto} [\mathbf{pt}_{\bar{X}}] \times [\bar{Y}] \overset{\beta_*}{\mapsto} b' \overset{\alpha_*}{\mapsto} [\mathbf{pt}_{\bar{X}}] \times [\bar{Y}] + \dots \overset{\gamma_*}{\mapsto} [\bar{Y}]$$

for some $b' \in \operatorname{Ch}(\bar{X} \times \bar{Y})$, where $\dots$ stands for a sum of $c \times c'$ with $\dim c = \operatorname{codim} c' > 0$ so that the whole sum is an arbitrary element of $\operatorname{Ch}_{\dim Y}(\bar{X} \times \bar{Y})$ mapped to $[\bar{Y}]$ under γ_*.

The diagonal class $\Delta \in \operatorname{Ch}(\bar{X} \times \bar{Y} \times \bar{X} \times \bar{Y})$ is the external product of the diagonal classes $\Delta_X \in \operatorname{Ch}(\bar{X} \times \bar{X})$ and $\Delta_Y \in \operatorname{Ch}(\bar{Y} \times \bar{Y})$. Multiplying decompositions (17) of Δ_X and Δ_Y, we get a decomposition of Δ. This decomposition of Δ possesses a unique summand ending with $[\mathbf{pt}_{\bar{X}}] \times [\bar{Y}]$. This unique summand starts with $[\bar{X}] \times [\mathbf{pt}_{\bar{Y}}]$. Moreover, any other summand ends with $c \times c'$ such that $\dim c > 0$ or $\operatorname{codim} c' > 0$. The resulting decomposition of $\alpha = (a \times [\bar{X}] \times [\bar{Y}]) \cdot \Delta$ also possesses a unique summand ending with $[\mathbf{pt}_{\bar{X}}] \times [\bar{Y}]$. This unique summand starts now with $a' := a \cdot ([\bar{X}] \times [\mathbf{pt}_{\bar{Y}}])$. Any other summand still ends with $c \times c'$, where $\dim c > 0$ or $\operatorname{codim} c' > 0$. Therefore, by (15), we must have $\deg(a' \cdot b') = 1$ in order to get the right image of b' under α_*.

Let pr be the projection $\bar{X} \times \bar{Y} \to \bar{X}$. It follows that $\deg(a'' \cdot b'') = 1$, where

$$a'' := pr_*(a') \in \operatorname{Ch}\bar{X} \quad \text{and} \quad b'' := pr_*\big(([\bar{X}] \times [\mathbf{pt}_{\bar{Y}}]) \cdot b'\big) \in \operatorname{Ch}\bar{X}.$$

Since a'' is $F(Y)$-rational and b'' is $F(X \times Y)$-rational, it follows by Remark 4 that $\dim a'' \geq \operatorname{cdim}_p X_{F(Y)}$. Since $\dim a'' = \dim a' = \dim a - \dim Y = d - \dim Y$, we get that $d \geq \operatorname{cdim}_p X_{F(Y)} + \dim Y$. The "$\geq$" part of equality (10) is proved.

The proof of the "$\leq$" part, given in [8, Lemma 3.4] for projective homogeneous X and Y, also works in our current settings. For reader's convenience, let us reproduce it. As in [8, Lemma 3.4], we prove the more general inequality

$$\operatorname{cdim}_p(X \times Y) \leq \operatorname{cdim}_p X + \operatorname{cdim}_p Y_{F(X)}$$

without any p-incompressibility assumption (on $X_{F(Y)}$, on $Y_{F(X)}$, or on $X \times Y$).

We set $x := \operatorname{cdim}_p X$ and $y := \operatorname{cdim}_p Y_{F(X)}$. Since the variety X is nice, we can find F-rational $a_X \in \operatorname{Ch}_x \bar{X}$ and $F(X)$-rational $b_X \in \operatorname{Ch}^x \bar{X}$ with $\deg(a_X \cdot b_X) = 1$. Similarly, since the variety $Y_{F(X)}$ is nice, we can find $F(X)$-rational $a_Y \in \operatorname{Ch}_y \bar{Y}$ and $F(X)(Y)$-rational $b_Y \in \operatorname{Ch}^y \bar{Y}$ with $\deg(a_Y \cdot b_Y) = 1$. Let $\alpha_Y \in \operatorname{Ch}_{\dim X + y}(X \times Y)$ be an F-rational preimage of a_Y under the pull-back along the morphism $\bar{Y}_{\bar{F}(X)} \to \bar{X} \times \bar{Y}$ induced by the generic point of X. We set

$$a := (a_X \times [\bar{Y}]) \cdot \alpha_Y \in \operatorname{Ch}_{x+y}(\bar{X} \times \bar{Y}) \quad \text{and} \quad b := b_X \times b_Y \in \operatorname{Ch}^{x+y}(\bar{X} \times \bar{Y}).$$

The element a is F-rational, the element b is $F(X \times Y)$-rational. We have the relation $\deg(a \cdot b) = \deg(a_X \cdot b_X) \cdot \deg(a_Y \cdot b_Y) = 1$ showing by Remark 4 that $\operatorname{cdim}_p(X \times Y) \leq x + y$. $\square$

References

[1] Biswas, I., Dhillon, A., and Hoffmann, N. On the essential dimension of coherent sheaves. arXiv:1306.6432v2 [math.AG] (2 Dec 2014), 17 pages.

[2] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[3] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[4] Karpenko, N. A. Incompressibility of products by Grassmannians of isotropic subspaces. Linear Algebraic Groups and Related Structures (preprint server) 541 (2014, Sept 23), 12 pages.

[5] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[6] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.

[7] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[8] Karpenko, N. A. Incompressibility of products of Weil transfers of generalized Severi-Brauer varieties. *Math. Z.* (2014), 1–11. DOI 10.1007/s00209-014-1393-4.

[9] Karpenko, N. A., and Merkurjev, A. S. Canonical p-dimension of algebraic groups. *Adv. Math. 205*, 2 (2006), 410–433.

[10] Karpenko, N. A., and Merkurjev, A. S. Essential dimension of finite p-groups. *Invent. Math. 172*, 3 (2008), 491–508.

[11] Karpenko, N. A., and Merkurjev, A. S. On standard norm varieties. *Ann. Sci. Éc. Norm. Supér. (4) 46*, 1 (2013), 175–214.

[12] Karpenko, N. A., and Reichstein, Z. A numerical invariant for linear representations of finite groups. Linear Algebraic Groups and Related Structures (preprint server) 534 (2014, May 15, revised: 2014, June 18), 24 pages.

[13] Köck, B. Chow motif and higher Chow theory of G/P. *Manuscripta Math. 70*, 4 (1991), 363–372.

[14] Lötscher, R., MacDonald, M., Meyer, A., and Reichstein, Z. Essential dimension of algebraic tori. *J. Reine Angew. Math. 677* (2013), 1–13.

[15] Lötscher, R., MacDonald, M., Meyer, A., and Reichstein, Z. Essential p-dimension of algebraic groups whose connected component is a torus. *Algebra Number Theory 7*, 8 (2013), 1817–1840.

[16] Merkurjev, A. S. Essential dimension. In *Quadratic Forms – Algebra, Arithmetic, and Geometry*, vol. 493 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2009, pp. 299–326.

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
E-mail address: karpenko *at* ualberta.ca, *web page*: www.ualberta.ca/~karpenko