FORMAL TORSORS UNDER REDUCTIVE GROUP SCHEMES

BENEDICTUS MARGAUX

Laboratoire de Recherche "Princesse Stéphanie"
Monte Carlo 51840, Monaco.
Email: benedictus.margaux@gmail.com

ABSTRACT. We consider the algebraization problem for torsors over a proper formal scheme under a reductive group scheme. Our results apply to the case of semisimple group schemes (which is addressed in detail).

Keywords: Torsors, formal schemes, reductive group schemes, algebraization.

MSC: 14D15, 14L30.

1. Introduction

Throughout this paper R will be a complete noetherian local ring with maximal ideal $\mathfrak{m}$. We put $R_n = R/\mathfrak{m}^{n+1}$ for each $n \geq 0$. The natural map $R \to \varprojlim R_n$ is a ring isomorphism and we will henceforth identify these two rings.

For the theory of formal schemes over R, we refer the reader to [EGAI, §10], [Ha, §II.9] and [St, Tag 0AHW, §79].[1] Let X be a proper R–scheme, and let $\widehat{X}$ be the associated formal scheme. Grothendieck's existence theorem provides an equivalence of categories between the category of coherent sheaves over X and the category of coherent sheaves on the formal scheme $\widehat{X}$ [EGAIII, 5.1.4], [I, §8.4]. The restriction to locally trivial coherent sheaves of constant rank r yields a natural equivalence between the category of GL_r–torsors over X and the category of $\widehat{\mathrm{GL}}_r$–torsors over $\widehat{X}$.

The purpose of the paper is to extend this statement to a larger class of affine group schemes over X which includes semisimple group schemes. This question has been also studied by Baranovsky [B, §3], but only for group schemes arising from R–group schemes by base change.

Conventions on vector groups and linear groups.

We use mainly the terminology and notation of Grothendieck-Dieudonné [EGAI, §9.4 and 9.6], which agrees with that of Demazure-Grothendieck used in [SGA3, Exp. I.4]

Let S be a scheme and let $\mathcal{E}$ be a quasi-coherent sheaf over S. For each morphism $f : T \to S$, we denote by $\mathcal{E}_{(T)} = f^*(\mathcal{E})$ the inverse image of $\mathcal{E}$ by the morphism f.

[1] Since the numbering of the Stacks Project [St] evolves over time, we also provide the relevant tags.

Recall that the S–scheme $\mathbf{V}(\mathcal{E}) = \operatorname{Spec}\big(\operatorname{Sym}^\bullet(\mathcal{E})\big)$ is affine over S and represents the S–functor $T \mapsto \operatorname{Hom}_{\mathcal{O}_T}(\mathcal{E}_{(T)}, \mathcal{O}_T)$ [EGAI, 9.4.9].

We assume now that $\mathcal{E}$ is locally free of finite rank and denote by $\mathcal{E}^\vee$ its dual. In this case the affine S–scheme $\mathbf{V}(\mathcal{E})$ is of finite presentation (ibid, 9.4.11); also the S–functor $T \mapsto H^0(T, \mathcal{E}_{(T)}) = \operatorname{Hom}_{\mathcal{O}_T}(\mathcal{O}_T, \mathcal{E}_{(T)})$ is representable by the affine S–scheme $\mathbf{V}(\mathcal{E}^\vee)$ which is also denoted by $\mathbf{W}(\mathcal{E})$ [SGA3, I.4.6].

The above considerations apply to the locally free coherent sheaf $\mathcal{E}nd(\mathcal{E}) = \mathcal{E}^\vee \otimes_{\mathcal{O}_S} \mathcal{E}$ over S so that we can consider the affine S–scheme $\mathbf{V}\big(\mathcal{E}nd(\mathcal{E})\big)$ which is an S–functor in associative commutative and unital algebras [EGAI, 9.6.2]. Now we consider the S–functor $T \mapsto \operatorname{Aut}_{\mathcal{O}_T}(\mathcal{E}_{(T)})$. It is representable by an open S–subscheme of $\mathbf{V}\big(\mathcal{E}nd(\mathcal{E})\big)$ which is denoted by $\mathrm{GL}(\mathcal{E})$ (*loc. cit.*, 9.6.4).

We set $\mathrm{GL}_{r,S} = \mathrm{GL}(\mathcal{O}_S^r)$ for each $r \geq 1$. If $S = Spec(A)$ is affine, then $\mathcal{E} = \mathcal{O}_S^r$ corresponds to the A-module $E = A^r$. In this case we will use the notation $\mathrm{GL}_r(E)$ instead of $\mathrm{GL}_{r,S}$. Finally, for scheme morphisms $Y \to X \to S$, we denote by $\prod\limits_{X/S}(Y/X)$ the S–functor defined by

$$\Big(\prod_{X/S}(Y/X)\Big)(T) = Y(X \times_S T)$$

for each S–scheme T. Recall that if $\prod\limits_{X/S}(Y/X)$ is representable by an S-scheme, this scheme is called the Weil restriction of Y to S.

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2. Formal torsors

Let R be as above, and let X be a proper R–scheme. We start with the following key observation about limits.

Lemma 2.1. *Let $f : Y \to X$ be a separated morphism of finite type. Then the natural map*

$$\Big(\prod_{X/R}(Y/X)\Big)(R) \to \varprojlim_n \Big(\prod_{X/R}(Y/X)\Big)(R_n) = \varprojlim_n \Big(\prod_{X_n/R_n}(Y_n/X_n)\Big)(R_n)$$

is bijective.

Proof. The last equality follows from the fact that $\prod\limits_{X/S}(Y/X)$ commutes with base change. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_R(X,Y) \arrow[r] & \varprojlim_n \operatorname{Hom}_{R_n}(X_n, Y_n) \\
\Big(\prod\limits_{X/R}(Y/X)\Big)(R) \arrow[r] \arrow[u, hook] & \varprojlim_n \Big(\prod\limits_{X_n/R_n}(Y_n/X_n)\Big)(R_n) \arrow[u, hook]
\end{tikzcd}$$

According to [St, Tag 0898, 29.28.3], the top horizontal map is bijective so that the bottom horizontal map is injective. Let $(s_n : X_n \to Y_n)_{n\geq 0}$ be a coherent family of sections. It lifts to a (unique) morphism $s : X \to Y$. Then the morphism $g = f \circ s : X \to X$ is such that $g_n = id_{X_n}$ for all $n \geq 0$. Since the map $\mathrm{Hom}_R(X, X) \to \varprojlim_n \mathrm{Hom}_{R_n}(X_n, X_n)$ is bijective, we conclude that $g = id_X$ whence s is a section of $Y \to X$. We have shown the surjectivity of the bottom map. $\square$

Let $\mathfrak{G}$ be an affine X–group scheme of finite presentation. We set $X_n = X \times_R R_n$ and $\mathfrak{G}_n = \mathfrak{G} \times_X X_n$ for each $n \geq 0$. We denote by $\widehat{\mathfrak{G}} = (\mathfrak{G}_n)_{n\geq 0}$ the formal group scheme over $\widehat{X}$ attached to $\mathfrak{G}$.

A formal $\widehat{\mathfrak{G}}$-torsor $\widehat{\mathfrak{P}}$ is the data of a $\mathfrak{G}_n$–torsor $\mathfrak{P}_n$ over X_n for each $n \geq 0$ together with compatible $\mathfrak{G}_{n+1}$–isomorphisms $\theta_n : \mathfrak{P}_{n+1} \times_{R_{n+1}} R_n \xrightarrow{\sim} \mathfrak{P}_n$. If $\mathfrak{P}$ is $\mathfrak{G}$–torsor, $\widehat{\mathfrak{P}}$ is formal $\widehat{\mathfrak{G}}$-torsor and this assignment is faithful in the following sense.

Lemma 2.2. *Let $\mathfrak{P}$, $\mathfrak{Q}$ be two $\mathfrak{G}$–torsors. The natural map* $\mathrm{Isom}_{\mathfrak{G}}(\mathfrak{P}, \mathfrak{Q}) \to \mathrm{Isom}_{\widehat{\mathfrak{G}}}(\widehat{\mathfrak{P}}, \widehat{\mathfrak{Q}})$ *is bijective.*

Proof. Up to replacing $\mathfrak{G}$ (resp. $\mathfrak{Q}$) by the twisted R–group scheme ${}^{\mathfrak{P}}\mathfrak{G}$ (resp. $\mathfrak{P}^{op} \wedge^{\mathfrak{G}} \mathfrak{Q}$), we may assume that $\mathfrak{P} = \mathfrak{G}$. In this case, we have $\mathrm{Isom}_{\mathfrak{G}}(\mathfrak{P}, \mathfrak{Q}) = \mathfrak{Q}(X)$ so that our original question is reduced to showing that the natural map

$$\mathfrak{Q}(X) \to \varprojlim_n \mathfrak{Q}_n(X_n)$$

is bijective. Locally for the fppf topology, $\mathfrak{Q}$ is isomorphic to $\mathfrak{G}$. According to the permanence properties of faithfully flat descent $\mathfrak{Q}$ is affine of finite presentation over X [EGAIV, 2.7.1.(vi) and (xiii)]. So Lemma 2.1 applies and shows that the above map is bijective. $\square$

2.1. **Algebraizable torsors.** We say that a formal $\widehat{\mathfrak{G}}$–torsor $\widehat{\mathfrak{P}}$ is *algebraizable* if it arises from a $\mathfrak{G}$–torsor $\mathfrak{P}$. Lemma 2.2 shows that if such a $\mathfrak{P}$ exists, it is unique up to isomorphism.

Lemma 2.3. *Let $\mathfrak{G}$ and $\mathfrak{G}'$ be two X-group schemes which are affine and of finite presentation. Assume that $\mathfrak{G}$ is flat and that $i : \mathfrak{G} \to \mathfrak{G}'$ is a monomorphism of X-group schemes with the property that the fppf quotient $\mathfrak{G}'/\mathfrak{G}$ is representable by an affine X–scheme $\mathfrak{Q}$.*

Let $\widehat{\mathfrak{F}}$ be a $\widehat{\mathfrak{G}}$–torsor and denote by $\widehat{\mathfrak{F}}' = i_*(\widehat{\mathfrak{F}})$ the corresponding $\widehat{\mathfrak{G}}'$–torsor. Then $\widehat{\mathfrak{F}}$ is algebraizable if and only if $\widehat{\mathfrak{F}}'$ is algebraizable.

Proof. It is clear that if $\widehat{\mathfrak{F}}$ is algebraizable then so is $\widehat{\mathfrak{F}}'$. Conversely, assume that the $\widehat{\mathfrak{G}}'$–torsor $\widehat{\mathfrak{F}}'$ is algebraizable i.e. it arises from a $\mathfrak{G}'$–torsor $\mathfrak{F}'$. We consider the affine X-scheme $\mathfrak{Z} = \mathfrak{F}'/\mathfrak{G} := \mathfrak{F}' \wedge^{\mathfrak{G}'} (\mathfrak{G}'/\mathfrak{G})$; the reduction of $\mathfrak{F}'$ to $\mathfrak{G}$ defined by faithfully flat descent. According to [SGA3, VI_B.9.2.(xiii).b], the X-scheme $\mathfrak{G}'/\mathfrak{G}$ is of finite presentation. Since $\mathfrak{Z}$ is locally isomorphic to $\mathfrak{G}'/\mathfrak{G}$ with respect to the fppf topology, the permanence properties of faithfully flat descent show that $\mathfrak{Z}$ is affine of finite presentation over X [EGAIV, 2.7.1.(vi) and (xiii)]. According to Lemma 2.1, the map $\mathfrak{Z}(X) \to \varprojlim_n \mathfrak{Z}_n(X_n)$ is bijective.

Each $\mathfrak{F}_n$ defines a point $z_n \in \mathfrak{Z}(R_n)$ in a coherent way so that we get a point $z \in \mathfrak{Z}(R)$. That point defines a reduction of the $\mathfrak{G}'$–torsor $\mathfrak{F}'$ to a $\mathfrak{G}$–torsor $\mathfrak{F}$ [Gd, III.3.2.1]. Since z maps to z_n, we have $\mathfrak{F}_{R_n} = \mathfrak{F}_n$ for each $n \geq 0$. Thus $\widehat{\mathfrak{F}}$ is algebraizable. □

3. Representations of group schemes

3.1. **The Chevalley case.** Let G be a reductive split $\mathbb{Z}$–group scheme and we denote by G_{ad} its adjoint quotient. We remind the reader that the functor of automorphisms of G is representable by a smooth $\mathbb{Z}$–group scheme $\mathrm{Aut}(G)$ [SGA3, XXIV.1]. Furthermore there is an exact sequence of $\mathbb{Z}$–group schemes

$$1 \to G_{ad} \xrightarrow{int} \mathrm{Aut}(G) \xrightarrow{\pi} \mathrm{Out}(G) \to 1$$

where $\mathrm{Out}(G)$ is a constant group scheme. In other words, $\mathrm{Out}(G)$ is the $\mathbb{Z}$–group scheme attached to the abstract group $\mathrm{Out}(G)(\mathbb{Z})$. In the semisimple case $\mathrm{Out}(G)$ is finite (and in particular $\mathrm{Aut}(G)$ is affine). This is not the case in general. For example $\mathrm{Aut}(\mathbb{G}_m^2)$ is the constant $\mathbb{Z}$–group scheme attached to the abstract group $\mathrm{GL}_2(\mathbb{Z})$.

Let Γ be a finite subgroup of $\mathrm{Out}(G)(\mathbb{Z})$. We get a monomorphism of $\mathbb{Z}$–group schemes $\Gamma_{\mathbb{Z}} \to \mathrm{Out}(G)$ and consider the $\mathbb{Z}$–group scheme

$$\mathrm{Aut}_\Gamma(G) = \mathrm{Aut}(G) \times_{\mathrm{Out}(G)} \Gamma_{\mathbb{Z}}.$$

obtained by pullback. The above yields the exact sequence

$$1 \to G_{ad} \to \mathrm{Aut}_\Gamma(G) \xrightarrow{\pi} \Gamma_{\mathbb{Z}} \to 1.$$

Since $\Gamma_{\mathbb{Z}}$ and G_{ad} are smooth affine over $\mathbb{Z}$, so is $\mathrm{Aut}_\Gamma(G)$ [SGA3, VI_B9.2.(viii)].

Lemma 3.1. *There exists a free $\mathbb{Z}$–module of finite type E, and a closed immersion $\mathbb{Z}$–group scheme homomophism $i : G \rtimes \mathrm{Aut}_\Gamma(G) \to \mathrm{GL}(E)$ such that the fppf quotient sheaf $\mathrm{GL}(E)/G$ (resp. $\mathrm{GL}(E)/\big(G \rtimes \mathrm{Aut}_\Gamma(G)\big)$, $\mathrm{GL}(E)/G_{ad}$) is representable by a smooth affine $\mathbb{Z}$–scheme.*

Proof. Since $G \rtimes \mathrm{Aut}_\Gamma(G)$ is an affine smooth $\mathbb{Z}$–group scheme, there exists a free $\mathbb{Z}$-module of finite rank E and a faithful linear representation $\rho : G \rtimes \mathrm{Aut}_\Gamma(G) \to \mathrm{GL}(E)$ which is a closed immersion [BT, 1.4.5].

The fppf sheaf $\mathrm{GL}(E)/\big(G \rtimes \mathrm{Aut}_\Gamma(G)\big)$ is representable by a $\mathbb{Z}$-scheme [A, Th. IV.4.B] which is smooth and separated [SGA3, VI_B.9.2.(x) and (xii)]. The $\mathbb{Z}$–group scheme $G \rtimes G_{ad}$ is reductive. According to [CTS, 6.12.ii], the fppf sheaf $\mathrm{GL}(E)/(G \rtimes G_{ad})$ is representable by an affine smooth $\mathbb{Z}$–scheme and so are $\mathrm{GL}(E)/G$ and $\mathrm{GL}(E)/G_{ad}$. Since the map $\mathrm{GL}(E)/(G \rtimes G_{ad}) \to \mathrm{GL}(E)/\big(G \rtimes \mathrm{Aut}_\Gamma(G)\big)$ is a $\Gamma_{\mathbb{Z}}$-torsor, it is a finite étale cover. It follows that $\mathrm{GL}(E)/\big(G \rtimes \mathrm{Aut}_\Gamma(G)\big)$ is affine [St, Tag 01YN, lemma 29.13.3]. Similarly the $\mathbb{Z}$-scheme $\mathrm{GL}(E)/\mathrm{Aut}_\Gamma(G)$ is affine. □

3.2. **An isotriviality condition.** In this section, we assume that the base scheme S is noetherian and we are given a reductive S–group scheme $\mathfrak{G}$ of constant type. Thus, there exists a Chevalley $\mathbb{Z}$–group scheme G such that $\mathfrak{G}$ is locally isomorphic to G_S for the étale topology [SGA3, XXII.2.3, 2.5]. Also the fppf sheaf $\underline{\mathrm{Isom}}(G_S, \mathfrak{G})$ is representable by a $\mathrm{Aut}(G)_S$–torsor $\mathrm{Isom}(G_S, \mathfrak{G})$ defined in [SGA3, XXIV.1.8]. The contracted product $\mathrm{Isomext}(G_S, \mathfrak{G}) := \mathrm{Isom}(G_S, \mathfrak{G}) \wedge^{\mathrm{Aut}(G)_S} \mathrm{Out}(G)_S$ is a $\mathrm{Out}(G)_S$–torsor (*ibid*, 1.10) which encodes the isomorphism class of the quasi-split form of $\mathfrak{G}$.

Proposition 3.2. *We assume that the* $\mathrm{Out}(G)_S$*–torsor* $\mathrm{Isomext}(G_S, \mathfrak{G})$ *is isotrivial, i.e. there exists a finite étale cover* S'/S *such that* $\mathrm{Isomext}(G_S, \mathfrak{G})(S') \neq \emptyset$*. Then there exists a locally free coherent* $\mathcal{O}_S$*–module* $\mathcal{E}$*, and a closed immersion* S*–group scheme homomorphism* $i : \mathfrak{G} \to \mathrm{GL}(\mathcal{E})$ *such that the fppf quotient sheaf* $\mathrm{GL}(\mathcal{E})/\mathfrak{G}$ *is representable by a smooth affine* S*–scheme.*

Remark 3.3. (a) If G is semisimple, $\mathrm{Out}(G)$ is a finite constant group so that the isotriviality condition is obviously satisfied.

(b) If S is a normal connected scheme, the isotriviality condition is satisfied since $\mathrm{Isomext}(G_S, \mathfrak{G}) \to S$ is a $\mathrm{Out}(G)_S$–cover [SGA3, X.6.2 and 5.14].

Proof. The noetherian assumption reduces the problem to the connected case (in particular S is non-empty by convention [St, Tag 004R, 5.7.1]). We consider the $\mathrm{Aut}(G)_S$-torsor $\mathfrak{E} = \mathrm{Isom}(G_S, \mathfrak{G})$ defined above; we have $\mathfrak{G} = {}^{\mathfrak{E}}(G_S)$, i.e. $\mathfrak{G}$ is the twist of G_S by the $\mathrm{Aut}(G)_S$–torsor $\mathfrak{E}$.

The isotriviality assumption for the $\mathrm{Out}(G)_S$–torsor $\mathfrak{F} = \mathfrak{E} \wedge^{\mathrm{Aut}(G)_S} \mathrm{Out}(G)_S$ means that there exists a finite étale cover S'/S such that $\mathfrak{F}(S') \neq \emptyset$. Grothendieck's theory of the algebraic fundamental group [SGA1] permits to assume that S' is connected and that $S' \to S$ is a Θ_S–torsor where Θ is a finite abstract group.

We have a bijection $H^1(\Theta, \mathrm{Out}(G)(S')) \xrightarrow{\sim} H^1(S'/S, \mathrm{Out}(G))$ [Gi, end of §2.2]. Since S' is connected, we have $\mathrm{Out}(G)(\mathbb{Z}) = \mathrm{Out}(G)(S')$ so that the action of Θ on $\mathrm{Out}(G)(S')$ is trivial. We have then a bijection

$$\mathrm{Hom}_{gr}\Big(\Theta, \mathrm{Out}(G)(\mathbb{Z})\Big)/\mathrm{Out}(G)(\mathbb{Z}) \xrightarrow{\sim} H^1(\Theta, \mathrm{Out}(G)(S')).$$

It follows that the class of the $\mathrm{Out}(G)_S$–torsor $\mathfrak{F}$ is given by the conjugacy class of a homomorphism $\rho : \Theta \to \mathrm{Out}(G)(\mathbb{Z})$.

Let $\Gamma = \mathrm{Im}(\rho)$, it is a finite subgroup of $\mathrm{Out}(G)(\mathbb{Z})$. We consider the $\mathbb{Z}$–group scheme $\mathrm{Aut}_\Gamma(G) = \pi^{-1}(\Gamma)$ as in the previous section. The isomorphism $\mathrm{Aut}(G)_S/\mathrm{Aut}_\Gamma(G)_S \xrightarrow{\sim} \mathrm{Out}(G)_S/\Gamma_S$ induces an isomorphism $\mathfrak{E}/\mathrm{Aut}_\Gamma(G)_S \xrightarrow{\sim} \mathfrak{F}/\Gamma_S$. The reduction of the $\mathrm{Out}(G)_S$–torsor $\mathfrak{F}$ to Γ_S defines then a reduction of the $\mathrm{Aut}(G)_S$–torsor $\mathfrak{E}$ to a $\mathrm{Aut}_\Gamma(G)_S$–torsor $\mathfrak{E}_\sharp$ [Gd, III.3.2.1]. □

Remark 3.4. (a) If G is semisimple, we can take in the proof $\Gamma = \mathrm{Out}(G)(\mathbb{Z})$. We thus find a $\mathcal{O}_S$–coherent sheaf $\mathcal{E}$ as desired which is $\mathfrak{G} \rtimes \mathrm{Aut}(\mathfrak{G})$-equivariant.

(b) Thomason has proven stronger statements than (1) for embedding group schemes in linear group schemes [T, §3].

4. Main statement

4.1. The following generalization of Grothendieck existence theorem strengthens Baranovsky's result [B, Th. 3.1].

Theorem 4.1. *Let R be a complete noetherian local ring. Let X be a proper R–scheme and let $\widehat{X}$ be the associated formal scheme. Let G be a Chevalley $\mathbb{Z}$–group scheme and let $\mathfrak{G}$ be an X-form of G_X. Assume that the* $\mathrm{Out}(G)_X$*–torsor* $\mathrm{Isomext}(G_X, \mathfrak{G})$ *is isotrivial. Then.*

(1) The assignment $\mathfrak{P} \mapsto \widehat{\mathfrak{P}}$ induces an equivalence of categories between the category of $\mathfrak{G}$–torsors of X and that of $\widehat{\mathfrak{G}}$–torsors over $\widehat{X}$.

(2) Assume that $\mathfrak{G}$ is semisimple. For $\mathfrak{H} = \mathfrak{G}, \mathrm{Aut}(\mathfrak{G}), \mathfrak{G} \rtimes \mathrm{Aut}(\mathfrak{G})$ the assignment $\mathfrak{P} \mapsto \widehat{\mathfrak{P}}$ induces an equivalence of categories between the category of $\mathfrak{H}$–torsors of X and that of $\widehat{\mathfrak{H}}$–torsors over $\widehat{X}$.

Proof. (1) By Lemma 2.2, we have only to show algebraization. The R–scheme X is proper, namely separated, of finite type, and universally closed. Since R is noetherian, X is locally noetherian. Also the morphism $X \to \mathrm{Spec}(R)$ is quasi-compact [St, 28.39.10] so that X is quasi-compact. The scheme X is quasi-compact and locally noetherian, hence is noetherian by definition [St, Tag 01OU, 27.5.1]. Without lost of generality we may assume that X is connected.

Proposition 3.2 provides a closed immersion $i : \mathfrak{G} \to \mathrm{GL}(\mathcal{E})$ where $\mathcal{E}$ is a locally free coherent $\mathcal{O}_X$–module and such that the fppf quotient sheaf $\mathrm{GL}(\mathcal{E})/\mathfrak{G}$ is representable by a smooth affine X–scheme. Lemma 2.3 reduces the algebraization problem to the case of $\mathrm{GL}(\mathcal{E})$. Since X is connected, $\mathcal{E}$ is locally free of rank r. We consider the GL_r–torsor $\mathfrak{Q} = \mathrm{Isom}(\mathcal{O}_X^r, \mathcal{E})$ over X. Torsion by $\mathfrak{Q}$ (resp. $\widehat{\mathfrak{Q}}$) induces an equivalence of categories between the category of GL_r–torsors (resp. $\widehat{\mathrm{GL}}_r$–torsors) and that of $\mathrm{GL}(\mathcal{E})$–torsors (resp. $\widehat{\mathrm{GL}(\mathcal{E})}$–torsors). It follows that the algebraization question is equivalent for GL_r–torsors and for $\mathrm{GL}(\mathcal{E})$–torsors. Grothendieck's existence theorem states that GL_r–torsors over $\widehat{X}$ are algebraizable. Thus algebraization holds for $\mathrm{GL}(\mathcal{E})$ and for $\mathfrak{G}$.

(2) Remark 3.4.(a) shows that the representation $\mathfrak{G} \to \mathrm{GL}(\mathcal{E})$ arises from a representation $\mathfrak{G} \rtimes \mathrm{Aut}(\mathfrak{G}) \to \mathrm{GL}(\mathcal{E})$. The same argument applies then to $\mathfrak{G} \rtimes \mathrm{Aut}(\mathfrak{G})$ and $\mathrm{Aut}(\mathfrak{G})$. □

4.2. Examples and applications. Let $d \geq 1$ be a positive integer. If we consider the case of $\mathfrak{G} = \mathrm{PGL}_n$ and use the dictionary given in [G, §7] between PGL_d-torsors and Azumaya algebras of degree d, we get an algebraization statement for Azumaya algebras of degree d we obtain the following.

Corollary 4.2. *There is an equivalence of categories between Azumaya algebras over X (of degree d) and formal degree d Azumaya algebras over $\widehat{X}$ (of degree d).* □

Similarly, by considering the case of the Chevalley $\mathbb{Z}$–group scheme of type G_2, we obtain an equivalence of categories octonion algebras over X and formal octonion algebras over $\widehat{X}$ [Co, App. B].

More generally for the group scheme $\mathrm{Aut}(G)$ of a semisimple Chevalley $\mathbb{Z}$–group G we have the following fact as special case of Theorem 4.1.(2).

Corollary 4.3. *There is an equivalence of categories between the groupoid of X–forms of G_X and that of formal $\widehat{X}$-forms of $\widehat{G}_X$.* $\square$

In particular, we obtain the following fact.

Corollary 4.4. *Assume that we are given a formal $\widehat{X}$-group scheme $\widehat{\mathfrak{G}}$ such that each $\mathfrak{G}_n$ is an X_n–form of G_{X_n}. Then $\widehat{\mathfrak{G}}$ is algebraizable in a semisimple X–group scheme $\mathfrak{G}$ which is a X-form of G_X.* $\square$

References

[A] S. Anantharaman, *Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1*, Mémoires de la Société Mathématique de France **33** (1973), 5-79.

[B] V. Baranovsky, *Algebraization of bundles on non-proper schemes*, Trans. Amer. Math. Soc. **362** (2010), 427-439.

[BT] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local : II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Publications Mathématiques de l'IHÉS **60** (1984), 5-184.

[CTS] J.–L. Colliot–Thélène, J.–J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Annalen **244** (1979), 105–134.

[Co] B. Conrad, *Non-split reductive groups over $\mathbb{Z}$*, Autour des schémas en groupes, vol II, Panoramas et Synthèses **46** (2015), 193-253.

[EGAI] A. Grothendieck, J.-A. Dieudonné, *Eléments de géométrie algébrique. I*, Grundlehren der Mathematischen Wissenschaften 166; Springer-Verlag, Berlin, 1971.

[EGAIII] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique II*, Publications mathématiques de l'I.H.É.S. no 11 and 17 (1961-1963).

[EGAIV] A. Grothendieck (avec la collaboration de J. Dieudonné), *Eléments de Géométrie Algébrique IV*, Publications mathématiques de l'I.H.É.S. no 20, 24, 28 and 32 (1964 - 1967).

[Gi] P. Gille, *Sur la classification des schémas en groupes semi-simples*, "Autour des schémas en groupes, III", Panoramas et Synthèses **47** (2015), 39-110.

[Gd] J. Giraud, *Cohomologie non-abélienne*, Springer (1970).

[G] A. Grothendieck, *Le groupe de Brauer I : Algèbres d'Azumaya et interprétations diverses*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, 46-66.

[Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Springer.

[I] L. Illusie, *Grothendieck's existence theorem in formal geometry*, with a letter of Jean-Pierre Serre, Math. Surveys Monogr. **123** (2005), Fundamental algebraic geometry, 179-233, Amer. Math. Soc., Providence, RI.

[SGA1] *Séminaire de Géométrie algébrique de l'I.H.É.S., 1960-1961, Revêtements étales et groupes fondamental, dirigé par A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

[SGA3] *Séminaire de Géométrie algébrique de l'I.H.É.S., 1963-1964, Schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

[St] Stacks Project, https://stacks.math.columbia.edu

[T] R. W. Thomason, *Equivariant resolution, linearization, and Hilbert's fourteenth problem over arbitrary base schemes*, Adv. in Math. **65** (1987), 16-34.