Some Observations on Inverses of Band Matrices and Low Rank Perturbations of Triangular Matrices

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1 Introduction

One of the many fields, in which P. Rozsa's contributions play important roles, is that of band matrices and their inverses. See e.g. [8], [9], [10],[6]. This field is not only of a certain mathematical beauty, but is also influenced by applications. For example E. Asplund, whose paper [1] is often cited as the first one to treat inverses of band matrices, mentions that his paper was initiated by the paper [2] of S. Asplund, dealing with the numerical solution of boundary value problems.

The same could be said of these notes, which grew out of intensive discussions concerning certain models of metabolisms, see [4]. Here it was of importance to exhibit in some matrices square submatrices of full rank, or equivalently, describe all singular submatrices, which have to be avoided for that purpose. The matrices involved turned out to be inverses of Hessenberg matrices or slightly perturbed Hessenberg matrices.

In this connection we will treat here the following problem:

Problem 1 Characterize all nonsingular n-by-n matrices B, whose inverses $A = B^{-1}$ are rank one perturbations of lower triangular matrices, i.e.,

$$A = L + uv^T,$$

where L is lower triangular and $u, v \in K^n$.

We see easily that B can be of very different forms. If L is nonsingular, then by the Sherman-Morrison formula, see, e.g., [5, p. 51], we get

$$B = (L + uv^T)^{-1} = L^{-1} - sL^{-1}uv^TL^{-1}, \qquad s = (1 + v^TL^{-1}u)^{-1}.$$

Hence B is of the same form as A. On the other hand, if L is strictly lower triangular, then B is an unreduced lower Hessenberg matrix, containing many zero entries.

It is not too difficult to discuss also the more general problem of perturbations of rank p:

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Problem 2 Characterize all nonsingular n-by-n matrices B, whose inverses $A = B^{-1}$ are rank p perturbations of lower triangular matrices, i.e.,

$$A = L + UV^T,$$

where L is lower triangular and U, V are n-by-p matrices.

Solutions to these two problems are given in section 4. They are intimately connected to the description of inverses of band matrices. In section 3 we give two new characterizations, which we feel to be slightly simpler than those in the original paper [1] by E. Asplund. They are given in terms of a few rank conditions.

Section 2 contains introductory material.

2 Notations and Definitions

All matrices involved are over the field K, where $K = R, C$. A square n-by-n matrix $H = (h_{ij})$ is a lower (upper) Hessenberg matrix if the relation $h_{ij} = 0$ for $j > i + 1 (i > j + 1)$ holds.

It is *unreduced* if in addition $h_{j,j+1} \neq 0, j = 1, \ldots, n-1$ $(h_{j+1,j} \neq 0, j = 1, \ldots, n-1)$.

It is a *band matrix of width* p if $h_{ij} = 0$ for $j > i + p$, and it is a *strict band matrix of width* p, if in addition $h_{i,j} \neq 0, \quad 1 \leq i, j \leq n, j - i = p$. Observe here that formally also negative bandwidths are possible. But for a nonsingular matrix the bandwidth is necessarily nonnegative.

The following result for the inverse of a block matrix is well-known, see e.g., [7].

If

$$G = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{2.1}$$

is nonsingular, where G_2 and G_3 are square, and G_2 is nonsingular, then the Schurcomplement

$$S = G_3 - G_4 G_2^{-1} G_1 \tag{2.2}$$

is nonsingular and

$$G^{-1} = \begin{pmatrix} -S^{-1}G_4G_2^{-1} & S^{-1} \\ G_2^{-1} + G_2^{-1}G_1S^{-1}G_4G_2^{-1} & -G_2^{-1}G_1S^{-1} \end{pmatrix} \tag{2.3}$$

Observe that we can also write it in the form

$$G^{-1} = \begin{pmatrix} 0 & 0 \\ G_2^{-1} & 0 \end{pmatrix} + UV^T \tag{2.4}$$

where

$$U = \begin{pmatrix} S^{-1} \\ -G_2^{-1}G_1S^{-1} \end{pmatrix}$$

and

$$V^T = (G_4G_2^{-1} \qquad I).$$

For an n-by-n matrix A we denote by $A(r, s)$ the r-by-s submatrix of A formed by the first r rows and last s columns of A. We have the following rank relation:

$$\operatorname{rank}(A) \leq \operatorname{rank}(A(r,s)) + (n - r) + (n - s) = 2n - (r + s) + \operatorname{rank}(A(r,s)), \tag{2.5}$$