SMASHING SUBCATEGORIES AND THE TELESCOPE CONJECTURE – AN ALGEBRAIC APPROACH

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Abstract. We give a new characterization of smashing subcategories in a compactly generated triangulated category and prove a modified version of Ravenel's telescope conjecture in this setting. Our results apply in particular to the stable homotopy category. Our approach, however, is purely algebraic; it is based on an analysis of pure injective objects in a compactly generated triangulated category.

Introduction

Let $\mathcal{C}$ be a compactly generated triangulated category, for example the stable homotopy category. Thus $\mathcal{C}$ is a triangulated category with arbitrary coproducts, and $\mathcal{C}$ is generated by a set of compact objects (an object X in $\mathcal{C}$ is *compact* if the representable functor $\mathrm{Hom}(X,-)$ preserves coproducts). Recall that a full triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$ is *localizing* if $\mathcal{B}$ is closed under taking coproducts. A localizing subcategory $\mathcal{B}$ is *smashing* if the right adjoint of the inclusion $\mathcal{B} \to \mathcal{C}$ preserves coproducts.

Theorem A. *Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{C}$, and denote by Φ the class of maps between compact objects in $\mathcal{C}$ which are sent to zero under the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$. Then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is smashing;*
(2) *an object X in $\mathcal{C}$ belongs to $\mathcal{B}$ if and only if every map $C \to X$ from a compact object C factors through a map $C \to D$ in Φ;*
(3) *an object X in $\mathcal{C}$ satisfies $\mathrm{Hom}(\mathcal{B}, X) = 0$ if and only if $\mathrm{Hom}(\Phi, X) = 0$.*

Given any class Φ of maps in $\mathcal{C}$, we say that a localizing subcategory $\mathcal{B}$ is *generated* by Φ if $\mathcal{B}$ is the smallest localizing subcategory of $\mathcal{C}$ such that every map in Φ is sent to zero under the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$.

Corollary. *Every smashing subcategory is generated by a set of maps between compact objects.*

The above statement is a modified version of the following "telescope conjecture" which is based on conjectures of Ravenel [19, 1.33] and Bousfield [3, 3.4]:

Every smashing subcategory is generated by a set of identity maps between compact objects.

This conjecture is known to be false [20]. In fact, Keller gives an example of a smashing subcategory which contains no non-zero compact object [11].

Our analysis of smashing subcategories is based on the concept of purity for compactly generated categories. Let us call a map $X \to Y$ in $\mathcal{C}$ a *pure monomorphism* if the induced map $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$ is a monomorphism for all compact objects C. An object X is called *pure-injective* if every pure monomorphism $X \to Y$ splits. These definitions are motivated by analogous concepts for the category of modules over a ring [4]. In this context one frequently studies the indecomposable pure-injective modules; they form

the Ziegler spectrum of the ring [24]. We shall see that the isomorphism classes of indecomposable pure-injective objects in $\mathcal{C}$ form a set which we denote by $\operatorname{Sp}\mathcal{C}$.

Theorem B. *Let $\mathcal{B}$ be a smashing subcategory of $\mathcal{C}$, and let $\mathbf{U}$ be the set of objects X in $\operatorname{Sp}\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{B}, X) = 0$. Then the following holds for any object X in $\mathcal{C}$:*

(1) *$X \in \mathcal{B}$ if and only if $\operatorname{Hom}(X, \mathbf{U}) = 0$;*
(2) *$\operatorname{Hom}(\mathcal{B}, X) = 0$ if and only if there is a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $Y_i \in \mathbf{U}$ for all i.*

We obtain the following consequence if we put $\mathcal{B} = 0$.

Corollary. *Every object X in $\mathcal{C}$ admits a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $Y_i \in \operatorname{Sp}\mathcal{C}$ for all i. In particular, $\operatorname{Hom}(X, Y) = 0$ for all $Y \in \operatorname{Sp}\mathcal{C}$ implies $X = 0$.*

The concept of purity is closely related to the occurence of phantom maps. Recall that a map $X \to Y$ is a *phantom map* if the induced map $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y)$ is zero for all compact objects C. From the existence of pure-injective envelopes in $\mathcal{C}$ we derive for every object X the existence of a universal phantom map ending in X and a universal pure monomorphism starting in X.

Theorem C. *For every object X in $\mathcal{C}$ there exists, up to isomorphism, a unique triangle*

$$X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \xrightarrow{\gamma} X'[1]$$

having the following properties:

(A1) *a map $\phi\colon Y \to X$ is a phantom map if and only if ϕ factors through α;*
(A2) *every endomorphism ϕ of X' satisfying $\alpha = \alpha \circ \phi$ is an isomorphism;*
(B1) *a map $\phi\colon X \to Y$ is a pure monomorphism if and only if β factors through ϕ;*
(B2) *every endomorphism ϕ of X'' satisfying $\beta = \phi \circ \beta$ is an isomorphism.*

Our main tool in this paper is a functor $h\colon \mathcal{C} \to \mathcal{A}$ which has the following universal property:

(1) *$h\colon \mathcal{C} \to \mathcal{A}$ is a cohomological functor into a Grothendieck category which preserves coproducts;*
(2) *any functor $h'\colon \mathcal{C} \to \mathcal{A}'$ as in (1) has a unique factorization $h' = f \circ h$ such that $f\colon \mathcal{A} \to \mathcal{A}'$ is exact and preserves coproducts.*

In Section 1 of this paper we exploit the fact that h induces an equivalence between the full subcategory of pure-injective objects in $\mathcal{C}$ and the full subcategory of injective objects in $\mathcal{A}$. We continue in Section 2 with the problem of extending cohomological functors. For instance, we prove the following result where $\mathcal{C}_0$ denotes the full triangulated subcategory which is formed by the compact objects in $\mathcal{C}$.

Theorem D. *Every cohomological functor $f\colon \mathcal{C}_0 \to \mathcal{A}$ into a Grothendieck category $\mathcal{A}$ extends, up to isomorphism, uniquely to a cohomological functor $f'\colon \mathcal{C} \to \mathcal{A}$ which preserves coproducts. Moreover, if $\mathcal{A}$ is the category of abelian groups, then f' preserves products if and only if $f' \simeq \operatorname{Hom}(X, -)$ for some compact object X in $\mathcal{C}$.*

In Section 3 we derive from the universal property of $h\colon \mathcal{C} \to \mathcal{A}$ a strong relation between localizing subcategories in $\mathcal{C}$ and localizing subcategories in $\mathcal{A}$. This interplay between triangulated and Grothendieck categories is crucial for our characterization of smashing subcategories. The final Section 4 is devoted to an explicit construction of the localization functor corresponding to a smashing subcategory. This construction specializes to the classical Bousfield localization provided that the localizing subcategory is generated by compact objects.

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1. PURITY

1.1. **Pure-exactness.** Let $\mathcal{C}$ be a triangulated category [22, 23] and suppose that arbitrary coproducts exist in $\mathcal{C}$. An object X in $\mathcal{C}$ is called *compact* if for every family $(Y_i)_{i\in I}$ in $\mathcal{C}$ the canonical map $\coprod_i \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \coprod_i Y_i)$ is an isomorphism. We denote by $\mathcal{C}_0$ the full subcategory of compact objects in $\mathcal{C}$ and observe that $\mathcal{C}_0$ is a triangulated subcategory of $\mathcal{C}$. Following [17], the category $\mathcal{C}$ is called *compactly generated* provided that the isomorphism classes of objects in $\mathcal{C}_0$ form a set, and $\mathrm{Hom}(C, X) = 0$ for all C in $\mathcal{C}_0$ implies $X = 0$ for every object X in $\mathcal{C}$. Examples of compactly generated triangulated categories arise in stable homotopy theory, algebraic geometry, or in representation theory.

Definition 1.1. Let $\mathcal{C}$ be a compactly generated triangulated category.

(1) A map $X \to Y$ in $\mathcal{C}$ is said to be a *pure monomorphism* if the induced map $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$ is a monomorphism for all compact objects C in $\mathcal{C}$.
(2) An object X in $\mathcal{C}$ is called *pure-injective* if every pure monomorphism $\phi\colon X \to Y$ splits, i.e. there exist a map $\phi'\colon Y \to X$ such that $\phi' \circ \phi = \mathrm{id}_X$.
(3) A triangle $X \to Y \to Z \to X[1]$ is called *pure-exact* if the induced sequence $0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y) \to \mathrm{Hom}(C, Z) \to 0$ is exact for all compact objects C in $\mathcal{C}$.

The preceding definition is motivated by analogous definitions for the category of modules over a ring [4]. However, contrary to the concept for modules, a pure monomorphism in $\mathcal{C}$ is usually not a monomorphism in the categorical sense. For the sake of completeness we include the following definition.

Definition 1.2. Let $\mathcal{C}$ be a compactly generated triangulated category.

(1) A map $Y \to Z$ in $\mathcal{C}$ is said to be a *pure epimorphism* if the induced map $\mathrm{Hom}(C, Y) \to \mathrm{Hom}(C, Z)$ is an epimorphism for all compact objects C in $\mathcal{C}$.
(2) An object Z in $\mathcal{C}$ is called *pure-projective* if every pure epimorphism $\psi\colon Y \to Z$ splits, i.e. there exist a map $\psi'\colon Z \to Y$ such that $\psi \circ \psi' = \mathrm{id}_Z$.

The concept of purity is closely related to the occurence of phantom maps. Recall that a map $X \to Y$ is a *phantom map* provided that the induced map $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y)$ is zero for all compact objects C in $\mathcal{C}$.

Lemma 1.3. *For a triangle $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\chi} X[1]$ the following are equivalent:*

(1) *ϕ is a phantom map;*
(2) *ψ is a pure monomorphism;*
(3) *χ is a pure epimorphism;*
(4) *the shifted triangle $Y \to Z \to X[1] \to Y[1]$ is pure-exact.*

Proof. Clear, since the induced sequence $\mathrm{Hom}(C, X) \to \mathrm{Hom}(C, Y) \to \mathrm{Hom}(C, Z) \to \mathrm{Hom}(C, X[1])$ is exact for every $C \in \mathcal{C}_0$. □

Lemma 1.4. *The following conditions are equivalent for an object X in $\mathcal{C}$:*

(1) *X is pure-injective;*
(2) *if $\phi\colon Y \to X$ is a phantom map, then $\phi = 0$;*

(3) *if* $\phi\colon V \to W$ *is a pure monomorphism, then every map* $V \to X$ *factors through* ϕ.

Proof. (1) $\Leftrightarrow$ (2) follows immediately from the preceding lemma, and the direction (3) $\Rightarrow$ (1) is also clear. To prove (1) $\Rightarrow$ (3), let $\phi\colon V \to W$ be a pure monomorphism and let $\psi\colon V \to X$ be a map with X pure-injective. We obtain a commutative diagram

$$\begin{array}{ccccccc} U & \longrightarrow & V & \xrightarrow{\phi} & W & \longrightarrow & U[1] \\ \| & & \downarrow{\scriptstyle\chi} & & \downarrow & & \| \\ U & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & U[1] \end{array}$$

such that both rows are triangles. The map $U \to V$ is a phantom map since ϕ is a pure monomorphism, and it follows that $U \to X$ is a phantom map. Therefore the map $X \to Y$ is a pure monomorphism which splits since X is pure-injective. It follows that χ factors through ϕ. □

1.2. **Modules.** Let $\mathcal{A}$ be any additive category. An $\mathcal{A}$-*module* is by definition an additive functor $\mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$ into the category Ab of abelian groups, and we denote for $\mathcal{A}$-modules M and N by $\mathrm{Hom}(M, N)$ the class of natural transformations $M \to N$. A sequence $L \to M \to N$ of maps between $\mathcal{A}$-modules is *exact* if the sequence $L(X) \to M(X) \to N(X)$ is exact for all X in $\mathcal{A}$. An $\mathcal{A}$-module M is *finitely generated* if there exists an exact sequence $\mathrm{Hom}(-, X) \to M \to 0$ for some X in $\mathcal{A}$, and M is *finitely presented* if there exists an exact sequence $\mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y) \to M \to 0$ with X and Y in $\mathcal{A}$. Note that $\mathrm{Hom}(M, N)$ is a set for every finitely generated $\mathcal{A}$-module M by Yoneda's lemma. The finitely presented $\mathcal{A}$-modules form an additive category with cokernels which we denote by $\operatorname{mod}\mathcal{A}$. It is well-known that $\operatorname{mod}\mathcal{A}$ is abelian if and only if every map $Y \to Z$ in $\mathcal{A}$ has a *weak kernel* $X \to Y$, i.e. the sequence $\mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y) \to \mathrm{Hom}(-, Z)$ is eaxct. In particular, $\operatorname{mod}\mathcal{A}$ is abelian if $\mathcal{A}$ is triangulated.

Suppose now that $\mathcal{A}$ is skeletally small. Then the $\mathcal{A}$-modules form together with the natural transformations an abelian Grothendieck category which we denote by $\operatorname{Mod}\mathcal{A}$. We shall use the following well-known consequence of Yoneda's lemma: For every object X in $\mathcal{A}$ the corresponding representable functor $H_X = \mathrm{Hom}(-, X)$ is a projective $\mathcal{A}$-module. Moreover, a representative set of objects in $\{H_X \mid X \in \mathcal{A}\}$ forms a generating set of objects in $\operatorname{Mod}\mathcal{A}$. In addition, we shall also need the fact that $\operatorname{Mod}\mathcal{A}$ has injective envelopes [6].

Our main tool for studying a compactly generated triangulated category $\mathcal{C}$ is the *Yoneda functor*

$$h_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \operatorname{Mod}\mathcal{C}_0, \quad X \mapsto H_X = \mathrm{Hom}(-, X)|_{\mathcal{C}_0}.$$

1.3. **Brown representability.** Recall that a functor $f\colon \mathcal{C} \to \mathcal{A}$ from a triangulated category $\mathcal{C}$ into an abelian category $\mathcal{A}$ is *cohomological* if for every triangle $X \to Y \to Z \to X[1]$ in $\mathcal{C}$ the sequence $f(X) \to f(Y) \to f(Z) \to f(X[1])$ is exact. Examples of cohomological functors are the representable functors $\mathrm{Hom}(X, -)\colon \mathcal{C} \to \mathrm{Ab}$ and $\mathrm{Hom}(-, X)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ for any X in $\mathcal{C}$. The Brown representability theorem characterizes the representable cohomological functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ for a compactly generated triangulated category $\mathcal{C}$.

Theorem (Brown). *Let* $f\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *be a cohomological functor such that the canonical map* $f(\coprod_i X_i) \to \prod_i f(X_i)$ *is an isomorphism for every family* $(X_i)_{i\in I}$ *of objects in* $\mathcal{C}$. *Then* $f \simeq \mathrm{Hom}(-, X)$ *for some object* X *in* $\mathcal{C}$.

Proof. See Theorem 3.1 in [17]. □

The existence of arbitrary products in $\mathcal{C}$ is a well-known consequence of the Brown representability theorem.

Lemma 1.5. *The category $\mathcal{C}$ has arbitrary products.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$ and let $f = \prod_i \mathrm{Hom}(-, X_i)$. Clearly, f is a cohomological functor which preserves coproducts. Thus $f \simeq \mathrm{Hom}(-, X)$ by the Brown representability theorem, and it is easily checked that $X = \prod_i X_i$ in $\mathcal{C}$. □

1.4. **Pure-injectives.** Our analysis of pure-injective objects in a compactly generated triangulated category $\mathcal{C}$ is based on some properties of the Yoneda functor $h_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}_0$. We need two lemmas. Recall that a module M is *fp-injective* if $\mathrm{Ext}^1(N, M) = 0$ for every finitely presented module N.

Lemma 1.6. *The $\mathcal{C}_0$-module H_X is fp-injective for every X in $\mathcal{C}$.*

Proof. A finitely presented $\mathcal{C}_0$-module N has a projective presentation

$$H_A \longrightarrow H_B \longrightarrow H_C \longrightarrow N \longrightarrow 0$$

coming from a triangle $A \to B \to C \to A[1]$ in $\mathcal{C}$ with objects in $\mathcal{C}_0$. Thus one can compute Ext^1 as the cohomology of the complex

$$\mathrm{Hom}(H_C, H_X) \longrightarrow \mathrm{Hom}(H_B, H_X) \longrightarrow \mathrm{Hom}(H_A, H_X).$$

This is, however, isomorphic to $\mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$, so it is exact. Therefore $\mathrm{Ext}^1(N, H_X) = 0$ and H_X is fp-injective. □

Lemma 1.7. *Let M be an injective $\mathcal{C}_0$-module. Then there exists, up to isomorphism, a unique object X in $\mathcal{C}$ having the following properties:*

(1) *$M \simeq H_X$;*
(2) *$h_{\mathcal{C}}$ induces an isomorphism $\mathrm{Hom}(Y, X) \to \mathrm{Hom}(H_Y, H_X)$ for all Y in $\mathcal{C}$.*

Proof. Let $f = \mathrm{Hom}(-, M) \circ h_{\mathcal{C}}$. Then f is a cohomological functor since $h_{\mathcal{C}}$ is cohomological and $\mathrm{Hom}(-, M)$ is exact. Moreover, $h_{\mathcal{C}}$ preserves coproducts and $\mathrm{Hom}(-, M)$ induces an isomorphism $\mathrm{Hom}(\coprod_i N_i, M) \simeq \prod_i \mathrm{Hom}(N_i, M)$ for every family $(N_i)_{i\in I}$ of $\mathcal{C}_0$-modules. Therefore $f \simeq \mathrm{Hom}(-, X)$ for some object X in $\mathcal{C}$ by the Brown representability theorem. The induced map $\mathrm{Hom}(X, X) \simeq f(X) = \mathrm{Hom}(H_X, M)$ sends id_X to a map $\phi\colon H_X \to M$ which is an isomorphism since

$$H_X(C) = \mathrm{Hom}(C, X) \simeq \mathrm{Hom}(H_C, M) \simeq M(C)$$

for every compact object C by Yoneda's lemma. The inverse $\phi^{-1}\colon M \to H_X$ induces an isomorphism

$$\mathrm{Hom}(Y, X) \simeq \mathrm{Hom}(H_Y, M) \simeq \mathrm{Hom}(H_Y, H_X)$$

which is precisely the map induced by $h_{\mathcal{C}}$. This finishes the proof. □

The following theorem collects a number of characterizing properties of pure-injective objects.

Theorem 1.8. *The following conditions are equivalent for an object X in $\mathcal{C}$:*

(1) *X is pure-injective;*
(2) *$H_X = \mathrm{Hom}(-, X)|_{\mathcal{C}_0}$ is an injective $\mathcal{C}_0$-module;*
(3) *the map $\mathrm{Hom}(Y, X) \to \mathrm{Hom}(H_Y, H_X)$, $\phi \mapsto \mathrm{Hom}(-, \phi)|_{\mathcal{C}_0}$, is an isomorphism for all Y in $\mathcal{C}$;*

(4) *if* $\phi\colon Y \to X$ *is a phantom map, then* $\phi = 0$*;*
(5) *for every set* I *the summation map* $X^{(I)} \to X$ *factors through the canonical map* $X^{(I)} \to X^I$*.*

Proof. (1) $\Rightarrow$ (2) Let $H_X \to M$ be an injective envelope in $\operatorname{Mod}\mathcal{C}_0$. It follows from Lemma 1.7 that $M \simeq H_Y$ for some object Y in $\mathcal{C}$, and the map $H_X \to M \simeq H_Y$ is of the form H_ϕ for some $\phi\colon X \to Y$. Clearly, ϕ is a pure monomorphism, and ϕ splits since X is pure-injective. Thus H_X is a direct summand of M and therefore injective.

(2) $\Rightarrow$ (3) Use Lemma 1.7.

(3) $\Rightarrow$ (4) If $\phi\colon Y \to X$ is a phantom map, then $\operatorname{Hom}(-,\phi)|_{\mathcal{C}_0} = 0$. Thus it follows from (3) that $\phi = 0$.

(4) $\Rightarrow$ (1) Use Lemma 1.4.

(2) $\Rightarrow$ (5) Suppose that $M = H_X$ is an injective $\mathcal{C}_0$-module. It follows that the summation map $M^{(I)} \to M$ factors through the canonical monomorphism $M^{(I)} \to M^I$. The corresponding map $H_{X^I} \to H_X$ is of the form H_ϕ for some map $\phi\colon X^I \to X$ by (3), and it follows that the composition of ϕ with $X^{(I)} \to X^I$ is the summation map.

(5) $\Rightarrow$ (2) $M = H_X$ is an fp-injective $\mathcal{C}_0$-module by Lemma 1.6 which is injective if the summation map $M^{(I)} \to M$ factors through the canonical monomorphism $M^{(I)} \to M^I$ for every set I by [14, Theorem 2.6]. □

We discuss a number of consequences.

Corollary 1.9. *The Yoneda functor* $\mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$ *induces an equivalence between the full subcategory of pure-injective objects in* $\mathcal{C}$ *and the full subcategory of injective objects in* $\operatorname{Mod}\mathcal{C}_0$*.*

Proof. The Yoneda functor sends pure-injectives to injectives by Theorem 1.8, and it is fully faithful and dense by Lemma 1.7. □

The isomorphism classes of indecomposable injective objects in $\operatorname{Mod}\mathcal{C}_0$ form a set since every indecomposable injective $\mathcal{C}_0$-module arises as an injective envelope of a finitely generated $\mathcal{C}_0$-module. It follows that the indecomposable pure-injective objects in $\mathcal{C}$ form a set which we denote by $\operatorname{Sp}\mathcal{C}$.

Corollary 1.10. *Every object* X *in* $\mathcal{C}$ *admits a pure monomorphism* $X \to \prod_{i\in I} Y_i$ *with* $Y_i \in \operatorname{Sp}\mathcal{C}$ *for all* i*. In particular,* $\operatorname{Hom}(X,Y) = 0$ *for all* $Y \in \operatorname{Sp}\mathcal{C}$ *implies* $X = 0$*.*

Proof. We observe first that the indecomposable injective $\mathcal{C}_0$-modules cogenerate $\operatorname{Mod}\mathcal{C}_0$. In fact, one could take the injective envelopes of all simple modules. Now let X be an object in $\mathcal{C}$ and choose a monomorphism $H_X \to M$ in $\operatorname{Mod}\mathcal{C}_0$ such that $M = \prod_i M_i$ is a product of indecomposable injective $\mathcal{C}_0$-modules. It follows from Lemma 1.7 that this map comes from a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $M_i \simeq H_{Y_i}$ for all i, and each Y_i is indecomposable pure-injective by Theorem 1.8. □

Remark 1.11. The set $\operatorname{Sp}\mathcal{C}$ carries two natural topologies. A subset $\mathbf{U}$ of $\operatorname{Sp}\mathcal{C}$ is *Ziegler-closed* if and only if $\mathbf{U} = \{X \in \operatorname{Sp}\mathcal{C} \mid \operatorname{Hom}(\phi, X) = 0 \text{ for all } \phi \in \Phi\}$ for some class Φ of maps in $\mathcal{C}_0$; see [12]. A subset $\mathbf{U}$ of $\operatorname{Sp}\mathcal{C}$ is *Zariski-open* if and only if there exists some class Φ of maps in $\mathcal{C}_0$ such that $\mathbf{U} = \{X \in \operatorname{Sp}\mathcal{C} \mid \operatorname{Hom}(\phi, X) = 0 \text{ for some } \phi \in \Phi\}$; see [6]. We refer to [15] for a detailed discussion of both topologies in the context of modules over a ring.

A map $\phi\colon X \to Y$ in $\mathcal{C}$ is said to be a *pure-injective envelope* of X if Y is pure-injective and a composition $\psi \circ \phi$ with a map $\psi\colon Y \to Z$ is a pure-monomorphism if and only if ψ is a pure monomorphism.

Lemma 1.12. *The following are equivalent for a pure monomorphism $\phi\colon X \to Y$ in $\mathcal{C}$:*

(1) *ϕ is a pure-injective envelope of X;*
(2) *Y is pure-injective and every endomorphism ψ of Y satisfying $\psi \circ \phi = \phi$ is an isomorphism;*
(3) *$H_\phi\colon H_X \to H_Y$ is an injective envelope in $\operatorname{Mod}\mathcal{C}_0$.*

Proof. Straightforward. □

Corollary 1.13. *Every object X in $\mathcal{C}$ admits a pure-injective envelope $\phi\colon X \to Y$. If $\phi'\colon X \to Y'$ is another pure-injective envelope, then there exists an isomorphism $\psi\colon Y \to Y'$ such that $\phi' = \psi \circ \phi$.*

Proof. The assertion is a consequence of Theorem 1.8 and the existence of injective envelopes in $\operatorname{Mod}\mathcal{C}_0$. □

We are now in a position to prove Theorem C. In fact, the existence of a universal phantom map $X' \to X$ ending in a fixed object X follows from the existence of a pure-injective envelope $X \to X''$.

Proof of Theorem C. Let X be an object in $\mathcal{C}$ and complete the pure-injective envelope $\beta\colon X \to X''$ to a triangle

$$X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \xrightarrow{\gamma} X'[1].$$

The map α is a phantom map by Lemma 1.3 since β is a pure monomorphism, and the property (4) in Theorem 1.8 implies that α is a universal phantom map ending in X since X'' is pure-injective. On the other hand, β is a universal pure monomorphism starting in X by Lemma 1.4 since X'' is pure-injective. Finally, the minimality of α and β follows from Lemma 1.12. □

It is interesting to observe that the full subcategory of pure-injective objects in $\mathcal{C}$ is completely determined by the full subcategory $\mathcal{C}_0$ of compact objects in $\mathcal{C}$.

Corollary 1.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated triangulated categories, and suppose that there exists an equivalence $f\colon \mathcal{C}_0 \to \mathcal{D}_0$ between the full subcategories of compact objects in $\mathcal{C}$ and $\mathcal{D}$. Then f induces an equivalence between the full subcategories of pure-injective objects in $\mathcal{C}$ and $\mathcal{D}$.*

Proof. The functor $h_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$ induces an equivalence between the full subcategory of pure-injectives in $\mathcal{C}$ and the full subcategory of injective $\mathcal{C}_0$-modules by Corollary 1.9. The assertion now follows since an equivalence $f\colon \mathcal{C}_0 \to \mathcal{D}_0$ induces an equivalence $\operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$. □

1.5. **Pure-injective modules.** The concept of purity has been studied extensively by algebraists. Pure-exactness and pure-injectivity for modules over a ring have been introduced by Cohn [4], and we refer to [10] for a modern treatment of this subject.

Let us recall briefly the relevant definitions. Let Λ be an associative ring with identity. We consider the category $\operatorname{Mod}\Lambda$ of (right) Λ-modules. A sequence $0 \to X \to Y \to Z \to 0$ of maps in $\operatorname{Mod}\Lambda$ is *pure-exact* if the induced sequence $0 \to \operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y) \to \operatorname{Hom}(C, Z) \to 0$ is exact for all finitely presented Λ-modules C. The map $X \to Y$ in such a sequence is called a *pure monomorphism*. Note that any pure-exact sequence is automatically an exact sequence in the usual sense. A module X is *pure-injective* if every pure monomorphism $X \to Y$ splits.

Suppose now that Λ is a *quasi-Frobenius ring*, i.e. projective and injective Λ-modules coincide. In this case the *stable category* $\underline{\operatorname{Mod}}\,\Lambda$ is triangulated; e.g. see [9]. Recall that the objects in $\underline{\operatorname{Mod}}\,\Lambda$ are those of $\operatorname{Mod}\Lambda$, and for two Λ-modules X, Y one defines $\underline{\operatorname{Hom}}(X,Y)$ to be $\operatorname{Hom}(X,Y)$ modulo the subgroup of maps which factor through a projective Λ-module. Note that the projection functor $\operatorname{Mod}\Lambda \to \underline{\operatorname{Mod}}\,\Lambda$ preserves products and coproducts. Thus $\underline{\operatorname{Mod}}\,\Lambda$ has arbitrary coproducts, and it is not difficult to check that an object X in $\underline{\operatorname{Mod}}\,\Lambda$ is compact if and only if $X \simeq Y$ in $\underline{\operatorname{Mod}}\,\Lambda$ for some finitely presented Λ-module Y. Therefore $\underline{\operatorname{Mod}}\,\Lambda$ is compactly generated.

Proposition 1.15. *A Λ-module X is pure-injective if and only if X is a pure-injective object in $\underline{\operatorname{Mod}}\,\Lambda$.*

Proof. We use the following characterization of pure-injectivity for Λ-modules which is due to Jensen and Lenzing [10, Proposition 7.32]: A Λ-module X is pure-injective if and only if for every set I the summation map $\sigma_I\colon X^{(I)} \to X$ factors through the canonical map $\iota_I\colon X^{(I)} \to X^I$. We now combine this characterization with the characterization of pure-injectivity in $\underline{\operatorname{Mod}}\,\Lambda$ from Theorem 1.8. Thus any pure-injective Λ-module is a pure-injective object in $\underline{\operatorname{Mod}}\,\Lambda$. To prove the converse, let X be a pure-injective object in $\underline{\operatorname{Mod}}\,\Lambda$ and fix a set I. Thus there exists a map $\phi\colon X^I \to X$ in $\operatorname{Mod}\Lambda$ such that $\sigma_I - \phi\circ\iota_I$ factors through a projective Λ-module P, i.e. $\sigma_I - \phi\circ\iota_I = \beta\circ\alpha$ for some map $\alpha\colon X^{(I)} \to P$. The map α factors through the monomorphism ι_I since P is injective, i.e. $\alpha = \alpha'\circ\iota_I$ for some map α', and therefore

$$\sigma_I = \beta\circ\alpha + \phi\circ\iota_I = (\beta\circ\alpha' + \phi)\circ\iota_I.$$

Thus σ_I factors through ι_I, and this finishes the proof. □

For some further discussion of the relation between pure-injectives in $\operatorname{Mod}\Lambda$ and $\underline{\operatorname{Mod}}\,\Lambda$ we refer to [13, 2].

2. Cohomological and exact functors

2.1. **Extending functors.** Let $\mathcal{C}$ be any triangulated category. We recall the following well-known property of the Yoneda functor $h\colon \mathcal{C} \to \operatorname{mod}\mathcal{C}$, $X \mapsto \operatorname{Hom}(-,X)$.

Lemma 2.1. *Every additive functor $f\colon \mathcal{C} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ extends, up to isomorphism, uniquely to a right exact functor $f'\colon \operatorname{mod}\mathcal{C} \to \mathcal{A}$ such that $f = f'\circ h$. The functor f' is exact if and only if f is a cohomological functor.*

Proof. Any finitely presented $\mathcal{C}$-module M has a projective presentation

$$\operatorname{Hom}(-,X) \xrightarrow{\operatorname{Hom}(-,\phi)} \operatorname{Hom}(-,Y) \longrightarrow M \longrightarrow 0,$$

and we obtain a right exact functor $f'\colon \operatorname{mod}\mathcal{C} \to \mathcal{A}$ if we define $f'(M) = \operatorname{Coker} f(\phi)$. Clearly, $f = f'\circ h$ holds by construction. Exactness of f' implies that f is cohomological, since h is cohomological. Suppose now that f is cohomological. Taking projective presentations of the modules in an exact sequence $0 \to L \to M \to N \to 0$ in $\operatorname{mod}\mathcal{C}$, one uses the snake lemma to prove that $0 \to f'(L) \to f'(M) \to f'(N) \to 0$ is exact in $\mathcal{A}$. This finishes the proof. □

Suppose now that $\mathcal{C}$ is a skeletally small triangulated category and consider the Yoneda functor $h\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}$. Recall that an abelian category $\mathcal{A}$ with arbitrary coproducts is a *Grothendieck category* if $\mathcal{A}$ has a set of generators and taking filtered colimits preserves exactness. In the following we shall not need the fact that $\mathcal{A}$ has a set of generators.

Lemma 2.2. *Every additive functor $f\colon \mathcal{C} \to \mathcal{A}$ into a Grothendieck category $\mathcal{A}$ extends, up to isomorphism, uniquely to a right exact functor $f'\colon \operatorname{Mod}\mathcal{C} \to \mathcal{A}$ which preserves coproducts and satisfies $f = f' \circ h$. The functor f' is exact if and only if f is a cohomological functor.*

Proof. Any $\mathcal{C}$-module M has a projective presentation

$$\coprod_i \operatorname{Hom}(-, X_i) \xrightarrow{(\operatorname{Hom}(-,\phi_{ij}))} \coprod_j \operatorname{Hom}(-, Y_j) \longrightarrow M \longrightarrow 0$$

which is given by a family of maps $\phi_{ij}\colon X_i \to Y_j$ in $\mathcal{C}$. We obtain a functor $f'\colon \operatorname{Mod}\mathcal{C} \to \mathcal{A}$ if we define $f'(M)$ as the cokernel of the map $(f(\phi_{ij}))\colon \coprod_i f(X_i) \to \coprod_j f(Y_j)$ in $\mathcal{A}$. It is easily checked that f' preserves colimits, and that $f = f' \circ h$. The restriction $f'|_{\operatorname{mod}\mathcal{C}}$ is exact if and only if f is cohomological by the preceding lemma. However, any exact sequence $0 \to L \to M \to N \to$ in $\operatorname{Mod}\mathcal{C}$ can be written as a filtered colimit of exact sequences $0 \to L_i \to M_i \to N_i \to$ in $\operatorname{mod}\mathcal{C}$, and therefore f' is exact if and only if f is cohomological. □

We are now in a position to prove the first part of Theorem D. To this end suppose that $\mathcal{C}$ is compactly generated and consider the Yoneda functor $h_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$.

Proposition 2.3. *Every cohomological functor $f\colon \mathcal{C}_0 \to \mathcal{A}$ into a Grothendieck category $\mathcal{A}$ extends, up to isomorphism, uniquely to a cohomological functor $f'\colon \mathcal{C} \to \mathcal{A}$ which preserves coproducts.*

Proof. We denote by $f^*\colon \operatorname{Mod}\mathcal{C}_0 \to \mathcal{A}$ the colimit preserving functor which extends f, and define $f' = f^* \circ h_{\mathcal{C}}$. Clearly, f' is cohomological, preserves coproducts, and $f'|_{\mathcal{C}_0} = f$. Suppose there is another functor $f''\colon \mathcal{C} \to \mathcal{A}$ with these properties. We construct a natural transformation $\eta\colon f' \to f''$ as follows. If X is coproduct of compact objects in $\mathcal{C}$, then we obtain a unique isomorphism $\eta_X\colon f'(X) \to f''(X)$ since f' and f'' preserve coproducts. Now let $X = X_0$ be an arbitrary object in $\mathcal{C}$. We can choose pure-exact triangles $X_{i+1} \to P_i \to X_i \to X_{i+1}[1]$ with P_i being a coproduct of compact objects for $i = 0, 1$, and we obtain a sequence of maps $P_1 \to P_0 \to X$ in $\mathcal{C}$ such that $H_{P_1} \to H_{P_0} \to H_X \to 0$ is exact. This gives a commutative diagram

$$\begin{array}{ccccccc} f'(P_1) & \longrightarrow & f'(P_0) & \longrightarrow & f'(X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta_{P_1}} & & \downarrow{\scriptstyle \eta_{P_0}} & & & & \\ f''(P_1) & \longrightarrow & f''(P_0) & \longrightarrow & f''(X) & & \end{array}$$

where the upper row is exact since f^* is exact. Thus there is a unique map $\eta_X\colon f'(X) \to f''(X)$ since the composition $P_1 \to P_0 \to X$ is zero. Now let $\mathcal{B}$ be the full subcategory formed by the objects X in $\mathcal{C}$ such that η_X is an isomorphism. Clearly, $\mathcal{B}$ contains $\mathcal{C}_0$, and it is triangulated since f' and f'' are cohomological. Furthermore, $\mathcal{B}$ is closed under taking coproducts since f' and f'' preserve coproducts. Thus $\mathcal{B} = \mathcal{C}$ by [17, Lemma 3.2], and therefore $\eta\colon f' \to f''$ is an isomorphism. □

The following consequence generalizes a result from [5].

Corollary 2.4. *Let $f\colon \mathcal{C} \to \mathcal{A}$ be a cohomological functor into a Grothendieck category $\mathcal{A}$. Suppose also that f preserves coproducts. Then there exists, up to isomorphism, a unique exact functor $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \mathcal{A}$ which preserves coproducts and satisfies $f = f' \circ h_{\mathcal{C}}$.*

Proof. Let $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \mathcal{A}$ be the colimit preserving functor which extends $f|_{\mathcal{C}_0}$. We have $f \simeq f' \circ h_{\mathcal{C}}$ by the preceding theorem since both functors are cohomological and preserve coproducts. This gives the uniqeness of f'. □

Corollary 2.5. *The following are equivalent for a map $\phi\colon X \to Y$ in $\mathcal{C}$:*

(1) *ϕ is a phantom map;*
(2) *$f(\phi) = 0$ for every cohomological functor $f\colon \mathcal{C} \to \mathcal{A}$ into a Grothendieck category $\mathcal{A}$ which preserves coproducts;*
(3) *the induced map $\operatorname{Hom}(Y,Q) \to \operatorname{Hom}(X,Q)$ is zero for every (indecomposable) pure-injective object Q in $\mathcal{C}$.*

2.2. **Adjoint functors.** We study pairs of adjoint functors between compactly generated triangulated categories. This is based on properties of adjoint functors between module categories. We start with some notation. Let $f\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between skeletally small additive categories. Then we denote by $f_*\colon \operatorname{Mod}\mathcal{B} \to \operatorname{Mod}\mathcal{A}$, $X \mapsto X \circ f$, the corresponding restriction functor, and $f^*\colon \operatorname{Mod}\mathcal{A} \to \operatorname{Mod}\mathcal{B}$ denotes the left adjoint of f_*. Note that $f^*(\operatorname{Hom}(-,X)) = \operatorname{Hom}(-,f(X))$ for every X in $\mathcal{A}$. Moreover, f^* is right exact and preserves coproducts.

Lemma 2.6. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between compactly generated triangulated categories. Suppose also that f preserves coproducts, and that the right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ of f preserves coproducts. Then f induces a functor $f_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ which makes the following diagrams commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle h_{\mathcal{C}}} & & \downarrow{\scriptstyle h_{\mathcal{D}}} \\ \operatorname{Mod}\mathcal{C}_0 & \xrightarrow{(f_0)^*} & \operatorname{Mod}\mathcal{D}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{D} & \xrightarrow{g} & \mathcal{C} \\ \downarrow{\scriptstyle h_{\mathcal{D}}} & & \downarrow{\scriptstyle h_{\mathcal{C}}} \\ \operatorname{Mod}\mathcal{D}_0 & \xrightarrow{(f_0)_*} & \operatorname{Mod}\mathcal{C}_0 \end{array}$$

Moreover, the functor $(f_0)^*$ is exact.

Proof. Given a compact object X in $\mathcal{C}$, it is well-known that $f(X)$ is compact since g preserves coproducts. This follows from the following sequence of canonical isomorphisms for every family $(Y_i)_{i\in I}$ of objects in $\mathcal{D}$:

$$\begin{aligned} \coprod_i \operatorname{Hom}(f(X),Y_i) &\simeq \coprod_i \operatorname{Hom}(X,g(Y_i)) \simeq \operatorname{Hom}(X,\coprod_i g(Y_i)) \\ &\simeq \operatorname{Hom}(X,g(\coprod_i Y_i)) \simeq \operatorname{Hom}(f(X),\coprod_i Y_i). \end{aligned}$$

Therefore f induces a functor $f_0\colon \mathcal{C}_0 \to \mathcal{D}_0$. The composition $h_{\mathcal{D}} \circ f$ is a cohomological functor which preserves coproducts. Thus there exists a unique exact functor $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$ commuting with coproducts and satisfying $h_{\mathcal{D}} \circ f = f' \circ h_{\mathcal{C}}$ by Corollary 2.4. We claim that $(f_0)^* = f'$. In fact, both functors are right exact, preserve coproducts, and coincide on the full subcategory of finitely generated projective objects in $\operatorname{Mod}\mathcal{C}_0$. The assertion follows since every object M in $\operatorname{Mod}\mathcal{C}_0$ has a projective presentation

$$\coprod_i H_{X_i} \xrightarrow{(\phi_{ij})} \coprod_j H_{Y_j} \longrightarrow M \to 0$$

with X_i and Y_j in $\mathcal{C}_0$ for all i and j.

To prove $h_{\mathcal{C}} \circ g = (f_0)_* \circ h_{\mathcal{D}}$, observe that for every C in $\mathcal{C}_0$ and X in $\mathcal{D}$ we have

$$h_{\mathcal{C}} \circ g(X)(C) = \operatorname{Hom}(C,g(X)) \simeq \operatorname{Hom}(f(C),X) = H_X(f(C)) = (f_0)_* \circ h_{\mathcal{D}}(X)(C).$$

□

2.3. **Flat modules.** Let $\mathcal{A}$ be a skeletally small additive category. Recall that there exists a *tensor product*

$$\operatorname{Mod}\mathcal{A} \times \operatorname{Mod}\mathcal{A}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad (M, N) \to M \otimes_{\mathcal{A}} N$$

where for any $\mathcal{A}$-module M, the tensor functor $M \otimes_{\mathcal{A}} -$ is determined by the fact that it preserves colimits and $M \otimes_{\mathcal{A}} \operatorname{Hom}(X, -) \simeq M(X)$ for all X in $\mathcal{A}$. An $\mathcal{A}$-module M is *flat* if the tensor functor $M \otimes_{\mathcal{A}} -$ exact, and we denote by $\operatorname{Flat}\mathcal{A}$ the full subcategory of flat $\mathcal{A}$-modules. Recall that an $\mathcal{A}$-module M is flat if and only if M is a filtered colimit of representable functors [18]. Therefore $\operatorname{Flat}\mathcal{A}$ is equivalent to the category of *ind-objects* over $\mathcal{A}$ in the sense of Grothendieck and Verdier [8]. In particular, $\operatorname{Flat}\mathcal{A}$ is a category with filtered colimits, and every functor $f\colon \mathcal{A} \to \mathcal{B}$ into a category $\mathcal{B}$ with filtered colimits extends uniquely to a functor $f'\colon \operatorname{Flat}\mathcal{A} \to \mathcal{B}$ preserving filtered colimits and satisfying $f'(\operatorname{Hom}(-, X)) = f(X)$ for all X in $\mathcal{A}$.

Suppose now that $\mathcal{A}$ is triangulated. Then we have the following characterization of flat $\mathcal{A}$-modules.

Lemma 2.7. *The following are equivalent for an additive functor $M\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Ab}$:*

(1) *M is a flat $\mathcal{A}$-module;*
(2) *M is a cohomological functor;*
(3) *M is a fp-injective $\mathcal{A}$-module.*

Proof. (1) $\Leftrightarrow$ (2) M is flat if and only if the restriction $M \otimes_{\mathcal{A}} -|_{\operatorname{mod}\mathcal{A}^{\mathrm{op}}}$ is exact since every exact sequence in $\operatorname{Mod}\mathcal{A}^{\mathrm{op}}$ can be written as a filtered colimit of exact sequences in $\operatorname{mod}\mathcal{A}^{\mathrm{op}}$. Thus M is flat if and only if $M \otimes_{\mathcal{A}} -|_{\mathcal{A}^{\mathrm{op}}}$ is a cohomological functor by Lemma 2.1. The assertion now follows since $M \simeq M \otimes_{\mathcal{A}} -|_{\mathcal{A}^{\mathrm{op}}}$.

(2) $\Leftrightarrow$ (3) Use the argument from the proof of Lemma 1.6. □

We combine the preceding lemma with our results about cohomological functors on compactly generated triangulated categories.

Theorem 2.8. *Let $\mathcal{C}$ be a compactly generated triangulated category. Then the following categories are pairwise equivalent:*

(1) *the category of cohomological functors $\mathcal{C} \to \mathrm{Ab}$ which preserve coproducts;*
(2) *the category of cohomological functors $\mathcal{C}_0 \to \mathrm{Ab}$;*
(3) *the category of ind-objects over $(\mathcal{C}_0)^{\mathrm{op}}$.*

Proof. Combine Proposition 2.3 and Lemma 2.7. □

The finitely presented modules over a ring Λ are characterized by the fact that the corresponding tensor functor $M \otimes_{\Lambda} -$ preserves arbitrary products of (finitely generated projective) Λ^{op}-modules; e.g. see [21]. This leads to a characterization of cohomological functors $\mathcal{C} \to \mathrm{Ab}$ which preserve products; it is the second part of Theorem D.

Proposition 2.9. *The following are equivalent for a cohomological functor $f\colon \mathcal{C} \to \mathrm{Ab}$ which preserves coproducts:*

(1) *$f(\prod_i X_i) \simeq \prod_i f(X_i)$ for every family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$;*
(2) *$f(\prod_i X_i) \simeq \prod_i f(X_i)$ for every family $(X_i)_{i\in I}$ of compact objects in $\mathcal{C}$;*
(3) *$f \simeq \operatorname{Hom}(C, -)$ for some compact object C in $\mathcal{C}$.*

Proof. The directions (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear. Therefore suppose that f preserves products of compact objects. The functor f extends uniquely to a colimit preserving functor $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Ab}$ by Corollary 2.4. We have $f' \simeq - \otimes_{\mathcal{C}_0} M$ for some flat $(\mathcal{C}_0)^{\mathrm{op}}$-module M, and M is finitely presented since f' preserves products of finitely generated projective $\mathcal{C}_0$-modules. Any flat module is finitely presented if and only if it is finitely generated projective, and therefore $M \simeq \operatorname{Hom}(C,-)$ for some C in $\mathcal{C}_0$. We obtain

$$f(Y) = f'(H_Y) \simeq H_Y \otimes_{\mathcal{C}_0} \operatorname{Hom}(C,-) \simeq H_Y(C) = \operatorname{Hom}(C,Y)$$

for every Y in $\mathcal{C}$, and therefore $f \simeq \operatorname{Hom}(C,-)$. □

2.4. **Pure-semisimplicity.** A compactly generated triangulated category $\mathcal{C}$ is *pure-semisimple* if every pure monomorphism in $\mathcal{C}$ splits; equivalently if every object in $\mathcal{C}$ is pure-injective. Bass has characterized the rings for which every flat module is projective. This can be generalized to rings with several objects and then describes when every flat $\mathcal{C}_0$-module is a projective $\mathcal{C}_0$-module, see [10, Theorem B.12]. On the other hand, noetherian rings can be characterized by the fact that every fp-injective module is injective, and this generalizes to rings with several objects as well, see [10, Theorem B.17]. We obtain therefore the following characterization of pure-semisimplicity, since the Yoneda functor $\mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$ identifies every object in $\mathcal{C}$ with a $\mathcal{C}_0$-module which is flat and fp-injective by Lemma 1.6 and Lemma 2.7.

Theorem 2.10. *The following are equivalent for a compactly generated triangulated category $\mathcal{C}$:*

(1) *$\mathcal{C}$ is pure-semisimple;*
(2) *every object in $\mathcal{C}$ is a coproduct of indecomposable objects with local endomorphism ring;*
(3) *every compact object is a finite coproduct of indecomposable objects with local endomorphism ring, and, given a sequence*

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \dots$$

of non-isomorphisms between indecomposable compact objects, the composition $\phi_n \circ \dots \circ \phi_2 \circ \phi_1$ is zero for n sufficiently large;
(4) *the Yoneda functor $h_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$, $X \mapsto \operatorname{Hom}(-,X)|_{\mathcal{C}_0}$, is fully faithful;*
(5) *$\mathcal{C}$ has filtered colimits.*

3. Localization

3.1. **Localization.** Let $\mathcal{C}$ be a compactly generated triangulated category. Recall that a full triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$ is *localizing* if $\mathcal{B}$ is closed under taking coproducts. The *quotient category* $\mathcal{C}/\mathcal{B}$ is, by definition, the quotient category $\mathcal{C}[\Sigma^{-1}]$ (in the sense of [7]) with respect to the class Σ of maps $Y \to Z$ which admit a triangle $X \to Y \to Z \to X[1]$ with X in $\mathcal{B}$. The following lemma collects some basic facts about the formation of the quotient category $\mathcal{C}/\mathcal{B}$.

Lemma 3.1. *Let $\mathcal{B}$ be a localizing subcategory of a compactly generated triangulated category $\mathcal{C}$.*

(1) *The quotient functor $f\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ has a right adjoint $g\colon \mathcal{C}/\mathcal{B} \to \mathcal{C}$.*
(2) *The inclusion functor $\mathcal{B} \to \mathcal{C}$ has a right adjoint $e\colon \mathcal{C} \to \mathcal{B}$.*
(3) *For every object X in $\mathcal{C}$ there is a triangle $g \circ f(X)[-1] \xrightarrow{\alpha_X} e(X) \xrightarrow{\beta_X} X \xrightarrow{\gamma_X} g \circ f(X)$ which is functorial in X.*

We shall need the following lemma about $\mathcal{C}[\Sigma^{-1}]$.

Lemma 3.2. *Let $\mathcal{C}$ be any additive category with coproducts. Suppose that Σ is a class of maps in $\mathcal{C}$ which admits a calculus of left fractions. If $\coprod_i \sigma_i \in \Sigma$ for every family $(\sigma_i)_{i\in I}$ in Σ, then the quotient category $\mathcal{C}[\Sigma^{-1}]$ has coproducts and the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves coproducts.*

Proof. Recall from [7] that the objects in $\mathcal{C}[\Sigma^{-1}]$ are those of $\mathcal{C}$, and that the maps $X \to Y$ in $\mathcal{C}[\Sigma^{-1}]$ are equivalence classes of *left fractions* $X \xrightarrow{\phi} Z \xleftarrow{\sigma} Y$ with $\sigma \in \Sigma$. Now let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}[\Sigma^{-1}]$. We claim that the coproduct $\coprod_i X_i$ in $\mathcal{C}$ is also a coproduct in $\mathcal{C}[\Sigma^{-1}]$. Thus we need to show that for every object Y, the canonical map $\alpha\colon \operatorname{Hom}(\coprod_i X_i, Y) \to \prod_i \operatorname{Hom}(X_i, Y)$ between Hom-sets in $\mathcal{C}[\Sigma^{-1}]$ is bijective.

To check surjectivity, let $(X_i \xrightarrow{\phi_i} Z_i \xleftarrow{\sigma_i} Y)_{i\in I}$ be a family of left fractions. We obtain a commutative diagram

$$\begin{array}{ccccc} \coprod_i X_i & \xrightarrow{\coprod_i \phi_i} & \coprod_i Z_i & \xleftarrow{\coprod_i \sigma_i} & \coprod_i Y \\ & & \downarrow & & \downarrow{\scriptstyle \pi_Y} \\ & & Z & \xleftarrow{\sigma} & Y \end{array}$$

where $\pi_Y\colon \coprod_i Y \to Y$ is the summation map and $\sigma \in \Sigma$. It is easily checked that

$$(X_i \to Z \xleftarrow{\sigma} Y) \sim (X_i \xrightarrow{\phi_i} Z_i \xleftarrow{\sigma_i} Y)$$

for all $i \in I$, and therefore α sends $\coprod_i X_i \to Z \xleftarrow{\sigma} Y$ to the family $(X_i \xrightarrow{\phi_i} Z_i \xleftarrow{\sigma_i} Y_i)_{i\in I}$.

To check injectivity, let $\coprod_i X_i \xrightarrow{\phi} Z \xleftarrow{\sigma} Y$ be a left fraction such that $(X_i \xrightarrow{\phi_i} Z \xleftarrow{\sigma} Y) \sim 0$ for all i. Thus there are maps $\psi_i\colon Z \to Z_i$ with $\psi_i \circ \phi_i = 0$ and $\psi_i \circ \sigma \in \Sigma$ for all i. We may assume that Σ is saturated so that $\psi_i \in \Sigma$ for all i. We obtain a commutative diagram

$$\begin{array}{ccccccc} \coprod_i X_i & \xrightarrow{\coprod_i \phi_i} & \coprod_i Z & \xrightarrow{\pi_Z} & Z & \xleftarrow{\sigma} & Y \\ & & \downarrow{\scriptstyle \coprod_i \psi_i} & & \downarrow{\scriptstyle \tau} & & \\ & & \coprod_i Z_i & \longrightarrow & Z' & & \end{array}$$

with $\tau \in \Sigma$. Thus $\sigma' = \tau \circ \sigma \in \Sigma$, and we have

$$(\coprod_i X_i \xrightarrow{0} Z' \xleftarrow{\sigma'} Y) \sim (\coprod_i X_i \xrightarrow{\phi} Z \xleftarrow{\sigma} Y)$$

since $\phi = \pi_Z \circ \coprod_i \phi_i$. Therefore α is also injective, and this completes the proof. □

Proof of Lemma 3.1. (1) The quotient functor preserves coproducts by Lemma 3.2, since Σ is closed under taking coproducts. Given an object X in $\mathcal{C}/\mathcal{B}$, the composition $\operatorname{Hom}(-, X) \circ f$ is a cohomological functor which preserve coproducts. Thus there exists Y in $\mathcal{C}$ with $\operatorname{Hom}(-, X) \circ f \simeq \operatorname{Hom}(-, Y)$ by the Brown representability theorem. We put $g(X) = Y$, and it is easily checked that this gives a right adjoint $g\colon \mathcal{C}/\mathcal{B} \to \mathcal{C}$ for f.

(2) Completing the canonical map $\gamma_X\colon X \to g \circ f(X)$ to a triangle $g \circ f(X)[-1] \to Y \to X \xrightarrow{\gamma_X} g \circ f(X)$ for every X in $\mathcal{C}$ gives a functor $e\colon \mathcal{C} \to \mathcal{B}$ if we put $e(X) = Y$. Clearly, e is a right adjoint for the inclusion $\mathcal{B} \to \mathcal{C}$.

(3) follows from (2). □

We shall use several times the following fact.

Lemma 3.3. *The natural transformation* $\mathrm{id} \to g \circ f$ *induces a functorial isomorphism* $\mathrm{Hom}(g \circ f(X), Y) \to \mathrm{Hom}(X, Y)$ *for all* X *and* Y *such that* $\mathrm{Hom}(\mathcal{B}, Y) = 0$.

Proof. Apply $\mathrm{Hom}(-, Y)$ to the triangle in part (3) of Lemma 3.1. □

Given any class $\mathcal{X}$ of objects or maps in $\mathcal{C}$, we say that an object Y in $\mathcal{C}$ is $\mathcal{X}$*-local* if $\mathrm{Hom}(\mathcal{X}, Y) = 0$. The full subcategory of $\mathcal{X}$-local objects is denoted by $\mathcal{X}^\perp$. We include for later reference the following lemma.

Lemma 3.4. (1) *The functor* g *induces an equivalence between* $\mathcal{C}/\mathcal{B}$ *and* $\mathcal{B}^\perp$.
(2) *An object* X *in* $\mathcal{C}$ *belongs to* $\mathcal{B}$ *if and only if* $\mathrm{Hom}(X, \mathcal{B}^\perp) = 0$.

Proof. Use Lemma 3.1. □

3.2. **Smashing subcategories.** We assume now that $\mathcal{B}$ is a localizing subcategory of a compactly generated triangulated category $\mathcal{C}$. We denote by $\mathcal{D} = \mathcal{C}/\mathcal{B}$ the quotient category and keep the notation of Lemma 3.1. We suppose in addition that $\mathcal{B}$ is a *smashing* subcategory, i.e. the right adjoint of the inclusion $\mathcal{B} \to \mathcal{C}$ preserves coproducts. Note that this is equivalent to the fact that the right adjoint of the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$ preserves coproducts.

Lemma 3.5. $\mathcal{C}/\mathcal{B}$ *is a compactly generated triangulated category.*

Proof. $\mathcal{C}/\mathcal{B}$ has coproducts by Lemma 3.2, and the argument in the proof of Lemma 2.6 shows that $f(\mathcal{C}_0) \subseteq \mathcal{D}_0$. Suppose now that $\mathrm{Hom}(D, X) = 0$ for all D in $\mathcal{D}_0$ and some X in $\mathcal{D}$. Then $\mathrm{Hom}(C, g(X)) \simeq \mathrm{Hom}(f(C), X) = 0$ for all C in $\mathcal{C}_0$, and therefore $X = 0$, since $\mathcal{C}$ is compactly generated and g is faithful. Thus $\mathcal{C}/\mathcal{B}$ is compactly generated. □

We now exhibit more closely the properties of the functor $(f_0)^*\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$ from Lemma 2.6. To this end let $\mathcal{T} = \mathrm{Ker}(f_0)^*$ and $\mathcal{S} = \mathcal{T} \cap \operatorname{mod}\mathcal{C}_0$.

Lemma 3.6. *The canonical map* $(f_0)^* \circ (f_0)_*(M) \to M$ *is an isomorphism for all* M *in* $\operatorname{Mod}\mathcal{D}_0$.

Proof. We apply Lemma 2.6. We have $(f_0)^* \circ (f_0)_* \simeq \mathrm{id}$ on the full subcategory of injective $\mathcal{D}_0$-modules, since $f \circ g \simeq \mathrm{id}$, and $h_{\mathcal{D}}$ induces an equivalence between the full subcategory of pure-injectives in $\mathcal{D}$ and the full subcategory of injectives in $\operatorname{Mod}\mathcal{D}_0$ by Corollary 1.9. The composition $(f_0)^* \circ (f_0)_*$ is left exact and therefore $(f_0)^* \circ (f_0)_* \simeq \mathrm{id}$ on $\operatorname{Mod}\mathcal{D}_0$ since every $\mathcal{D}_0$-module M fits into an exact sequence $0 \to M \to Q_0 \to Q_1$ with Q_i injective for $i = 0, 1$. □

Remark 3.7. The statement of the preceding lemma is equivalent to the fact that $(f_0)^*\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$ induces an equivalence between the quotient category $\operatorname{Mod}\mathcal{C}_0/\mathcal{T}$ and $\operatorname{Mod}\mathcal{D}_0$; see [6].

Lemma 3.8. *A* $\mathcal{C}_0$*-module* M *belongs to* $\mathcal{T}$ *if and only if* M *is a filtered colimit of modules in* $\mathcal{S}$.

Proof. If M is a filtered colimit of modules in $\mathcal{S}$, then $M \in \mathcal{T}$ since $(f_0)^*$ preserves filtered colimits. To prove the converse, we denote by $t\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{C}_0$ the functor which is obtained from the functorial exact sequence $0 \to t(M) \to M \to (f_0)_* \circ (f_0)^*(M)$. It follows directly from Lemma 3.6 that t induces a right adjoint $\operatorname{Mod}\mathcal{C}_0 \to \mathcal{T}$ for the inclusion $\mathcal{T} \to \operatorname{Mod}\mathcal{C}_0$. Moreover, t preserves filtered colimits since $(f_0)^*$ and $(f_0)_*$ have this property. Now let $M \in \operatorname{mod}\mathcal{C}_0$, and write $t(M) = \varinjlim M_i$ as a filtered colimit of finitely generated submodules. For all i, we have $M_i \in \mathcal{T}$ since $\mathcal{T}$ is closed under taking submodules, and $M_i \in \operatorname{mod}\mathcal{C}_0$ since $\mathcal{C}_0$ has weak kernels and therefore finitely

generated submodules of finitely presented modules are again finitely presented. It follows that $t(M)$ is a filtered colimit of modules in $\mathcal{S}$. Given any module M in $\mathcal{T}$, we can write $M = \varinjlim M_i$ as a filtered colimit of finitely presented modules. Thus $M = t(\varinjlim M_i) \simeq \varinjlim t(M_i)$ is a filtered colimit of modules in $\mathcal{S}$. □

We denote by Φ the collection of maps ϕ in $\mathcal{C}_0$ such that $f(\phi) = 0$.

Lemma 3.9. $\mathcal{S} = \{\operatorname{Im} H_\phi \mid \phi \in \Phi\}$.

Proof. We observe first that every finitely presented $\mathcal{C}_0$-module M with projective presentation $H_X \to H_Y \to M \to 0$ is of the form $M = \operatorname{Im} H_\phi$ where ϕ is the map occuring in the triangle $X \to Y \xrightarrow{\phi} Z \to X[1]$. Now let ϕ be any map in $\mathcal{C}_0$. We have $f(\phi) = 0$ if and only if $h_{\mathcal{D}} \circ f(\phi) = 0$ since $h_{\mathcal{D}}$ is faithful on $\mathcal{D}_0$. Therefore $f(\phi) = 0$ if and only if $(f_0)^* \circ h_{\mathcal{C}}(\phi) = 0$ if and only if $(f_0)^*(\operatorname{Im} H_\phi) = 0$ since $(f_0)^*$ is exact. It follows that $\mathcal{S} = \{\operatorname{Im} H_\phi \mid \phi \in \Phi\}$. □

Lemma 3.10. *The following are equivalent for X in $\mathcal{C}$:*

(1) $X \in \mathcal{B}$;
(2) $H_X \in \mathcal{T}$;
(3) *every map $C \to X$ with $C \in \mathcal{C}_0$ factors through some $C \to D$ in Φ.*

Proof. (1) $\Leftrightarrow$ (2) The assertion follows from the fact that $(f_0)^* \circ h_{\mathcal{C}} = h_{\mathcal{D}} \circ f$, since $h_{\mathcal{C}}$ and $h_{\mathcal{D}}$ send non-zero objects to non-zero objects.

(2) $\Rightarrow$ (3) We can write $H_X = \varinjlim \operatorname{Im} H_{\phi_i}$ with $\phi_i \in \Phi$ for all i, by Lemma 3.8 and Lemma 3.9. Now let $\phi\colon C \to X$ be any map with $C \in \mathcal{C}_0$. We have $\operatorname{Hom}(C, X) = H_X(C) = \varinjlim \operatorname{Im} H_{\phi_i}(C)$ and obtain therefore a factorization $C \to C_i \xrightarrow{\phi_i} D_i \to X$ of ϕ for some i. The composition $C \to C_i \xrightarrow{\phi_i} D_i$ belongs to Φ since $\phi_i \in \Phi$, and this proves (3).

(3) $\Rightarrow$ (2) We can write $H_X = \varinjlim M_i$ as a filtered colimit of finitely generated submodules, and it is sufficient to show that $M_i \in \mathcal{T}$ for all i. To this end suppose that M_i is a quotient of H_C for some $C \in \mathcal{C}_0$. The corresponding map $H_C \to M_i \to H_X$ is of the form H_ϕ for some $\phi\colon C \to X$. Let $C \xrightarrow{\phi'} D \xrightarrow{\phi''} X$ be a factorization of ϕ with $\phi' \in \Phi$. It follows that M_i is a quotient of $\operatorname{Im} H_{\phi'}$. Thus $M_i \in \mathcal{T}$ since $\operatorname{Im} H_{\phi'} \in \mathcal{T}$, and this proves (2). □

3.3. **Proof of the main results.** We keep the notation of Lemma 3.1.

Proof of Theorem A. (1) $\Rightarrow$ (2) This is proved in Lemma 3.10.

(2) $\Rightarrow$ (3) Suppose (2) and let X be an object in $\mathcal{C}$. If $\operatorname{Hom}(\mathcal{B}, X) = 0$, then $\operatorname{Hom}(\Phi, X) = 0$ by Lemma 3.3. Now suppose that $\operatorname{Hom}(\Phi, X) = 0$ and consider the triangle $g \circ f(X)[-1] \xrightarrow{\alpha_X} e(X) \xrightarrow{\beta_X} X \xrightarrow{\gamma_X} g \circ f(X)$. Every map $\phi\colon C \to e(X)$ with C in $\mathcal{C}_0$ has a factorization $C \xrightarrow{\phi'} D \xrightarrow{\phi''} e(X)$ with $\phi' \in \Phi$, and therefore $\beta_X \circ \phi = 0$. Thus ϕ factors through α_X. The same argument shows that ϕ'' factors through α_X, and therefore $\phi = 0$ since $\operatorname{Hom}(\phi', g \circ f(X)[-1]) = 0$ by Lemma 3.3. We conclude that $e(X) = 0$ and therefore $\operatorname{Hom}(\mathcal{B}, X) = 0$.

(3) $\Rightarrow$ (1) The functor $g\colon \mathcal{C}/\mathcal{B} \to \mathcal{C}$ identifies $\mathcal{C}/\mathcal{B}$ with the full subcategory of objects X in $\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{B}, X) = 0$. It follows that $\operatorname{Hom}(\Phi, \coprod_i X_i) = 0$ for every family $(X_i)_{i \in I}$ with $\operatorname{Hom}(\Phi, X_i) = 0$ for all i, since Φ belongs to $\mathcal{C}_0$. Thus g preserves coproducts and therefore $\mathcal{B}$ is smashing. □

The corollary of Theorem A is now an immediate consequence. In fact, a smashing subcategory $\mathcal{B}$ of $\mathcal{C}$ is generated by the class Φ of maps in $\mathcal{C}_0$ which are sent to zero under the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$. To see this, suppose that $\mathcal{A}$ is a localizing subcategory of $\mathcal{C}$ such that Φ is sent to zero under the quotient functor $C \to \mathcal{C}/\mathcal{A}$. Observe that $X \in \mathcal{A}$ if and only if $\operatorname{Hom}(X, \mathcal{A}^\perp) = 0$ by Lemma 3.4. We have $\mathcal{A}^\perp \subseteq \Phi^\perp$ by Lemma 3.3 since Φ is sent to zero under $\mathcal{C} \to \mathcal{C}/\mathcal{A}$, and $\Phi^\perp = \mathcal{B}^\perp$ by Theorem A. It follows from the preceding observation that $\mathcal{B} \subseteq \mathcal{A}$, and therefore $\mathcal{B}$ is generated by Φ.

We proceed now with the proof of Theorem B. To this end suppose that $\mathcal{B}$ is a smashing subcategory of $\mathcal{C}$. We need the following observation.

Lemma 3.11. *The right adjoint $\mathcal{C}/\mathcal{B} \to \mathcal{C}$ of the quotient functor sends pure-exact triangles to pure-exact triangles, and pure-injectives to pure-injectives.*

Proof. The first assertion follows directly from the adjointness formula and the fact that the quotient functor preserves compactness. The second assertion follows from the characterization of pure-injectivity in Theorem 1.8, and the fact that $\mathcal{C}/\mathcal{B} \to \mathcal{C}$ preserves products and coproducts. □

Proof of Theorem B. We identify $\mathcal{D} = \mathcal{C}/\mathcal{B}$ via g with the full subcategory of objects X in $\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{B}, X) = 0$. In particular, this identifies $\operatorname{Sp}\mathcal{D}$ with $\mathbf{U} = \{X \in \operatorname{Sp}\mathcal{C} \mid \operatorname{Hom}(\mathcal{B}, X) = 0\}$.

(1) Clearly, $X \in \mathcal{B}$ implies $\operatorname{Hom}(X, \mathbf{U}) = 0$. Conversely, $\operatorname{Hom}(X, \mathbf{U}) = 0$ implies $\operatorname{Hom}(g \circ f(X), \mathbf{U}) = 0$ by Lemma 3.3, and this implies $g \circ f(X) = 0$ since $\mathbf{U}$ cogenerates $\mathcal{D}$ by Corollary 1.10. Thus $f(X) = 0$ since g is faithful, and therefore X belongs to $\mathcal{B}$.

(2) Suppose first that $\operatorname{Hom}(\mathcal{B}, X) = 0$. We apply Corollary 1.10 and get a pure monomorphism $X \to \prod_i Y_i$ in $\mathcal{D}$ with $Y_i \in \mathbf{U}$ for all i. The inclusion $\mathcal{D} \to \mathcal{C}$ preserves pure monomorphisms by Lemma 3.11, and this proves one direction. Now suppose that we have a pure monomorphism $X \to Y$ in $\mathcal{C}$ with $\operatorname{Hom}(\mathcal{B}, Y) = 0$. It follows from part (3) in Theorem A that $\operatorname{Hom}(\mathcal{B}, X) = 0$, and therefore the proof of Theorem B is complete. □

4. The localization functor

Let $\mathcal{B}$ be a localizing subcategory of a triangulated category $\mathcal{C}$. Recall that a map $X \to Y$ in $\mathcal{C}$ is a *$\mathcal{B}$-localization* of X if Y is $\mathcal{B}$-local and if the induced map $\operatorname{Hom}(Y, Y') \to \operatorname{Hom}(X, Y')$ is bijective for every $\mathcal{B}$-local object Y'. In this section we give an explicit construction of the $\mathcal{B}$-localization provided that $\mathcal{B}$ is smashing. This construction specializes to the usual Bousfield localization if $\mathcal{B}$ is generated by compact objects. We need to recall the following definition from [1]. Let $\mathcal{Y}$ be a class of objects in $\mathcal{C}$. Then a map $X \to Y$ is a *left $\mathcal{Y}$-approximation* of X if Y belongs to $\mathcal{Y}$ and if the induced map $\operatorname{Hom}(Y, Y') \to \operatorname{Hom}(X, Y')$ is surjective for every Y' in $\mathcal{Y}$. In general, a left $\mathcal{Y}$-approximation is far from being unique. Therefore one defines a map $\phi\colon X \to Y$ to be *left minimal* if every endomorphism ψ of Y such that $\phi = \psi \circ \phi$ is an isomorphism.

Suppose now that $\mathcal{C}$ is a compactly generated triangulated category, and let Φ be a class of maps in $\mathcal{C}_0$. We construct for any object X in $\mathcal{C}$ a left $\Phi^\perp$-approximation $\gamma_{X,\Phi}\colon X \to X_\Phi$. To this end we define inductively maps $\alpha_n\colon X_n \to X_{n+1}$ for every $n \geq 0$. By definition, set $X_0 = X$. Let Ψ_n be a representative set of maps $\psi\colon C \to X_n$ which factor through some map $C \to D$ in Φ. We obtain α_n if we complete the canonical map $\coprod_{\psi \in \Psi_n} C \to X_n$ to a triangle

$$\coprod_{\psi \in \Psi_n} C \longrightarrow X_n \xrightarrow{\alpha_n} X_{n+1} \longrightarrow (\coprod_{\psi \in \Psi_n} C)[1].$$

We denote by X_Φ the homotopy colimit $\operatorname{hocolim} X_n$ of the sequence

$$X = X_0 \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots .$$

More precisely, X_Φ is obtained from the triangle

$$\coprod_n X_n \xrightarrow{\mathrm{id}-\alpha} \coprod_n X_n \longrightarrow \operatorname{hocolim} X_n \longrightarrow (\coprod_n X_n)[1].$$

We denote by $\gamma_{X,\Phi}\colon X \to X_\Phi$ the canonical map from X_0 into $\operatorname{hocolim} X_n$, but this map is only unique up to a non-unique isomorphism since the construction involves the completion of various maps to triangles. Given any map $\psi\colon X \to Y$ in $\mathcal{C}$, we obtain a sequence of commuting diagrams

$$\begin{array}{ccccccccc} X & \xrightarrow{\mathrm{id}} & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \dots \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi_0} & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \\ Y & \xrightarrow{\mathrm{id}} & Y_0 & \xrightarrow{\beta_0} & Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \dots \end{array}$$

and we denote by $\psi_\Phi\colon X_\Phi \to Y_\Phi$ the homotopy colimit $\operatorname{hocolim} \psi_n$.

Proposition 4.1. *The map $\gamma_{X,\Phi}\colon X \to X_\Phi$ is a left $\Phi^\perp$-approximation.*

Proof. We need to show that X_Φ is Φ-local. To this end let $\phi\colon C \to D$ be a map in Φ. The canonical maps $\xi_n\colon X_n \to \operatorname{hocolim} X_n$ induce an isomorphism

$$\varinjlim \operatorname{Hom}(D, X_n) \longrightarrow \operatorname{Hom}(D, \operatorname{hocolim} X_n)$$

since D is compact; e.g. see [16, Lemma 1.5]. Therefore any map $\psi\colon D \to X_\Phi$ has a factorization $\psi = \xi_n \circ \psi'$ for some $n \in \mathbb{N}$. The construction of α_n implies $\alpha_n \circ \psi' \circ \phi = 0$, and therefore $\psi \circ \phi = 0$. Thus X_Φ is Φ-local. We have $Y_\Phi = Y$ for every Φ-local object Y, and therefore every map $\psi\colon X \to Y$ with $Y \in \Phi^\perp$ factors through $\gamma_{X,\Phi}$ via $\psi_\Phi\colon X_\Phi \to Y_\Phi = Y$. Thus $\gamma_{X,\Phi}$ is a left $\Phi^\perp$-approximation. □

Suppose now that $\mathcal{B}$ is a smashing subcategory of $\mathcal{C}$. Let Φ be a class of maps in $\mathcal{C}_0$ such that $\mathcal{B}^\perp = \Phi^\perp$. For instance, by Theorem A, one could take all maps in $\mathcal{C}_0$ which are sent to zero under the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$.

Theorem 4.2. *The map $\gamma_{X,\Phi}\colon X \to X_\Phi$ has a decomposition*

$$(\gamma', \gamma'')\colon X \longrightarrow X_\Phi = Y' \coprod Y''$$

such that γ' is left minimal and $\gamma'' = 0$. In this case, the map $\gamma'\colon X \to Y'$ is a $\mathcal{B}$-localization of X.

Proof. We infer from Lemma 3.1 that there exists a $\mathcal{B}$-localization $\gamma_X\colon X \to Y$ of X. Using this fact and the preceding proposition, we obtain maps $\alpha\colon Y \to X_\Phi$ and $\beta\colon X_\Phi \to Y$ such that $\gamma_{X,\Phi} = \alpha \circ \gamma_X$ and $\gamma_X = \beta \circ \gamma_{X,\Phi}$. We have $\beta \circ \alpha = \mathrm{id}_Y$ since γ_X is a $\mathcal{B}$-localization of X, and we obtain therefore a decomposition $\gamma_{X,\Phi} = (\gamma', \gamma'')$ such that γ' is isomorphic to γ_X and $\gamma'' = 0$. This finishes the proof. □

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