Resolvent estimates for boundary value problems on large intervals via the theory of discrete approximations

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1 Introduction

Resolvent estimates and spectral properties of linear boundary value problems are essential for establishing the well-posedness and asymptotic stability of solutions to time dependent nonlinear evolution equations. For example, the stability of traveling waves in one space dimension is determined by the spectral properties of second order differential operators

$$Pu = Au_{xx} + Bu_x + Cu,\ A \in \mathbb{C}^{l,l}, B, C \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{l,l}), \tag{1.1}$$

see e.g. the monographs [8, 23] and the papers [10, 16]. Depending on the properties of the matrix A the time dependent system

$$u_t = Pu, \quad x \in \mathbb{R},\ t \geq 0 \tag{1.2}$$

is of hyperbolic, parabolic or coupled hyperbolic-parabolic type. The spectrum of the operator P from (1.1) will depend on this type and on the function spaces used. In general, it may contain continuous spectrum as well as isolated eigenvalues. In order to determine isolated eigenvalues it is essential to analyze the solvability and derive solution estimates for the resolvent equation

$$(sI - P)u = f \quad \text{in } \mathbb{R}, \tag{1.3}$$

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where the range of s-values should at least contain the positive half-plane. There are several alternatives for solving this problem in real applications. One alternative is to reduce the eigenvalue problem to finding the zeros of the Evans function [16, 14, 4] and to approximate the Evans function numerically (see [9] and the citations therein). Another alternative is to truncate equation (1.3) to a large but compact finite interval $J = [x_-, x_+]$ and to impose appropriate boundary conditions

$$\begin{aligned} f &= (sI - P)u \quad \text{in } J = [x_-, x_+], && (1.4a) \\ 0 &= Ru = R_-^I u(x_-) + R_-^{II} u_x(x_-) + R_+^I u(x_+) + R_+^{II} u_x(x_+). && (1.4b) \end{aligned}$$

The purpose of this paper is to analyze, within a general framework, the error introduced by this truncation and by the choice of the finite boundary conditions. We will not consider the subsequent discretization of the boundary value problem (1.4) by standard methods such as finite differences or finite element methods.

Our general approach is as follows. We view the truncation to a finite interval J as a *projection* from a space of functions on $\mathbb{R}$, e.g. the Sobolev space $H^2(\mathbb{R})$, to a space of functions on J. The truncated differential operator will be connected with a two point boundary operator R as above and the projection will be chosen such that homogeneous boundary conditions are imposed. In this way the convergence of the truncated operators to the original operator as $J \to \mathbb{R}$ can be interpreted as the error obtained by commuting a differential operator with the projection to a finite interval (precise definitions and details are given in Section 2).

Viewing approximation problems in this way is also the underlying idea in the theory of discrete approximations developed in the 1970s by F. Stummel [17, 18, 19], R. D. Grigorieff [5, 6, 7], and G. Vainikko [21]. This theory has found numerous applications to the analysis of finite element or finite difference approximations as well as perturbations of coefficients and bounded domains in differential equations (Stummel [20]).

In our approach we first transform equation (1.2) to a first order system

$$Lz := z_x - M(x, s)z = h \quad \text{in } \mathbb{R}. \tag{1.5}$$

Our main result (Theorem 1 in Section 2) gives sufficient conditions on the

boundary operators and on the asymptotic behavior of the matrices $M(x,s)$ such that regular convergence holds in the sense of discrete approximations (see Vainikko [21]) and within the framework described above. We note that regular convergence has important implications for the convergence of linear and nonlinear eigenvalue problems (see for example [22]) as well as existence, uniqueness and convergence of solutions to nonlinear problems.
As a particular instance we show in Section 3 the regular convergence of a truncated version of a linear coupled hyperbolic-parabolic system of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + B \begin{pmatrix} u \\ v \end{pmatrix}_x + C \begin{pmatrix} u \\ v \end{pmatrix} =: P \begin{pmatrix} u \\ v \end{pmatrix}. \tag{1.6}$$

Under appropriate assumptions on the boundary operator that takes into account the mixed nature of the problem we prove resolvent estimates that are uniform in s-values in a compact set as well as in the size of the large interval J.
More general applications, in particular convergence theorems for isolated eigenvalues can be found in the Master's Thesis [15]. We also note that the approach in [15] is more general in the sense that all results hold for x_+ and $-x_-$ sufficiently large, while in this paper, for the ease of readability, we restrict to a nested sequence of growing intervals $J_n = [x_-^n, x_+^n]$. Due to its abstract nature, our result also applies to purely hyperbolic and purely parabolic problems (see [15]). For example, the convergence results and resolvent estimates from [2, sec. 3, 4] can be deduced from the regular convergence result in this paper.
Finally, we show that the stability problem for pulses in the FitzHugh-Nagumo equations of nerve signalling leads to a system of the form (1.6) that can be included into our theory.

2 Boundary value problems and discrete approximations

In this section we consider the connection of the boundary value problem on the whole real line and of its finite interval approximations in the setting of discrete approximations. We mainly follow the monograph [21] and first recall some basic definitions. Since we are particularly interested in the application to boundary value problems we will give the definitions in a simplified form.

If $(x_n)_{n\in\mathbb{N}}$ is some sequence in a topological space we write $x_n \to x$ $(n \in \mathbb{N})$ iff $\lim_{n\to\infty} x_n = x$. Furthermore by $\mathbb{N}' \subset \mathbb{N}$, $\mathbb{N}'' \subset \mathbb{N}$, etc. we always denote unbounded subsets of $\mathbb{N}$. Therefore $(x_n)_{n\in\mathbb{N}'}$ denotes the corresponding subsequence of $(x_n)_{n\in\mathbb{N}}$ and $x_n \to x$ $(n \in \mathbb{N}')$ means the convergence of the subsequence to x.

Let E and F denote separable Banach-spaces and let $(E_i)_{i\in\mathbb{N}}$ and $(F_i)_{i\in\mathbb{N}}$ denote sequences of separable Banach-spaces. Let $(p_i)_{i\in\mathbb{N}}$ denote a sequence of linear bounded operators $p_i \in L[E, E_i]$ with the property

$$\|p_i e\|_{E_i} \to \|e\|_E \ (i \in \mathbb{N}) \ \forall e \in E. \tag{2.7}$$

Similarly let $(q_i)_{i\in\mathbb{N}}$ be a sequence of linear bounded operators $q_i \in L[F, F_i]$ which also satisfies the property (2.7).

We say that a sequence $(e_i)_{i\in\mathbb{N}}$ with $e_i \in E_i$

- $\mathcal{P}$-converges to $e \in E$ iff $\|p_i e - e_i\|_{E_i} \to 0$ $(i \in \mathbb{N})$, written as $e_i \xrightarrow{\mathcal{P}} e$ $(i \in \mathbb{N})$,
- is $\mathcal{P}$-compact iff for every $\mathbb{N}' \subset \mathbb{N}$ there is $\mathbb{N}'' \subset \mathbb{N}'$ and $e \in E$ with $e_i \xrightarrow{\mathcal{P}} e$ $(i \in \mathbb{N}'')$.

Similarly we define $\mathcal{Q}$-convergence and $\mathcal{Q}$-compactness.

Now let $(A_i)_{i\in\mathbb{N}}$ with $A_i \in L[E_i, F_i]$ be a sequence of bounded linear operators and let $A \in L[E, F]$. We say that

- A_i $\mathcal{PQ}$-converges to A (written $A_i \xrightarrow{\mathcal{PQ}} A$ $(i \in \mathbb{N})$) iff $e_i \xrightarrow{\mathcal{P}} e$ $(i \in \mathbb{N})$ implies $A_i e_i \xrightarrow{\mathcal{Q}} Ae$ $(i \in \mathbb{N})$,
- A_i $\mathcal{PQ}$-converges stably to A iff
  1. $A_i \xrightarrow{\mathcal{PQ}} A$ $(i \in \mathbb{N})$ and
  2. there is $i_0 \in \mathbb{N}$ and $C > 0$ such that for all $i \geq i_0$ the inverse $A_i^{-1} \in L[F_i, E_i]$ exists and satisfies

$$\left\|A_i^{-1}\right\|_{F_i \to E_i} \leq C, \ \forall i \geq i_0,$$

- A_i $\mathcal{PQ}$-converges regularly to A iff
  1. $A_i \xrightarrow{\mathcal{PQ}} A$ and
  2. if $(e_i)_{i\in\mathbb{N}}$, $e_i \in E_i$, is a bounded sequence such that the sequence $(A_i e_i)_{i\in\mathbb{N}}$ is $\mathcal{Q}$-compact then $(e_i)_{i\in\mathbb{N}}$ is $\mathcal{P}$-compact.

Now we consider the special case of approximating boundary value problems on the whole real line. Let $(J_i)_{i\in\mathbb{N}}$, $J_i = [x_-^i, x_+^i]$, be a sequence of compact intervals with $x_-^i \le -1$, $x_+^i \ge 1$ and $\lim_{i\to\infty} x_\pm^i = \pm\infty$. For $i \in \mathbb{N}$ consider the separable complex Banach spaces

$$E = H^1(\mathbb{R}, \mathbb{C}^l),\ E_i = H^1(J_i, \mathbb{C}^l),$$

and

$$F = L_2(\mathbb{R}, \mathbb{C}^l),\ F_i = L_2(J_i, \mathbb{C}^l) \times \mathbb{C}^l,$$

with the usual L_2 and H^1-norms for E, F, and E_i $i \in \mathbb{N}$. On F_i we use the norm $\|(h,\eta)\|_{F_i} = \|h\|_{L_2(J_i)} + |\eta|$. Furthermore consider the families of linear and continuous mappings $\mathcal{P} = (p_i)_{i\in\mathbb{N}}$ and $\mathcal{Q} = (q_i)_{i\in\mathbb{N}}$ defined by

$$p_i : \begin{array}{ccc} E & \to & E_i \\ z & \mapsto & z|_{J_i} \end{array} \text{ and } q_i : \begin{array}{ccc} F & \to & F_i \\ h & \mapsto & (h|_{J_i}, 0) \end{array}.$$

Assume the matrix-valued function $M \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{l,l})$ is asymptotically constant with $\lim_{x\to\pm\infty} M(x) = M_\pm$ and define the differential operator

$$L : E \to F,\ z \mapsto Lz := z_x - M(\cdot)z.$$

Similarly, on finite intervals consider the sequence of linear bounded operators

$$L_i : E_i \to F_i,\ z \mapsto L_i z := \begin{pmatrix} z_x - M(\cdot)z \\ Rz \end{pmatrix}$$

where $R : E_i \to \mathbb{C}^l$ is a two point boundary operator given by

$$R := R_- z(x_-^i) + R_+ z(x_+^i)$$

with constant matrices $R_\pm$.

Theorem 1. *Consider a linear differential operator L as above and assume that the matrices M_+ and M_- are hyperbolic, i.e. there are no purely imaginary eigenvalues of $M_\pm$. Let $V_-^{II} \in \mathbb{C}^{l,p}$ be a basis of the stable subspace of M_- and let $V_+^I \in \mathbb{C}^{l,q}$ be a basis of the unstable subspace of M_+ and assume $p+q=l$. Finally assume that the matrices R_- and R_+ satisfy*

$$\det\begin{pmatrix} R_- V_-^{II} & R_+ V_+^I \end{pmatrix} \neq 0. \tag{2.8}$$

Then

$$L_i \xrightarrow{\mathcal{PQ}} L \ \ \textit{regularly as } i \to \infty.$$

Proof. For $z_i \in E_i$ holds

$$\|L_i z_i\|_{F_i} = \|z_{i,x} - M(\cdot) z_i\|_{L_2(J_i)} + \left|R_- z_i(x_-^i) + R_+ z_i(x_+^i)\right| \\ \leq \text{const}\, \|z_i\|_{E_i} \tag{2.9}$$

by Sobolev's inequality and the boundedness of M in L_∞. Furthermore for every $z \in E$ holds

$$\|L_i p_i z - q_i L z\|_{F_i} \\ = \|(z|_{J_i})_x - M(\cdot) z|_{J_i} - (z_x - M(\cdot) z)|_{J_i}\|_{L_2(J_i)} + \left|R_- z(x_-^i) + R_+ z(x_+^i)\right| \\ \xrightarrow[i\to\infty]{} 0 \tag{2.10}$$

since the first summand is zero and $z(x_-^i), z(x_+^i) \to 0$ $(i \in \mathbb{N})$ by Sobolev's embedding theorem.

It is well known that (2.9) and (2.10) imply $L_i \xrightarrow{\mathcal{PQ}} L$ $(i \in \mathbb{N})$ [21, Satz §2(8)].

It remains to prove the regularity of the convergence. Let $(z_i)_{i\in\mathbb{N}}$, $z_i \in E_i$, be a bounded sequence such that $(L_i z_i)_{i\in\mathbb{N}}$ is $\mathcal{Q}$-compact. Let $\mathbb{N}' \subset \mathbb{N}$ be a subsequence. There is $\mathbb{N}'' \subset \mathbb{N}'$ and $h \in F$ with

$$L_i z_i = \begin{pmatrix} z_{i,x} - M(\cdot) z_i \\ R z_i \end{pmatrix} =: \begin{pmatrix} h_i \\ s_i \end{pmatrix} \xrightarrow{\mathcal{Q}} h \ (i \in \mathbb{N}'')$$

by the $\mathcal{Q}$-compactness, i.e.

$$\|h_i - h|_{J_i}\|_{H^1} + |s_i| \to 0 \text{ as } i \to \infty. \tag{2.11}$$

The hyperbolicity assumption for M_+ and M_- implies that the differential operator $z \mapsto z_x - M(\cdot)z$ has exponential dichotomies on $\mathbb{R}_+$ and on $\mathbb{R}_-$. See the Appendix for the definition and some general properties of exponential dichotomies. We denote the data of the dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ by (K_+, β_+, π_+) and (K_-, β_-, π_-) respectively. Note that [12, Lemma 3.4] shows that the projections are asymptotically constant, i.e.

$$\lim_{x\to\pm\infty} \pi_\pm(x) = \bar{\pi}_\pm \text{ with } \mathcal{R}(I - \bar{\pi}_+) = \mathcal{R}(V_+^I) \text{ and } \mathcal{R}(\bar{\pi}_-) = \mathcal{R}(V_-^{II}). \quad (2.12)$$

Because of the exponential dichotomy it follows from Theorem 10 in the Appendix that the element $z_i \in E_i$ can be written as

$$z_i(x) = \begin{cases} S(x,0)\pi_+(0)z_i(0) + S(x,x_+^i)(I - \pi_+(x_+^i))z_i(x_+^i) + \rho_+^i(x), & x \geq 0 \\ S(x,0)(I - \pi_-(0))z_i(0) + S(x,x_-^i)\pi_-(x_-^i)z_i(x_-^i) + \rho_-^i(x), & x \leq 0 \end{cases} \quad (2.13)$$

where $S(\cdot,\cdot)$ is the solution operator for $z \mapsto z_x - M(\cdot)z$ and ρ_+^i, ρ_-^i are given by

$$\rho_+^i(x) = \int_0^{x_+^i} G_+(x,y)h_i(y)dy, \; x_+^i \geq x \geq 0$$

$$\rho_-^i(x) = \int_{x_-^i}^0 G_-(x,y)h_i(y)dy, \; x_-^i \leq x \leq 0.$$

Here G_+ and G_- denote the Green's functions

$$G_+(x,y) = \begin{cases} S(x,y)\pi_+(y), & 0 \leq y \leq x \\ S(x,y)(\pi_+(y) - I), & 0 \leq x < y \end{cases}$$

and

$$G_-(x,y) = \begin{cases} S(x,y)\pi_-(y), & y \leq x \leq 0 \\ S(x,y)(\pi_-(y) - I), & x < y \leq 0. \end{cases}$$

By Sobolev's inequality the sequence $(z_i(0))_{i\in\mathbb{N}''}$ is bounded and there is a subsequence $\mathbb{N}''' \subset \mathbb{N}''$ and $\eta \in \mathbb{C}^l$ with

$$z_i(0) \to \eta \; (i \in \mathbb{N}''').$$

Let

$$z(x) := \begin{cases} S(x,0)\pi_+(0)\eta + \rho_+(x), & x > 0 \\ S(x,0)(I - \pi_-(0))\eta + \rho_-(x), & x < 0 \end{cases} \quad (2.14)$$

where

$$\rho_+(x) := \int_0^\infty G_+(x,y)h(y)dy,\ x \geq 0$$

and

$$\rho_-(x) := \int_{-\infty}^0 G_-(x,y)h(y)dy,\ x \leq 0.$$

Using the exponential dichotomy we obtain

$$\int_{x_-^i}^0 \left|S(x,0)(I-\pi_-(0))(z_i(0)-\eta)\right|^2 dx \leq \frac{K_-^2}{2\beta_-}|z_i(0)-\eta|^2 \to 0\ (i \in \mathbb{N}''') \tag{2.15a}$$

as well as

$$\int_0^{x_+^i} \left|S(x,0)\pi_+(0)(z_i(0)-\eta)\right|^2 dx \to 0\ (i \in \mathbb{N}'''). \tag{2.15b}$$

Furthermore using the exponential dichotomy and the definitions of $\rho_-^i(x)$ and $\rho_-(x)$ it follows with the Cauchy-Schwarz-inequality

$$\begin{aligned}
&\int_{x_-^i}^0 \left|\rho_-^i(x)-\rho_-(x)\right|^2 dx \\
&\qquad \leq 2\int_{x_-^i}^0 \int_{x_-^i}^0 K_-^2 e^{-\beta_-|x-y|}dy \int_{x_-^i}^0 e^{-\beta_-|x-y|}\left|h_i(y)-h(y)\right|^2 dy\, dx \\
&\qquad\quad + 2\int_{x_-^i}^0 \left|\int_{-\infty}^{x_-^i} K_- e^{-\beta_-|x-y|}|h(y)|dy\right|^2 dx.
\end{aligned}$$

Using Fubini the first term can be bounded from above by

$$\frac{8K_-^2}{\beta_-^2}\int_{x_-^i}^0 \left|h_i(y)-h(y)\right|^2 dy$$

which converges to zero by (2.11). The second summand converges to zero as one sees by using the Cauchy-Schwarz-inequality and Fubini together with $h \in L_2(\mathbb{R},\mathbb{C}^l)$. Therefore one finds

$$\int_{x_-^i}^0 \left|\rho_-^i(x)-\rho_-(x)\right|^2 dx \to 0\ (i \in \mathbb{N}''') \tag{2.15c}$$

and by analogous arguments

$$\int_0^{x_+^i} \left|\rho_+^i(x)-\rho_+(x)\right|^2 dx \to 0\ (i \in \mathbb{N}'''). \tag{2.15d}$$

Our next aim is to show

$$\int_{x_-^i}^{0} \left|S(x,x_-^i)\pi_-(x_-^i)z_i(x_-^i)\right|^2 dx \to 0 \; (i \in \mathbb{N}''') \tag{2.15e}$$

and

$$\int_0^{x_+^i} \left|S(x,x_+^i)(I-\pi_+(x_+^i))z_i(x_+^i)\right|^2 dx \to 0 \; (i \in \mathbb{N}'''). \tag{2.15f}$$

Because of

$$|S(x,x_-^i)\pi_-(x_-^i)z_i(x_-^i)|^2 \le K_-^2 e^{-2\beta_-|x-x_-^i|}|\pi_-(x_-^i)z_i(x_-^i)|^2$$

and

$$|S(x,x_+^i)(I-\pi_+(x_+^i))z_i(x_+^i)|^2 \le K_+^2 e^{-2\beta_+|x-x_+^i|}|\pi_+(x_+^i)z_i(x_+^i)|^2$$

it suffices to prove

$$|\pi_-(x_-^i)z_i(x_-^i)| \to 0 \,(i \in \mathbb{N}''') \text{ and } |(I-\pi_+(x_+^i))z_i(x_+^i)| \to 0 \,(i \in \mathbb{N}''').$$

These will be shown by using the boundary conditions. By (2.13) we obtain

$$\begin{aligned} Rz_i =& R_-\pi_-(x_-^i)z_i(x_-^i) + R_+(I-\pi_+(x_+^i))z_i(x_+^i) \\ &+ R_-S(x_-^i,0)(I-\pi_-(0))z_i(0) + R_+S(x_+^i,0)\pi_+(0)z_i(0) \\ &+ R_-\int_{x_-^i}^{0} G_-(x_-^i,y)h_i(y)dy + R_+\int_0^{x_+^i} G_+(x_+^i,y)h_i(y)dy. \end{aligned} \tag{2.16}$$

From the boundedness of $(z_i(0))_{i\in\mathbb{N}}$ easily follows the convergence

$$R_-S(x_-^i,0)(I-\pi_-(0))z_i(0) + R_+S(x_+^i,0)\pi_+(0)z_i(0) \to 0 (i \in \mathbb{N}''').$$

Next we prove

$$R_-\int_{x_-^i}^{0} G_-(x_-^i,y)h_i(y)dy \to 0 (i \in \mathbb{N}''')$$

and

$$R_+\int_0^{x_+^i} G_+(x_+^i,y)h_i(y)dy \to 0 (i \in \mathbb{N}''').$$

To see these choose $\tilde{h} \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C}^l)$ with $\left\|h - \tilde{h}\right\| \leq \frac{\varepsilon}{2}$. Then there is $i_0 \in \mathbb{N}$ with $\left\|h_i - \tilde{h}|_{J_i}\right\|_{L_2(J_i)} \leq \varepsilon$ for all $i \in \mathbb{N}'''$ with $i \geq i_0$. It follows

$$\begin{aligned}
\left|R_- \int_{x_-^i}^0 G_-(x_-^i, y) h_i(y) dy\right|^2 \leq |R_-|^2 \left(\int_{x_-^i}^0 K_- e^{-\beta_-|x_-^i - y|} |h_i(y)| dy\right)^2 \\
\leq 2|R_-|^2 \left(\int_{x_-^i}^0 K_- e^{-\beta_-|x_-^i - y|} |\tilde{h}(y) - h_i(y)| dy\right)^2 \\
+ 2|R_-|^2 \left(\int_{x_-^i}^0 K_- e^{-\beta_-|x_-^i - y|} |\tilde{h}(y)| dy\right)^2.
\end{aligned}$$

The second term converges to zero as $i \to \infty$ since $\tilde{h} \in \mathcal{C}_0^\infty$ and for the first term one finds

$$\begin{aligned}
&\left(\int_{x_-^i}^0 K_- e^{-\beta_-|x_-^i - y|} |\tilde{h}(y) - h_i(y)| dy\right)^2 \\
&\qquad \leq \frac{K_-^2}{2\beta_-} \int_{x_-^i}^0 \left|\tilde{h}(y) - h_i(y)\right|^2 dy \leq \frac{K_-^2}{2\beta_-} \varepsilon \quad \forall i \geq i_0, i \in \mathbb{N}'''
\end{aligned}$$

where we used the Cauchy-Schwarz-inequality. Since $\varepsilon > 0$ was arbitrary equations (2.16) and (2.11) imply

$$R_- \pi_-(x_-^i) z_i(x_-^i) + R_+(I - \pi_+(x_+^i)) z_i(x_+^i) \to 0 \; (i \in \mathbb{N}''').$$

From the convergence of the projectors (2.12) combined with the determinant condition (2.8) we obtain

$$\pi_-(x_-^i) z_i(x_-^i) \to 0 \; (i \in \mathbb{N}''')$$

and

$$(I - \pi_+(x_+^i)) z_i(x_+^i) \to 0 \; (i \in \mathbb{N}''')$$

hence (2.15e) and (2.15f) hold.

Summarizing (2.15a)–(2.15f) shows

$$\|z_i - z|_{J_i}\|_{L_2(J_i)} \to 0 \; (i \in \mathbb{N}'''). \tag{2.17}$$

To finish the proof we still have to show $z \in H^1$ and $z_i \xrightarrow{\mathcal{P}} z \, (i \in \mathbb{N}''')$.

For this define $w := M(\cdot) z + h \in L_2(\mathbb{R}, \mathbb{C}^l)$ and let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{C}^l)$ be arbitrary. Then there is $i_0 \in \mathbb{N}$ with $J_i \supset \operatorname{supp}(\phi)$ $\forall i \geq i_0$. Using the definition of w and the equality $z_{i,x} = M(\cdot)z_i + h_i$ one can estimate for all $i \geq i_0$

$$
\begin{aligned}
&\left|(w,\phi)_{L_2(\mathbb{R})} + (z,\phi')_{L_2(\mathbb{R})}\right| \\
&\qquad \leq \left|(w|_{J_i},\phi)_{L_2(J_i)} - (z_{i,x},\phi)_{L_2(J_i)}\right| + \left|(z|_{J_i} - z_i,\phi')_{L_2(J_i)}\right| \\
&\qquad\qquad \leq \text{const}\, \|\phi\|_{H^1} \left(\|z|_{J_i} - z_i\|_{L_2(J_i)} + \|h|_{J_i} - h_i\|_{L_2(J_i)} \right).
\end{aligned}
$$

Since we know $\|h|_{J_i} - h_i\|_{L_2(J_i)} \to 0$ $(i \in \mathbb{N}''')$ from equation (2.11) and furthermore $\|z|_{J_i} - z_i\|_{L_2(J_i)} \to 0$ $(i \in \mathbb{N}''')$ by (2.17) it follows that $z_x = w$ holds in the distributional sense and thus $z \in H^1(\mathbb{R})$.
Finally

$$
\begin{aligned}
\|z|_{J_i} - z_i\|_{H^1}^2 &= \|z|_{J_i} - z_i\|_{L_2}^2 + \|z_x|_{J_i} - z_{i,x}\|_{L_2}^2 \\
&\leq \|z|_{J_i} - z_i\|_{L_2}^2 + \|(Mz+h)|_{J_i} - (Mz_i + h_i)\|_{L_2}^2 \to 0 \ (i \in \mathbb{N}''')
\end{aligned}
$$

where we used the definition of $w = z_x$ again. □

3 Application to coupled hyperbolic-parabolic systems

Consider the coupled hyperbolic-parabolic system

$$
\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} =: P \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.18}
$$

with $u(t,x) \in \mathbb{C}^n$, $v(t,x) \in \mathbb{C}^m$, and assume that the coefficients A, B_{ij}, and C_{ij} satisfy the following assumptions.

(PA) $A \in \mathbb{C}^{n,n},\ A + A^* \geq \alpha I > 0$,

(CA) B_{ij}, C_{ij} are continuous and asymptotically constant matrix-valued functions

$$
\begin{aligned}
\lim_{x\to+\infty} B_{ij}(x) &= \lim_{x\to-\infty} B_{ij}(x) =: B_{ij,\infty}, \\
\lim_{x\to+\infty} C_{ij}(x) &= \lim_{x\to-\infty} C_{ij}(x) =: C_{ij,\infty},
\end{aligned}
$$

(HA) $B_{22} = diag(b_1, \ldots, b_m)$, $|b_i(x)| \geq \gamma > 0 \;\forall i, x \in \mathbb{R}$.

(SA) $s \in \sigma\left(-\omega^2 \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + i\omega \begin{pmatrix} B_{11,\infty} & B_{12,\infty} \\ B_{21,\infty} & B_{22,\infty} \end{pmatrix} + \begin{pmatrix} C_{11,\infty} & C_{12,\infty} \\ C_{21,\infty} & C_{22,\infty} \end{pmatrix}\right)$
implies $\mathrm{Re} s < -\delta < 0$.

Remark. *One can show that the convergence results below generalize to the case where the limits $x \to -\infty$ and $x \to +\infty$ in condition* **(CA)** *are different. However, this requires a considerably longer proof and we refer to [15] for the details.*

The operator P is defined by the right hand side of (3.18)

$$P : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m).$$

In the following we restrict the infinite interval to a compact interval $J = [x_-, x_+]$ and impose certain boundary conditions. We consider the operator

$$P|_J : H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m) \to L_2(J, \mathbb{C}^n) \times L_2(J, \mathbb{C}^m)$$

which is defined in the same way as P, together with a boundary operator

$$R : H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m) \to \mathbb{C}^{2n+m}$$

defined as follows

$$R \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} R_-^I & R_-^{II} & R_-^{III} \end{pmatrix} \begin{pmatrix} u(x_-) \\ u_x(x_-) \\ v(x_-) \end{pmatrix} + \begin{pmatrix} R_+^I & R_+^{II} & R_+^{III} \end{pmatrix} \begin{pmatrix} u(x_+) \\ u_x(x_+) \\ v(x_+) \end{pmatrix}.$$

The resolvent equation for P reads

$$(sI - P) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{in } L_2(\mathbb{R}, \mathbb{C}^{n+m}) \tag{3.19}$$

with $f \in L_2(\mathbb{R}, \mathbb{C}^n)$, $g \in L_2(\mathbb{R}, \mathbb{C}^m)$, $u \in H^2(\mathbb{R}, \mathbb{C}^n)$, and $v \in H^1(\mathbb{R}, \mathbb{C}^m)$. The finite interval analog on $J = [x_-, x_+]$ is given by

$$\begin{pmatrix} sI - P|_J \\ R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \\ \eta \end{pmatrix} \quad \text{in } L_2(J, \mathbb{C}^{n+m}) \times \mathbb{C}^{2n+m}, \tag{3.20}$$

where $u \in H^2(J,\mathbb{C}^n)$, $v \in H^1(J,\mathbb{C}^m)$, $f \in L_2(J,\mathbb{C}^n)$, $g \in L_2(J,\mathbb{C}^m)$ and $\eta \in \mathbb{C}^{2n+m}$.

Both systems (3.19) and (3.20) will be transformed by using the variables

$$(u,v)^T \rightsquigarrow z := (u, Au_x, v)^T. \tag{3.21}$$

For the all line problem (3.19) this transformation leads to the first equation

$$L(\cdot,s)z := z_x - M(\cdot,s)z = h, \tag{3.22}$$

which holds in the space $H^1(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^n)$. For the finite interval problem (3.20) we obtain

$$L_J(s)z := \begin{pmatrix} L|_J(\cdot,s) \\ R_1 \end{pmatrix} z = \begin{pmatrix} z_x - M(\cdot,s)z \\ R_1 z \end{pmatrix} = \begin{pmatrix} h \\ \eta \end{pmatrix} \tag{3.23}$$

as an equality in $H^1(J,\mathbb{C}^n) \times L_2(J,\mathbb{C}^n) \times L_2(J,\mathbb{C}^m) \times \mathbb{C}^{2n+m}$.

The matrix-valued function $M(\cdot,s)$ in (3.22) and in (3.23) is given by

$$M(\cdot,s) = \begin{pmatrix} 0 & A^{-1} & 0 \\ B_{12}B_{22}^{-1}C_{21} + sI - C_{11} & -B_{11}A^{-1} + B_{12}B_{22}^{-1}B_{21}A^{-1} & -C_{12} - B_{12}B_{22}^{-1}(sI - C_{22}) \\ -B_{22}^{-1}C_{21} & -B_{22}^{-1}B_{21}A^{-1} & B_{22}^{-1}(sI - C_{22}) \end{pmatrix}$$

and the function h reads

$$h = \begin{pmatrix} 0 \\ -f + B_{12}B_{22}^{-1}g \\ -B_{22}^{-1}g \end{pmatrix} \in H^1(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^n) \times L_2(\mathbb{R},\mathbb{C}^m)$$

for the all line problem (3.22). The same definition for the finite interval problem (3.23) leads to $h \in H^1(J,\mathbb{C}^n) \times L_2(J,\mathbb{C}^n) \times L_2(J,\mathbb{C}^m)$.

Finally the boundary operator R_1 is given by

$$R_1 z = \begin{pmatrix} R_-^I & R_-^{II}A^{-1} & R_-^{III} \end{pmatrix} z(x_-) + \begin{pmatrix} R_+^I & R_+^{II}A^{-1} & R_+^{III} \end{pmatrix} z(x_+).$$

By assumption **(CA)** the limits $M_\infty(s) := \lim_{x\to\pm\infty} M(x,s)$ exist. The following lemma is easily verified.

Lemma 2. *For every $s,\kappa \in \mathbb{C}$ the following conditions are equivalent*

1. $s \in \sigma\left(\kappa^2\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix} + \kappa\begin{pmatrix} B_{11,\infty} & B_{12,\infty} \\ B_{21,\infty} & B_{22,\infty}\end{pmatrix} + \begin{pmatrix} C_{11,\infty} & C_{12,\infty} \\ C_{21,\infty} & C_{22,\infty}\end{pmatrix}\right)$ *and*

2. $\det\left(\kappa I - M_\infty(s)\right) = 0$.

An important consequence of this result is the next corollary.

Corollary 3. *For every* $s \in \{\mathrm{Re} s > -\delta\}$ *the matrices* $M_\infty(s)$ *are hyperbolic and the operators*

$$L(\cdot, s) : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to H^1(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

are Fredholm of index zero.

Proof. The first part immediately follows from Lemma 2 together with the spectral assumption **(SA)**.
The second part follows from Lemma 11 and its Corollary 12 in the appendix.

□

An important connection between the original operator and the first order operator is that their Fredholm properties coincide. We formulate this in the next lemma, again with the proof given in the appendix.

Lemma 4. *The operator*

$$sI - P : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

is Fredholm if and only if the operator

$$L(\cdot, s) : H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m) \to H^1(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

is Fredholm. Furthermore the Fredholm-indices coincide.

Let $V_-(s) \in \mathbb{C}^{2n+m,p}$ denote a basis of the stable subspace of $M_\infty(s)$ and let $V_+(s) \in \mathbb{C}^{2n+m,q}$ denote a basis of the unstable subspace of $M_\infty(s)$. Because of the assumption **(CA)** it is $p+q = 2n+m$. Define the determinant function

$$D(s) := \det\Big[\begin{pmatrix} R_-^I & R_-^{II}A^{-1} & R_-^{III}\end{pmatrix} V_-(s), \begin{pmatrix} R_+^I & R_+^{II}A^{-1} & R_+^{III}\end{pmatrix} V_+(s)\Big]. \tag{3.24}$$

The condition $D(s) \neq 0$ will give a sufficient criterion for appropriate artificial boundary conditions when we truncate to finite intervals.

3.1 Regular convergence for hyperbolic-parabolic problems

Let E, E_i, F, and F_i be as in Section 2 with $l = 2n + m$. Furthermore consider the spaces

$$\begin{aligned}
\tilde{E} &= H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m),\\
\tilde{E}_i &= H^2(J_i, \mathbb{C}^n) \times H^1(J_i, \mathbb{C}^m),\ i \in \mathbb{N},\\
\tilde{F} &= L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m),\\
\tilde{F}_i &= L_2(J_i, \mathbb{C}^n) \times L_2(J_i, \mathbb{C}^m) \times \mathbb{C}^{2n+m},\ i \in \mathbb{N},
\end{aligned}$$

and the mappings

$$\begin{aligned}
\tilde{p}_i &: \tilde{E} \to \tilde{E}_i, \quad (u, v)^T \mapsto (u|_{J_i}, v|_{J_i})^T,\ i \in \mathbb{N},\\
\tilde{q}_i &: \tilde{F} \to \tilde{F}_i, \quad (f, g)^T \mapsto (f|_{J_i}, g|_{J_i}, 0)^T,\ i \in \mathbb{N}.
\end{aligned}$$

The families of mappings $\tilde{\mathcal{P}} = (\tilde{p}_i)_{i\in\mathbb{N}}$ and $\tilde{\mathcal{Q}} = (\tilde{q}_i)_{i\in\mathbb{N}}$ are continuous and bounded and satisfy (2.7).
Finally we consider the bounded linear operators

$$\mathcal{A}_i(s) := \begin{pmatrix} sI - P|_{J_i} \\ R \end{pmatrix} : \tilde{E}_i \to \tilde{F}_i \tag{3.25}$$

and

$$\mathcal{A}(s) := (sI - P) : \tilde{E} \to \tilde{F}. \tag{3.26}$$

Our main result is the following theorem.

Theorem 5. *Let $s \in \{\mathrm{Re} s > -\delta\}$ and assume $D(s) \neq 0$ with $D(s)$ defined in (3.24). Then the sequence of bounded linear operators $(\mathcal{A}_i(s))_{i\in\mathbb{N}}$ $\tilde{\mathcal{P}}\tilde{\mathcal{Q}}$-converges regularly to the operator $\mathcal{A}(s) := (sI - P) : \tilde{E} \to \tilde{F}$.*

Proof. First we show the $\tilde{\mathcal{P}}\tilde{\mathcal{Q}}$-convergence. Let $i \in \mathbb{N}$ and $(u, v)^T \in \tilde{E}_i$ be arbitrary. Then because of Sobolev's inequality and the boundedness of R holds

$$\left\| \mathcal{A}_i(s) \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\tilde{F}_i} \leq \left\| (sI - P|_{J_i}) \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{L_2} + \left| R \begin{pmatrix} u \\ v \end{pmatrix} \right| \leq C_0 \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{H^2(J_i)\times H^1(J_i)}$$

with a constant C_0 independent of $i \in \mathbb{N}$.

Now let $(u,v)^T \in E$ be arbitrary. From the Sobolev-embedding-Theorem we obtain

$$\left\| \tilde{q}_i \mathcal{A}(s) \begin{pmatrix} u \\ v \end{pmatrix} - \mathcal{A}_i(s) \tilde{p}_i \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{\tilde{F}_i} \leq \left| R \begin{pmatrix} u|_{J_i} \\ v|_{J_i} \end{pmatrix} \right| \to 0 \ (i \in \mathbb{N})$$

so that by [21, §2 Satz(8)] the $\tilde{\mathcal{P}}\tilde{\mathcal{Q}}$-convergence follows.
Using the transformation (3.21) we rewrite the second order equation

$$(sI - P) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{in } \tilde{F}$$

as the first order system

$$L(\cdot, s) \begin{pmatrix} u \\ Au_x \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -f + B_{12} B_{22}^{-1} g \\ -B_{22}^{-1} g \end{pmatrix} \quad \text{in } F$$

and its finite interval approximation

$$\begin{pmatrix} sI - P|_{J_i} \\ R \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} f_i \\ g_i \\ \eta_i \end{pmatrix} \quad \text{in } \tilde{F}_i$$

as

$$L_{J_i}(\cdot, s) \begin{pmatrix} u_i \\ Au_{i,x} \\ v_i \end{pmatrix} = \begin{pmatrix} 0 \\ -f_i + B_{12} B_{22}^{-1} g_i \\ -B_{22}^{-1} g_i \\ \eta_i \end{pmatrix} \quad \text{in } F_i.$$

Define the mappings

$$\begin{aligned}
\iota_E : \tilde{E} \to E, \qquad & (u,v)^T \mapsto (u,\ Au_x,\ v)^T, \\
\iota_{E_i} : \tilde{E}_i \to E_i, \qquad & (u_i,\ v_i)^T \mapsto (u_i,\ Au_{i,x},\ v_i)^T, \\
\iota_F : \tilde{F} \to F, \qquad & (f,\ g)^T \mapsto (0,\ -f + B_{12} B_{22}^{-1} g,\ -B_{22}^{-1} g)^T, \\
\iota_{F_i} : \tilde{F}_i \to F_i, \qquad & (f_i,\ g_i,\ \eta_i)^T \mapsto (0,\ -f_i + B_{12} B_{22}^{-1} g_i,\ -B_{22}^{-1} g_i,\ \eta_i)^T.
\end{aligned}$$

The whole setting of spaces and mappings is shown schematically in Figure 1.

$$
\begin{tikzcd}
\tilde{E} \arrow[rrr, "\mathcal{A}(s)"] \arrow[dr, "\iota_E"'] \arrow[ddd, "\tilde{p}_i"'] & & & \tilde{F} \arrow[dl, "\iota_F"] \arrow[ddd, "\tilde{q}_i"] \\
& E \arrow[r, "L(\cdot,s)"] \arrow[d, "p_i"'] & F \arrow[d, "q_i"] & \\
& E_i \arrow[r, "L_{J_i}(s)"'] & F_i & \\
\tilde{E}_i \arrow[rrr, "\mathcal{A}_i(s)"'] \arrow[ur, "\iota_{E_i}"'] & & & \tilde{F}_i \arrow[ul, "\iota_{F_i}"]
\end{tikzcd}
$$

Figure 1: The setting of spaces and mappings. Note that the diagram is usually not commutative.

Let $(u_i, v_i)^T_{i\in\mathbb{N}}$ be a bounded sequence with $(u_i, v_i)^T \in \tilde{E}_i$ such that the sequence $\left(\mathcal{A}_i(s)\begin{pmatrix} u_i \\ v_i \end{pmatrix}\right)_{i\in\mathbb{N}}$ is $\tilde{\mathcal{Q}}$-compact. Then also the sequence

$$\left(L_{J_i}(s)\begin{pmatrix} u_i \\ Au_{i,x} \\ v_i \end{pmatrix}\right)_{i\in\mathbb{N}} = \left((\iota_{F_i} \circ \mathcal{A}_i(s))\begin{pmatrix} u_i \\ v_i \end{pmatrix}\right)_{i\in\mathbb{N}}$$

is $\mathcal{Q}$-compact, since $\iota_{F_i} \xrightarrow{\tilde{\mathcal{Q}}\mathcal{Q}} \iota_F$.

Let $\mathbb{N}' \subset \mathbb{N}$ be arbitrary. By Theorem 1 we know

$$L_{J_i}(\cdot, s) \xrightarrow{\mathcal{P}\mathcal{Q}} L(\cdot, s) \text{ regularly}$$

so that there is a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and $(u, w, v) \in E$ with

$$\begin{pmatrix} u_i \\ Au_{i,x} \\ v_i \end{pmatrix} \xrightarrow{\mathcal{P}} \begin{pmatrix} u \\ w \\ v \end{pmatrix} \quad (i \in \mathbb{N}''). \tag{3.27}$$

Furthermore by the $\tilde{\mathcal{Q}}$-compactness of $\left(\mathcal{A}_i(s)\begin{pmatrix} u_i \\ v_i \end{pmatrix}\right)_{i\in\mathbb{N}}$ there is $\mathbb{N}''' \subset \mathbb{N}''$ and $(f, g)^T \in \tilde{F}$ with

$$\mathcal{A}_i(s)\begin{pmatrix} u_i \\ v_i \end{pmatrix} \xrightarrow{\mathcal{Q}} \begin{pmatrix} f \\ F \end{pmatrix} \quad (i \in \mathbb{N}''').$$

By construction we thus obtain

$$L(\cdot,s)\begin{pmatrix}u\\ w\\ v\end{pmatrix} \overset{\mathcal{Q}}{\longleftarrow} L_i(\cdot,s)\begin{pmatrix}u_i\\ Au_{i,x}\\ v_i\end{pmatrix} \overset{\mathcal{Q}}{\longrightarrow} \begin{pmatrix}0\\ -f+B_{12}B_{22}^{-1}g\\ -B_{22}^{-1}g\end{pmatrix} \quad (i\in\mathbb{N}''')$$

which by uniqueness of the $\mathcal{Q}$-limit (see [21]) implies

$$L(\cdot,s)\begin{pmatrix}u\\ w\\ v\end{pmatrix} = \begin{pmatrix}0\\ -f+B_{12}B_{22}^{-1}g\\ -B_{22}^{-1}g\end{pmatrix}.$$

Using the differential equation we find $w = Au_x \in H^1(\mathbb{R},\mathbb{C}^n)$ and therefore $u \in H^2(\mathbb{R},\mathbb{C}^n)$, $v \in H^1(\mathbb{R},\mathbb{C}^m)$, and the equality

$$(sI-P)\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}f\\ g\end{pmatrix}$$

holds. Now the definition of the $\mathcal{P}$-convergence in (3.27) yields

$$\begin{aligned}\|u_i - u|_{J_i}\|_{H^1(J_i,\mathbb{C}^n)} + \|u_{i,x} - u|_{J_i}\|_{H^1(J_i,\mathbb{C}^n)} + \|v_i - v|_{J_i}\|_{H_1(J_i,\mathbb{C}^m)} \\ \le K\left\|\begin{pmatrix}u_i\\ Au_{i,x}\\ v_i\end{pmatrix} - \begin{pmatrix}u|_{J_i}\\ Au_x|_{J_i}\\ v|_{J_i}\end{pmatrix}\right\|_{E_i} \to 0\ (i\in\mathbb{N}''')\end{aligned}$$

and this finally shows

$$\begin{pmatrix}u_i\\ v_i\end{pmatrix} \overset{\tilde{\mathcal{P}}}{\longrightarrow} \begin{pmatrix}u\\ v\end{pmatrix} \ (i\in\mathbb{N}''').$$

Since $\mathbb{N}'$ was arbitrary the compactness of the sequence $(u_i,v_i)_{i\in\mathbb{N}}$ follows which proves the regularity of the convergence. □

Theorem 5 above is a far reaching result. We only show one consequence of the theorem, further applications can be found in [15]. For example, using an abstract result from [22] it is shown that eigenvalues and eigenvectors are well approximated in finite intervals and that exponential estimates hold. For the linear operator P we denote by $\rho(P)$ the resolvent-set of the operator.

Theorem 6. *Let $\Omega \subset \{\mathrm{Re} s > -\delta\} \cap \rho(P)$ be a compact set and assume*

$$D(s) \neq 0 \; \forall s \in \Omega.$$

Let $(J_i)_{i \in \mathbb{N}}$ be a sequence of compact intervals as before. Then there is an index $i_0 \in \mathbb{N}$ and a constant $K_0 > 0$ such that for all $s \in \Omega$ and all $i \geq i_0$ equation (3.20) has a unique solution $(u, v)^T \in H^2(J_i, \mathbb{C}^n) \times H^1(J_i, \mathbb{C}^m)$ for all $f \in L_2(J_i, \mathbb{C}^n)$, $g \in L_2(J_i, \mathbb{C}^m)$, $\eta \in \mathbb{C}^{2n+m}$.
Moreover the solution can be estimated by

$$\begin{aligned} &\|u\|_{H^2(J_i)} + \|v\|_{H^1(J_i)} + |u|_\Gamma + |u_x|_\Gamma + |v|_\Gamma \\ &\qquad\qquad \leq K_0 \big(\|f\|_{L_2(J_i)} + \|g\|_{L_2(J_i)} + |\eta| \big), \end{aligned} \tag{3.28}$$

where $|u|_\Gamma$ is the norm of u on the boundary of the interval defined by $|u|_\Gamma^2 = |u(x_-)|^2 + |u(x_+)|^2$.

Proof. Let $s_0 \in \Omega$ be arbitrary. Then the operator $\mathcal{A}(s_0)$ is invertible and therefore $\mathcal{N}(\mathcal{A}(s_0)) = \{0\}$. The Fredholm-alternative for boundary value problems implies that for all $i \in \mathbb{N}$ the operators $\mathcal{A}_i(s_0)$ are Fredholm of index 0.
By Theorem 5 we have

$$\mathcal{A}_i(s_0) \xrightarrow{\tilde{\mathcal{P}}\tilde{\mathcal{Q}}} \mathcal{A}(s_0) \;\text{ regularly }\; (i \in \mathbb{N})$$

so that [21, §2 Satz 60] is applicable and shows that the sequence of operators $(\mathcal{A}_i(s_0))_{i \in \mathbb{N}}$ converges regularly and stably to $\mathcal{A}(s_0)$.
Hence there is $i_0 = i_0(s_0) \in \mathbb{N}$ and $K_0 = K_0(s_0) > 0$ such that for all $i \geq i_0$ the inverses $\mathcal{A}_i(s_0)^{-1} \in L[F_i, E_i]$ exist and are bounded by

$$\left\| \mathcal{A}_i(s_0)^{-1} \right\|_{L[F_i, E_i]} \leq K_0.$$

Let $\varepsilon_0 = \varepsilon_0(s_0) := \frac{1}{2K_0}$, then for all $s \in K_{\varepsilon_0}(s_0) = \{s \in \mathbb{C} : |s - s_0| < \varepsilon_0\}$ and all $i \geq i_0$ holds

$$\|\mathcal{A}_i(s) - \mathcal{A}_i(s_0)\|_{L[E_i, F_i]} \leq |s - s_0| < \varepsilon_0 = \frac{1}{2K_0}. \tag{3.29}$$

By Lemma 8 the inverses $\mathcal{A}_i(s)^{-1} \in L[F_i, E_i]$ exist for all $i \geq i_0$ and all $s \in K_{\varepsilon_0}(s_0)$ and satisfy

$$\left\| \mathcal{A}_i(s)^{-1} \right\|_{L[F_i, E_i]} \leq 2K_0.$$

Since $s_0 \in \Omega$ was arbitrary one finds for every $s \in \Omega$ an open neighborhood of s in which similar estimates hold. Choosing a finite subcovering of this covering shows our assertion. $\square$

4 The FitzHugh-Nagumo System

In this section we show that the abstract theory applies to the FitzHugh-Nagumo System. The FitzHugh-Nagumo System arises in the modelling of electric pulses in nerve cells. It reads

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_{xx} + u - \frac{1}{3}u^3 - v \\ \Phi(u + a - bv) \end{pmatrix} \tag{4.30}$$

with positive parameters a, b, and Φ.

We consider the parameter-values $a = 0.7$, $b = 0.8$, $\Phi = 0.08$ which are a quite common choice in the literature (e.g. [11]). It is known that for these parameter-values the system has a stable and an unstable traveling wave solution, see for example Bates and Jones [1]. The traveling wave solutions are homoclinic connecting orbits of the stationary point $(u_\infty, v_\infty)^T \approx (-1.1994, -0.6243)^T$ and have a nonzero speed. The linearization about a traveling wave solution $\binom{u}{v} = \left(\binom{\bar{u}}{\bar{v}}(x - ct)\right)$ with speed $c \neq 0$ reads

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \tilde{P}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 - \bar{u}^2 & -1 \\ \Phi & -\Phi b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{4.31}$$

It is easy to check that the assumptions **(PA)**, **(CA)**, and **(HA)** are satisfied. To see that also **(SA)** is satisfied is more involved and we give a sufficient condition which is much easier to check.

Consider a general operator P of the form (3.18) which satisfies the assumptions **(PA)**, **(CA)**, and **(HA)**. Define

$$B_\infty := \begin{pmatrix} B_{11,\infty} & B_{12,\infty} \\ B_{21,\infty} & B_{22,\infty} \end{pmatrix} \text{ and } C_\infty := \begin{pmatrix} C_{11,\infty} & C_{12,\infty} \\ C_{21,\infty} & C_{22,\infty} \end{pmatrix}$$

and consider the following condition.

(SC) There is a Hermitian positive definite matrix $H \in \mathbb{C}^{n+m,n+m}$ of the form $H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$ such that

- $H_1 A + A^* H_1 > 0$,
- H_2 is a diagonal matrix,
- $HB_\infty = B_\infty^* H$,
- $HC_\infty + C_\infty^* H < -2\delta H$ for some $\delta > 0$.

Lemma 7. *The condition* **(SC)** *implies the spectral assumption* **(SA)**.

Proof. Let $\mu, \nu \in \mathbb{C}^{n+m}$. By **(SC)** it holds

$$-\mu^* \begin{pmatrix} H_1 A + A^* H_1 & 0 \\ 0 & 0 \end{pmatrix} \mu - \mu^*(HB_\infty - B_\infty^* H)\nu + \nu^*(HC_\infty + C_\infty^* H)\nu \leq -2\delta \nu^* H \nu. \quad (4.32)$$

Now let $\omega \in \mathbb{R}$ and assume that the symbol $\hat{P}(i\omega)$ has the eigenvector ν with eigenvalue $s \in \mathbb{C}$, i.e.

$$\hat{P}(i\omega)\nu = s\nu \text{ and let } \mu = i\omega\nu.$$

Then we have

$$\begin{aligned} 2\mathrm{Re}(s)\nu^* H \nu &= s\nu^* H \nu + (s\nu^* H \nu)^* \\ &= \nu^* H \hat{P}(i\Omega)\nu + (\nu^* H \hat{P}(i\Omega)\nu)^* \\ &= -\omega^2 \nu^* H \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \nu + i\omega\nu^* HB_\infty \nu + \nu^* HC_\infty \nu \\ &\quad + \left(-\omega^2 \nu^* H \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \nu\right)^* + (i\omega\nu^* HB_\infty \nu)^* + (\nu^* HC_\infty \nu)^* \\ &= -\mu^* \left(\begin{smallmatrix} H_1 A + A^* H_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \mu - \mu^*(HB_\infty - B_\infty^* H)\nu + \nu^*(HC_\infty + C_\infty^* H)\nu \\ &\leq -2\delta \nu^* H \nu \end{aligned}$$

and thus $\mathrm{Re}(s) < -\delta$. Hence the spectral assumption **(SA)** is proven. $\square$

With the help of the sufficient condition **(SC)** it is now easy to show that the FitzHugh-Nagumo system satisfies the assumptions from Section 3. Let $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \frac{1}{\Phi} \end{smallmatrix}\right) \in \mathbb{C}^{2,2}$. With this matrix we verify **(SC)**

- $1A + A^* 1 = 2 > 0$,
- $HB_\infty = \begin{pmatrix} c & 0 \\ 0 & \frac{c}{\Phi} \end{pmatrix} = B_\infty^* H$,
- $HC_\infty + C_\infty^* H = \begin{pmatrix} 2 - 2u_\infty^2 & 0 \\ 0 & -2b \end{pmatrix} < -2\delta H$ for some $\delta > 0$.

5 Appendix

In this appendix we state some basic results used throughout the text. We begin with a well known perturbation result.

Lemma 8. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be Banach spaces. Assume that* $A : X \to Y$ *is a linear homeomorphism.*
Then for every bounded linear operator $B : X \to Y$ *with*

$$\|B\|_{X\to Y} < \frac{1}{\|A^{-1}\|_{Y\to X}}$$

the operator $A + B : X \to Y$ *is a linear homeomorphism and*

$$\left\|(A+B)^{-1}\right\|_{Y\to X} \leq \left\|A^{-1}\right\|_{Y\to X} \frac{1}{1 - \|A^{-1}\|_{Y\to X}\,\|B\|_{X\to Y}}.$$

Exponential Dichotomies

Throughout this section we consider an ordinary differential operator of the form

$$Lz = z_x - M(x)z, \ x \in J, \tag{5.33}$$

where $M \in \mathcal{C}(J, \mathbb{C}^{l,l})$ is a continuous matrix-valued function on the closed interval $J = [x_-, x_+]$, $x_- < x_+ \in \mathbb{R} \cup \{-\infty, +\infty\}$. We denote by $S(\cdot,\cdot)$ the solution-operator for L. First we give the definition of an exponential dichotomy (see for example [3, 12, 2, 16]).

Definition 9. The operator L has an exponential dichotomy on the interval J if there are positive constants K, β, and for every $x \in J$ there is a projection $\pi(x) : \mathbb{C}^l \to \mathbb{C}^l$ such that

$$\begin{aligned} S(x,y)\pi(y) &= \pi(x)S(x,y) && \forall x, y \in J, \\ |S(x,y)\pi(y)| &\leq Ke^{-\beta(x-y)} && \forall x \geq y \in J, \\ |S(x,y)(I-\pi(y))| &\leq Ke^{-\beta(y-x)} && \forall x < y \in J. \end{aligned}$$

We call (K, β, π) the data of the dichotomy.

The data of the dichotomy are not unique in general. The benefit of exponential dichotomies lies in semi-infinite or infinite interval problems. If J contains an interval of the form $[x_0, \infty)$, the ranges of the projectors are unique and if it contains an interval of the form $(-\infty, x_0]$, the kernels of the projectors are unique. In particular, if the operator L has an exponential dichotomy on the whole real line, the projectors are uniquely determined. For results in this direction see Coppel [3] and Palmer [12].

Next we recall a result about the solvability and solution estimates for boundary value problems in the presence of an exponential dichotomy.

Theorem 10 ([2, Theorem A.1]). *Assume the operator L has an exponential dichotomy on J with data (K, β, π).*

Define the Green's function G with respect to π for all $x, y \in J$ by

$$G(x,y) = \begin{cases} S(x,y)\pi(y), & y \leq x, \\ S(x,y)(\pi(y) - I), & x < y. \end{cases} \tag{5.34}$$

Then for every $h \in L_2(J, \mathbb{C}^l)$, $\gamma_- \in \mathcal{R}(\pi(x_-))$, $\gamma_+ \in \mathcal{R}(I - \pi(x_+))$ there is a unique solution $z \in H^1(J, \mathbb{C}^l)$ of the boundary value problem

$$\begin{aligned} Lz &= h, \quad \text{in } L_2(J), \\ (I - \pi(x_+))z(x_+) &= \gamma_+, \\ \pi(x_-)z(x_-) &= \gamma_-. \end{aligned}$$

In the case $x_- = -\infty$ the boundary condition for $z(x_-)$ is hidden in the space and there is no explicit boundary condition. The same is true for the case $x_+ = +\infty$. The solution can be written in the form $z = z_{sp} + z_h$, where z_{sp} and z_h are given by

$$z_{sp}(x) = \int_J G(x,y)h(y)dy, \quad \text{and} \quad z_h(x) = S(x, x_-)\gamma_- + S(x, x_+)\gamma_+. \tag{5.35}$$

We also need a result about the Fredholm properties of ordinary differential operators on the whole real line. On bounded intervals Fredholm properties are easy to verify by integration, but on unbounded domains it is more involved. A general result about the connection of exponential dichotomies and Fredholm properties of differential operators was proven by K. J. Palmer in [12] and [13]. We will make explicit use of the result [12, Lemma 4.2] which is presented for bounded and continuously differentiable functions there. But the proof given in [12] directly carries over to the spaces L_2 and H^1 and we only state the result in the following lemma.

Lemma 11. *Let $M \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{l,l})$ be a bounded matrix-valued function so that the differential operator*

$$L(\cdot) : \begin{array}{ccc} H^1(\mathbb{R}, \mathbb{C}^l) & \to & L_2(\mathbb{R}, \mathbb{C}^l) \\ z & \mapsto & z_x - M(\cdot)z \end{array}$$

has an exponential dichotomy on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with projectors $\pi_\pm(\cdot)$. Then L is Fredholm and $f \in \mathcal{R}(L)$ if and only if

$$\int_{-\infty}^{\infty} u^*(t) f(t) dt = 0$$

for all solutions $u \in H^1(\mathbb{R}, \mathbb{C}^l)$ of the adjoint equation

$$L^{ad}(\cdot)u = u_x + M(\cdot)^* u = 0.$$

Furthermore the Fredholm index of L is $\dim \mathcal{R}(\pi_+(0)) + \dim \mathcal{R}(I - \pi_-(0)) - l$.

We always denoted by '$*$' the transposed conjugated matrix or vector.

Corollary 12. *Let $M = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$ be an $l \times l$ matrix-valued function with the same properties as in Lemma 11 and assume that $A \in \mathbb{C}^{r,l-r}$ is constant. Then the ordinary differential operator*

$$\tilde{L} : H^2(\mathbb{R}, \mathbb{C}^r) \times H^1(\mathbb{R}, \mathbb{C}^{l-r}) \to H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r}),\ z \mapsto z_x - M(\cdot)z,$$

is a Fredholm operator of the same Fredholm index as

$$L : H^1(\mathbb{R}, \mathbb{C}^l) \to L_2(\mathbb{R}, \mathbb{C}^l),\ z \mapsto z_x - M(\cdot)z.$$

Proof. Let $(u,v)^T \in \mathcal{N}(L) \subset H^1(\mathbb{R}, \mathbb{C}^l)$. By the structure of M we obtain $(u,v)^T \in \mathcal{N}(\tilde{L})$ and so $\mathcal{N}(\tilde{L}) = \mathcal{N}(L)$. By Lemma 11 $(f,g) \in \mathcal{R}(L)$ if and only if

$$\int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} f \\ g \end{pmatrix} \rangle dx = 0 \quad \forall \psi \in \mathcal{N}(L^{ad}),$$

where $L^{ad} : H^1(\mathbb{R}, \mathbb{C}^l) \to L_2(\mathbb{R}, \mathbb{C}^l)$ is the same operator as in Lemma 11. Obviously $\mathcal{R}(\tilde{L}) \subset \mathcal{R}(L)$. Let $(f,g) \in H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r})$ with

$$\int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} f \\ g \end{pmatrix} \rangle dx = 0 \quad \forall \psi \in \mathcal{N}(L^{ad}).$$

Thus there is $(z_1, z_2)^T \in H^1(\mathbb{R}, \mathbb{C}^l)$ with

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} z_{1,x} - Az_2 \\ z_{2,x} - Bz_1 - Cz_2 \end{pmatrix} \text{ in } L_2(\mathbb{R}, \mathbb{C}^l),$$

but this equality implies $z_{1,x} \in H^1(\mathbb{R}, \mathbb{C}^r)$ and so $z_1 \in H^2(\mathbb{R}, \mathbb{C}^r)$. Therefore $(f, g)^T \in \mathcal{R}(\tilde{L})$ and it follows that $\begin{pmatrix} f \\ g \end{pmatrix} \in H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r})$ is an element of $\mathcal{R}(\tilde{L})$
if and only if

$$\int_{-\infty}^{\infty} \langle \psi, \begin{pmatrix} f \\ g \end{pmatrix} \rangle dx = 0 \; \forall \psi \in \mathcal{N}(L^{ad}).$$

From this equivalence we find

$$\dim(H^1(\mathbb{R}, \mathbb{C}^r) \times L_2(\mathbb{R}, \mathbb{C}^{l-r}))/\mathcal{R}(\tilde{L}) = \dim \mathcal{N}(L^{ad}) = \dim L_2(\mathbb{R}, \mathbb{C}^l)/\mathcal{R}(L).$$

□

Proof of Lemma 4

First we show $\dim \mathcal{N}(sI - P) = \dim \mathcal{N}(L(\cdot, s))$.
Let $(u, v)^T \in \mathcal{N}(sI - P)$, then $(u, Au_x, v)^T \in \mathcal{N}(L(\cdot, s))$ and it follows

$$\dim \mathcal{N}(sI - P) \leq \dim \mathcal{N}(L(\cdot, s)).$$

Now let $(z_1, z_2, z_3)^T \in \mathcal{N}(L(\cdot, s))$. By the definition of $L(\cdot, s)$ it holds $z_{1,x} = A^{-1}z_2$ and therefore $z_2 = Az_{1,x}$. One easily finds $(z_1, z_3)^T \in \mathcal{N}(sI - P)$. Let $(z_1^i, z_2^i, z_3^i)^T$, $i = 1, \dots, l$, be linearly independent elements in $\mathcal{N}(L(\cdot, s))$. Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{C}^l$ with

$$\sum_i \alpha_i \begin{pmatrix} z_1^i \\ z_3^i \end{pmatrix} = 0.$$

Then by linearity $\sum_i \alpha_i (z_1^i, Az_{1,x}^i, z_3^i)^T = 0$, but since the differential equation shows $Az_{1,x}^i = z_2^i$, we conclude $\alpha = 0$ from the linear independence. Hence we find

$$\dim \mathcal{N}(sI - P) \geq \dim \mathcal{N}(L(\cdot, s)).$$

Second we show $\operatorname{codim} \mathcal{R}(sI - P) = \operatorname{codim} \mathcal{R}(L(\cdot, s))$.
Since $(0, -f + B_{12}B_{22}^{-1}g, -B_{22}^{-1}g)^T \in \mathcal{R}(L(\cdot, s))$ implies $(f, g)^T \in \mathcal{R}(sI - P)$ we obtain

$$\operatorname{codim} \mathcal{R}(sI - P) \leq \operatorname{codim} \mathcal{R}(L(\cdot, s)).$$

Now let (f^i, g^i, h^i), $i = 1, \dots, l$, be a cobasis of $\mathcal{R}(L(\cdot, s))$. Then the elements

$$\begin{pmatrix} -Af_x^i - g^i - B_{11}f^i - B_{12}h^i \\ B_{21}f^i - B_{22}h^i \end{pmatrix} \in L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)$$

are linearly independent elements of $[L_2(\mathbb{R}, \mathbb{C}^n) \times L_2(\mathbb{R}, \mathbb{C}^m)]/\mathcal{R}(sI - P)$:
Let $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{C}^l$ and assume there is $\begin{pmatrix} u \\ v \end{pmatrix} \in H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$ with

$$(sI - P)\begin{pmatrix} u \\ v \end{pmatrix} = \sum \alpha_i \begin{pmatrix} -Af_x^i - g^i - B_{11}f^i - B_{12}h^i \\ B_{21}f^i - B_{22}h^i \end{pmatrix}. \tag{5.36}$$

Consider $\begin{pmatrix} u \\ Au_x - A\sum \alpha_i f^i \\ v \end{pmatrix}$ which is an element of the product space $H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m)$. Then

$$\begin{aligned}
&L(s)\begin{pmatrix} u \\ Au_x - A\sum \alpha_i f^i \\ v \end{pmatrix} \\
&= \begin{pmatrix} u_x \\ Au_{xx} - A\sum \alpha_i f_x^i \\ v_x \end{pmatrix} - M(\cdot, s)\begin{pmatrix} u \\ Au_x - A\sum \alpha_i f^i \\ v \end{pmatrix} \\
&= \begin{pmatrix} \sum \alpha_i f^i \\ Au_{xx} + B_{11}u_x - B_{11}\sum \alpha_i f^i - A\sum \alpha_i f_x^i + C_{11}u + C_{12}v - sIu \\ v_x + B_{22}^{-1}\big(C_{21}u + B_{21}(u_x - \sum \alpha_i f^i) + (C_{22} - sI)v\big) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -B_{12}B_{22}^{-1}\big(C_{21}u + B_{21}(u_x - \sum \alpha_i f^i) + (C_{22} - sI)v\big) \\ 0 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \sum \alpha_i f^i \\ Au_{xx} + B_{11}u_x + B_{12}v_x + C_{11}u + C_{12}v - sIu \\ \sum \alpha_i h^i \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ -B_{11}\sum \alpha_i f^i - A\sum \alpha_i f^i_x - B_{12}\sum \alpha_i h^i \\ 0 \end{pmatrix} \\
&= \sum \alpha_i \begin{pmatrix} f^i \\ g^i \\ h^i \end{pmatrix} \in \mathcal{R}(L(\cdot, s)),
\end{aligned}
$$

where we used the differential equation (5.36). Since the elements form a cobasis of $\mathcal{R}(L(\cdot, s))$ it follows $\alpha = 0$. This shows

$$
\operatorname{codim} \mathcal{R}(sI - P) \geq \operatorname{codim} \mathcal{R}(L(\cdot, s)).
$$

□

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