NUMERICAL APPROXIMATION OF RELATIVE EQUILIBRIA FOR EQUIVARIANT PDES

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Abstract. We prove convergence results for the numerical approximation of relative equilibria of parabolic systems in one space dimension. These systems are are special examples of equivariant evolution equations. We use finite differences on a large interval with appropriately choosen boundary conditions. Moreover, we consider the approximation of isolated eigenvalues of finite multiplicity of the linear operator which arises from linearization at the equilibrium as well as the approximation of the corresponding invariant subspace. The results in this paper which are a generalization of the results in [23] are illustrated by numerical computations for the cubic quintic Ginzburg Landau equation.

1. Introduction and basic setting. Relative equilibria, i.e. solutions of partial differential equations which are equilibria in an appropriately comoving frame, frequently occur in systems with underlying symmetry [7],[16]. A basic class is formed by traveling waves which are stationary in a frame which moves with the velocity of the wave. The purpose of this paper is to analyze numerical methods for the approximation of relative equilibria of parabolic systems in one space dimension which are equivariant w.r.t. the action of a finite dimensional Lie group. The numerical computation of the relative equilibrium amounts to solving a boundary value problem on the real line and to compute extra parameters (from the Lie algebra). These determine the motion of the comoving frame in the original coordinates. We analyze the combined effect of discretization with finite differences and the truncation of the infinite boundary value problem to a finite but large interval. We prove the rate of convergence to the exact solution as well as error estimates for invariant subspaces corresponding to isolated eigenvalues of finite multiplicity of the linearization at the relative equilibrium.

A similar task has been accomplished for Galerkin approximations for traveling waves on cylindrical domains in [14]. In [17] a different approach for approximating homoclinic and heteroclinic orbits using Laguerre approximations has been proposed. In this paper we deal with the simplest numerical method - finite differences - and with a rather general set of problems - relative equilibria on the real line. The key difficulty is the simultaneous truncation of the real line to a finite interval together with the approximation of differential operators by finite differences. This is different to well known results for boundary value problems on finite intervals as presented e.g. in [1]. In this section we review the basic setting of the freezing approach which realizes the comoving frame and which has been developed in [5]. We fix the notations and provide an example. Moreover, we state the main spectral conditions on the linearization of continuous system at the relative equilibrium. In Section 2 we introduce the discretization and boundary conditions and state the main approximation results Theorem 2.5 (approximation of relative equilibria) and Theorem 2.6 (approximation of isolated eigenvalues and invariant subspaces of the linearization). In Section 3 we show the main linear result Theorem 3.9 which will be used in Section 4 to prove the main results. In Section 5 we illustrate the results of Theorem 2.5 and Theorem 2.6 by numerical computations for rotating pulses of the cubic quintic Ginzburg Landau equation.

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1.1. Equivariant evolution equations. We begin with a rather abstract setting for equivariant equations on Banach manifolds that covers the approaches in [5, 7, 21, 23] (see also [6]). It will be applied later in the examples 1.3 and 1.4 with affine Banach spaces.
Let M be a manifold modelled over some Banach space X and let N be a submanifold modelled over some dense subspace $Y \subset X$. Consider an evolutionary equation

$$u_t = F(u), \quad u(0) = u^0, \tag{1.1}$$

with a vector field $F : N \to TM$ where TM denotes the tangent bundle of M. In our applications (see Section 1.2) we will either have Banach spaces $X = M,\ Y = N$ or affine spaces $M = \tilde{v} + X,\ N = \tilde{v} + Y$ such that the tangent spaces always satisfy $T_u M = X,\ T_v N = Y$ for all $u \in M, v \in N$.
Further we assume that the system is equivariant w.r.t. a finite dimensional (possibly noncompact) Lie group G acting on M via

$$a : G \times M \to M,\ (\gamma, u) \mapsto a(\gamma)u,$$

where

$$a(\gamma_1 \circ \gamma_2)v = a(\gamma_1)[a(\gamma_2)v], \quad a(\mathbb{1})v = v, \quad \mathbb{1} = \text{unit element in } G,$$

which has a tangent action Ta in TM, i.e $Ta(\gamma) : T_v M \to T_{a(\gamma)v}M$. By equivariance we mean that the following relation holds

$$\begin{aligned} a(\gamma)(N) &\subset N \quad \forall \gamma \in G, \\ F(a(\gamma)u) &= Ta(\gamma)F(u), \quad \forall u \in N, \gamma \in G. \end{aligned}$$

We assume that for any $v \in M$ the map $a(\cdot)v :\ G \to M, \quad \gamma \mapsto a(\gamma)v$ is continuous and is continuously differentiable for any $v \in N$ with derivative denoted by

$$da(\gamma)v : T_\gamma G \to T_{a(\gamma)v}M, \quad \lambda \mapsto [da(\gamma)v]\lambda.$$

Finally, we denote the left translation by $L_\gamma :\ G \to G,\ g \mapsto \gamma \circ g$ and its derivative at g by $dL_\gamma(g) : T_g G \to T_{\gamma\circ g}G$.
In the following we give a constructive definition of relative equilibria which is appropriate from a numerical point of view [5].
Definition 1.1. *A solution $\bar{u}$ of (1.1) is called a relative equilibrium, if it has the form $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ where $\bar{\gamma} : [0,\infty) \to G$ is a smooth curve satisfying $\bar{\gamma}(0) = \mathbb{1}$ and $\bar{v} \in N$ does not depend on time.*
In general one calls the whole group orbit $\mathcal{O}(\bar{v}) = \{a(\gamma)\bar{v},\ \gamma \in G\}$ a relative equilibrium if it is invariant under the semi-flow [7],[16]. For our purpose it is more convenient to select a special time orbit within this group orbit since $\mathcal{O}(\bar{v})$ can always be generated by applying the group action a to the selected time orbit.
Using the ansatz $u(t) = a(\gamma(t))v(t)$ equation (1.1) can be transformed into a system for the unknowns $v(t) \in M,\ \gamma(t) \in G,\ \mu(t) \in T_{\mathbb{1}}G$ as follows (cf. [5],[20],[23])

$$v_t = F(v) - [da(\mathbb{1})v]\mu, \qquad v(0) = u^0 \tag{1.2a}$$
$$\gamma_t = dL_\gamma(\mathbb{1})\mu, \qquad \gamma(0) = \mathbb{1}. \tag{1.2b}$$

If a relative equilibrium $\bar{u}(t) = a(\bar{\gamma}(t))\bar{v}$ is given such that $da(\mathbb{1})\bar{v}\mu \in T_{\bar{v}}M$ for all $\mu \in T_{\mathbb{1}}G$ and $da(\mathbb{1})\bar{v} : T_{\mathbb{1}}G \to T_{\bar{v}}M$ has full rank, then it can be shown that there exists $\bar{\mu} \in T_{\mathbb{1}}G$ such that $(\bar{v}, \bar{\mu})$ is a stationary solution of (1.2), i.e.

$$0 = F(\bar{v}) - [da(\mathbb{1})\bar{v}]\bar{\mu} \tag{1.3}$$

and $\bar{\gamma}_t = dL_{\gamma}(\mathbb{1})\bar{\mu} = \exp(t\bar{\mu})$. Conversely if (1.3) holds for $(\bar{v}, \bar{\mu})$ then $a(\bar{\gamma}(t))\bar{v}$ is a relative equilibrium where $\bar{\gamma}$ solves $\gamma_t = dL_{\gamma}(\mathbb{1})\bar{\mu} = \exp(t\bar{\mu})$.

1.2. Parabolic equations. In the following we are concerned with the special case of a parabolic PDE

$$u_t = Au_{xx} + f(u, u_x), \quad x \in \mathbb{R}, t > 0, u(x,t) \in \mathbb{R}^m, \tag{1.4}$$

where $A \in \mathbb{R}^{m,m}$ is positive definite and $f \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ is of the form

$$f(u, v) = f_1(u)v + f_2(u), \quad f_1 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^{m,m}), f_2 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^m), \tag{1.5}$$

and f_1, f_2, f_1', f_2' are globally Lipschitz.

Remark 1.2. *The above conditions on f are also satisfied for "conservation laws" with special flux functions (e.g. Burgers equation). They imply that f_1', f_2' are globally bounded and with*

$$D_1 f(u, w) = f_1'(u)(w, \cdot) + f_2'(u), \quad D_2 f(u, w) = f_1(u),$$

we obtain for $u, w, \delta_u, \delta_w \in \mathbb{R}^m$

$$\begin{aligned} \|D_1 f(u + \delta_u, w + \delta_w) - D_1 f(u, w)\| &\le \text{const}(\|\delta_u\| + \|\delta_w\|), \\ \|D_2 f(u + \delta_u, w + \delta_w) - D_2 f(u, w)\| &\le \text{const}\,\|\delta_u\|. \end{aligned} \tag{1.6}$$

In this case F in (1.1) reads

$$F(u) = Au'' + \bar{f}(u, u'),$$

where $\bar{f}(u, u')(x) = f(u(x), u'(x))$. Let a function $\tilde{v} : \mathbb{R} \to \mathbb{R}^m$ be given with

$$F(\tilde{v}) = A\tilde{v}'' + f(\tilde{v}, \tilde{v}') \in \mathcal{L}_2. \tag{1.7}$$

Then we define $M = \tilde{v} + \mathcal{L}_2$, $N = \tilde{v} + \mathcal{H}^2$ and with the properties of f we obtain that the condition $F : \tilde{v} + \mathcal{H}^2 \to \mathcal{L}_2$ is satisfied.

We choose a basis $\{e^1, \ldots, e^p\}$ in the Lie algebra $T_{\mathbb{1}}G$, where p is the dimension of G, write $\mu = \sum_{i=1}^p \mu_i e^i$ and define $S^i(v) = -\,da(\mathbb{1})ve^i$. Then the stationary equation reads

$$0 = Av'' + S(v)\mu + \bar{f}(v, v'), \quad x \in \mathbb{R},\ v(x) \in \mathbb{R}^m, \mu \in \mathbb{R}^p, \tag{1.8}$$

where we use the short notation $S(v)\mu = \sum_{i=1}^p S^i(v)\mu_i$.

Throughout the paper we need two additional assumptions:

1. The operators S^i are linear differential operators of order ≤ 1 which can be written as

   $$S^i(v)(x) = S_0^i v(x) + S_1^i v'(x), \quad S_{0,1}^i \in \mathbb{R}^{m,m} \tag{1.9}$$

   and for which $S^i(\bar{v}) \in \mathcal{L}_2$ for $i = 1, \ldots, p$. The last condition is necessary for the existence of a solution $\bar{v} \in \tilde{v} + \mathcal{H}^2$ of (1.8).
2. There exist $\bar{v}_\pm \in \mathbb{R}^m$ such that

   $$\lim_{x \to \pm\infty} \bar{v}(x) = \bar{v}_\pm \tag{1.10}$$

In order to compensate for the additional p degrees of freedom which are obtained by introducing the parameter μ, we add a phase condition of the form

$$0 = \langle S^i(\hat{v}), v - \hat{v} \rangle_{\mathcal{L}_2}, \quad i = 1, \ldots, p, \tag{1.11}$$

where $\hat{v} \neq 0$ is a given template function which satisfies $\hat{v} - \bar{v} \in \mathcal{H}^1$. The use of this phase condition is also called the method of slices [20]. In practice one can use the initial approximation of $\bar{v}$ as a template function.
In the following we present two characteristic examples:
Example 1.3. *Consider* (1.4) *and let* $\tilde{v}$ *be a function with* $\|\tilde{v}(x) - v_\pm\| \leq \text{const}\, e^{\pm\varrho x}$ *where* $f(v_\pm, 0) = 0$. *Consider the shift action of* $G = \mathbb{R}$, *i.e.* $[a(\gamma)u](x) = u(x - \gamma)$ *on* $M = \tilde{v} + \mathcal{L}_2 \supset N = \tilde{v} + \mathcal{H}^2$. *Then we have* $[da(\mathbb{1})v]e^1 = -v_x$ *i.e.* $S_1^1 = I$,$S_0^1 = 0$ *and* (1.2a), (1.11) *reads*

$$\begin{aligned} v_t &= A v_{xx} + \lambda v_x + f(v, v_x), \\ 0 &= \langle \hat{v}', v - \hat{v} \rangle_{\mathcal{L}_2}. \end{aligned}$$

The relative equilibria are traveling waves $\bar{u}(x,t) = \bar{v}(x - \bar{\lambda}t)$ *with stationary points* $\lim_{x\to\pm\infty} \bar{v}(x) = v_\pm$.
Example 1.4. *Consider* (1.4) *for* $\tilde{v} = 0$, *i.e. for* $M = \mathcal{L}_2$ *and* $N = \mathcal{H}^2$. *Let the Lie group be* $G = S^1 \times \mathbb{R}$ *with* $(\rho, \tau) = \gamma \in G$ *and* $(\rho, \tau) \circ (\tilde{\rho}, \tilde{\tau}) = (\rho + \tilde{\rho}, \tau + \tilde{\tau})$. *Let the action* $a : G \times \mathcal{L}_2 \to \mathcal{L}_2$ *be given for* $u : \mathbb{R} \to \mathbb{R}^2$ *by*

$$[a(\gamma)u](x) = R_{-\rho} u(x - \tau), \quad R_\rho = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix}.$$

Then we have $[da(\mathbb{1})v]e^1 = -v_x$,$[da(\mathbb{1})v]e^2 = -R_{\frac{\pi}{2}} v$, *i.e.* $S_1^1 = I$, $S_0^2 = R_{\frac{\pi}{2}}$,$S_0^1 = S_1^2 = 0$ *and* (1.2a), (1.11) *read with* $\mu_\tau = \tau_t$, $\mu_\rho = \rho_t$

$$\begin{aligned} v_t &= A v_{xx} + \mu_\tau v_x + \mu_\rho R_{\frac{\pi}{2}} v + f(v, v_x), \\ 0 &= \langle \hat{v}', v - \hat{v} \rangle_{\mathcal{L}_2}, \quad 0 = \langle R_{\frac{\pi}{2}} v, v - \hat{v} \rangle_{\mathcal{L}_2}. \end{aligned}$$

The relative equilibria are rotating and traveling waves

$$\bar{u}(x,t) = R_{-\bar{\mu}_\rho t} \bar{v}(x - \bar{\mu}_\tau t). \tag{1.12}$$

If $\bar{v}$ is a front, i.e. $\bar{v}_- \neq \bar{v}_+$, then $\bar{v}$ and $R_{\frac{\pi}{2}}\bar{v}$ are not in $\mathcal{L}_2$. In this case, considering a rotating front, the condition $S^2(\bar{v}) = R_{\frac{\pi}{2}}\bar{v} \in \mathcal{L}_2$ is not satisfied and the approximation theorems of this paper are not applicable. However, in this case any small phase shift w.r.t. the rotation will lead to a difference between the shifted and the original solution which is not in $\mathcal{L}_2$ anymore, in contrast to the case when $\bar{v}$ is a pulse.

1.3. Spectral properties of the continuous system. The linearization of (1.8) w.r.t. v at $(\bar{v}, \bar{\mu})$ is given by

$$\Lambda v = A v'' + B v' + C v, \tag{1.13}$$

where

$$B(x) = D_2 f(\bar{v}(x), \bar{v}'(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_1^i, \quad C(x) = D_1 f(\bar{v}(x), \bar{v}'(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_0^i.$$

Assumption (1.10) implies with the invertibility of A that $\lim_{x\to\pm\infty}\bar{v}'(x)=0$. Thus Λ converges for $x\to\pm\infty$ to constant coefficient operators

$$\Lambda_\pm v = Av'' + B_\pm v' + C_\pm v, \qquad B_\pm = \lim_{x\pm\infty} B(x),\ C_\pm = \lim_{x\pm\infty} C(x). \tag{1.14}$$

The main spectral assumption on Λ is the following:
HYPOTHESIS 1.5.

1. **eigenvalue condition:** *The functions $S^i(\bar{v})\in\mathcal{L}_2$, $i=1,\dots,p$ are linearly independent and span the null space of $\Lambda:\mathcal{H}^2\to\mathcal{L}_2$, i.e.*

$$\mathcal{N}(\Lambda)=\operatorname{span}\{S^1(\bar{v}),\dots,S^p(\bar{v})\}.$$

Moreover, the algebraic and the geometric multiplicity of zero both equal p.

2. **spectral condition:**
The eigenvalue 0 lies in the connected component of $\mathbb{C}\setminus\{\Sigma_+\cup\Sigma_-\}$ that contains a right half plane, where

$$\Sigma_\pm=\{s\in\mathbb{C}\ :\det(-\kappa^2A+i\kappa B_\pm+C_\pm-sI)=0,\ \textit{for some}\ \kappa\in\mathbb{R}\}.$$

Note that spectral properties of Λ determine asymptotic stability with asymptotic phase of relative equilibria [10]. However, the spectral conditions given above are weaker than the standard stability conditions and allow to discuss the approximation of unstable equilibria as well.
EXAMPLE 1.6. *For Example 1.4 the operator Λ reads*

$$\Lambda v = Av'' + (\mu_\tau I + D_2 f(\bar{v},\bar{v}'))v' + (\mu_\rho R_{\frac{\pi}{2}} + D_1 f(\bar{v},\bar{v}'))v$$

and its null space is spanned by $\bar{v}'$ and $R_{\frac{\pi}{2}}\bar{v}$.
The condition that zero is semi-simple cannot be concluded from properties of the Lie group G. For the cubic Ginzburg-Landau equation which has the symmetry properties of Example 1.4 and is a special case of the system considered in Section 5 a principal vector exists for a special parameter combination [3], i.e. the algebraic multiplicity is 3 and the geometric multiplicity is 2.

2. Discretization. In order to compute numerical approximations of $(\bar{v},\bar{\mu})$ we define a discrete interval

$$J=[n_-,n_+]=\{n\in\mathbb{Z}:\ n_-\le n\le n_+,\ \text{where}\ n_\pm\in\mathbb{Z}\cup\{\pm\infty\}\ \}$$

and a corresponding equidistant grid with grid size $h>0$

$$J_h=\{x_n:\ x_n=nh,\ n\in J\}.$$

We write $J_h\to\mathbb{R}$ if $h\to0$ and simultaneously $h\cdot\min\{-n_-,n_+\}\to\infty$, i.e. $\pm n_\pm$ grows faster than h decreases, so that $[hn_-,hn_+]\to\mathbb{R}$.
We denote the Banach space of sequences in $\mathbb{R}^m$ which are indexed by J provided with the supremum norm $\|z\|_\infty=\sup_{n\in J}\|z_n\|$ by $\ell_\infty^J(\mathbb{R}^m)$ and write $\ell_\infty(\mathbb{R}^m)$ for $\ell_\infty^{\mathbb{Z}}(\mathbb{R}^m)$. If no confusion is possible we drop the argument $\mathbb{R}^m$ and write just ℓ_∞^J and ℓ_∞.
Let the standard finite difference operators on the extended grid

$$\hat{J}_h=\{x_n:\ x_n=nh,\ n\in\hat{J}=[n_--1,n_++1]\}$$

be given by

$$
\begin{aligned}
\delta_0 : \ell_\infty^{\hat{J}} &\to \ell_\infty^J, & (\delta_0 v)_n &= \frac{1}{2h}(v_{n+1} - v_{n-1}),\\
\delta_+ : \ell_\infty^{[n_-,n_++1]} &\to \ell_\infty^J, & (\delta_+ v)_n &= \frac{1}{h}(v_{n+1} - v_n),\\
\delta_- : \ell_\infty^{[n_--1,n_+]} &\to \ell_\infty^J, & (\delta_- v)_n &= \frac{1}{h}(v_n - v_{n-1}).
\end{aligned}
$$

For sequences $u, v \in \ell_\infty^J$, $J = [n_-, n_+]$ we define the inner product and discrete Sobolev norms by

$$
\begin{aligned}
\langle u, v\rangle_{J_h} &= \sum_{n=n_-}^{n_+} h u_n^T v_n, & \|u\|_{\mathcal{L}_{2,h}} &= \sqrt{\langle u, u\rangle_{J_h}},\\
\|u\|_{\mathcal{H}_h^1} &= \|u\|_{\mathcal{L}_{2,h}} + \|\delta_+ u\|_{\mathcal{L}_{2,h}}, & \|u\|_{\mathcal{H}_h^2} &= \|u\|_{\mathcal{H}_h^1} + \|\delta_+\delta_- u\|_{\mathcal{L}_{2,h}}.
\end{aligned}
$$

One has to keep in mind that the summation is done on different sets for the different difference operators. If necessary, we embed each $u \in \ell_\infty^J$ in ℓ_∞ by setting $u_n = 0$ for $n \in \mathbb{Z} \setminus J$ without further notice. For matrices $U = [U^1, ..., U^p]$, $V = [V^1, ..., V^p] \in (\ell_\infty^J(\mathbb{R}^m))^p$ we use the notation $\langle U, V\rangle_{J_h} = [\langle U^i, V^j\rangle_{J_h}]_{i=1,\dots,p}^{j=1,\dots,p} \in \mathbb{R}^{p,p}$. This notation is also adopted for the $\mathcal{L}_2$ inner product $\langle\cdot,\cdot\rangle_{\mathcal{L}_2}$ in an analogous fashion.
We discretize (1.8),(1.11) and add linear boundary conditions

$$
\mathcal{B}v = P_- v_{n_-} + Q_-(\delta_0 v)_{n_-} + P_+ v_{n_+} + Q_+(\delta_0 v)_{n_+}, \quad P_\pm, Q_\pm \in \mathbb{R}^{2m,m}. \tag{2.1}
$$

This leads to the discrete boundary value problem

$$
\begin{aligned}
A(\delta_+\delta_- v)_n + \hat{S}_n(v)\mu + f(v_n, (\delta_0 v)_n) &= 0, \quad n \in J, &\text{(2.2a)}\\
\mathcal{B}v &= \eta, &\text{(2.2b)}\\
\langle \hat{S}^i(\hat{v}_{|J_h}), v - \hat{v}_{|J_h}\rangle_{J_h} &= 0, \quad i = 1, \dots, p, &\text{(2.2c)}
\end{aligned}
$$

where $\hat{S}_n^i(v) = S_0^i v_n + S_1^i(\delta_0 v)_n \in \mathbb{R}^m$ and $\hat{S}_n(v)\mu = \sum_{i=1}^p \mu_i \hat{S}_n^i(v)$.
Linearizing the l.h.s. of (2.2a) at a solution $(\tilde{v}, \tilde{\mu}) \in \ell_\infty^{\hat{J}} \times \mathbb{R}^p$ of (2.2) leads to the following discrete approximation of Λ

$$
(\tilde{\Lambda} v)_n = A(\delta_+\delta_- v)_n + \tilde{B}_n(\delta_0 v)_n + \tilde{C}_n v_n, \quad n \in J, \tag{2.3}
$$

where

$$
\tilde{B}_n = D_2 f(\tilde{v}_n, (\delta_0\tilde{v})_n) + \sum_{i=1}^p \tilde{\mu}_i S_1^i, \quad \tilde{C}_n = D_1 f(\tilde{v}_n, (\delta_0\tilde{v})_n) + \sum_{i=1}^p \tilde{\mu}_i S_0^i.
$$

It is well known that a multiple eigenvalue will split into several eigenvalues under perturbations [11]. Therefore we approximate the whole invariant subspace which corresponds to zero instead of dealing with each eigenfunction separately. The spatial discretization of the invariant subspace equation together with boundary conditions and a normalizing condition leads for the unknowns (V, Σ) to the system

$$
\begin{aligned}
(\tilde{\Lambda} V)_n - V_n\Sigma &= 0, \quad n \in J, &\text{(2.4a)}\\
\mathcal{B}V &= 0, &\text{(2.4b)}\\
\langle \hat{X}_{|J_h}, V\rangle_{J_h} &= I. &\text{(2.4c)}
\end{aligned}
$$

Here $\hat{X} = [\hat{X}^1, ..., \hat{X}^p] : \mathbb{R} \to \mathbb{R}^{m,p}$ is a given normalizing function for which $\langle \hat{X}, S(\bar{v}) \rangle_{\mathcal{L}_2} = I$. The columns of $V \in (\ell_\infty^{\hat{J}})^p$ span the invariant subspace which corresponds to the eigenvalues of $\Sigma \in \mathbb{R}^{p,p}$. Thus for V to be a good approximation of the null space of Λ, the norm of Σ has to be small.

2.1. Main results. Before we can state the main approximation results we have to collect the necessary hypotheses on the boundary conditions and the phase condition. The template function $\hat{v}$ and the normalizing function $\hat{X}$ are assumed to be in the following class of functions:

DEFINITION 2.1. *We define a function $g : I \to \mathbb{R}^{m,p}$, $I \subset \mathbb{R}$ to be in $\mathcal{E}_\varrho(I, \mathbb{R}^{m,p})$ if there exists $K > 0$ such that $\forall x \in I$:*

$$\|g(x)\| \le K\mathrm{e}^{-\varrho|x|} \quad \textit{and} \quad \|g'(x)\| \le K\mathrm{e}^{-\varrho|x|}.$$

Similar to [26] we have the following Lemma.

LEMMA 2.2. *Let $g \in \mathcal{E}_\varrho(\mathbb{R}^+, \mathbb{R}^{m,p})$ and $\tilde{g} \in (\ell_\infty)^p$ be given with*

$$\|g(x_n) - \tilde{g}_n\| \le \mathrm{const}\, h \mathrm{e}^{-\varrho x_n}, \quad \forall n \in \mathbb{N}.$$

Then the following estimate holds

$$\|\int_0^\infty g(x)\, dx - h \sum_{n=0}^{n_+-1} \tilde{g}_n\| \le \mathrm{const}(h + \mathrm{e}^{-\varrho x_{n_+}}). \tag{2.5}$$

A well known determinant condition [4], which ensures the existence of the resolvent on finite intervals in the continuous case, uses the following determinant.

DEFINITION 2.3. *Define*

$$\mathcal{D} = \det\left(\begin{pmatrix} P_- & Q_- \end{pmatrix} \begin{pmatrix} Y_-^s \\ Y_-^s \Sigma_-^s \end{pmatrix} \quad \begin{pmatrix} P_+ & Q_+ \end{pmatrix} \begin{pmatrix} Y_+^u \\ Y_+^u \Sigma_+^u \end{pmatrix} \right), \tag{2.6}$$

where (Σ_-^s, Y_-^s), $(\Sigma_+^u, Y_+^u) \in \mathbb{R}^{m,m} \times \mathbb{R}^{m,m}$ solve the quadratic eigenvalue problems

$$AY\Sigma^2 + B_\pm Y\Sigma + C_\pm Y = 0 \tag{2.7}$$

with $\mathrm{Re}\, \sigma(\Sigma_-^s) < 0$ and $\mathrm{Re}\, \sigma(\Sigma_+^u) > 0$.

Then we can formulate the determinant condition and a consistency assumption for the boundary conditions as well as a regularity assumption for the phase condition.

HYPOTHESIS 2.4.

1. **boundary conditions:** *The boundary condition (2.2b) is satisfied at the stationary points $\bar{v}_\pm$, i.e.*

$$\eta = P_-\bar{v}_- + P_+\bar{v}_+$$

and the determinant $\mathcal{D}$ defined in (2.6) is nonzero.

2. **phase condition:** *The phase condition is satisfied by $\bar{v}$, i.e.*

$$\langle S(\hat{v}), \bar{v} - \hat{v} \rangle_{\mathcal{L}_2} = 0, \tag{2.8}$$

$\bar{v} - \hat{v} \in \mathcal{H}^1$, $S(\hat{v}) \in \mathcal{E}_\varrho(\mathbb{R}, \mathbb{R}^{m,p})$ and the $p \times p$ matrix

$$\langle S(\hat{v}), S(\bar{v}) \rangle_{\mathcal{L}_2} = \int_{\mathbb{R}} [S(\hat{v})](x)^T [S(\bar{v})](x)\, dx \tag{2.9}$$

is nonsingular.

The determinant condition on the boundary conditions may seem hard to check, but is satisfied for Dirichlet, Neumann and periodic boundary conditions (see the discussion in [4]). Condition (2.8) is a technical assumption, that prevents that the approximate solution converges to a shifted version of $\bar{v}$, which can be dropped. Then one has to deal with an additional phase shift, which can be estimated in terms of $\bar{v}$ and $\hat{v}$, but we do not pursue this here.

Now we can formulate the main approximation results of the paper.

THEOREM 2.5 (Approximation of the relative equilibrium). *Assume Hypotheses 1.5 and 2.4. Then there exist $\varrho > 0$, $T > 0$, $h_0 > 0$ such that for $h < h_0$ and $\pm hn_\pm > T$ the boundary value problem (2.2) has a unique solution $(\tilde{v}, \tilde{\mu})$ in a neighborhood $B_\varrho(\bar{v}, \bar{\mu}) = \{(v, \mu) \in \ell_\infty^{\hat{J}} \times \mathbb{R}^p : \|\bar{v}_{|J_h} - v\|_{\mathcal{H}_h^2} + \|\bar{\mu} - \mu\| < \varrho\}$. This solution obeys the following estimate for some $\alpha > 0$*

$$\|\bar{v}_{|J_h} - \tilde{v}\|_{\mathcal{H}_h^2} + \|\bar{\mu} - \tilde{\mu}\| \leq \mathrm{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}). \tag{2.10}$$

The second result is an approximation theorem for the zero eigenvalue of Λ and the corresponding invariant subspace.

THEOREM 2.6 (Approximation of the zero eigenvalue). *Assume Hypothesis 1.5 and $\hat{X} \in \mathcal{E}_\beta(\mathbb{R}, \mathbb{R}^{m,p})$ for some $\beta > 0$. Then there exist $\varrho > 0$, $T > 0$, $h_0 > 0$, such that for $h < h_0$ and $\pm hn_\pm > T$ the eigenvalue problem (2.4) has a unique solution (X, Σ) in a neighborhood $B_\varrho(S(\bar{v})_{|J_h}, 0) := \{(X, \Sigma) \in ((\ell_\infty^{\hat{J}})^p \times \mathbb{R}^{p,p} : \|S(\bar{v})_{|J_h} - X\|_{\mathcal{H}_h^2} + \|\Sigma\| < \varrho\}$. This solution obeys the following estimate for some $\alpha > 0$*

$$\|S(\bar{v})_{|J_h} - X\|_{\mathcal{H}_h^2} + \|\Sigma\| \leq \mathrm{const}(h^2 + \mathrm{e}^{-\alpha \min\{-n_-, n_+\}}). \tag{2.11}$$

REMARK 2.7.

- *The above result can be easily generalized to invariant subspaces which are related to other isolated eigenvalues of finite multiplicity.*
- *The stated results hold for $\|\cdot\|_\infty$ as well and estimates similar to (2.10) and (2.11) can be shown [23].*
- *Estimate (2.11) implies with standard perturbation results for eigenvalues [11] the convergence of the eigenvalues as $J_h \to \mathbb{R}$.*

3. The linear equation. Before we can state the main linear result in this section, we have to collect some results from the theory of exponential dichotomies [18],[19], which are needed in the sequel.

3.1. Exponential dichotomies and auxiliary results. Consider the linear differential operator

$$Lz = z' - M(\cdot)z, \quad x \in J \subset \mathbb{R},\ M(\cdot) : J \to \mathbb{R}^{k,k} \tag{3.1}$$

with solution operator $\mathcal{S}(x, \xi)$.

DEFINITION 3.1 (Exponential dichotomy, continuous case).
The operator L defined in (3.1) has an exponential dichotomy in the interval $J = [x_-, x_+]$, $x_\pm \in \mathbb{R} \cup \{\pm\infty\}$ with data (K, α, π), if there exist a bound $K > 0$, a rate $\alpha > 0$ and a function $\pi : J \ni x \mapsto \pi(x)$, $\pi(x)$ a projector, such that the following holds:

$$\mathcal{S}(x, \xi)\pi(\xi) = \pi(x)S(x, \xi) \tag{3.2}$$

and the Green's function

$$\mathcal{G}(x,\xi)=\begin{cases}\mathcal{S}(x,\xi)\pi(\xi), & x\geq\xi,\\ -\mathcal{S}(x,\xi)(I-\pi(\xi)), & x<\xi\end{cases}$$

satisfies the exponential estimate

$$\|\mathcal{G}(x,\xi)\|\leq K\mathrm{e}^{-\alpha|x-\xi|},\quad x,\xi\in J. \tag{3.3}$$

In the following remarks we recall some well known facts about exponential dichotomies (see [18], [2]).

REMARK 3.2. *If L has an exponential dichotomy on $\mathbb{R}^-=(-\infty,0]$ and $\mathbb{R}^+=[0,\infty)$ with data (K_-,α_-,π_-) and (K_+,α_+,π_+) respectively, then the kernel of L is given by*

$$\mathcal{N}(L)=\{\mathcal{S}(\cdot,0)z_0:\ z_0\in\mathcal{N}(\pi_-(0))\cap\mathcal{R}(\pi_+(0))\}.$$

REMARK 3.3. *We transform Λ via $z=(v,v')$ into a first order operator*

$$Lz=z'-M(\cdot)z,\qquad M(x)=\begin{pmatrix}0 & I\\ -A^{-1}C(x) & -A^{-1}B(x)\end{pmatrix}. \tag{3.4}$$

Hypothesis 1.5 ensures together with the positive definiteness of A that L possesses exponential dichotomies on $\mathbb{R}^\pm$ with data $(K^\pm,\alpha^\pm,\pi^\pm)$. Moreover, the null space of L is given by

$$\mathcal{N}(L)=\operatorname{span}\{\phi^1,\ldots,\phi^p\},\quad \phi^i=(S^i(\bar v),S^i(\bar v'))^T. \tag{3.5}$$

Note that the columns of $X_-^s=\begin{pmatrix}Y_-^s\\ Y_-^s\Sigma_-^s\end{pmatrix}$ and $X_+^u=\begin{pmatrix}Y_+^u\\ Y_+^u\Sigma_+^u\end{pmatrix}$, where (Y_-^s,Σ_-^s) and $(Y_+^u\Sigma_+^u)$ are the solutions of the characteristic equation (2.7) in Definition 2.3, span the stable invariant subspace of $M_-=\lim_{x\to-\infty}M(x)$ and the unstable invariant subspace of $M_+=\lim_{x\to+\infty}M(x)$ respectively.

REMARK 3.4. *The fact that L defined in (3.4) has exponential dichotomies on $\mathbb{R}^-,\mathbb{R}^+$ implies that $(\bar v',\bar v'')$ is actually exponentially decaying for $x\to\pm\infty$, i.e.*

$$\|\bar v(x)-v_\pm\|\leq\mathrm{const}\,\mathrm{e}^{\mp\varrho x}\quad \textit{as well as}\quad \|\bar v^{(k)}(x)\|\leq\mathrm{const}\,\mathrm{e}^{-\varrho|x|},\ k=1,2$$

for some $\varrho>0$. Thus we have $\bar v'\in\mathcal{E}_\varrho(\mathbb{R},\mathbb{R}^m)$ and $\lim_{x\to\infty}[S^i(\bar v')](x)=0$ for $i=1,...,p$. Moreover, from the eigenvalue condition we obtain $S^i(\bar v)\in\mathcal{H}^2$ and thus $\lim_{x\to\infty}[S^i(\bar v)](x)=0$, $i=1,...,p$.

Exponential dichotomies can be defined for a linear difference operator

$$(\hat Lz)_n=z_{n+1}-\hat M_nz_n,\quad n\in J,\ \hat M_n\in\mathbb{R}^{k,k} \tag{3.6}$$

in the following way. If the matrices $\hat M_n$ are invertible for all $n\in J$ then the map $\hat{\mathcal{S}}:J\times J\to\mathbb{R}^{k,k}$ given by

$$\hat{\mathcal{S}}(n,m)=\begin{cases}\hat M_{n-1}\cdot\dots\cdot\hat M_m, & \text{for } n>m\\ I, & \text{for n=m}\\ \hat M_n^{-1}\cdot\dots\cdot\hat M_{m-1}^{-1}, & \text{for } n<m\end{cases}$$

is a solution operator for (3.6), which has the cocycle property

$$\hat{\mathcal{S}}(n,l)\hat{\mathcal{S}}(l,m) = \hat{\mathcal{S}}(n,m) \quad \forall\, l,m,n \in J.$$

Definition 3.5 (Exponential dichotomy, discrete case).
The linear difference operator $\hat{L}$, defined in (3.6) has an exponential dichotomy with data (K,α,P) on $J \subset \mathbb{Z}$ if M_n is invertible for all $n \in J$ and there exist a bound $K > 0$, a rate $\alpha > 0$ and projectors P_n such that the following holds

$$\hat{\mathcal{S}}(n,m)P_m = P_n\hat{\mathcal{S}}(n,m) \tag{3.7}$$

and the Green's function

$$\hat{\mathcal{G}}(n,m) = \begin{cases} \hat{\mathcal{S}}(n,m)P_m, & \text{for } n \geq m, \\ -\hat{\mathcal{S}}(n,m)(I-P_m), & \text{for } n < m \end{cases} \tag{3.8}$$

satisfies the exponential estimate

$$\|\hat{\mathcal{G}}(n,m)\| \leq K\mathrm{e}^{-\alpha|n-m|}, \quad n,m \in J. \tag{3.9}$$

A connection between an exponential dichotomy for an ODE and the corresponding difference equation is given via the time h-map of the flow [26].
Lemma 3.6. *Let the linear differential operator L given by*

$$Lz = z' - Mz, \quad x \in J = \mathbb{R}^{\pm}, \quad M : J \to \mathbb{R}^{k,k}$$

have an exponential dichotomy with data (K,α,π) on $J = \mathbb{R}^{\pm}$ and define

$$\hat{\mathcal{S}}(n,m) = \mathcal{S}(x_n,x_m).$$

Then the difference operator

$$\hat{L}z = (z_{n+1} - \hat{\mathcal{S}}(n+1,n)z_n)_{n\in\hat{J}}$$

has an exponential dichotomy on $\hat{J} = \mathbb{Z}^{\pm}$ with data $(K,\alpha h,P)$, where $P_n = \pi(x_n)$.

3.2. The linear difference equation. The main result in this section deals with the existence of solutions $(Z,\Upsilon) \in (\ell_\infty^{[n_-,n_++1]}(\mathbb{R}^{2m}))^q \times \mathbb{R}^{p,q}$, $1 \leq q \leq p$ of the following linear inhomogeneous boundary value problem

$$Z_{n+1} - \hat{M}_nZ_n - \hat{V}_n\Upsilon = R^{de}, \quad n \in J = [n_-,n_+], \tag{3.10a}$$
$$(P_-Q_-)Z_{n_-} + (P_+Q_+)Z_{n_+} = R^{bc} \ \in \mathbb{R}^{2m,q}, \tag{3.10b}$$
$$\langle\hat{\Psi},Z\rangle_{J_h} = R^{pc} \ \in \mathbb{R}^{p,q}, \tag{3.10c}$$

where

$$\hat{M}_n = \mathcal{S}(x_{n+1},x_n) \in \mathbb{R}^{2m,2m}, \tag{3.11}$$

and $\mathcal{S}$ denotes the solution operator corresponding to L defined in (3.4), and $\hat{\Psi} \in \ell_\infty^J(\mathbb{R}^{2m,p})$ and $\hat{V}_n \in \mathbb{R}^p$ satisfy the following hypothesis.

Hypothesis 3.7.

1. **phase condition:** *There exists a continuous template function* $\Psi \in \mathcal{E}_{\varrho}(\mathbb{R}, \mathbb{R}^{2m,p})$ *such that* $\|\hat{\Psi} - \Psi_{|_{J_h}}\|_{\mathcal{L}_{2,h}} \to 0$ *as* $J_h \to \mathbb{R}$ *and* $\langle \Psi, \Phi \rangle_{\mathcal{L}_{2,h}} \in \mathbb{R}^{p,p}$ *is nonsingular, where*
$$\Phi = [\phi^1, \ldots, \phi^p], \tag{3.12}$$
and $\phi^i,\ i = 1, ..., p$ *are the functions which span the null space of* L, *which have been defined in* (3.5).
2. **nondegeneracy condition:** *The matrices* $\hat{V}_n$ *have the form*
$$\hat{V}_n = hV(x_n) + \mathcal{O}(h^2) \in \mathbb{R}^{2m,p}, \tag{3.13}$$
where $V \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{2m,p}),\ \|V\|_\infty < \infty,\ \operatorname{rank}(V) = p$ *and* $\mathcal{R}(L) \cap \mathcal{R}(V) = \{0\}$.

Remark 3.8. *Define* $\Xi = [\xi_1, \ldots, \xi_p]$, *where* $\xi_i,\ i = 1, ..., p$ *span the null space of the adjoint operator*
$$L^* : z \mapsto -z' - M^T(\cdot)z.$$
Then the eigenvalue condition in Hypothesis 1.5 *implies that the Melnikov integral*
$$\langle \Xi, V \rangle = \int_{\mathbb{R}} [\xi_1, \ldots, \xi_p](x)^T V(x)\, dx \ \in \mathbb{R}^{p,p}$$
is nonsingular.

It can be shown that Hypothesis 3.7 implies the invertibility of the operator $\begin{pmatrix} L & -V \\ \Psi & 0 \end{pmatrix}$ by the Bordering Lemma [12].

Now we can formulate the main linear existence result from which we obtain the existence of solutions of (3.10) as well as corresponding estimates.

Theorem 3.9. *Consider* (3.10) *and let Hypothesis* 3.7 *be satisfied. Then there exist* $h_0 > 0$, $T > 0$ *such that for* $h < h_0$ *and* $\pm hn_\pm > T$ *equation* (3.10) *has a unique solution* $(\tilde{Z}, \tilde{\Upsilon}) \in (\ell_\infty^{[n_-, n_+ + 1]}(\mathbb{R}^{2m}))^q \times \mathbb{R}^{p,q}$ *for any* $R^{de} \in (\ell_\infty^J(\mathbb{R}^{2m}))^q$, $R^{bc} \in \mathbb{R}^{2m,q}$, $R^{pc} \in \mathbb{R}^{p,q}$. *Furthermore, the following estimate holds*
$$\|\tilde{Z}\|_{\mathcal{H}_h^1} + \|\tilde{\Upsilon}\| \leq \text{const}(\frac{1}{h}\|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|). \tag{3.14}$$

Note that in the traveling wave case we have $p = 1$ and $q = 1$, whereas for the approximation of general equilibria we have $p = \dim G$ and $q = 1$ and for the approximation of the null space we have $p = q = \dim G$.

The existence of exponential dichotomies ensures that certain boundary value problems can be solved that arise later in the construction of solutions of (3.10). We use a slightly adapted version of Lemma 1.1.6 in [13] or Lemma 2.7 in [19] in which the dependency on h is addressed.

Lemma 3.10. *Let the linear difference operator*
$$\hat{L} : \ell_\infty^{[n_-, n_+ + 1]} \to \ell_\infty^J, z \mapsto (z_{n+1} - \hat{M}_n z_n)_{n \in J}$$
have an exponential dichotomy with data (K, β, P) *on* $J = [n_-, n_+] \subset \mathbb{Z}$, *where* $n_\pm = \pm\infty$ *is allowed.*

For each $r \in \ell_\infty^J$ there exists a unique solution $\tilde{z} \in \ell_\infty^{[n_-,n_++1]}$ of the inhomogeneous equation

$$\begin{aligned} (\hat{L}z)_n &= r_n, \quad n \in J, \\ P_{n_-} z_{n_-} &= \varrho_- \in \mathcal{R}(P_{n_-}), \qquad \text{if } n_- \in \mathbb{Z}, \\ (I - P_{n_+}) z_{n_+} &= \varrho_+ \in \mathcal{R}(I - P_{n_+}), \quad \text{if } n_+ \in \mathbb{Z}. \end{aligned}$$

It is given by

$$\begin{aligned} \tilde{z}_n &= b_n^-(\varrho_-) + b_n^+(\varrho_+) + \hat{s}_n(r), \quad n \in J, \\ \tilde{z}_{n_++1} &= \hat{M}_{n_+} \tilde{z}_{n_+} + r_{n_+}, \end{aligned}$$

where $\hat{s}$ is defined with the Green's function $\hat{\mathcal{G}}$ from (3.8) as follows:

$$\hat{s}_n(r) = \sum_{m=n_-}^{n_+-1} \hat{\mathcal{G}}(n, m+1)\, r_m, \quad n \in J \tag{3.15}$$

and

$$b_n^{\pm}(\varrho) = \begin{cases} \hat{\mathcal{S}}(n, n_\pm)\varrho, & \text{in case } \pm n_\pm < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, the following estimate holds for $n \in J$

$$\|\hat{s}_n(r)\| \le K\, C_\beta \|r\|_\infty, \quad \text{where} \quad C_\beta = \frac{1 + \mathrm{e}^{-\beta}}{1 - \mathrm{e}^{-\beta}}. \tag{3.16}$$

In addition, if $r \in \mathcal{L}_{2,h}(J)$ then

$$\|\hat{s}_n(r)\| \le K \sqrt{\frac{C_{2\beta}}{h}}\, \|r\|_{\mathcal{L}_{2,h}} \quad \forall n \in J \quad \text{and} \quad \|\hat{s}(r)\|_{\mathcal{L}_{2,h}} \le K C_\beta\, \|r\|_{\mathcal{L}_{2,h}}. \tag{3.17}$$

In case $\pm n_\pm < \infty$ we obtain for the boundary terms the estimates

$$\|b_n^{\pm}(\varrho_\pm)\| \le K \mathrm{e}^{-\beta|n-n_\pm|} \|\varrho_\pm\|, \; n \in J \quad \text{and} \quad \|b^{\pm}(\varrho_\pm)\|_{\mathcal{L}_{2,h}} \le K \sqrt{h C_{2\beta}} \|\varrho_\pm\|.$$

Note that C_β does not depend on the interval J but only on the dichotomy data and that $C_{h\alpha}$ is of order $\mathcal{O}(\frac{1}{h})$ for small h. Thus we can estimate the solutions $\tilde{z}$ for $\beta = \alpha h$ as follows:

Corollary 3.11. *If $\beta = \alpha h$ then the partial solutions $\tilde{z}$ defined in Lemma 3.10 obey the estimate*

$$\|\tilde{z}\|_{\mathcal{L}_{2,h}} \le \mathrm{const}(\frac{1}{h} \|r\|_{\mathcal{L}_{2,h}} + \|\varrho_-\| + \|\varrho_+\|). \tag{3.18}$$

In the following we transfer the proof in [13] to the discrete case along the lines of the method used in [26] and [27].

Proof of Theorem 3.9

From the properties of L (see Remark 3.3) one obtains that L has exponential dichotomies on $\mathbb{R}^\pm$ with data $(K_\pm, \alpha_\pm, \pi^\pm)$. Then Lemma 3.6 implies that the operator $\hat{L} : \ell_\infty \to \ell_\infty$ defined by

$$\hat{L}z = (z_{n+1} - \hat{M}_n z_n)_{n \in \mathbb{Z}}$$

possesses exponential dichotomies on $\mathbb{Z}^{\pm}$ with data $(K_{\pm}, \alpha_{\pm}h, P^{\pm})$ and

$$\mathcal{N}(\hat{L}) = \operatorname{span}\{\hat{\phi}^1, \dots, \hat{\phi}^p\}, \quad \text{where} \quad \hat{\phi}^i = \phi^i{}_{|J_h} = \begin{pmatrix} S^i(\bar{v}) \\ S^i(\bar{v}') \end{pmatrix}, \; i = 1, \dots, p. \tag{3.19}$$

We use Lemma 3.10 to define solutions of the boundary value problems

$$\begin{aligned} (\hat{L}Z)_n &= R_n^{de}, \quad n \in [n_-, -1] \\ P_{n_-}^- Z_{n_-} &= \rho_- \in (\mathcal{R}(P_{n_-}^-))^q, \\ (I - P_0^-)Z_0 &= Z_0^- \in (\mathcal{N}(P_0^-))^q \end{aligned}$$

and

$$\begin{aligned} (\hat{L}Z)_n &= R_n^{de}, \quad n \in [0, n_+] \\ P_0^+ Z_0 &= Z_0^+ \in (\mathcal{R}(P_0^+))^q, \\ (I - P_{n_+}^+)Z_{n_+} &= \rho_+ \in (\mathcal{N}(P_{n_+}^+))^q, \end{aligned}$$

where we use the notation $(\hat{L}Z)_n := [(\hat{L}Z^1)_n, \dots, (\hat{L}Z^q)_n] \in \mathbb{R}^{m,q}$. These are for each $R^{de} \in (\ell_\infty^J)^q$ defined by

$$\begin{aligned} \tilde{Z}_n^- &= \hat{s}_n^-(R^{de} + \hat{V}^\Upsilon) + \hat{\mathcal{S}}(n,0)Z_0^- + \hat{\mathcal{S}}(n, n_-)\rho_-, \quad n \in [n_-, 0], \\ \tilde{Z}_n^+ &= \hat{s}_n^+(R^{de} + \hat{V}^\Upsilon) + \hat{\mathcal{S}}(n,0)Z_0^+ + \hat{\mathcal{S}}(n, n_+)\rho_+, \quad n \in [0, n_+], \\ \tilde{Z}_{n_++1}^+ &= \hat{M}_{n_+}\tilde{Z}_{n_+}^+ + R_{n_+}^{de} \in \mathbb{R}^{2m,q}, \end{aligned}$$

where $\hat{V}_n^\Upsilon = \hat{V}_n \Upsilon \in \mathbb{R}^{2m,q}$.
Then $\tilde{Z} \in (\ell_\infty^{[n_-, n_++1]})^q$ defined by

$$\tilde{Z}_n = \begin{cases} \tilde{Z}_n^-, & \text{for } n \in J^- := [n_-, -1] \\ \tilde{Z}_n^+, & \text{for } n \in J^+ := [0, n_+ + 1] \end{cases} \tag{3.20}$$

is a solution of (3.10) if it solves the following system

$$\begin{aligned} \tilde{Z}_0^- &= \tilde{Z}_0^+ &\in \mathbb{R}^{2m,q}, && (3.21a) \\ (P_- Q_-)\tilde{Z}_{n_-} + (P_+ Q_+)\tilde{Z}_{n_+} &= R^{bc} &\in \mathbb{R}^{2m,q}, && (3.21b) \\ \langle \hat{\Psi}, \tilde{Z} \rangle_{J_h} &= R^{pc} &\in \mathbb{R}^{p,q}. && (3.21c) \end{aligned}$$

Note that the dependency on $\Upsilon \in \mathbb{R}^{p,q}$ is hidden in the definition of $\tilde{Z}_n^{\pm}$ and is yet to be determined.
We decompose $\mathbb{R}^{2m}$ as follows: Let $\mathcal{W}_1 = \mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-)$. From (3.19) we obtain $\dim(\mathcal{W}_1) = p$, thus we can complement $\mathcal{W}_1$ by subspaces $\mathcal{W}_2$ and $\mathcal{W}_3$ such that

$$\mathcal{R}(P_0^+) = \mathcal{W}_1 \oplus \mathcal{W}_2, \quad \mathcal{N}(P_0^-) = \mathcal{W}_1 \oplus \mathcal{W}_3.$$

Since $\dim(\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3) = 2m - p$ there exists a subspace $\mathcal{W}_4$ with $\dim \mathcal{W}_4 = p$ such that $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 = \mathbb{R}^{2m}$ is a complete decomposition of $\mathbb{R}^{2m}$.
We can change the projectors $P_0^{\pm}$ in such a way that (see [19], Prop. 2.3)

$$\mathcal{N}(P_0^+) = \mathcal{W}_3 \oplus \mathcal{W}_4, \quad \mathcal{R}(P_0^-) = \mathcal{W}_2 \oplus \mathcal{W}_4,$$

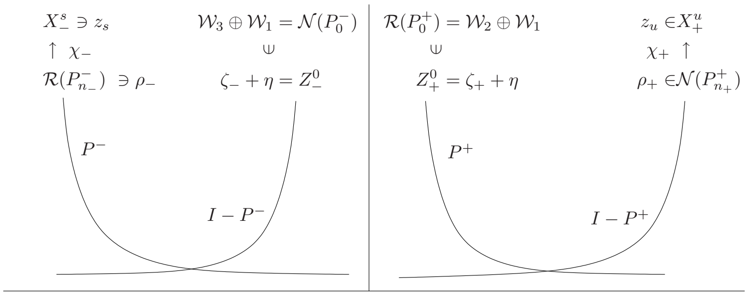


FIG. 3.1. *Overview over dichotomy estimates*

increasing the dichotomy constants $K_\pm$ to $2K_\pm(1+K_\pm)$ and without changing $\mathcal{N}(P_0^-)$,$\mathcal{R}(P_0^+)$ or the dichotomy exponents $\alpha_\pm$. From the choice of Z_0^-, Z_0^+ follows

$$\begin{aligned}(I-P_0^-)\tilde{Z}_0^- = Z_0^- \quad &\in (\mathcal{N}(P_0^-))^q = \mathcal{W}_3^q \oplus \mathcal{W}_1^q,\\ P_0^+\tilde{Z}_0^+ = Z_0^+ \quad &\in (\mathcal{R}(P_0^+))^q = \mathcal{W}_2^q \oplus \mathcal{W}_1^q.\end{aligned}$$

We use the ansatz $Z_0^- = \zeta_- + \eta_-$, $Z_0^+ = \zeta_+ + \eta_+$, where $\zeta_- \in \mathcal{W}_3^q$, $\zeta_+ \in \mathcal{W}_2^q$, $\eta_\pm \in \mathcal{W}_1$ and from (3.21a) we obtain $\eta_+ = \eta_- =: \eta$. Equation (3.21a) now reads

$$\zeta_- - \zeta_+ + \hat{\mathcal{S}}(0,n_-)\rho_- - \hat{\mathcal{S}}(0,n_+)\rho_+ + (\hat{s}_0^-(\hat{V}) - \hat{s}_0^+(\hat{V}))\Upsilon = \hat{s}_0^+(R^{de}) - \hat{s}_0^-(R^{de}).$$

We transform the boundary values ρ_-, ρ_+ to coordinates (z_s, z_u) which are independent of J as follows: Denote by E_-^s the projector onto X_-^s along X_-^u and by E_+^u the projector onto X_+^u along X_+^s, where $X_\pm^s$ are the stable subspaces and $X_\pm^u$ are the unstable subspaces of $M_\pm$ (cf. Definition (2.3)).
We define the transformations

$$\chi_- : \mathcal{R}(P_{n_-}^-) \to X_-^s,\ \varrho_- \mapsto z_s, \qquad \chi_+ : \mathcal{N}(P_{n_+}^+) \to X_+^u,\ \varrho_+ \mapsto z_u$$

by

$$\chi_- = I + E_-^s - P_{n_-}^-, \qquad \chi_+ = I - E_+^s + P_{n_+}^+.$$

Now we apply $\chi_\pm$ columnwise. From the roughness theorem for exponential dichotomies [18] we have $\lim\limits_{x\to\pm\infty} \pi^\pm(x) = E_\pm^s$. With $P_n^\pm = \pi^\pm(x_n)$ the invertibility of χ_- and χ_+ follows for $J_h \to \mathbb{R}$ together with the estimates

$$\|\chi_\pm^{-1}\| \le \frac{1}{1 - \|P_{n_\pm}^\pm - E_\pm^s\|} \le 2. \tag{3.22}$$

This implies for all $(z_s, z_u) \in X_-^s \times X_+^u$

$$\|(I - E_-^s)\chi_-^{-1} z_s\| \le \frac{\|P_{n_-}^- - E_-^s\|\|z_s\|}{1 - \|P_{n_-}^- - E_-^s\|}, \qquad \|E_+^s \chi_+^{-1} z_u\| \le \frac{\|P_{n_+}^+ - E_+^s\|\|z_u\|}{1 - \|P_{n_+}^+ - E_+^s\|}. \tag{3.23}$$

Define $\hat{\Phi} \in \ell_\infty(\mathbb{R}^{2m,p})$ by $\hat{\Phi}_n = \Phi(hn)$, where Φ has been defined in (3.12). Since the columns $\hat{\phi}_0^i,\ i = 1,\dots,p$ of $\hat{\Phi}_0 = \Phi(0)$ span a basis of $\mathcal{N}(P_0^-) \cap \mathcal{R}(P_0^+)$ (see Remark 3.2) we can write $\eta \in \mathcal{W}_1^q$ as $\eta = \hat{\Phi}_0\kappa$ for some $\kappa \in \mathbb{R}^{p,q}$.

Equation (3.21) can now be written as

$$T\begin{pmatrix}(\zeta_-,\zeta_+,\Upsilon)\\(z_s,z_u)\\\kappa\end{pmatrix}=\begin{pmatrix}\hat{s}_0^+(R^{de})-\hat{s}_0^-(R^{de})\\R^{bc}-\Big((P_+Q_+)\hat{s}_{n_+}^+(R^{de})-(P_-Q_-)\hat{s}_{n_-}^-(R^{de})\Big)\\R^{pc}-\langle\hat{\Psi}_{|_{J^-}},\hat{s}^-(R^{de})\rangle_{J_h}-\langle\hat{\Psi}_{|_{J^+}},\hat{s}^+(R^{de})\rangle_{J_h}\end{pmatrix},\qquad(3.24)$$

where $T:\ (\mathcal{W}_2^q\times\mathcal{W}_3^q\times\mathbb{R}^{p,q})\times(X_-^s)^q\times(X_+^u)^q\times\mathbb{R}^{p,q}\to(\mathcal{W}_2\oplus\mathcal{W}_3\oplus\mathcal{W}_4)^q\times\mathbb{R}^{2m,q}\times\mathbb{R}^{p,q}$ has the following structure:

$$T=\begin{pmatrix}\mathcal{X}&\sigma&0\\\Delta&\mathcal{Y}&\varrho\\\Theta&\Pi&\mathcal{Z}\end{pmatrix},$$

where

$$\begin{aligned}\mathcal{X}(\zeta_-,\zeta_+,\lambda)&=\zeta_--\zeta_++(\hat{s}_0^-(\hat{V})-\hat{s}_0^+(\hat{V}))\Upsilon,\\\sigma(z_s,z_u)&=\hat{\mathcal{S}}(0,n_-)[\chi_-^{-1}z_s^1,\dots,\chi_-^{-1}z_s^q]-\hat{\mathcal{S}}(0,n_+)[\chi_+^{-1}z_u^1,\dots,\chi_+^{-1}z_u^q],\\\Delta(\zeta_-,\zeta_+,\Upsilon)&=(P_-Q_-)(\hat{\mathcal{S}}(n_-,0)\zeta_-+\hat{s}_{n_-}^-(\hat{V})\Upsilon)+(P_+Q_+)(\hat{\mathcal{S}}(n_+,0)\zeta_++\hat{s}_{n_+}^+(\hat{V})\Upsilon),\\\mathcal{Y}(z_s,z_u)&=(P_-Q_-)[\chi_-^{-1}z_s^1,\dots,\chi_-^{-1}z_s^q]+(P_+Q_+)[\chi_+^{-1}z_u^1,\dots,\chi_+^{-1}z_u^q],\\\varrho(\kappa)&=((P_-Q_-)\hat{\mathcal{S}}(n_-,0)+(P_+Q_+)\hat{\mathcal{S}}(n_+,0))\hat{\Phi}_0\kappa,\\\Theta(\zeta_-,\zeta_+,\Upsilon)&=\langle\hat{\Psi}_{|_{J^-}},\hat{\mathcal{S}}(\cdot,0)\zeta_-+\hat{s}^-(\hat{V})\Upsilon\rangle_{J_h}+\langle\hat{\Psi}_{|_{J^+}},\hat{\mathcal{S}}(\cdot,0)\zeta_++\hat{s}^+(\hat{V})\Upsilon\rangle_{J_h},\\\Pi(z_s,z_u)&=\langle\hat{\Psi}_{|_{J^-}},\hat{\mathcal{S}}(\cdot,n_-)\chi_-^{-1}[z_s^1,\dots,z_s^q]\rangle_{J_h}+\langle\hat{\Psi}_{|_{J^+}},\hat{\mathcal{S}}(\cdot,n_+)\chi_+^{-1}[z_u^1,\dots,z_u^q]\rangle_{J_h},\\\mathcal{Z}(\kappa)&=\langle\hat{\Psi},\hat{\Phi}\rangle_{J_h}\kappa.\end{aligned}$$

We have to show the invertibility of T as well as an estimate of the inverse of T. The terms σ,ϱ can be estimated using (3.22), the exponential dichotomy of $\hat{L}$ and $\mathcal{R}(\hat{\Phi}_0)=\mathcal{R}(P_0^+)\cap\mathcal{N}(P_0^-)$ by

$$\|\sigma\|,\|\varrho\|\le\text{const}\,\mathrm{e}^{-\alpha h\min(-n_-,n_+)}\quad\to 0\quad\text{as } J_h\to\mathbb{R}.$$

The boundedness of the operators Δ,Π,Θ follows from the dichotomy estimates (3.9) and the properties of $\hat{V}$. The properties of $\hat{\Psi}$ in Hypothesis 3.7 ensure that the map $(\mathcal{L}_{2,h}(\mathbb{R}^{2m}))^q\to\mathbb{R}^{p,q},\ Z\mapsto\langle\hat{\Psi},Z\rangle_{J_h}$ is uniformly bounded.
Hypothesis 2.4 and (3.23) imply that $\mathcal{Y}$ has a uniformly bounded inverse. Lemma 2.2 and the properties of $\hat{V}$ imply that the other operators on the diagonal $\mathcal{X}$ and $\mathcal{Z}$ converge for $J_h\to\mathbb{R}$ to asymptotic operators $\bar{\mathcal{X}}$ and $\bar{\mathcal{Z}}$. These are defined as follows: Let $\mathcal{S}$ be the solution operator corresponding to L and $\pi^\pm$ the stable and unstable dichotomy projectors. We define

$$\bar{\mathcal{X}}(\zeta_-,\zeta_+,\Upsilon)=\zeta_--\zeta_++(s^-(V)(0)-s^+(V)(0))\Upsilon,$$

where

$$\begin{aligned}s^-(V)(x)&=\int_{-\infty}^x\mathcal{S}(x,\xi)\pi^-(\xi)V(\xi)\,d\xi-\int_x^0\mathcal{S}(x,\xi)(I-\pi^-(\xi))V(\xi)\,d\xi,\quad\text{for } x\le 0,\\s^+(V)(x)&=\int_0^x\mathcal{S}(x,\xi)\pi^+(\xi)V(\xi)\,d\xi-\int_x^\infty\mathcal{S}(x,\xi)(I-\pi^+(\xi))V(\xi)\,d\xi,\quad\text{for } x\ge 0\end{aligned}$$

and

$$\bar{\mathcal{Z}}(\kappa) = \langle \Psi, \Phi \rangle \kappa.$$

The invertibility of $\bar{\mathcal{Z}}$ is ensured by Hypothesis 3.7. Similarly the invertibility of $\bar{\mathcal{X}}$ follows from the nondegeneracy condition in Hypothesis 3.7 similar to [2], by multiplying the equation

$$0 = \zeta_- - \zeta_+ + (s^-(V)(0) - s^+(V)(0))\Upsilon$$

from the left by $\Xi(0)^T$ and using Remark 3.8.
Then we obtain that $\mathcal{X}$ and $\mathcal{Z}$ are invertible for h small enough and $\mp hn_\pm$ large enough with a uniform bound for the inverse.
Summing up the estimates for the right hand side in (3.24) and using estimate (3.16) for $\hat{s}^\pm$ in Lemma 3.10 as well as the properties of $\hat{\Psi}$ one obtains for $J_h \to \mathbb{R}$

$$\|\zeta_-\| + \|\zeta_+\| + \|\Upsilon\| + \|z_s\| + \|z_u\| + \|\kappa\| \leq \text{const}(\frac{1}{h}\|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|).$$

This implies with Corollary 3.11 and (3.22) for $\tilde{Z}^- \in (\ell_\infty^{J^-})^q$ and $\tilde{Z}^+ \in (\ell_\infty^{J^+})^q$

$$\|\tilde{Z}^\pm\|_{\mathcal{L}_{2,h}} \leq \text{const}(\frac{1}{h}\|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|).$$

A similar estimate for the contribution of $\tilde{Z}_{n_+ + 1}$ at $n_+ + 1$ leads to

$$\|\tilde{Z}\|_{\mathcal{L}_{2,h}} + \|\Upsilon\| \leq \text{const}(\frac{1}{h}\|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|).$$

This can be improved for h small enough to the $\|\cdot\|_{\mathcal{H}_h^1}$ estimate (3.14) using the difference equation (3.10a) and the following properties of $\hat{V}$ and $\hat{M}$:

$$\hat{V}_n = hV(x_n) + \mathcal{O}(h^2), \quad \hat{M}_n = I + \mathcal{O}(h),$$

which follow from (3.13) and (3.11). □

4. Proof of the main results. In the following section we use Theorem 3.9 as well as the following fixed point theorem (see [24], or [2, Lemma 3.1]) to prove the main approximation results.

THEOREM 4.1. *Let $F : \mathcal{Y} \supset B_\varrho(\bar{y}) \to \mathcal{Z}$ be a C^1 mapping between two Banach spaces $\mathcal{Y}$ and $\mathcal{Z}$ and let $(DF(\bar{y}))^{-1} \in L[\mathcal{Z}, \mathcal{Y}]$ exist. Assume the following estimates for $\kappa, \sigma > 0$*

$$\|DF(y) - DF(\bar{y})\|_{\mathcal{Y}\to\mathcal{Z}} \leq \kappa < \sigma \leq \frac{1}{\|DF(\bar{y})^{-1}\|_{\mathcal{Z}\to\mathcal{Y}}} \quad \forall y \in B_\varrho(\bar{y}), \tag{4.1}$$

$$\|F(\bar{y})\|_Z \leq (\sigma - \kappa)\varrho. \tag{4.2}$$

Then F has a unique zero y_0 in $B_\varrho(\bar{y}) = \{y : \|y - \bar{y}\|_\mathcal{Y} \leq \varrho\}$ and the following estimates hold

$$\|y_0 - \bar{y}\|_\mathcal{Y} \leq \frac{1}{(\sigma - \kappa)}\|F(\bar{y})\|_\mathcal{Z}, \tag{4.3}$$

$$\|y_1 - y_2\|_\mathcal{Y} \leq \frac{1}{(\sigma - \kappa)}\|F(y_1) - F(y_2)\|_\mathcal{Z} \quad \forall y_1, y_2 \in B_\varrho(\bar{y}). \tag{4.4}$$

4.1. Approximation of relative equilibria.

Proof of Theorem 2.5
Set $w = v - \bar{v}$, $\lambda = \mu - \bar{\mu}$. Then a solution of (2.2) is given by $(\bar{w} + \bar{v}, \bar{\lambda} + \bar{\mu})$, where $(\bar{w}, \bar{\lambda})$ is a zero of the operator $F : \ell_\infty^{\hat{J}} \times \mathbb{R}^p \to \ell_\infty^J \times \mathbb{R}^{2m} \times \mathbb{R}^p$, given by

$$F(w,\lambda) = \begin{pmatrix} \Big((\bar{\Lambda} w)_n + \hat{S}_n(\bar{v}_{|J_h})\lambda + g_n(w,\lambda) + (R^{de}_{J_h})_n \Big)_{n\in J} \\ \mathcal{B}w + R^{bc}_{J_h} \\ \langle \hat{S}(\hat{v}_{|J_h}), w\rangle_{J_h} + R^{pc}_{J_h} \end{pmatrix},$$

where

$$\begin{aligned} (\bar{\Lambda} w)_n &= A(\delta_+\delta_- w)_n + B_n(\delta_0 w)_n + C_n w_n, \qquad B_n = B(x_n),\ C_n = C(x_n), \\ g_n(w,\lambda) &= f(\bar{v}_n + w_n, \delta_0(\bar{v}+w)_n) - f(\bar{v}_n, \delta_0\bar{v}_n) \\ &\quad - D_1 f(\bar{v}_n, \bar{v}_n')w_n - D_2 f(\bar{v}_n, \bar{v}_n')\delta_0 w_n + \hat{S}_n(w)\lambda, \quad n \in J \end{aligned}$$

and

$$\begin{aligned} R^{de}_{J_h} &= A\delta_+\delta_-\bar{v} + \hat{S}(\bar{v})\bar{\mu} + f(\bar{v}, \delta_0\bar{v}), \qquad R^{bc}_{J_h} = \mathcal{B}\bar{v}_{|J_h} - (P_-\bar{v}_- + P_+\bar{v}_+), \\ R^{pc}_{J_h} &= \langle \hat{S}(\hat{v}_{|J_h}), (\bar{v} - \hat{v})_{|J_h}\rangle_{J_h}. \end{aligned}$$

We will apply Theorem 4.1 to F with $\mathcal{Y} = (\ell_\infty^{\hat{J}}) \times \mathbb{R}^p$, $\mathcal{Z} = (\ell_\infty^J) \times \mathbb{R}^{2m} \times \mathbb{R}^p$, at the approximative zero $\bar{y} = 0$ with norms

$$\|(v,\mu)\|_{\mathcal{Y}} = \|v\|_{\mathcal{H}_h^2} + \|\mu\|, \qquad \|(R^{de}, R^{bc}, R^{pc})\|_{\mathcal{Z}} = \|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|.$$

We have with the properties of $\bar{v}$ (see Remark 3.4)

$$\begin{aligned} \|R^{de}_{J_h}\|_{\mathcal{L}_{2,h}} &\le \|A\|\|\delta_+\delta_-\bar{v} - \bar{v}''\|_{\mathcal{L}_{2,h}} + \|f(\bar{v},\delta_0\bar{v}) - f(\bar{v},\bar{v}')\|_{\mathcal{L}_{2,h}} \\ &\quad + \|\hat{S}(\bar{v}_{|J_h}) - S(\bar{v})\|_{\mathcal{L}_{2,h}}\|\bar{\mu}\| \\ &\le \text{const}\, h^2. \end{aligned}$$

Similarly, we obtain with the assumptions on the phase condition in Hypothesis 2.4 and Lemma 2.2

$$\begin{aligned} \|R^{pc}_{J_h}\| &\le \|\langle S(\hat{v})_{|J_h}, (\bar{v}-\hat{v})_{|J_h}\rangle_{J_h}\| + \|\langle \hat{S}(\hat{v}_{|J_h}) - S(\hat{v})_{|J_h}, (\bar{v}-\hat{v})_{|J_h}\rangle_{J_h}\| \\ &\le \text{const}(h^2 + \mathrm{e}^{-\alpha h \min\{n_-,n_+\}}). \end{aligned}$$

Using the consistency of the boundary conditions in Hypothesis 2.4 we have

$$\begin{aligned} \|R^{bc}_{J_h}\| &\le \|P_-\|\|\bar{v}_{n_-} - \bar{v}_-\| + \|Q_-\|(\|\delta_0\bar{v}_{n_-} - \bar{v}'(x_{n_-})\| + \|\bar{v}'(x_{n_-})\|) \\ &\quad + \|P_+\|\|\bar{v}_{n_+} - \bar{v}_+\| + \|Q_+\|(\|\delta_0\bar{v}_{n_+} - \bar{v}'(x_{n_+})\| + \|\bar{v}'(x_{n_+})\|) \\ &\le \text{const}(h^2 + \mathrm{e}^{-\alpha h \min\{n_-,n_+\}}). \end{aligned}$$

These estimates together with $g(0,0) = 0$ imply the consistency estimate

$$\|F(0,0)\|_Z \le \|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\| \le \text{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-,n_+\}}).$$

With

$$DF(\tilde{w},\tilde{\lambda})(w,\lambda) = \begin{pmatrix} (\bar{\Lambda}w)_n + \hat{S}_n(\bar{v}_{|J_h})\lambda + Dg_n(\tilde{w},\tilde{\lambda})(w,\lambda) \\ \mathcal{B}w \\ \langle \hat{S}(\hat{v}_{|J_h}), w\rangle_{J_h} \end{pmatrix},$$

$Dg(0,0) = 0$, the properties of f (see Remark 1.2) and the discrete Sobolev embedding $\|v\|_\infty \leq \text{const}\,\|v\|_{\mathcal{H}^1_h}$ we arrive at

$$\begin{aligned} \|(DF(\tilde{w},\tilde{\lambda}) - DF(0,0))(w,\lambda)\|_{\mathcal{Z}} &= \|Dg(\tilde{w},\tilde{\lambda})(w,\lambda)\|_{\mathcal{L}_{2,h}} \\ &\leq \text{const}(\|\tilde{w}\|_{\mathcal{H}^2_h} + \|\tilde{\lambda}\|)(\|w\|_{\mathcal{H}^2_h} + \|\lambda\|). \end{aligned}$$

It remains to estimate the inverse of $DF(0,0)$. With the definitions

$$z_n = (v_n, w_n),\ \ w_n = (\delta_- v)_n \quad \text{and} \quad \hat{R}^{de}_n = \begin{pmatrix} 0 \\ hR^{de}_n \end{pmatrix}, \tag{4.5}$$

the equation

$$DF(0,0)(w,\lambda) = (R^{de}, R^{bc}, R^{pc})^T \tag{4.6}$$

is transformed into

$$\tilde{T}(z,\lambda) = (\hat{R}^{de}, R^{bc}, R^{pc})^T, \tag{4.7}$$

where $\tilde{T} : \ell^{[n_-,n_++1]}_\infty(\mathbb{R}^{2m}) \times \mathbb{R}^p \to \ell^J_\infty(\mathbb{R}^{2m}) \times \mathbb{R}^{2m} \times \mathbb{R}^p$ is given by

$$\tilde{T}(z,\lambda) = \begin{pmatrix} (\tilde{N}_n z_{n+1} - \tilde{K}_n z_n - \tilde{U}_n\lambda)_{n\in J} \\ P_- v_{n_-} + Q_-(w_{n_-} + \frac{h}{2}\delta_+ w_{n_-}) + P_+ v_{n_+} + Q_+(w_{n_+} + \frac{h}{2}\delta_+ w_{n_+}) \\ \langle \tilde{\Psi}, z\rangle_{J_h} \end{pmatrix}, \tag{4.8}$$

with

$$\tilde{N}_n = \begin{pmatrix} I & -hI \\ 0 & A + \frac{h}{2}B_n \end{pmatrix},\ \tilde{K}_n = \begin{pmatrix} I & 0 \\ -hC_n & A - \frac{h}{2}B_n \end{pmatrix},\ \tilde{U}_n = -\begin{pmatrix} 0 \\ h\hat{S}(\bar{v}_{|J_h})_n \end{pmatrix}$$

and

$$\tilde{\Psi} = \begin{pmatrix} \hat{S}(\hat{v}_{|J_h}) \\ 0 \end{pmatrix}.$$

In order to use the properties of the continuous operator L we consider the following approximation of $\tilde{T}$

$$T(z,\lambda) = \begin{pmatrix} (Nz_{n+1} - N\hat{M}_n z_n - U_n\lambda)_{n\in J} \\ (P_-\ Q_-)z_{n_-} + (P_+\ Q_+)z_{n_+} \\ \langle \Psi_{|J_h}, z\rangle_{J_h} \end{pmatrix},$$

where

$$N = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},\ U_n = -\begin{pmatrix} 0 \\ h[S(\bar{v})](x_n) \end{pmatrix},\ \Psi(x) = \begin{pmatrix} [S(\hat{v})](x) \\ 0 \end{pmatrix}. \tag{4.9}$$

From the definition of $\hat{M}_n$ we get

$$\hat{M} = \mathcal{S}(x_{n+1}, x_n) = I + hM(x_n) + \mathcal{O}(h^2)$$

as well as the estimates

$$\begin{aligned}\|\tilde{N}_n - N\| &\le \mathrm{const}\, h,\\ \|N\hat{M}_n - \tilde{K}_n + \tilde{N}_n - N\| &\le \mathrm{const}\, h^2,\\ \|\tilde{U} - U\|_{\mathcal{L}_{2,h}} &\le \mathrm{const}\, h^2.\end{aligned}$$

These estimates imply that T is a perturbation of $\tilde{T}$ of order h w.r.t. the norms $\|(\hat{R}^{de}, R^{bc}, R^{pc})\|_h^* = \frac{1}{h}\|\hat{R}^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|$ and $\|(z,\lambda)\|_{\mathcal{H}_h^1}^* = \|z\|_{\mathcal{H}_h^1} + \|\lambda\|$ as follows. We have

$$\begin{aligned}&\|((\tilde{N}_n - N)z_{n+1} - (\tilde{K}_n - N\hat{M}_n)z_n + (\tilde{U}_n - U_n)\lambda)_{n\in J}\|_{\mathcal{L}_{2,h}}^2\\ &\quad\le \mathrm{const} \sum_{n=n_-}^{n_+} h\Big(h^2\|\tilde{N}_n - N\|^2\|\delta_+ z_n\|^2 + \|N\hat{M}_n - \tilde{K}_n + \tilde{N}_n - N\|^2\|z_n\|^2\\ &\qquad\qquad + \|\tilde{U}_n - U_n\|^2\|\lambda\|^2\Big)\\ &\quad\le \mathrm{const}\, h^4\|\delta_+ z\|_{\mathcal{L}_{2,h}}^2 + h^4\|z\|_{\mathcal{L}_{2,h}}^2 + \|\tilde{U} - U\|_{\mathcal{L}_{2,h}}^2\|\lambda\|^2 \le \mathrm{const}\, h^4(\|z\|_{\mathcal{H}_h^1} + \|\lambda\|)^2\end{aligned}$$

and with Lemma 2.2

$$\|\Psi_{|J_h} - \tilde{\Psi}\|_{J_h} \le \|\hat{S}(\hat{v}_{|J_h}) - S(\hat{v})_{|J_h}\|_{J_h} \le \mathrm{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}).$$

Thus we obtain

$$\begin{aligned}\|(\tilde{T} - T)(z,\lambda)\|_h^* &= \frac{1}{h}\big\|\big((\tilde{N}_n - N)z_{n+1} - (\tilde{K}_n - N\hat{M}_n)z_n + (\tilde{U}_n - U_n)\lambda\big)_{n\in J}\big\|_{\mathcal{L}_{2,h}}\\ &\quad + \|Q_-(\frac{h}{2}\delta_+ w_{n_-}) + Q_+(\frac{h}{2}\delta_+ w_{n_+})\| + \|\langle\Psi_{|J_h} - \tilde{\Psi}, z\rangle_{J_h}\|\\ &\le \mathrm{const}\, h(\|z\|_{\mathcal{H}_h^1} + \|\lambda\|)\\ &\quad + \frac{h}{2}(\|Q_-\|\|\delta_+ w_{n_-}\| + \|Q_+\|\|\delta_+ w_{n_+}\|) + \mathrm{const}\, h^2\|z\|_{\mathcal{L}_{2,h}}\\ &\le \mathrm{const}(h + \mathrm{e}^{-\alpha h\min\{-n_-, n_+\}})(\|z\|_{\mathcal{H}_h^1} + \|\lambda\|).\end{aligned}$$

Define

$$V(x) = \begin{pmatrix} 0 \\ -A^{-1}[S(\bar{v})](x)\end{pmatrix} \tag{4.10}$$

and set $\hat{V}_n = N^{-1}U_n = hV(x_n)$. Then the equation

$$T(z,\lambda) = (\hat{R}^{de}, R^{bc}, R^{pc})^T \tag{4.11}$$

can be equivalently written as

$$\begin{aligned} z_{n+1} - \hat{M}_n z_n - \hat{V}_n\lambda &= N^{-1}\hat{R}_n^{de},\\ (P_-\ Q_-)z_{n_-} + (P_+\ Q_+)z_{n_+} &= R^{bc},\\ \langle\Psi_{|J_h}, z\rangle_{J_h} &= R^{pc}.\end{aligned} \tag{4.12}$$

In order to apply the linear Theorem 3.9 to (4.12) we show that Hypothesis 3.7 is satisfied. The nondegeneracy condition in Hypothesis 3.7 follows from the fact that the null space of Λ is spanned by $S^i(\bar{v}),\ i = 1, ..., p$ and (4.10).
The assumptions on the phase condition in Hypothesis 3.7 follow with the definition of Ψ in (4.9) directly from the corresponding assumptions in Hypothesis 2.4.
By applying Theorem 3.9 to (4.12) and multiplying with the bounded matrix N we obtain the invertibility of T as well as an uniform bound for the inverse

$$\|T^{-1}(R^{de}, R^{bc}, R^{pc})\|^*_{\mathcal{H}_h^1} \leq \text{const}(\frac{1}{h}\|\hat{R}^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|).$$

By perturbation arguments the invertibility of $\tilde{T}$ follows with the same bound for a possibly different constant. These estimates can be transformed back into (2.10) by using the structure of the matrices $\tilde{N}_n, \tilde{K}_n, \tilde{U}_n$ (see (4.9)) as well as the structure of the right hand side $\hat{R}^{de}$ (see (4.5)) together with the estimate $\|z\|_{\mathcal{H}_h^1} \leq \text{const}\,\|v\|_{\mathcal{H}_h^2}$. □

4.2. Approximation of the spectrum.

The proof of Theorem 2.6 follows the same lines as the proof of Theorem 2.5.

Proof of Theorem 2.6
As in the previous proof we insert $X = V - S(\bar{v})_{|J_h}$ into (2.4) and obtain $F(X, \Sigma) = 0$ where

$$F(X,\Sigma) = \begin{pmatrix} \Big((\tilde{\Lambda}X)_n - [S(\bar{v})](x_n)\Sigma - X_n\Sigma + (R^{de}_{J_h})_n\Big)_{n\in J} \\ \mathcal{B}X + R^{bc}_{J_h} \\ \langle \hat{X}_{|J_h}, X\rangle_{J_h} + R^{pc}_{J_h} \end{pmatrix}$$

with $R^{de}_{J_h} = \tilde{\Lambda}S(\bar{v})_{|J_h}, \quad R^{bc}_{J_h} = \mathcal{B}S(\bar{v})_{|J_h}, \quad R^{pc}_{J_h} = \langle \hat{X}_{|J_h}, S(\bar{v})_{|J_h}\rangle_{J_h} - I.$
We will apply Theorem 4.1 to F with $\mathcal{Y} = (\ell_\infty^{\hat{J}})^p \times \mathbb{R}^{p,p},\ \mathcal{Z} = (\ell_\infty^J)^p \times \mathbb{R}^{2m,p} \times \mathbb{R}^{p,p}$, at the approximative zero $\bar{y} = 0$ with norms

$$\|(X,\Sigma)\|_{\mathcal{Y}} = \|X\|_{\mathcal{H}_h^2} + \|\Sigma\|, \qquad \|(R^{de}, R^{bc}, R^{pc})\|_{\mathcal{Z}} = \|R^{de}\|_{\mathcal{L}_{2,h}} + \|R^{bc}\| + \|R^{pc}\|.$$

In the same way as in the proof of Theorem 2.5 we have

$$\|R^{bc}_{J_h}\| + \|R^{pc}_{J_h}\| \leq \text{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}})$$

and

$$\begin{aligned} \|R^{de}_{J_h}\|_{\mathcal{L}_{2,h}} &\leq \|\tilde{\Lambda}S(\bar{v})_{|J_h} - (\Lambda S(\bar{v}))_{|J_h}\|_{\mathcal{L}_{2,h}} + \|(\Lambda S(\bar{v}))_{|J_h}\|_{\mathcal{L}_{2,h}} \\ &\leq \text{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}). \end{aligned}$$

The properties of $S(\bar{v})$ (see Remark 3.4) imply $P_- S(\bar{v}_-) + P_+ S(\bar{v}_+) = 0$ as well as $\lim_{x\to\infty}[S(\bar{v}')](x) = 0$. These estimates lead to

$$\begin{aligned} \|F(0,0)\|_Z &\leq \|\tilde{\Lambda}S(\bar{v})_{|J_h}\|_{\mathcal{L}_{2,h}} + \|\mathcal{B}S(\bar{v})_{|J_h}\| + \|\langle \hat{X}_{|J_h}, S(\bar{v})_{|J_h}\rangle_{J_h} - I\| \\ &\leq \text{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}). \end{aligned}$$

With

$$DF(\tilde{X},\tilde{\Sigma})(X,\Sigma) = \begin{pmatrix} ((\tilde{\Lambda}X)_n - [S(\bar{v})](x_n)\Sigma - \tilde{X}_n\Sigma - X_n\tilde{\Sigma})_{n\in J} \\ \mathcal{B}X \\ \langle \hat{X}_{|_{J_h}}, X\rangle_{J_h} \end{pmatrix}$$

we obtain

$$\begin{aligned} \|(DF(\tilde{X},\tilde{\Sigma}) - DF(0,0))(X,\Sigma)\|_{\mathcal{Z}} &= \|(\tilde{X}_n\Sigma + X_n\tilde{\Sigma})_{n\in J}\|_{\mathcal{L}_{2,h}} \\ &\le \mathrm{const}(\|\tilde{X}\|_{\mathcal{H}_h^2} + \|\tilde{\Sigma}\|)(\|X\|_{\mathcal{H}_h^2} + \|\Sigma\|). \end{aligned}$$

It remains to estimate the inverse of $DF(0,0)$. In the same way as before we transform (4.6) into (4.7), where $\tilde{T} : (\ell_\infty^{[n_-,n_++1]}(\mathbb{R}^{2m}))^p \times \mathbb{R}^{p,p} \to (\ell_\infty^J(\mathbb{R}^{2m}))^p \times \mathbb{R}^{2m,p} \times \mathbb{R}^{p,p}$ is given by (4.8) with

$$\tilde{N}_n = \begin{pmatrix} I & -hI \\ 0 & A + \frac{h}{2}\tilde{B}_n \end{pmatrix}, \; \tilde{K}_n = \begin{pmatrix} I & 0 \\ -h\tilde{C}_n & A - \frac{h}{2}\tilde{B}_n \end{pmatrix}, \; \tilde{U}_n = -\begin{pmatrix} 0 \\ h[S(\bar{v})](x_n) \end{pmatrix},$$
$$\tilde{\Psi} = \begin{pmatrix} \hat{X} \\ 0 \end{pmatrix}.$$

Again we solve (4.11) in order to solve (4.7) where $U_n = \tilde{U}_n$ and $\Psi = \tilde{\Psi}$. With the properties of f (see (1.6)) and (2.10) we obtain

$$\begin{aligned} \|\tilde{B}_n - B(x_n)\| &\le \|D_2 f(\tilde{v}_n, (\delta_0\tilde{v})_n) - D_2 f(\bar{v}_n, \bar{v}'_n)\| + \sum_{i=1}^p |\tilde{\mu}_i - \bar{\mu}_i| \|S_1^i\| \\ &\le \mathrm{const}(h^2 + \mathrm{e}^{-\alpha h \min\{n_-,n_+\}}), \end{aligned}$$

and similarly

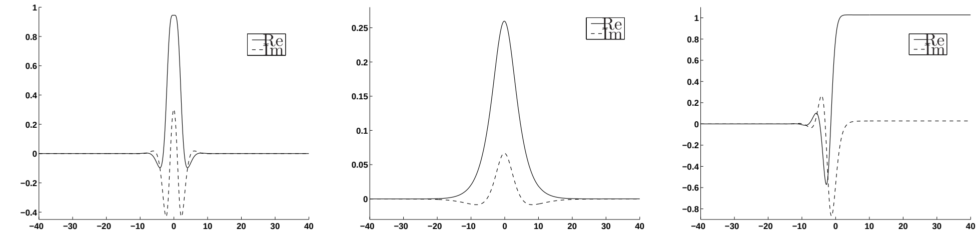
$$\|\tilde{C}_n - C(x_n)\| \le \mathrm{const}(h^2 + \mathrm{e}^{-\alpha h \min\{-n_-,n_+\}}).$$

This implies that T is a perturbation of $\tilde{T}$ of order h. As in the previous proof, the assumptions on the phase condition in Hypothesis 3.7 follow with the definition of Ψ directly from the assumptions on $\hat{X}$. The rest of the arguments is the same as in the proof of Theorem 2.5. □

5. Numerical results. We illustrate the approximation results on the cubic quintic Ginzburg-Landau equation, a reaction-diffusion equation for which exact solutions are known. We compare the order of approximation for different grid sizes h and intervals J with the expected behavior from Theorems 2.5 and 2.6.
The cubic quintic Ginzburg Landau equation [22], [25], [8] reads

$$u_t = a u_{xx} + \delta u + g(u), \quad g(u) = \beta|u|^2 u + \gamma|u|^4 u, \; \delta \in \mathbb{R}, \; a, \beta, \gamma \in \mathbb{C}. \tag{5.1}$$

It shows a variety of coherent structures, like stable pulse solutions, fronts, sources, sinks, etc..
For numerical computations we write (5.1) in real variables; then equation (5.1) has the equivariance properties given in Example 1.4. For certain parameter values this equation possesses stable and unstable rotating pulses [22] as well as rotating and traveling fronts. All these solutions can be written in the form (1.12) where for the



(a) stable pulse (b) unstable pulse (c) front

FIG. 5.1. *stable/unstable pulse and stable front*

rotating pulses we have $\bar{\mu}_\tau = 0$. Depending on the choice of initial conditions a different type of solution is selected.
Using Painlevé methods, some exact solutions have been constructed explicitly in [15]. The explicit expression for the unstable pulse reads (in complex notation)

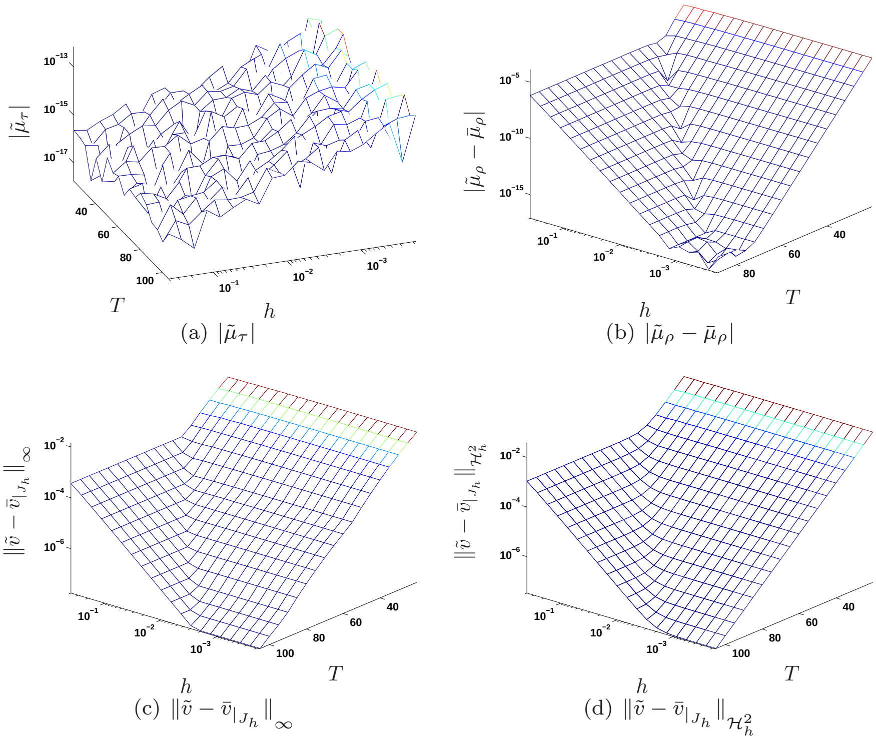
$$u(x,t) = \mathrm{e}^{-i\bar{\mu}_\rho t}\ \bar{v}(x - \bar{\mu}_\tau t), \tag{5.2}$$

where

$$\bar{v}(\xi) = u_0 \mathrm{e}^{ia_0\theta_0\xi}(\cosh(k\xi) - \cosh(\rho))^{ia_0}\sqrt{\frac{k\sinh(\rho)}{\cosh(k\xi)+\cosh(\rho)}}\ .$$

The parameters $u_0, a_0, \theta_0, \rho, \bar{\mu}_\rho, \bar{\mu}_\tau, k$ can be computed explicitly from a, δ, β and γ by the formulae given in [15]. These solutions are relative equilibria in the sense of Definition 1.1 (cf. Example 1.4). For the parameter set $a = 1,\ \delta = -0.1,\ \beta = 3 + i,\ \gamma = -2.75 + i$, which has been used in [22], we have computed numerically a stable pulse as well as a rotating front on the interval $[-40, 40]$ with mesh size $h = 0.1$ using Dirichlet and Neumann boundary conditions, respectively, using the method described in [5],[23],[6]. An approximation of an unstable pulse on the same grid with Dirichlet boundary conditions has been computed by starting a Newton iteration with the explicit solution defined in (5.2). All these solutions are depicted in Figure 5.1.

5.1. Approximation of the unstable pulse. We compare the approximation error of the solution of the boundary value problem (2.2) with the estimates in Theorem 2.5. For the unstable pulse the exact solution is explicitly given by (5.2). Figure 5.2 shows the approximation error of the pulse for Dirichlet boundary conditions. The grid size h is varied exponentially in $[10^{-4}, 10^{-1}]$ and the size T of the symmetric interval J linearly in $[20, 80]$. As shown in Figure 5.2 the parameters μ_τ, μ_ρ converge much faster than the wave form $\tilde{v}$ to the exact values.
The rate of convergence of μ_ρ to $\bar{\mu}_\rho$ is of order 4 in h and the exponential rate in T is $\alpha \approx 0.5$. In contrast, μ_τ reaches quickly the range of machine precision where rounding errors dominate and the bad conditioning of the equations in the Newton iteration becomes prominent. The wave $\tilde{v}$ itself converges as predicted with quadratic order in h and with $\alpha \approx 0.16$ in T. This can be observed in $\|\cdot\|_{\mathcal{H}_h^2}$ as well as in $\|\cdot\|_\infty$ (see Figures 5.2(c), 5.2(d)). In all cases the overall behavior matches the predictions made in Theorem 2.5.

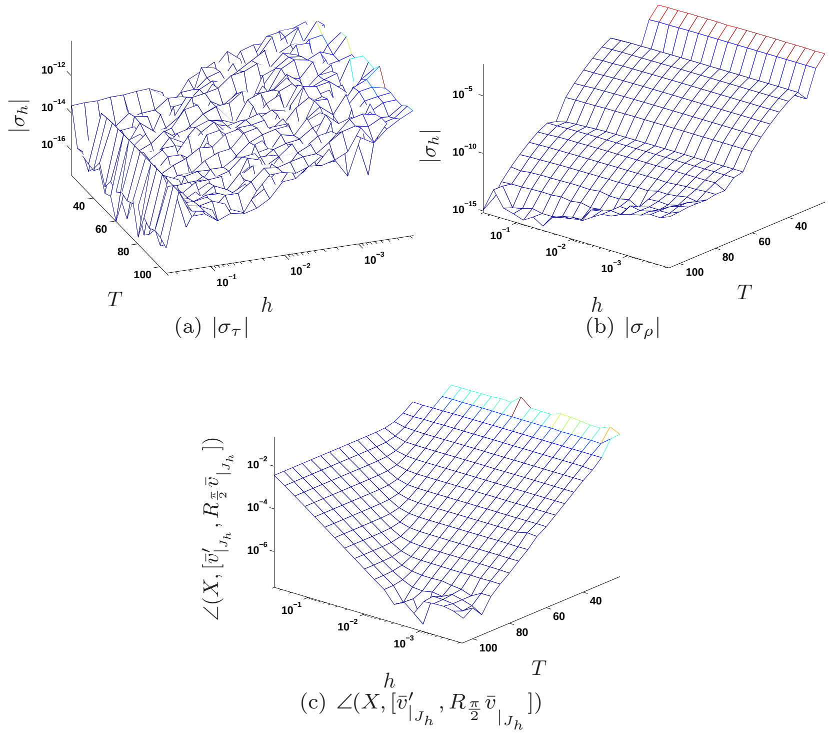


(a) $|\tilde{\mu}_\tau|$ (b) $|\tilde{\mu}_\rho - \bar{\mu}_\rho|$

(c) $\|\tilde{v} - \bar{v}_{|J_h}\|_\infty$ (d) $\|\tilde{v} - \bar{v}_{|J_h}\|_{\mathcal{H}_h^2}$

FIG. 5.2. *Approximation error for the unstable pulse*

5.2. Approximation of discrete eigenvalues. In order to solve the eigenvalue problem (2.4) we use a Newton iteration, starting from $\hat{X} = [\bar{v}'_{|J_h}, R_{\frac{\pi}{2}}\bar{v}_{|J_h}]$. Alternatively one can use an Arnoldi iteration to compute the two eigenvalues of smallest magnitude of the corresponding generalized eigenvalue problem. This leads to similar results. The angle between the approximating subspace X and $\hat{X}$ (see [9]) and the absolute values of the two eigenvalues near zero are shown in Figure 5.3 for the unstable pulse. Here σ_τ denotes the eigenvalue which belongs to the approximation of the translational eigenfunction $\bar{v}'$ and σ_ρ is the eigenvalue which belongs to the approximation of the eigenfunction $R_{\frac{\pi}{2}}\bar{v}$. It can be seen that the eigenvalue σ_τ is in the range of machine precision, thus the errors increase for decreasing h, since the condition of the eigenvalue problem gets worse. The error in the rotational eigenvalue σ_ρ is nearly constant for different h, but decreases for increasing T, as expected. For very small h and large T the increase in error due to the conditioning becomes visible as well. The angle between the invariant subspace which belongs to σ_τ and σ_ρ and the span of the exact eigenfunctions $\bar{v}'$ and $R_{\frac{\pi}{2}}\bar{v}$ of Λ restricted to the grid shows the expected behavior. It decreases quadratically in h and exponentially in T with a rate of $\alpha \approx -0.32$ until the range of machine precision is reached.
Note that Theorems 2.5 and 2.6 are not applicable to the rotating front. In this case $R_{\frac{\pi}{2}}\bar{v}$ is not in $\mathcal{L}_2$ (cf. Example 1.4).



(a) $|\sigma_\tau|$ (b) $|\sigma_\rho|$

(c) $\angle(X, [\bar{v}'_{|J_h}, R_{\frac{\pi}{2}}\bar{v}_{|J_h}])$

FIG. 5.3. *Approximation error for the double zero eigenvalue*

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