A Model Function for Non-Autonomous Bifurcations of Maps

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Abstract

In this paper, we introduce a class of one-dimensional non-autonomous dynamical systems that allows an explicit study of its orbits as well as of the solutions of the associated variational equation. Furthermore, the solution operators also have explicit representations. In a special case, the model function can be transformed into the non-autonomous Beverton-Holt equation. We use this function for analyzing various notions of non-autonomous transcritical and pitchfork bifurcations that have been recently developed in the literature.

Keywords: Non-autonomous bifurcation theory, Transcritical bifurcation, Pitchfork bifurcation, Non-autonomous discrete time dynamical systems, Beverton-Holt equation.

AMS subject classification: 37B55, 34C23, 70G60, 37N25.

1 Introduction

The occurrence of a bifurcation in a parameter dependent dynamical system results in changes of its structural behavior. For autonomous systems the mathematical equivalent of such structural changes is quite clear and can be expressed in terms of topological equivalence, cf. [7]. But for non-autonomous systems such generally accepted notion seems to be unavailable yet. It will be important to test the

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suggested notions on systems from real application, such as population models in mathematical biology.

Consider a non-autonomous discrete time dynamical system depending on one parameter

$$x_{n+1} = g_n(x_n, \lambda), \quad x_n \in \mathbb{R}^k,\ \lambda \in \mathbb{R},\ n \in \mathbb{Z}, \tag{1}$$

where $k \geq 1$ and $(g_n)_{n\in\mathbb{Z}}$ is a family of smooth maps with respect to both x and λ. Let $\bar{x} = 0$ be a fixed point of $g_n(\cdot, \lambda)$ for all λ and all $n \in \mathbb{Z}$.

In case the fixed point $\bar{x}$ undergoes a bifurcation at a critical parameter $\bar{\lambda}$, the behavior of this system changes structurally. In an autonomous version of (1), i.e. $g_n = g$ for all n, bifurcation results are well understood, cf. [4, 7, 11], but results for the non-autonomous case are currently being developed. For example, a survey of non-autonomous bifurcation phenomena is given in [5, 9] and generalizations of the autonomous transcritical and pitchfork bifurcation are proposed in [8] and [10].

In this paper we set up a scalar model function that can be used to study various notions of non-autonomous bifurcations explicitly. This model function is based on the construction in [3]. For every $q \in \mathbb{N}$ the scalar function is defined as

$$g_n(x, \lambda) := \frac{\lambda x}{\left(1 + \frac{b_n q}{\lambda} x^q\right)^{1/q}}. \tag{2}$$

It has the remarkable property that the solutions $(x_n)_{n\in\mathbb{N}}$ of (1) can be given explicitly. Furthermore, the solution of the associated variational equation as well as the corresponding solution operator have an explicit form, too.

In case $q = 1$, this function can be transformed into the so-called Beverton-Holt equation; a function that originates from population biology. It describes the density of a population in a fluctuating environment, cf. [1, 6]. Using our approach, we find an explicit representation of the solutions of the Beverton-Holt equation.

We show in Section 3 that in case $q = 1$, the fixed point $\bar{x} = 0$ of our model function undergoes a non-autonomous transcritical bifurcation at the critical parameter $\bar{\lambda} = 1$. We perform the analysis for two related concepts of natural generalizations of the transcritical bifurcation from the literature, cf. [8, 10]. In [8] a bifurcation[1] is characterized by the change from a pullback attracting fixed point, losing its stability at a critical parameter $\bar{\lambda}$, to a pullback stable complete trajectory. The author of [10] determines a bifurcation by a qualitative change of the domain of attraction of the fixed point $\bar{x}$ from a trivial to a non-trivial object in the limit $\lambda \to \bar{\lambda}$. We indicate that the model function in case $q = 1$ exhibits a transcritical bifurcation, according to both definitions in [8, 10].

In Section 4, a similar analysis for the model function (2) in case $q = 2$ shows that the fixed point $\bar{x} = 0$ undergoes at $\bar{\lambda} = 1$ a non-autonomous pitchfork bifurcation.

Finally, in Section 5, we introduce a second model function that has, in contrast to (2), an n dependent linear part in its Taylor series. We show that this function also exhibits a non-autonomous transcritical bifurcation.

[1] We restrict the presentation to the case in which a stable fixed point $\bar{x}$ exists for $\lambda < \bar{\lambda}$.

2 A model function for polynomial rates in non-autonomous dynamical systems

Let $q \in \mathbb{N}$ be a fixed natural number, and let $(b_n)_{n\in\mathbb{Z}}$ be a bi-infinite sequence such that $b_n \in \mathbb{R}^+$ for all $n \in \mathbb{Z}$. Consider the non-autonomous discrete time dynamical system

$$x_{n+1} = g_n(x_n, \lambda), \quad n \in \mathbb{Z}, \tag{3}$$

depending on one parameter $\lambda \in \mathbb{R}^+$, where the family of maps g_n is defined in (2). Note that these functions have at 0 the Taylor series

$$g_n(x, \lambda) = \lambda x - b_n x^{q+1} + \mathcal{O}(x^{2q+1}),$$

and additionally it holds in case q is even

$$g_n(-x, \lambda) = -g_n(x, \lambda). \tag{4}$$

This class of functions is of particular interest, since we find an explicit representation of the orbits of (3) as well as of the solution of the associated variational equation

$$u_{n+1} = D_x g_n(x_n, \lambda) u_n, \quad n \in \mathbb{Z}. \tag{5}$$

Here

$$D_x g_n(x, \lambda) = \frac{\lambda}{\left(1 + \frac{b_n q}{\lambda} x^q\right)^{1+1/q}}$$

denotes the derivative of g_n w.r.t. the variable x.

Definition 1 *Let $n \geq m$. The evolution operator of (3) is defined as*

$$\Psi(n, m)(x, \lambda) := g_{n-1}(\cdot, \lambda) \circ \ldots \circ g_m(x, \lambda).$$

Furthermore, the solution operator of the variational equation (5) along a solution $(x_n)_{n\in\mathbb{Z}}$ of (3) is given by

$$\Phi(n, m)(\lambda) := D_x g_{n-1}(x_{n-1}, \lambda) \cdot \ldots \cdot D_x g_m(x_m, \lambda).$$

First, we give explicit formulae for the evolution operator as well as for the orbit.

Proposition 2 *The evolution operator of (3) has for $n \geq m$, $x > 0$ the explicit representation*

$$\Psi(n, m)(x, \lambda) = \frac{\lambda^{n-m} x}{\left(1 + q x^q \sum_{i=m}^{n-1} b_i \lambda^{(i-m)q-1}\right)^{1/q}}. \tag{6}$$

The orbit with starting point $x_1 = \frac{\lambda}{\gamma^{1/q}}$ has for $n \in \mathbb{N}$ the explicit form

$$x_n = \frac{\lambda^n}{\left(\gamma + q \sum_{i=1}^{n-1} b_i \lambda^{iq-1}\right)^{1/q}}. \tag{7}$$

Proof: We prove (6) by induction. Let $m \in \mathbb{N}$ be fixed, then $\Psi(m,m)(x,\lambda) = x$. Inductively, we get

$$\begin{aligned}
& g_n(\cdot,\lambda) \circ \Psi(n,m)(x,\lambda) \\
&= \lambda \Psi(n,m)(x,\lambda) \left(1 + \frac{b_n q}{\lambda} \big(\Psi(n,m)(x,\lambda)\big)^q \right)^{-1/q} \\
&= \frac{\lambda^{n-m+1} x}{\left(1 + q x^q \sum_{i=m}^{n-1} b_i \lambda^{(i-m)q-1}\right)^{1/q}} \left(1 + \frac{b_n q}{\lambda} \frac{\lambda^{(n-m)q} x^q}{1 + q x^q \sum_{i=m}^{n-1} b_i \lambda^{(i-m)q-1}} \right)^{-1/q} \\
&= \lambda^{n-m+1} x \left(1 + q x^q \sum_{i=m}^{n-1} b_i \lambda^{(i-m)q-1} + b_n q x^q \lambda^{(n-m)q-1} \right)^{-1/q} \\
&= \lambda^{n-m+1} x \left(1 + q x^q \sum_{i=m}^{n} b_i \lambda^{(i-m)q-1} \right)^{-1/q} \\
&= \Psi(n+1,m)(x,\lambda).
\end{aligned}$$

Applying this result, it follows for $n \in \mathbb{N}$

$$\begin{aligned}
x_n &= \Psi(n,1)(x_1,\lambda) = \Psi(n,1)\left(\frac{\lambda}{\gamma^{1/q}}, \lambda\right) \\
&= \lambda^{n-1} \frac{\lambda}{\gamma^{1/q}} \left(1 + q \frac{\lambda^q}{\gamma} \sum_{i=1}^{n-1} b_i \lambda^{(i-1)q-1} \right)^{-1/q} = \frac{\lambda^n}{\left(\gamma + q \sum_{i=1}^{n-1} b_i \lambda^{iq-1}\right)^{1/q}}.
\end{aligned}$$

■

For even q, we immediately find a representation of x_n for negative starting points, by combining Proposition 2 and (4):

$$x_1 = -\frac{\lambda}{\gamma^{1/q}} \quad \Rightarrow \quad x_n = -\frac{\lambda^n}{\left(\gamma + q \sum_{i=1}^{n-1} b_i \lambda^{iq-1}\right)^{1/q}}, \quad n \in \mathbb{N}.$$

Note that in case q is odd and $x_1 = \frac{\lambda}{\gamma^{1/q}} < 0$, the representations (6) and (7) only hold true, as long as we do not approach the singularity, i.e.

$$q x_1^q \sum_{i=1}^{n-1} b_i \lambda^{(i-1)q-1} > -1$$

which is equivalent to

$$q \sum_{i=1}^{n-1} b_i \lambda^{iq-1} < -\gamma.$$

The inverse of the solution operator also has an explicit representation.

Proposition 3 *The evolution operator of (3) has for $n < m$ the explicit representation*

$$\Psi(n,m)(x,\lambda) = \frac{x}{\left(\lambda^{(m-n)q} - qx^q \sum_{i=n}^{m-1} b_i \lambda^{(i-n)q-1}\right)^{1/q}},$$

if $n < m$ and x are chosen such that the denominator is positive.

The next proposition states corresponding results for the variational equation.

Proposition 4 *Let x_n be the orbit, defined in (7), then the solution of the variational equation (5) with starting point $u_1 = \frac{\lambda}{\gamma^{1+1/q}}$ has the form*

$$u_n = \frac{\lambda^n}{\left(\gamma + q \sum_{i=1}^{n-1} b_i \lambda^{iq-1}\right)^{1+1/q}}, \quad n \in \mathbb{N}.$$

Furthermore, the solution operator is for $n \geq m \geq 1$ given by

$$\Phi(n,m)(\lambda) = \frac{\lambda^{n-m} \left(\gamma + q \sum_{i=1}^{m-1} b_i \lambda^{iq-1}\right)^{1+1/q}}{\left(\gamma + q \sum_{i=1}^{n-1} b_i \lambda^{iq-1}\right)^{1+1/q}}.$$

Proof: Similar to the proof of Proposition 2 the assertion follows by induction. ■

2.1 Connection between the Beverton-Holt equation and the model function

We demonstrate that the model function can be transformed in case $q = 1$ into the (non-autonomous) Beverton-Holt equation

$$f_n(x,\mu) := \frac{\mu K_n x}{K_n + (\mu - 1)x}, \quad n \in \mathbb{N}, \tag{8}$$

where $(K_n)_{n\in\mathbb{N}}$ is a give sequence of positive numbers. This function arises in population biology when studying population densities in a fluctuating environment, cf. [1, 2, 6].

In case $q = 1$, our model function has the form

$$g_n(x,\lambda) := \frac{\lambda x}{1 + \frac{b_n}{\lambda} x} \tag{9}$$

and by Proposition 2 the iterates with starting point $x_1 = \frac{\lambda}{\gamma}$ have the explicit representation

$$x_n = \frac{\lambda^n}{\gamma + \sum_{i=1}^{n-1} b_i \lambda^{i-1}}, \quad n \in \mathbb{N}. \tag{10}$$

Indeed, by choosing

$$b_n = \frac{(\mu - 1)\mu}{K_n} \quad \text{and} \quad \lambda = \mu,$$

the model function (9) is transformed into the Beverton-Holt equation (8). Using this approach, we find from (10) the following explicit solution of $y_{n+1} = f_n(y_n, \mu)$ with starting point $y_1 = \frac{\mu}{\gamma}$:

$$y_n = \frac{\mu^n}{\gamma + \sum_{i=1}^{n-1} \frac{(\mu-1)}{K_i} \mu^i}, \quad n \in \mathbb{N}.$$

Since γ can be chosen freely, each orbit of the Beverton-Holt equation is of this form.

3 The non-autonomous transcritical bifurcation

In this section, we show that in case $q = 1$ the model function (9) undergoes a non-autonomous transcritical bifurcation. Here we study two slightly different concepts for the transcritical bifurcation, developed in [8] for scalar ODEs and in [10] for maps.

First we define a non-autonomous transcritical bifurcation for maps, following the ideas of [8]. For this purpose we introduce the notion of pullback attraction.

Definition 5 *A trajectory $(y_n)_{n\in\mathbb{Z}}$ is called **pullback attracting** within its domain of attraction D, if*

$$\lim_{n\to-\infty} \operatorname{dist}(\Psi(m,n)K, y_m) = 0$$

for each $m \in \mathbb{Z}$ and every compact set $K \subset D$.
Here $\operatorname{dist}(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$ denotes the Hausdorff semi-distance.

*The trajectory $(y_n)_{n\in\mathbb{Z}}$ is called **locally pullback attracting** within D, if for every $m \in \mathbb{Z}$ there exists a $\delta(m) > 0$ such that if $K(\cdot) \subset D$ is compact and*

$$\lim_{n\to-\infty} \operatorname{dist}(K(n), y_n) < \delta(m)$$

then

$$\lim_{n\to-\infty} \operatorname{dist}(\Psi(m,n)K(n), y_m) = 0$$

holds.

Next, we state the definition of a non-autonomous discrete time transcritical bifurcation, analogously to the definition for ODEs, introduced in [8].

Definition 6 *The system $x_{n+1} = f_n(x_n, \lambda)$ undergoes a **local transcritical bifurcation** at $\bar{x} = 0$, $\lambda = \bar{\lambda}$, if there exist $\lambda_- < \bar{\lambda} < \lambda_+$ and an $\varepsilon > 0$ such that:*

- *For $\lambda \in (\lambda_-, \bar\lambda)$, $\bar x = 0$ is pullback attracting within $[0, \varepsilon)$ and locally pullback attracting within $(-\varepsilon, 0]$. Furthermore, there exists an asymptotically unstable trajectory $\bar x_n(\lambda)$ within $(-\varepsilon, 0]$ such that*

$$\bar x_n(\lambda) \to 0 \quad \text{as } \lambda \nearrow \bar\lambda.$$

- *For $\lambda = \bar\lambda$, the fixed point $\bar x = 0$ is asymptotically unstable but still pullback attracting within $[0, \varepsilon)$.*

- *For $\lambda \in (\bar\lambda, \lambda_+)$, $\bar x = 0$ is asymptotically unstable, and there exists a pullback attracting trajectory $\bar x_n(\lambda)$ within $(0, \varepsilon)$ satisfying*

$$\bar x_n(\lambda) \to 0 \quad \text{as } \lambda \searrow \bar\lambda.$$

The following theorem states a bifurcation result for our model function.

Theorem 7 *Assume that the sequence $(b_n)_{n\in\mathbb{Z}}$ is bounded from below and from above:*

$$0 < b_- \le b_n \le b_+ \quad \text{for all } n \in \mathbb{Z}. \tag{11}$$

Then the model function (2) exhibits in case $q = 1$ a non-autonomous transcritical bifurcation at the parameter $\bar\lambda = 1$.

Proof: Let $\lambda < \bar\lambda := 1$ and $m \in \mathbb{Z}$ be fixed. It holds

$$\begin{aligned} \lim_{n\to-\infty} \Psi(m,n)(x,\lambda) &= \lim_{n\to-\infty} \frac{\lambda^{m-n}x}{1 + x\sum_{i=n}^{m-1} b_i\lambda^{i-n-1}} \\ &= \lim_{n\to-\infty} \frac{\lambda^{m-n}x}{1 + x\sum_{i=0}^{m-1-n} b_{i+n}\lambda^{i-1}} \\ &= 0 \text{ for } \begin{cases} x \ge 0, \\ x < 0 \text{ such that } -\frac{1}{x} > b_+\sum_{i=0}^{\infty}\lambda^{i-1}. \end{cases} \end{aligned}$$

Thus, $\bar x = 0$ is for $\lambda < 1$ a locally pullback stable fixed point.

Furthermore, choosing for $\lambda < 1$ the starting point $x_1 = \frac{\lambda}{\gamma}$, where $\gamma = -\sum_{i=1}^{\infty} b_i\lambda^{i-1}$, we find by Proposition 2 the trajectory

$$\bar x_n^-(\lambda) = \frac{\lambda^n}{-\sum_{i=1}^{\infty} b_i\lambda^{i-1} + \sum_{i=1}^{n-1} b_i\lambda^{i-1}} = -\frac{1}{\sum_{i=0}^{\infty} b_{i+n}\lambda^{i-1}}. \tag{12}$$

This trajectory is unstable, since for $\varepsilon > 0$ and $x_1 = \frac{\lambda}{\gamma-\varepsilon}$ we get the trajectory

$$x_n = \frac{\lambda^n}{-\varepsilon - \sum_{i=1}^{\infty} b_i\lambda^{i-1} + \sum_{i=1}^{n-1} b_i\lambda^{i-1}} = \frac{1}{-\frac{\varepsilon}{\lambda^n} - \sum_{i=0}^{\infty} b_{i+n}\lambda^{i-1}}$$

which converges towards 0 as $n \to \infty$. Note that $\bar x_n^-(\lambda) \to 0$ as $\lambda \nearrow 1$.

For $\lambda = 1$, the fixed point $\bar{x} = 0$ is only in $[0, \varepsilon)$ pullback stable (with a polynomial rate of convergence). Let $m \in \mathbb{Z}$ be fixed and $x > 0$, then

$$\lim_{n\to-\infty} \Psi(m,n)(x,1) = \lim_{n\to-\infty} \frac{x}{1 + x\sum_{i=n}^{m-1} b_i} = \frac{1}{\frac{1}{x} + \sum_{i=-\infty}^{m-1} b_i} = 0.$$

For $\lambda > 1$ the fixed point $\bar{x}$ is asymptotically unstable. But there exists a second trajectory

$$\bar{x}_n^+(\lambda) := \frac{\lambda^n}{\sum_{i=-\infty}^{0} b_i\lambda^{i-1} + \sum_{i=1}^{n-1} b_i\lambda^{i-1}} = \frac{\lambda^n}{\sum_{i=-\infty}^{n-1} b_i\lambda^{i-1}} = \frac{1}{\sum_{i=-\infty}^{-1} b_{i+n}\lambda^{i-1}}, \quad (13)$$

which is pullback stable, because for fixed $m \in \mathbb{Z}$ and $x > 0$, we get

$$\begin{aligned} \lim_{n\to-\infty} \Psi(m,n)(x,\lambda) &= \lim_{n\to-\infty} \frac{\lambda^{m-n}}{1 + x\sum_{i=n}^{m-1} b_i\lambda^{i-n-1}} \\ &= \lim_{n\to-\infty} \frac{\lambda^m}{\frac{\lambda^n}{x} + \sum_{i=n}^{m-1} b_i\lambda^{i-1}} = \bar{x}_m^+(\lambda). \end{aligned}$$

Additionally, $\bar{x}_n^+(\lambda) \to 0$ as $\lambda \searrow 1$.

Therefore, the fixed point 0 undergoes, according to Definition 6, a transcritical bifurcation at the parameter $\bar{\lambda} = 1$.

Note that due to Proposition 3, the explicit representations of $\bar{x}_n^{\pm}(\lambda)$, introduced in (12) and (13), hold for all $n \in \mathbb{Z}$.

■

For an illustration of the stable and unstable trajectories, see Figure 1.

The author of [10] introduces an alternative characterization of the transcritical bifurcation. He detects bifurcations by a qualitative change of the domain of attraction from a trivial to a non-trivial object in the limit $\lambda \to \bar{\lambda}$. The all-time attraction radius of a trajectory $(\bar{x}_n(\lambda))_{n\in\mathbb{Z}}$ is defined as

$$\mathcal{A}_{\bar{x}}^{\lambda} := \sup\left\{\eta > 0 : \lim_{n\to\infty} \sup_{m\in\mathbb{Z}} |\Psi(m+n,m)(y,\lambda) - \bar{x}_{m+n}(\lambda)| = 0 \forall y \in B_\eta(\bar{x}_m(\lambda))\right\},$$

with the interval $B_\eta(x) = (x - \eta, x + \eta)$. Similarly, the all-time repulsive radius is

$$\mathcal{R}_{\bar{x}}^{\lambda} := \sup\left\{\eta > 0 : \lim_{n\to\infty} \sup_{m\in\mathbb{Z}} |\Psi(m-n,m)(y,\lambda) - \bar{x}_{m-n}(\lambda)| = 0 \forall y \in B_\eta(\bar{x}_m(\lambda))\right\}.$$

The change of the attraction-radii is a necessary condition for the occurrence of a non-autonomous bifurcation:

Proposition 8 *Assume the sequence $(b_n)_{n\in\mathbb{Z}}$ to be bounded, cf. (11). Then we get for the all-time attraction and the all-time repulsive radius of the model function in case $q = 1$*

$$\lim_{\lambda\nearrow\bar{\lambda}} \mathcal{A}_0^{\lambda} = 0 \quad \textit{and} \quad \lim_{\lambda\searrow\bar{\lambda}} \mathcal{R}_0^{\lambda} = 0.$$

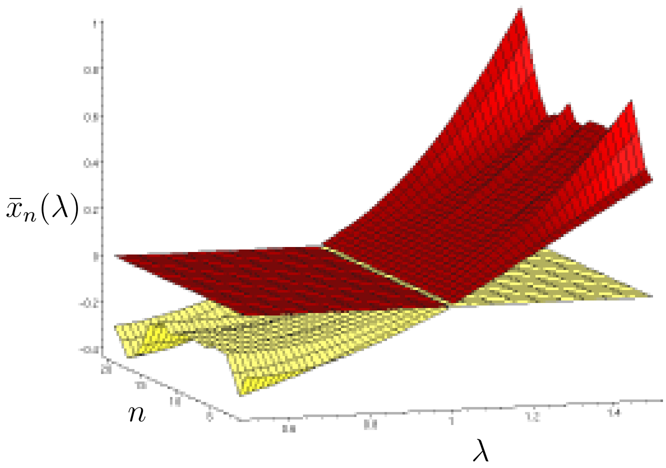

Figure 1: *Pullback attracting trajectories (in red) and asymptotically unstable trajectories (in yellow) of the model function (9), plotted over* λ *and* n*. The sequence* $(b_n)_{n\in\mathbb{Z}} \in [\frac{1}{2}, \frac{3}{2}]^{\mathbb{Z}}$ *is chosen randomly.*

Proof: For our model function, the zero-solution is all-time attractive for $\lambda \in (\lambda_-, \bar{\lambda})$ and all-time repulsive for $\lambda \in (\bar{\lambda}, \lambda_+)$.

For $\lambda < 1$, we obtain using Proposition 2 for $m \in \mathbb{Z}$ and $n > 0$

$$\Psi(m+n,m)(y,\lambda) = \frac{\lambda^{m+n-m}}{1 + y\sum_{i=m}^{n+m-1} b_i \lambda^{i-m-1}} = \frac{\lambda^n}{1 + y\sum_{i=0}^{n-1} b_{i+m}\lambda^{i-1}}. \tag{14}$$

This expression converges towards 0 as $n \to \infty$ as long as y lies above the unstable trajectory $\bar{x}_m^-(\lambda)$, i.e.

$$y > \bar{x}_m^-(\lambda) = -\frac{1}{\sum_{i=0}^{\infty} b_{i+m}\lambda^{i-1}} \quad \text{for all } m \in \mathbb{Z}.$$

Note that this condition guarantees that the denominator in (14) is positive. Thus

$$\mathcal{A}_0^\lambda = \inf_{m\in\mathbb{Z}} \frac{1}{\sum_{i=0}^{\infty} b_{i+m}\lambda^{i-1}}.$$

Obviously, $\mathcal{A}_0^\lambda \to 0$ as $\lambda \nearrow 1$.

Applying Proposition 3 for $n > 0$, $m \in \mathbb{Z}$ and $\lambda > 1$, we get

$$\Psi(m-n,m)(y,\lambda) = \frac{y}{\lambda^{m-m+n} - y\sum_{i=m-n}^{m-1} b_i\lambda^{i-m+n-1}} = \frac{\lambda^{-n}y}{1 - y\sum_{i=-n}^{-1} b_{i+m}\lambda^{i-1}}.$$

If y lies below the stable trajectory $\bar{x}_m^+(\lambda)$, i.e.

$$y < \bar{x}_m^+(\lambda) = \frac{1}{\sum_{i=-\infty}^{-1} b_{i+m}\lambda^{i-1}} \quad \text{for all } m \in \mathbb{Z},$$

$\Psi(m-n,m)(y,\lambda)$ converges towards 0 as $n \to \infty$. It follows

$$\mathcal{R}_0^\lambda = \inf_{m\in\mathbb{Z}} \frac{1}{\sum_{i=-\infty}^{-1} b_{i+m}\lambda^{i-1}}$$

and $\mathcal{R}_0^\lambda \to 0$ as $\lambda \searrow 1$.

$\blacksquare$

Furthermore, [10, Theorem 5.1] states technical estimates on the linear, quadratic and high-order terms of (9) to determine the specific nature of the bifurcation. These estimates can be verified for our model function (9) and guarantee the occurrence of a non-autonomous transcritical bifurcation.

4 The non-autonomous pitchfork bifurcation

The next bifurcation to consider is the non-autonomous pitchfork bifurcation. We show that our model function exhibits this bifurcation in case $q = 2$.

Analogously to the definition for ODEs introduced in [8], we define this bifurcation for discrete time systems.

Definition 9 *The system $x_{n+1} = f_n(x_n, \lambda)$ undergoes a **local pitchfork bifurcation** at $\bar{x} = 0$, $\lambda = \bar{\lambda}$, if there exist $\lambda_- < \bar{\lambda} < \lambda_+$ and an $\varepsilon > 0$ such that:*

- *For all $\lambda \in (\lambda_-, \bar{\lambda}]$ the fixed point $\bar{x} = 0$ is pullback attracting within $(-\varepsilon, \varepsilon)$.*
- *For $\lambda \in (\bar{\lambda}, \lambda_+)$, the fixed point $\bar{x} = 0$ is asymptotically unstable. Furthermore, there exist two bounded trajectories $\bar{x}_n^+(\lambda)$ and $\bar{x}_n^-(\lambda)$ that are pullback attracting in $(0, \varepsilon)$ and $(-\varepsilon, 0)$, respectively. These trajectories converge uniformly on finite intervals of $\mathbb{Z}$:*

$$\bar{x}_n^\pm(\lambda) \to 0 \quad \text{as } \lambda \searrow \bar{\lambda}.$$

In case $q = 2$, the model function is defined as

$$g_n(x,\lambda) = \frac{\lambda x}{\sqrt{1 + \frac{2b_n}{\lambda}x^2}} \tag{15}$$

and its Taylor series at 0 is given by

$$g_n(x,\lambda) = \lambda x - b_n x^3 + \frac{3b_n^2}{2\lambda}x^5 + \mathcal{O}(x^7). \tag{16}$$

The iterates x_n with starting point $x_1 = \frac{\lambda}{\sqrt{\gamma}}$, $\gamma > 0$ have by Proposition 2 the form

$$x_n = \frac{\lambda^n}{\sqrt{\gamma + 2\sum_{i=1}^{n-1} b_i \lambda^{2i-1}}}, \quad n \in \mathbb{N}.$$

We obtain the following bifurcation result for the model function (15).

Theorem 10 *Assume the sequence $(b_n)_{n\in\mathbb{Z}}$ to be bounded, cf. (11). Then the model function (15) exhibits a non-autonomous pitchfork bifurcation at the parameter $\bar{\lambda} = 1$.*

Proof: Obviously, $\bar{x} = 0$ is a fixed point of $g_n(\cdot, \lambda)$ for all λ.

For $\lambda \leq 1$, the fixed point $\bar{x}$ is pullback attracting, since we get for fixed $m \in \mathbb{Z}$

$$\begin{aligned}\lim_{n\to-\infty} \Psi(m,n)(x,\lambda) &= \lim_{n\to-\infty} \frac{\lambda^{m-n}x}{\sqrt{1+2x^2\sum_{i=n}^{m-1} b_i\lambda^{(i-n)2-1}}} \\ &= \lim_{n\to-\infty} \frac{\lambda^{m}x}{\sqrt{\lambda^{2n}+2x^2\sum_{i=n}^{m-1} b_i\lambda^{2i-1}}} \\ &= 0.\end{aligned}$$

For $\lambda > 1$ the fixed point $\bar{x}$ is asymptotically unstable, but two new pullback attracting trajectories

$$\bar{x}_n^{\pm}(\lambda) := \pm\frac{\lambda^n}{\sqrt{2\sum_{i=-\infty}^{n-1} b_i\lambda^{2i-1}}} = \pm\frac{1}{\sqrt{2\sum_{i=-\infty}^{-1} b_{i+n}\lambda^{2i-1}}}, \qquad n \in \mathbb{Z}$$

are born at the critical parameter $\bar{\lambda}$. It holds for fixed $n \in \mathbb{Z}$

$$\begin{aligned}\lim_{m\to-\infty} \Psi(n,m)x &= \lim_{m\to-\infty} \frac{\lambda^{n-m}x}{\sqrt{1+2x^2\sum_{i=m}^{n-1} b_i\lambda^{(i-m)2-1}}} \\ &= \frac{\operatorname{sign}(x)}{\sqrt{2\sum_{i=-\infty}^{-1} b_{i+n}\lambda^{2i-1}}} \\ &= \bar{x}_n^{\operatorname{sign}(x)}(\lambda).\end{aligned}$$

Note that due to our assumption (11), $\lim_{\lambda\searrow 1} \bar{x}_n^{\pm}(\lambda) = 0$ uniformly on bounded intervals of $\mathbb{Z}$.

Thus, by Definition 9 our model function exhibits in case $q = 2$ a non-autonomous pitchfork bifurcation.

■

An illustration of the stable and unstable trajectories is shown in Figure 2.

In the language of [10] the model function undergoes a bifurcation, since for the all-time repulsion radius we get

$$\lim_{\lambda\searrow\bar{\lambda}} \mathcal{R}_0^{\lambda} = 0.$$

Furthermore, one can verify the conditions, given in [10, Theorem 6.1] on the linear, cubic and remaining part of (16), to assure that this function undergoes a pitchfork bifurcation.

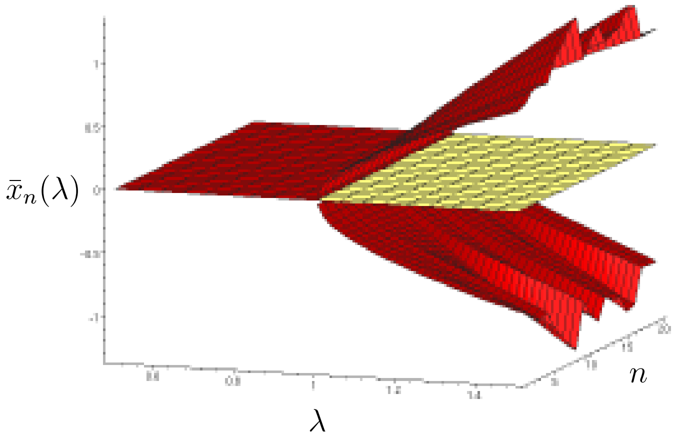

Figure 2: *Pullback attracting trajectories (in red) and asymptotically unstable trajectories (in yellow) of the model function (15), plotted over* λ *and* n*. The sequence* $(b_n)_{n\in\mathbb{Z}} \in [\frac{1}{2}, \frac{3}{2}]^{\mathbb{Z}}$ *is chosen randomly.*

5 A model function with a non-autonomous linear part

In this section we introduce a second model function

$$h_n(x) := \frac{b_n x}{1 + b_n x}, \tag{17}$$

which at 0 has the Taylor expansion

$$h_n(x) = b_n x - b_n^2 x^2 + b_n^3 x^3 + \mathcal{O}(x^4).$$

Here the sequence $(b_n)_{n\in\mathbb{N}}$ is given such that $b_n > 0$ for all $n \in \mathbb{N}$. In contrast to the model function, introduced in Section 2, also the linear part of h_n depends on the sequence $(b_n)_{n\in\mathbb{N}}$. Nevertheless, it is possible to state explicit formulae for the orbit and the solution operator, analogously to Proposition 2 and 4.

For a special parameter-dependent choice of $(b_n)_{n\in\mathbb{Z}}$, we show at the end of this section that the model function (17) also exhibits a transcritical bifurcation, according to Definition 6.

We first derive explicit expressions for the orbit

$$x_{n+1} = h_n(x_n), \quad n \in \mathbb{Z} \tag{18}$$

and for the corresponding evolution operator, cf. Definition 1. Similar to Proposition 2, one can prove the following proposition inductively.

Proposition 11 *The evolution operator of (18) has for $n \geq m$ the explicit representation*

$$\Psi(n,m)(x) = \frac{x\prod_{i=m}^{n-1} b_i}{1 + x\sum_{i=m}^{n-1}\prod_{j=m}^{i} b_j},$$

where $n \geq m$ and x have to be chosen, such that the denominator is positive. The orbit with starting point $x_1 = \frac{1}{\gamma}$ has for $n \in \mathbb{N}$ the explicit form

$$x_n = \frac{\prod_{i=1}^{n-1} b_i}{\gamma + \sum_{i=1}^{n-1}\prod_{j=1}^{i} b_j}. \tag{19}$$

Next, we introduce the explicit form for the solution of the variational equation

$$u_{n+1} = Dh(x_n)u_n = \frac{b_n}{(1 + b_n x_n)^2}u_n, \quad n \in \mathbb{N} \tag{20}$$

and for its solution operator Φ.

Proposition 12 *Let x_n be the orbit, defined in (19), then the solution of the variational equation (20) with starting point $u_1 = \frac{1}{\gamma^2}$ has the form*

$$u_n = \frac{\prod_{i=1}^{n-1} b_i}{\left(\gamma + \sum_{i=1}^{n-1}\prod_{j=1}^{i} b_j\right)^2}, \quad n \in \mathbb{N}.$$

Furthermore, the solution operator is for $n \geq m \geq 1$ given by

$$\Phi(n,m) = \frac{\prod_{\ell=m}^{n-1} b_\ell \left(\gamma + \sum_{i=1}^{m-1}\prod_{j=1}^{i} b_j\right)^2}{\left(\gamma + \sum_{i=1}^{n-1}\prod_{j=1}^{i} b_j\right)^2}.$$

For the forthcoming bifurcation analysis, we assume that the sequence $(b_n(\lambda))_{n\in\mathbb{Z}}$ depends on the parameter λ in the following sense. For each $\lambda \in [0,2]$, assume $b_n(\lambda) > 0$ for all $n \in \mathbb{Z}$ and

$$\lim_{n\to\infty}\prod_{i=m}^{m+n}\frac{b_i(\lambda)}{\lambda} = 1 \tag{21}$$

holds uniformly for all $m \in \mathbb{Z}$.

Theorem 13 *Assume (21). Then the system*

$$x_{n+1} = h_n(x_n, \lambda), \quad n \in \mathbb{Z},$$

where h_n is defined as

$$h_n(x,\lambda) = \frac{b_n(\lambda)x}{1 + b_n(\lambda)x}$$

exhibits a non-autonomous transcritical bifurcation at the parameter $\bar{\lambda} = 1$.

Proof: Obviously, $\bar{x} = 0$ is a fixed point of $h_n(\cdot, \lambda)$. This fixed point is locally pullback attracting for $\lambda < 1$ and asymptotically unstable for $\lambda \geq 1$.

Furthermore, for $\lambda < 1$

$$\bar{x}_n^-(\lambda) := -\frac{1}{\sum_{i=0}^{\infty} \prod_{j=0}^{i} b_{j+n}(\lambda)}, \quad n \in \mathbb{Z}$$

defines an unstable trajectory. Note that $\bar{x}_n^-(\lambda) \to 0$ as $\lambda \nearrow 1$.

For $\lambda > 1$

$$\bar{x}_n^+(\lambda) := \frac{1}{\sum_{i=-\infty}^{n-1} \prod_{j=i+1}^{n-1} \frac{1}{b_j(\lambda)}}, \quad n \in \mathbb{Z}$$

is a pullback attracting trajectory and $\bar{x}_n^+(\lambda) \to 0$ as $\lambda \searrow 1$. Thus, according to Definition 6, the system

$$x_{n+1} = h_n(x_n, \lambda)$$

undergoes a transcritical bifurcation at the parameter $\bar{\lambda} = 1$.

■

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References

[1] S. Elaydi and R. J. Sacker. Global stability of periodic orbits of non-autonomous difference equations and population biology. *J. Differential Equations*, 208(1):258–273, 2005.

[2] C. Haskell and R. J. Sacker. The stochastic Beverton-Holt equation and the M. Neubert conjecture. *J. Dynam. Differential Equations*, 17(4):825–844, 2005.

[3] T. Hüls. A model function for polynomial rates in discrete dynamical systems. *Applied Mathematics Letters*, 17(1):1–5, 2004.

[4] Y. Ilyashenko and W. Li. *Nonlocal Bifurcations.* American Mathematical Society, Providence, RI, 1999.

[5] P. E. Kloeden and S. Siegmund. Bifurcations and continuous transitions of attractors in autonomous and nonautonomous systems. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 15(3):743–762, 2005.

[6] V. L. Kocic. A note on the nonautonomous Beverton-Holt model. *J. Difference Equ. Appl.*, 11(4-5):415–422, 2005.

[7] Y. A. Kuznetsov. *Elements of Applied Bifurcation Theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, New York, third edition, 2004.

[8] J. A. Langa, J. C. Robinson, and A. Suárez. Bifurcations in non-autonomous scalar equations. *To appear in: Journal of Differential Equations.*

[9] J. A. Langa, J. C. Robinson, and A. Suárez. Stability, instability, and bifurcation phenomena in non-autonomous differential equations. *Nonlinearity*, 15(3):887–903, 2002.

[10] M. Rasmussen. Towards a bifurcation theory for nonautonomous difference equations. *To appear in Journal of Difference Equations and Applications*, 2005.

[11] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, volume 2 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 2003.