Homoclinic Orbits of Non-Autonomous Maps and their Approximation

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Abstract

We consider homoclinic orbits in non-autonomous discrete time dynamical systems of the form

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z},$$

where it is assumed that an n independent fixed point exists. A numerical method for computing finite approximations of transversal homoclinic orbits is introduced and a detailed error analysis is presented. The non-autonomous setup requires special tools. We prove that the analytic condition of transversality of the orbit corresponds to a transversal intersection of the corresponding invariant fiber bundles. The approximation method and the validity of the error estimate is illustrated by an example.

Keywords: Non-autonomous discrete time dynamical systems, Non-autonomous homoclinic orbits, Invariant fiber bundles, Transversality, Numerical approximation, Approximation error.

1 Introduction

A milestone in the development of dynamical systems was the famous theorem of Smale [16], stating that the dynamics in a neighborhood of a homoclinic orbit is chaotic. This result initiated further studies on dynamical systems, cf. [11] for a historical overview on the study of homoclinic phenomena. Approximation results

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for homoclinic orbits are developed in [2] and [5] for continuous and discrete time dynamical systems, respectively. In [10] a degenerate case is considered, where the stable and unstable manifold intersect tangentially. For heteroclinic orbits with non-hyperbolic end points, approximation and bifurcation results are stated in [8, 3] and [9]. Finally, an overview on numerical methods and examples for degenerate connecting orbits is presented in [4].

In several "real" systems, coming, for example, from applications in physics or mathematical biology the limitation to autonomous systems is too restrictive. In general, these systems are non-autonomous. The generalization of autonomous results to the non-autonomous context and the development of new non-autonomous methods, having no autonomous analog, is an important area of research.

This paper contributes to this task by proving an approximation theorem for homoclinic orbits in non-autonomous discrete time dynamical systems of the form

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}. \tag{1}$$

We assume that 0 is a fixed point for all n, cf. [1, 15, 14], in order to apply the theory of invariant fiber bundles. A homoclinic orbit $\bar{x}_{\mathbb{Z}} = (\bar{x}_n)_{n\in\mathbb{Z}}$ with respect to this fixed point is a solution of (1) fulfilling $\lim_{n\to\pm\infty} \bar{x}_n = 0$. Furthermore, the functions $(f_n)_{n\in\mathbb{Z}}$ are assumed to be *closely related* in the sense that the variational equation

$$u_{n+1} = Df_n(\bar{x}_n)u_n, \quad n \in \mathbb{Z}$$

possesses an exponential dichotomy. The precise assumptions are stated in Section 2.

In Section 3, we show that transversality of the orbit, see Definition 1, corresponds to a transversal intersection of the corresponding fiber bundles. Then we have all tools at hand to prove an approximation theorem for non-autonomous homoclinic orbits, providing precise error estimates. The main idea is to introduce a boundary value problem, the solution of which is a finite approximation of the homoclinic orbit. More precisely, we introduce in Section 4 the operator

$$\Gamma_J(x_J) = \left(\left(x_{n+1} - f_n(x_n)\right)_{n=n_-,\dots,n_+-1}, b(x_{n_-}, x_{n_+}) \right),$$

where b is an appropriately chosen boundary operator, and compute a finite approximation of $\bar{x}_{\mathbb{Z}}$ by solving

$$\Gamma_J(x_J) = 0. \tag{2}$$

For the numerical computations in Section 5, we solve (2), applying Newton's method. Note that the Newton matrix

$$\begin{pmatrix} -Df_{n_-}(x_{n_-}) & I & & & \\ & \ddots & & \ddots & \\ & & \ddots & & \ddots \\ & & & -Df_{n_+-1}(x_{n_+-1}) & I \\ D_1 b(x_{n_-}, x_{n_+}) & 0 & \dots & 0 & D_2 b(x_{n_-}, x_{n_+}) \end{pmatrix}$$

has a sparse structure. Employing this structure allows very efficient numerical computations, also of very long orbit segments.

For an illustration, we use Hénon's map $h(x, a, b)$ in Section 5, shifted such that 0 is a fixed point for all a, b. We construct the non-autonomous family in the form $f_n(\cdot) = h(\cdot, a_n, 0.3)$, where $a_{\mathbb{Z}}$ is a random sequence. For the resulting non-autonomous discrete time dynamical system, we compute a solution of (2) numerically and illustrate the validity of our estimates for the approximation error.

2 Basic setup

First, we introduce our basic assumptions to guarantee that the linearized system possesses an exponential dichotomy. We refer to Appendix A.1 for a short introduction of exponential dichotomies.

Consider the non-autonomous discrete time dynamical system

$$x_{n+1} = f_n(x_n), \quad x_n \in \mathbb{R}^k \text{ for all } n \in \mathbb{Z}, \tag{3}$$

and demand the following assumptions to be satisfied.

A1 Let $f_n \in \mathcal{C}^\infty(\mathbb{R}^k, \mathbb{R}^k)$ be a diffeomorphism for all $n \in \mathbb{Z}$.

A2 Let $f_n(0) = 0$ for all $n \in \mathbb{Z}$.

Let $J = [n_-, n_+] \cap \mathbb{Z}$ be a discrete interval, where the cases $n_- = -\infty$ and $n_+ = \infty$ are included. We denote the space of bounded sequences on J w.r.t. $\|\cdot\|$ by

$$X_J := \left\{ u_J = (u_n)_{n\in J} \in (\mathbb{R}^k)^J : \sup_{n\in J} \|u_n\| < \infty \right\}.$$

Definition 1 *A* ***homoclinic orbit*** *is a solution $\bar{x}_{\mathbb{Z}}$ of (3), satisfying*

$$\lim_{n\to\pm\infty} \bar{x}_n = 0.$$

Each point $\bar{x}_n$ is called a ***homoclinic point****.*

The orbit $\bar{x}_{\mathbb{Z}}$ is ***transversal*** *if*

$$u_{n+1} = Df_n(\bar{x}_n)u_n, \ n \in \mathbb{Z} \text{ for } u_{\mathbb{Z}} \in X_{\mathbb{Z}} \quad \Longleftrightarrow \quad u_{\mathbb{Z}} = 0.$$

A3 A homoclinic orbit $\bar{x}_{\mathbb{Z}}$ of (3) exists.

A4 Let the homoclinic orbit $\bar{x}_{\mathbb{Z}}$ be transversal according to Definition 1.

A5 The matrix $A := Df_0(0)$ is hyperbolic.

By assumption **A5**, the linear difference equation

$$u_{n+1} = Df_0(0)u_n, \quad n \in \mathbb{Z} \tag{4}$$

possesses an exponential dichotomy on $\mathbb{Z}$ with data $(1, \beta, P^s, P^u)$, cf. Appendix A.1, Definition 15. In order to prove an exponential dichotomy for the *variational equation*, associated to (3)

$$u_{n+1} = Df_n(\bar{x}_n)u_n, \quad n \in \mathbb{Z} \tag{5}$$

further assumptions on f_n, $n \in \mathbb{Z}$ have to be required.

A6 Let $B_n := Df_n(0) - Df_0(0)$ and

$$\|B_n\| \leq \min\left\{\frac{1}{\|A^{-1}\|}, \ \frac{1}{4}\frac{1-\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}}, \ \frac{1}{2}\mathrm{e}^{-\beta}\frac{e^{\delta}-1}{\mathrm{e}^{-\delta}+1}\right\}$$

for all $n \in \mathbb{Z}$ and a $0 < \delta < \beta$.

Furthermore, we state the following uniform Lipschitz condition.

A7 A neighborhood V of 0 and a constant $L > 0$ exist, such that

$$\|Df_n(x) - Df_n(y)\| \leq L\|x - y\|, \quad \forall x, y \in V$$

holds uniformly for all $n \in \mathbb{Z}$.

Lemma 2 *Assume **A1**–**A3**, **A5**–**A7**. Then the difference equation*

$$u_{n+1} = Df_n(0)u_n, \quad n \in \mathbb{Z} \tag{6}$$

possesses an exponential dichotomy on $\mathbb{Z}$ with data $(K, \alpha, P_n^s, P_n^u)$, where

$$\|P^s - P_n^s\| \leq \frac{1}{2} \quad \textit{for all } n \in \mathbb{Z}.$$

Furthermore, the variational equation (5) has an exponential dichotomy on $\mathbb{Z}^-$ with data $(K^-, \alpha^-, Q_n^{-s}, Q_n^{-u})$ and on $\mathbb{Z}^+$ with data $(K^+, \alpha^+, Q_n^{+s}, Q_n^{+u})$, with projectors of equal rank and

$$\lim_{n\to\pm\infty} \|P_n^s - Q_n^{\pm s}\| = 0. \tag{7}$$

Proof: By assumption **A5**, (4) has an exponential dichotomy. Applying the Roughness-Theorem 16, we get by assumption **A6** an exponential dichotomy for the perturbed equation

$$u_{n+1} = (Df_0(0) + B_n)u_n = Df_n(0)u_n$$

and estimates for the projectors follow from (39):

$$\|P^s - P_n^s\| \leq 2\frac{1+\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}} \sup_{m\in\mathbb{Z}} \|B_m\| \leq 2\frac{1+\mathrm{e}^{-\beta}}{1-\mathrm{e}^{-\beta}} \ \frac{1}{4} \ \frac{1-\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}} = \frac{1}{2}.$$

Since we have $\bar{x}_n \to 0$ as $n \to \pm\infty$ due to assumption **A3**, and

$$\|Df_n(0) - Df_n(\bar{x}_n)\| \leq L\|\bar{x}_n\| \to 0 \quad \text{as } n \to \pm\infty$$

by assumption **A7**, the variational equation (5) possesses an exponential dichotomy on $\mathbb{Z}^-$ and on $\mathbb{Z}^+$ with projectors of equal rank fulfilling (7). This is a consequence of the Roughness-Theorem 16, cf. [5, Proposition 2.5].

■

We introduce the solution operator for the difference equation (3) as well as for the associated variational equation (5).

Definition 3 *The **solution operator of (3)** is defined as*

$$\Psi(n,m)(x) := \begin{cases} f_{n-1} \circ \ldots \circ f_m(x), & \textit{for } n > m, \\ x, & \textit{for } n = m, \\ f_n^{-1} \circ \ldots \circ f_{m-1}^{-1}(x), & \textit{for } n < m. \end{cases}$$

*Similarly, we define the **solution operator of the variational equation** $u_{n+1} = Df_n(x_n)u_n$ along a solution $x_\mathbb{Z}$ of (3):*

$$\Phi(n,m) := \begin{cases} Df_{n-1}(x_{n-1}) \cdot \ldots \cdot Df_m(x_m), & \textit{for } n > m, \\ I, & \textit{for } n = m, \\ \big(Df_n(x_n)\big)^{-1} \cdot \ldots \cdot \big(Df_{m-1}(x_{m-1})\big)^{-1}, & \textit{for } n < m. \end{cases}$$

Note that these solution operators are closely related. Let $x_\mathbb{Z}$ be a solution of (3) then

$$D\Psi(n,m)(x_m) = \Phi(n,m)$$

holds for all n, $m \in \mathbb{Z}$.

3 Transversality

In an autonomous version of (3), i.e. $f_n = f$ for all $n \in \mathbb{Z}$, a homoclinic orbit lies in the intersection of the stable and the unstable manifold of the fixed point 0. It is well known (cf. [12]) that this orbit is transversal according to Definition 1, if these manifolds intersect transversally.

In this section, we generalize this result to our non-autonomous context.

3.1 Invariant fiber bundles

First we introduce the non-autonomous equivalent of the autonomous invariant manifold – the so called invariant fiber bundles, cf. [13].

Definition 4 *The* ***stable and unstable global fiber bundles*** *of the fixed point* 0 *are defined as*

$$\boldsymbol{S}^{\pm} := \left\{(n,x) \in \mathbb{Z} \times \mathbb{R}^k : \lim_{m\to\pm\infty} \Psi(m,n)(x) = 0\right\}$$

and the ***global stable and unstable n-fibers*** *are*

$$\boldsymbol{S}_n^{\pm} := \{x \in \mathbb{R}^k : (n,x) \in \boldsymbol{S}^{\pm}\}. \tag{8}$$

Furthermore, the ***local stable and unstable fiber bundles*** *w.r.t. a neighborhood* $U \subset \mathbb{R}^k$ *of* 0 *are defined as*

$$\begin{aligned} S^+ &= \{(n,x) \in \boldsymbol{S}^+ : \Psi(m,n)(x) \in U \text{ for all } m \geq n\}, \\ S^- &= \{(n,x) \in \boldsymbol{S}^- : \Psi(m,n)(x) \in U \text{ for all } m \leq n\}. \end{aligned}$$

The ***local n-fibers*** *are defined similar to (8).*

Next, we introduce a local graph representation for invariant fiber bundles.

Theorem 5 *Let* $P_n^{s,u}$ *be the dichotomy projectors of (6).*

Then an open, convex neighborhood U *of* 0 *and functions* $s^{\pm} : U \times \mathbb{Z} \to \mathbb{R}^k$ *that are continuously differentiable w.r.t. the first argument exist, fulfilling*

$$\begin{aligned} s^{\pm}(n,0) &= 0 \quad \text{for all } n \in \mathbb{Z}, \\ \lim_{x\to 0} D_1 s^{\pm}(n,x) &= 0 \quad \text{uniformly for all } n \in \mathbb{Z}, \\ s^+(n,x) &= s^+(n,P_n^s x) \in \mathcal{R}(P_n^u) \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in U, \\ s^-(n,x) &= s^-(n,P_n^u x) \in \mathcal{R}(P_n^s) \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in U \end{aligned}$$

such that the following local graph representations

$$\begin{aligned} S^+ &= \left\{(n, x + s^+(n,x)) \in \mathbb{Z} \times \mathbb{R}^k : x \in \mathcal{R}(P_n^s) \cap U\right\}, \\ S^- &= \left\{(n, x + s^-(n,x)) \in \mathbb{Z} \times \mathbb{R}^k : x \in \mathcal{R}(P_n^u) \cap U\right\} \end{aligned}$$

of locally stable and unstable fiber bundles hold. Furthermore the invariance condition

$$(n_0, x_0) \in S^{\pm} \Rightarrow \big(n, \Psi(n,n_0)(x_0)\big) \in S^{\pm} \tag{9}$$

is satisfied for all $n \in \mathbb{Z}$ *as long as* $\Psi(n,n_0)(x_0) \in U$.

The n-fibers of $S^{\pm}$ *have the local graph representation*

$$\begin{aligned} S_n^+ &:= \left\{h_n^+(x) : x \in \mathcal{R}(P_n^s) \cap U\right\}, \\ S_n^- &:= \left\{h_n^-(x) : x \in \mathcal{R}(P_n^u) \cap U\right\}, \end{aligned} \tag{10}$$

where $h_n^{\pm} : \mathcal{R}(P_n^{s,u}) \to \mathbb{R}^k$, $h_n^{\pm}(x) = x + s^{\pm}(n,x)$.

Finally, the global n-fibers are given by

$$\boldsymbol{S}_n^+ = \bigcup_{n\leq m} \Psi(n,m) S_m^+, \quad \boldsymbol{S}_n^- = \bigcup_{n\geq m} \Psi(n,m) S_m^-.$$

For a proof and for existence results of invariant fiber bundles in a sufficiently small convex neighborhood U, we refer to [15], [13] and [14].

Note that the invariance condition (9) for the n-fiber is

$$x_0 \in S^{\pm}_{n_0} \quad \Rightarrow \quad \Psi(n,n_0)(x_0) \in S^{\pm}_n \tag{11}$$

for all $n \in \mathbb{Z}$ as long as $\Psi(n,n_0)(x_0) \in U$.

We show how tangent spaces of fiber bundles transform into each other by the linearized flow. Let $T_y S$ be the tangent space of the fiber S at the point y.

Lemma 6 *Assume **A1**–**A3**, **A5**–**A7** and let $y_0 \in S^+_0$, $y_n = \Psi(n,0)(y_0) \in U$. Denote by Φ the solution operator of the associated variational equation. Then*

$$T_{y_0} S^+_0 = \Phi(0,n) T_{y_n} S^+_n.$$

Proof: Let $y_0 \in S^+_0$ and $\Psi(n,0)(y_0) \in S^+_n \cap U$. Using the local graph representation of local stable fiber bundle, cf. Theorem 5, an $\eta \in \mathcal{R}(P^s_n)$ exists, such that $y_n = \Psi(n,0)(y_0) = h^+_n(\eta)$. Note that the map $\Psi(0,n) \circ h^+_n : \mathcal{R}(P^s_n) \to S^+_0$ is continuously differentiable. Thus

$$\begin{aligned} T_{y_0} S^+_0 &= \mathcal{R}\left(D\big[\Psi(0,n) \circ h^+_n(\eta)\big]\right) \\ &= D\Psi(0,n)(y_n)\mathcal{R}\big(Dh^+_n(\eta)\big) \\ &= \Phi(0,n) T_{y_n} S^+_n. \end{aligned}$$

■

3.2 Transversally intersecting fiber bundles

In Theorem 8 we prove that the tangent space of the stable n-fiber consists of those points $u \in \mathbb{R}^k$ that stay bounded under forward iteration with respect to the variational equation (5), i.e. $\|\Phi(m,n)u\| < \infty$ for all $m \geq n$, cf. [12, Proposition 5.4] for the autonomous counterpart. First, we prove a technical but important lemma, generalizing the ideas of [12, Section 5].

Proposition 7 *Assume **A1**–**A3**, **A5**–**A7**. Consider the matrix valued initial value problem*

$$Z_{n+1} = Df_n\big(\Psi(n,N)(\bar{x}_N)\big) Z_n, \quad n \geq N, \tag{12}$$

$$P^s_N Z_N = P^s_N, \tag{13}$$

where U is an open, convex neighborhood of 0, given as in Theorem 5. Denote by $N \in \mathbb{N}$ a sufficiently large constant, chosen such that $\bar{x}_N \in U$. Let $\bar{x}_N = h^+_N(\eta)$, $\eta \in \mathcal{R}(P^s_N)$, where h^+_N is defined as in Theorem 5.

Then the boundary value problem (12), (13) possesses a unique bounded solution

$$\bar{Z}_n = D\big(\Psi(n,N) \circ h^+_N(\eta)\big), \quad n \geq N. \tag{14}$$

Proof: Obviously, $\bar{Z}_n = D\big(\Psi(n,N) \circ h_N^+(\eta)\big)$ is for $n \geq N$ a solution of (12), (13). On the other hand, Lemma 18 guarantees the existence and uniqueness of a bounded solution Y_n, $n \geq N$ of (12), (13). For establishing boundedness of $\bar{Z}_n$, we have to prove $\bar{Z}_n = Y_n$, for all $n \geq N$, which follows from

$$\Psi(n,N) \circ h_N^+(\eta + \ell) - \Psi(n,N) \circ h_N^+(\eta) - Y_n \ell = \mathcal{O}(\|\ell\|^2). \tag{15}$$

We carry out these computations in three steps.

Step 1: Let $x_n(\eta) := \Psi(n,N) \circ h_N^+(\eta)$, then

$$\|x_n(\eta_1) - x_n(\eta_2)\| \leq 2K\|\eta_1 - \eta_2\| \quad \text{for all } \eta_1, \eta_2 \in \mathcal{R}(P_N^s) \cap U. \tag{16}$$

Step 2: For $n \geq N$

$$\begin{aligned} w_n &:= \Psi(n,N) \circ h_N^+(\eta + \ell) - \Psi(n,N) \circ h_N^+(\eta) - Y_n \ell \\ &= x_n(\eta + \ell) - x_n(\eta) - Y_n \ell \end{aligned} \tag{17}$$

is a solution of

$$w_{n+1} = Df_n(0) w_n + r_n(\eta, \ell) \tag{18}$$

with initial condition

$$P_N^s w_N = 0, \tag{19}$$

where

$$r_n(\eta, \ell) = g_n(x_n(\eta + \ell)) - g_n(x_n(\eta)) - Dg_n(x_n(\eta)) Y_n \ell, \tag{20}$$

$$g_n(x) = f_n(x) - Df_n(0)x. \tag{21}$$

Step 3: For $n \geq N$, we get the estimate $\|w_n\| \leq C\|\ell\|^2$.

Proof of step 1: Let $\gamma = \frac{1}{2} K^{-1} \frac{1-e^{-\alpha}}{1+e^{-\alpha}}$. By Assumption **A7** it holds for $x \in B_{\gamma/L}(0)$

$$\|Dg_n(x)\| \overset{(21)}{=} \|Df_n(x) - Df_n(0)\| \leq L\|x\| \leq \gamma$$

for all $n \in \mathbb{Z}$, therefore

$$\|g_n(x) - g_n(y)\| \leq \gamma \|x - y\| \quad \text{for all } x, y \in B_{\gamma/L}(0),\ n \in \mathbb{Z}.$$

Let $\eta_1,\ \eta_2 \in \mathcal{R}(P_N^s) \cap U$, $v_n = x_n(\eta_1) - x_n(\eta_2)$, then

$$\begin{aligned} v_{n+1} &= \Psi(n+1,N) \circ h_N(\eta_1) - \Psi(n+1,N) \circ h_N(\eta_2) \\ &= f_n(x_n(\eta_1)) - f_n(x_n(\eta_2)) \\ &\overset{(21)}{=} Df_n(0)\big(x_n(\eta_1) - x_n(\eta_2)\big) + g_n(x_n(\eta_1)) - g_n(x_n(\eta_2)). \end{aligned}$$

Thus, v_n is a solution of the inhomogeneous equation

$$v_{n+1} = Df_n(0)v_n + \big(g_n(x_n(\eta_1)) - g_n(x_n(\eta_2))\big) \tag{22}$$

with initial value

$$P_N^s v_N = P_N^s\big(x_N(\eta_1) - x_N(\eta_2)\big) = P_N^s\big(h_N^+(\eta_1) - h_N^+(\eta_2)\big) = P_N^s(\eta_1 - \eta_2) = \eta_1 - \eta_2. \tag{23}$$

Due to Theorem 5, Lemma 18, and assumption **A7**, a unique bounded solution $(v_n)_{n\geq N}$ of (22), (23) exists, fulfilling for all $n \geq N$, where N is chosen sufficiently large, the estimate

$$\begin{aligned}\|v_n\| &\leq K\|\eta_1 - \eta_2\| + K\frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}} \sup_{m\geq N} \big\|g_m(x_m(\eta_1)) - g_m(x_m(\eta_2))\big\| \\ &\leq K\|\eta_1 - \eta_2\| + K\frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}}\gamma \sup_{m\geq N} \|v_m\|.\end{aligned}$$

Finally, we get from the definition of γ

$$\sup_{n\geq N} \|v_n\| \leq 2K\|\eta_1 - \eta_2\|.$$

Proof of step 2: The proof of step 2 immediately follows from direct computations and will be omitted.

Proof of step 3: First, we prove an estimate of $r_n(\eta, \ell)$, defined in (20):

$$\begin{aligned}\|r_n(\eta,\ell)\| &\leq \Big\| \int_0^1 Dg_n\big(x_n(\eta) + \theta\big(x_n(\eta+\ell) - x_n(\eta)\big)\big)d\theta\ [x_n(\eta+\ell) - x_n(\eta)] \\ &\quad - Dg_n(x_n(\eta))Y_n\ell\Big\| \\ &= \Big\| \int_0^1 Dg_n\big(x_n(\eta) + \theta(x_n(\eta+\ell) - x_n(\eta))\big) - Dg_n(x_n(\eta))d\theta \\ &\quad \cdot[x_n(\eta+\ell) - x_n(\eta)] + Dg_n(x_n(\eta))\big(x_n(\eta+\ell) - x_n(\eta) - Y_n\ell\big)\Big\| \\ &\leq \Big| \int_0^1 \big\|Dg_n\big(x_n(\eta) + \theta(x_n(\eta+\ell) - x_n(\eta))\big) - Dg_n(x_n(\eta))\big\|d\theta\Big| \\ &\quad \cdot\|x_n(\eta+\ell) - x_n(\eta)\| + \|Dg_n(x_n(\eta))\|\|w_n\|.\end{aligned}$$

In step 1, we derived for sufficiently large n the estimates $\|x_n(\eta+\ell) - x_n(\eta)\| \leq 2K\|\ell\|$, $\|Dg_n(x_n(\eta))\| \leq \gamma$. By (21) and assumption **A7** we get for sufficiently small ℓ

$$\|r_n(\eta,\ell)\| \leq 2K^2L\|\ell\|^2 + \gamma\|w\|,$$

where $\|w\| = \sup_{m\geq N}\|w_m\|$.

Since $(w_n)_{n\geq N}$ is a bounded solution of the inhomogeneous initial value problem (18), (19), this solution is unique (Lemma 18) and fulfills the estimate

$$\|w\| \leq K\frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}} \sup_{m\geq N} \|r_m(\eta,\ell)\| \leq K\frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}}\big(2K^2L\|\ell\|^2+\gamma\|w\|\big).$$

Inserting the definition of γ, we finally have

$$\|w\| \leq 4K^3L\frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}}\|\ell\|^2.$$

■

Applying this result, we prove a transversality theorem for fiber bundles.

Theorem 8 *Assume* ***A1***–***A3***, ***A5***–***A7***. *It holds for all* $n \in \mathbb{Z}$

$$T_{\bar{x}_n}S_n^+ = \left\{u \in \mathbb{R}^k : \sup_{m\geq n} \|\Phi(m,n)u\| < \infty\right\}. \tag{24}$$

Proof: Since $\bar{x}_{\mathbb{Z}}$ is assumed to be a homoclinic orbits w.r.t. the fixed point 0, there exists an N such that $\bar{x}_N \in U$. Therefore, we find an $\eta \in \mathcal{R}(P_N^s)$ such that $\bar{x}_N = h_N^+(\eta)$, thus $T_{\bar{x}_N}S_N^+ = \mathcal{R}(Dh_N^+(\eta))$.

By Proposition 7 $(\bar{Z}_n\zeta)_{n\geq N}$ is for all $\zeta \in \mathbb{R}^k$ a bounded solution of

$$u_{n+1} = Df_n(\bar{x}_n)u_n, \quad n \geq N.$$

Consequently, $\bar{Z}_n\zeta \in \mathcal{R}(Q_N^{+s})$, since

$$\mathcal{R}(Q_n^{+s}) = \left\{u \in \mathbb{R}^k : \sup_{m\geq n} \|\Phi(m,n)u\| < \infty\right\}$$

holds, cf. Lemma 17. Using the representation of $\bar{Z}_N$ given in (14) it follows that $\mathcal{R}\big(Dh_N^+(\eta)\big) \subset \mathcal{R}(Q_N^{+s})$. For sufficiently small η (which corresponds to a sufficiently large choice of N) we get

$$\operatorname{rank}\big(Dh_N^+(\eta)\big) = \operatorname{rank}\big(Dh_N^+(0)\big) = \operatorname{rank}(P_N^s) = \operatorname{rank}(Q_N^{+s}).$$

Therefore, $\mathcal{R}\big(Dh_N^+(\eta)\big) = \mathcal{R}(Q_N^{+s})$.

Finally, we apply Lemma 6 for each $n \in \mathbb{Z}$:

$$\begin{aligned} T_{\bar{x}_n}S_n^+ &= \Phi(n,N)T_{\bar{x}_N}S_N^+ = \Phi(n,N)\left\{u \in \mathbb{R}^k : \sup_{m\geq N} \|\Phi(m,N)u\| < \infty\right\} \\ &= \left\{u \in \mathbb{R}^k : \sup_{m\geq n} \|\Phi(m,N)\Phi(N,n)u\| < \infty\right\} \\ &= \left\{u \in \mathbb{R}^k : \sup_{m\geq n} \|\Phi(m,n)u\| < \infty\right\}. \end{aligned}$$

■

Analogously, we get for sufficiently large N

$$\mathcal{R}(Dh^-_{-N}(\eta)) = \mathcal{R}(P^u_{-N}) = \left\{ u \in \mathbb{R}^k : \sup_{m \le -N} \|\Phi(m, -N)u\| < \infty \right\},$$

and for all $n \in \mathbb{Z}$ it follows that

$$T_{\bar{x}_n} S^-_n = \left\{ u \in \mathbb{R}^k : \sup_{m \le n} \|\Phi(m, n)u\| < \infty \right\}.$$

If the stable and unstable fiber bundle intersect transversally at $\bar{x}_0$, then they intersect transversally at each point of the orbit.

Lemma 9 *Assume* ***A1–A3, A5–A7****. Let $T_{\bar{x}_0} S^-_0 \cap T_{\bar{x}_0} S^+_0 = \{0\}$. Then it holds for each $n \in \mathbb{Z}$*

$$T_{\bar{x}_n} S^-_n \cap T_{\bar{x}_n} S^+_n = \{0\}.$$

Proof: Since f_n, $n \in \mathbb{Z}$ are diffeomorphisms, we get by Lemma 6

$$\begin{aligned} T_{\bar{x}_n} S^-_n \cap T_{\bar{x}_n} S^+_n &= \Phi(n, 0) T_{\bar{x}_0} S^-_0 \cap \Phi(n, 0) T_{\bar{x}_0} S^+_0 \\ &= \Phi(n, 0) \big(T_{\bar{x}_0} S^-_0 \cap T_{\bar{x}_0} S^+_0 \big) = \{0\}. \end{aligned}$$

■

Finally, we prove that at a homoclinic point, transversality of the orbit, cf. Definition 1, is equivalent to a transversal intersection of the corresponding stable and unstable fiber bundle.

Lemma 10 *Assume* ***A1–A3, A5–A7****. Then the orbit $\bar{x}_{\mathbb{Z}}$ is transversal according to Definition 1 if and only if*

$$T_{\bar{x}_0} S^-_0 \cap T_{\bar{x}_0} S^+_0 = \{0\}.$$

Proof: From Theorem 8 we know

$$\begin{aligned} T_{\bar{x}_0} S^-_0 &= \left\{ u \in \mathbb{R}^k : \sup_{n \le 0} \|\Phi(n, 0)u\| < \infty \right\}, \\ T_{\bar{x}_0} S^+_0 &= \left\{ u \in \mathbb{R}^k : \sup_{n \ge 0} \|\Phi(n, 0)u\| < \infty \right\}. \end{aligned}$$

It holds

$$\begin{aligned} & u_0 \in T_{\bar{x}_0} S^-_0 \cap T_{\bar{x}_0} S^+_0 \\ \iff & \sup_{n \in \mathbb{Z}} \|\Phi(n, 0)u_0\| < \infty \\ \iff & u_{\mathbb{Z}} = \big(\Phi(n, 0)u_0\big)_{n \in \mathbb{Z}} \in X_{\mathbb{Z}}. \end{aligned}$$

Thus

$$T_{\bar{x}_0} S^-_0 \cap T_{\bar{x}_0} S^+_0 = \{0\} \quad \iff \quad \Big[\big(\Phi(n, 0)u_0\big)_{n \in \mathbb{Z}} = u_{\mathbb{Z}} \in X_{\mathbb{Z}} \Leftrightarrow u_{\mathbb{Z}} = 0 \Big],$$

which proves the assertion.

■

4 Approximation of homoclinic orbits

In this section, we present an approximation result for a non-autonomous homoclinic orbit $\bar{x}_{\mathbb{Z}}$. Obviously, a homoclinic orbit is a zero of the operator $\Gamma : X_{\mathbb{Z}} \to X_{\mathbb{Z}}$, defined as

$$\Gamma(x_{\mathbb{Z}}) = \big(x_{n+1} - f_n(x_n)\big)_{n\in\mathbb{Z}}.$$

For an approximation, we restrict Γ to a finite interval $J = [n_-, n_+]$ and introduce a boundary condition that reflects the limit condition $\lim_{n\to\pm\infty} \bar{x}_n = 0$. Let

$$\Gamma_J(x_J) := \Big(\big(x_{n+1} - f_n(x_n)\big)_{n\in\tilde{J}}, b(x_{n_-}, x_{n_+})\Big), \tag{25}$$

where $\tilde{J} = [n_-, n_+ - 1]$ and $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ is an appropriately chosen boundary operator.

Definition 11 *The boundary operator b is of **order** (p_-, p_+), if a constant $C > 0$ exists, such that the following estimate holds true for sufficiently large $-n_-$, n_+*

$$\|b(\bar{x}_{n_-}, \bar{x}_{n_+}) - b(0,0)\| \le C\left(\|\bar{x}_{n_-}\|^{p_-} + \|\bar{x}_{n_+}\|^{p_+}\right).$$

Assuming **A1**–**A6**, the variational equation

$$u_{n+1} = Df_n(\bar{x}_n)u_n, \quad n \in \mathbb{Z}$$

possesses by Lemma 2 separated exponential dichotomies on $\mathbb{Z}^-$ with data $(K^-, \alpha^-, Q_n^{-s}, Q_n^{-u})$ and on $\mathbb{Z}^+$ with data $(K^+, \alpha^+, Q_n^{+s}, Q_n^{+u})$.

Furthermore, we denote by $(K, \alpha, P_n^s, P_n^u)$ the dichotomy data of

$$u_{n+1} = Df_n(0)u_n, \quad n \in \mathbb{Z}.$$

We impose the following assumptions on the boundary operator:

A8 Let $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$, $b(0,0) = 0$ and

$$R_{n_\pm} := \begin{pmatrix} D_1 b(0,0)_{|\mathcal{R}(P^s_{n_-})} & D_2 b(0,0)_{|\mathcal{R}(P^u_{n_+})}\end{pmatrix}$$

is for sufficiently large $-n_-$, n_+ invertible and has a uniformly bounded inverse, i.e. there exist a $C > 0$ and an $N \in \mathbb{N}$ such that

$$\|R_{n_\pm}^{-1}\| \le C \quad \text{for all } -n_-, n_+ \ge N.$$

4.1 Approximation Theorem

In order to prove the existence of a unique solution of $\Gamma_J(x_J) = 0$ in a sufficiently small neighborhood of $\bar{x}_{|J}$, we first state a solution result for the linearized operator Γ'_J.

Lemma 12 *Assume* ***A1***–***A8***. *Then there exist constants N and σ, such that for any $-n_-, n_+ \geq N$ and any $(y_{\tilde{J}}, r) \in (X_{\tilde{J}} \times \mathbb{R}^k)$, the inhomogeneous equation*

$$\Gamma'_J(\bar{x}_{|J})u_J = (y_{\tilde{J}}, r) \tag{26}$$

has a unique solution $u_J \in X_J$, $J = [n_-, n_+]$. Furthermore, we have the estimate

$$\|u_J\| \leq \sigma^{-1}\big(\|y_{\tilde{J}}\| + \|r\|\big), \tag{27}$$

where the constant σ is independent of J and the right hand side.

Proof: First, we note that (26) has the explicit form

$$\begin{aligned} u_{n+1} - Df_n(\bar{x}_n)u_n &= y_n, \quad n \in \tilde{J}, & (28)\\ D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})u_{n_-} + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})u_{n_+} &= r. & (29)\end{aligned}$$

Applying Green's function

$$G^{\pm}(n,m) := \begin{cases} \Phi(n,m)Q_m^{\pm s}, & \text{for } n \geq m, \\ -\Phi(n,m)Q_m^{\pm u}, & \text{for } n < m, \end{cases} \qquad n, m \in \mathbb{Z}^{\pm},$$

we construct two solutions for (28):

$$\begin{aligned} z_n^-(y_{\tilde{J}}) &= \sum_{i=n_-}^{-1} G^-(n, i+1)y_i, \quad n_- \leq n \leq 0, & (30)\\ z_n^+(y_{\tilde{J}}) &= \sum_{i=0}^{n_+-1} G^+(n, i+1)y_i, \quad 0 \leq n \leq n_+. & (31)\end{aligned}$$

The dichotomy estimates immediately show that there exists some constant $C > 0$, such that $\|z_n^{\pm}(y_{\tilde{J}})\| \leq C\|y_{\tilde{J}}\|$.

In order to find a solution of (28), defined on the whole interval J, we construct arbitrary half-solutions

$$\begin{aligned} u_n^- &= \Phi(n,0)\eta + z_n^-(y_{\tilde{J}}), \quad n_- \leq n \leq 0, \quad \eta \in \mathbb{R}^k, & (32)\\ u_n^+ &= \Phi(n,0)\zeta + z_n^+(y_{\tilde{J}}), \quad 0 \leq n \leq n_+, \quad \zeta \in \mathbb{R}^k & (33)\end{aligned}$$

and choose η and ζ such that

$$u_0^- = u_0^+$$

holds, which is equivalent to

$$\zeta - \eta = Z(y_{\tilde{J}}), \quad \text{where } Z(y_{\tilde{J}}) := z_0^-(y_{\tilde{J}}) - z_0^+(y_{\tilde{J}}). \tag{34}$$

Inserting $u_J^{\pm}$ into the boundary condition (29) gives

$$D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})\Phi(n_-, 0)\eta + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})\Phi(n_+, 0)\zeta = R(y_{\tilde{J}}, r), \tag{35}$$

where $R(y_{\tilde{J}}, r) := r - D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+}) z^-_{n_-}(y_{\tilde{J}}) - D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+}) z^+_{n_+}(y_{\tilde{J}})$.

We introduce the following decomposition of η and ζ:

$$
\begin{array}{rclrclrcl}
\eta & = & \eta_0 + \Phi(0, n_-)\eta_-, & \eta_0 & = & Q_0^{-u}\eta, & \eta_- & = & Q_{n_-}^{-s}\Phi(n_-, 0)\eta, \\
\zeta & = & \zeta_0 + \Phi(0, n_+)\zeta_+, & \zeta_0 & = & Q_0^{+s}\zeta, & \zeta_+ & = & Q_{n_+}^{+u}\Phi(n_+, 0)\zeta.
\end{array}
$$

For transforming η_- and ζ_+ into spaces, independent of n we consider

$$
\begin{array}{rcl}
V_{n_-} & := & I + P^s - Q_{n_-}^{-s} : \mathcal{R}(Q_{n_-}^{-s}) \to \mathcal{R}(P^s), \\
W_{n_+} & := & I + P^u - Q_{n_+}^{+u} : \mathcal{R}(Q_{n_+}^{+u}) \to \mathcal{R}(P^u),
\end{array}
$$

where P^s, P^u are the (constant) dichotomy projectors of $u_{n+1} = Df_0(0)u_n$. Note that these transformations are by Lemma 2 invertible:

$$
\|P^s - Q_{n_-}^{-s}\| \le \|P^s - P^s_{n_-}\| + \|P^s_{n_-} - Q_{n_-}^{-s}\| \le \frac{1}{2} + \frac{1}{4} = \frac{3}{4}
$$

for $-n_-$ sufficiently large and therefore

$$
\|V_{n_-}^{-1}\| \le \frac{1}{1 - \|P^s - Q_{n_-}^{-s}\|} \le 4.
$$

A similar result holds for W_{n_+}.

With $\bar{\eta}_- = V_{n_-}\eta_-$, $\bar{\zeta}_+ = W_{n_+}\zeta_+$, (34) and (35) are equivalent to

$$
\underbrace{\begin{pmatrix} I^- & I^+ & \Omega^1_{n_\pm} & \Omega^2_{n_\pm} \\ \Delta^1_{n_\pm} & \Delta^2_{n_\pm} & \Theta^1_{n_\pm} & \Theta^2_{n_\pm} \end{pmatrix}}_{=:A_{n_\pm}} \begin{pmatrix} \eta_0 \\ \zeta_0 \\ \bar{\eta}_- \\ \bar{\zeta}_+ \end{pmatrix} = \begin{pmatrix} Z(y_{\tilde{J}}) \\ R(y_{\tilde{J}}, r) \end{pmatrix}, \tag{36}
$$

where

$$
\begin{array}{rclrcl}
I^- & = & -I_{|\mathcal{R}(Q_0^{-u})}, & I^+ & = & I_{|\mathcal{R}(Q_0^{+s})}, \\
\Omega^1_{n_\pm} & = & -\Phi(0, n_-)V^{-1}_{n_-\,|\mathcal{R}(P^s)}, & \Omega^2_{n_\pm} & = & \Phi(0, n_+)W^{-1}_{n_+\,|\mathcal{R}(P^u)}, \\
\Delta^1_{n_\pm} & = & D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})\Phi(n_-, 0)_{|\mathcal{R}(Q_0^{-u})}, & \Delta^2_{n_\pm} & = & D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})\Phi(n_+, 0)_{|\mathcal{R}(Q_0^{+s})}, \\
\Theta^1_{n_\pm} & = & D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})V^{-1}_{n_-\,|\mathcal{R}(P^s)}, & \Theta^2_{n_\pm} & = & D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})W^{-1}_{n_+\,|\mathcal{R}(P^u)}.
\end{array}
$$

From the dichotomy estimates we immediately get

$$
\lim_{n_\pm \to \pm\infty} \|\Omega^{1,2}_{n_\pm}\| = 0 \quad \text{and} \quad \lim_{n_\pm \to \pm\infty} \|\Delta^{1,2}_{n_\pm}\| = 0.
$$

Furthermore, our transversality assumption **A4** guarantees that $\begin{pmatrix} I^- & I^+ \end{pmatrix}$ is invertible, cf. Lemma 17. Finally, $\begin{pmatrix} \Theta^1_{n_\pm} & \Theta^2_{n_\pm} \end{pmatrix}$ is by the non-degeneracy assumption **A8** invertible and has a uniformly bounded inverse for $-n_-$, n_+ sufficiently large, therefore $A_{n_\pm}$ has a uniformly bounded inverse, too.

Inserting the unique solution η, ζ of (36) back into (32), (33), we find the unique solution of (28), (29)

$$u_n := \begin{cases} u_n^-, & \text{for } n_- \le n < 0, \\ u_n^+, & \text{for } 0 \le n \le n_+. \end{cases}$$

Finally, we prove the claimed estimate (27). With a generic constant $C > 0$, we get

$$\|Z(y_{\tilde J})\| \le C\|y_{\tilde J}\| \quad \text{and} \quad \|R(y_{\tilde J}, r)\| \le \|r\| + C\|y_{\tilde J}\|$$

and since V_{n_-} and W_{n_+} have bounded inverses

$$\| \begin{pmatrix} \eta_0 & \zeta_0 & \eta_- & \zeta_+ \end{pmatrix}^T \| \le C\big(\|y_{\tilde J}\| + \|r\|\big).$$

It holds for $n_+ \ge n \ge 0$

$$\begin{array}{rcl} \|u_n\| & \le & \|\Phi(n,0)\zeta\| + \|z_n^+(y_{\tilde J})\| \\ & \le & \|\Phi(n,0)Q_0^{+s}\|\|\zeta_0\| + \|\Phi(n,n_+)Q_{n_+}^{+u}\|\|\zeta_+\| + C\|y_{\tilde J}\| \\ & \le & K^+\mathrm{e}^{-\alpha^+ n}\|\zeta_0\| + K^+\mathrm{e}^{-\alpha^+(n_+-n)}\|\zeta_+\| + C\|y_{\tilde J}\| \\ & \le & C\big(\|y_{\tilde J}\| + \|r\|\big) \end{array}$$

and a similar computation for $n_- \le n < 0$ proves

$$\|u_J\| \le \sigma^{-1}\big(\|y_{\tilde J}\| + \|r\|\big)$$

for all intervals $J = [n_-, n_+]$, $-n_-, n_+ \ge N$, where σ does not depend on J and the right hand side.

■

With this linear result, we have all tools at hand to prove our main theorem.

Theorem 13 *Assume **A1**–**A8**. There exist two constants δ, $C > 0$, such that the approximating system $\Gamma_J(x_J) = 0$ possesses a unique solution*

$$x_J \in B_\delta(\bar x_{|J}) \quad \textit{for all } J = [n_-, n_+],$$

where $-n_-$, $n_+ \ge N$. The approximation error can be estimated as

$$\|\bar x_{|J} - x_J\| \le C\|b(\bar x_{n_-}, \bar x_{n_+})\|.$$

Proof: We apply Appendix A.2, Lemma 19 with the setting

$$Y = (X_J, \|\cdot\|), \quad Z = \big(X_{\tilde J} \times \mathbb{R}^k, \|\cdot\| + \|\cdot\|\big), \quad F = \Gamma_J, \quad y_0 = \bar x_{|J}.$$

By Lemma 12 the linearized system $\Gamma'_J(\bar x_{|J})u_J = (y_{\tilde J}, r)$ has for all $(y_{\tilde J}, r) \in (X_{\tilde J}, \mathbb{R}^k)$ a unique solution u_J, fulfilling $\|u_J\| \le \sigma^{-1}(\|y_{\tilde J}\| + \|r\|)$. It follows

$$\|\Gamma'_J(\bar x_{|J})^{-1}\| \le \sigma^{-1}.$$

For proving assumption (40) of Lemma 19, we consider

$$\|\Gamma_J'(x_J) - \Gamma_J'(\bar{x}_{|J})\| \le \sup_{n \in \tilde{J}} \|Df_n(x_n) - Df_n(\bar{x}_n)\| + \Lambda_{n_\pm},$$

where $\Lambda_{n_\pm} = \|D_1 b(x_{n_-}, x_{n_+}) - D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})\| + \|D_2 b(x_{n_-}, x_{n_+}) - D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})\|$. Since $b \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$, a $\delta_1 > 0$ exists, such that $\Lambda_{n_\pm} \le \frac{\sigma}{4}$ for $\|x_{n_\pm} - \bar{x}_{n_\pm}\| \le \delta_1$. Let $\sup_{n \in J} \|x_n - \bar{x}_n\| \le \delta_2 = \frac{\sigma}{4L}$. From assumption **A7**, we get for $n \in J$ the uniform estimate $\|Df_n(x_n) - Df_n(\bar{x}_n)\| < \frac{\sigma}{4}$. Thus (40) holds with $\delta = \min\{\delta_1, \delta_2\}$ and $\kappa = \frac{\sigma}{2}$.

Assumption (41) can also be verified:

$$\begin{aligned} \|\Gamma_J(\bar{x}_{|J})\| &= \left\| \big(\bar{x}_{n+1} - f_n(\bar{x}_n)\big)_{n \in \tilde{J}} \right\| + \|b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \\ &\le C \left(\|\bar{x}_{n_-}\|^{p_-} + \|\bar{x}_{n_+}\|^{p_+} \right) \\ &\le \frac{\sigma}{2} \delta \end{aligned}$$

for sufficiently large $-n_-$, n_+.

By Lemma 19 a unique solution of $\Gamma_J(x_J) = 0$ exists in $B_\delta(\bar{x}_{|J})$ for J sufficiently large, and an estimate for the approximation error follows from (43):

$$\begin{aligned} \|\bar{x}_{|J} - x_J\| &\le \frac{1}{\sigma - \kappa} \|\Gamma_J(\bar{x}_{|J}) - \Gamma_J(x_J)\| \\ &= \frac{2}{\sigma} \|b(\bar{x}_{n_-}, \bar{x}_{n_+})\|. \end{aligned}$$

■

4.2 Choice of the boundary operator

For numerical computations we introduce a projection boundary operator $b_{\mathrm{proj}} : \mathbb{R}^{2k} \to \mathbb{R}^k$ defined as

$$b_{\mathrm{proj}}(x, y) := \begin{pmatrix} Y_s^T x \\ Y_u^T y \end{pmatrix}, \quad x, y \in \mathbb{R}^k,$$

where the columns of Y_s and Y_u form an orthogonal basis of the stable and unstable subspace of $Df_0(0)^T$. Note that the stable and unstable subspace of $Df_0(0)^T$ is orthogonal to $\mathcal{R}(P^u)$ and $\mathcal{R}(P^s)$, respectively.

Lemma 14 *The projection boundary operator b_{proj} fulfills the assumptions, stated in* ***A8****.*

Proof: Obviously, $b_{\mathrm{proj}} \in C^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ and $b_{\mathrm{proj}}(0,0) = 0$. It remains to show that

$$R_{n_\pm} = \begin{pmatrix} D_1 b_{\mathrm{proj}}(0,0)_{|\mathcal{R}(P^s_{n_-})} & D_2 b_{\mathrm{proj}}(0,0)_{|\mathcal{R}(P^u_{n_+})} \end{pmatrix}$$

is invertible and has a uniformly bounded inverse.

Let $x \in \mathcal{R}(P^s_{n_-})$. Applying Lemma 2, we get

$$\begin{aligned} \frac{1}{2} &\geq \|P^s_{n_-} - P^s\| = \|P^s_{n_-} - (I - P^u)\| \\ &\geq \frac{\|P^s_{n_-}x - x + P^u x\|}{\|x\|} = \frac{\|P^u x\|}{\|x\|}, \end{aligned}$$

therefore, $\|P^u x\| \leq \frac{1}{2}\|x\|$ and consequently $\|P^s x\| \geq \frac{1}{2}\|x\|$.

Since $\mathcal{R}(Y_s^T) \perp \mathcal{R}(P^u)$ and P^s, P^u are complementary projectors, there exists an $\varphi > 0$ such that for all $x \in \mathcal{R}(P^s_{n_-})$

$$\|Y_s^T x\| = \|Y_s^T P^s x\| \geq \varphi \|P^s x\| \geq \varphi \frac{1}{2}\|x\|$$

holds. With a similar argument for Y_u^T it follows that $R_{n_\pm}$ is invertible and has a uniformly bounded inverse.

$\blacksquare$

5 Example

For numerical computations, we consider the well known Hénon-map

$$x \mapsto h(x, a, b) = \begin{pmatrix} 1 + x_2 - a x_1^2 \\ b x_1 \end{pmatrix}, \tag{37}$$

cf. [6, 7]. The classical values for the parameters are $a = 1.4$ and $b = 0.3$. This map possesses for $a > -\frac{(b-1)^2}{4}$ the fixed point

$$\xi(a, b) = \begin{pmatrix} z(a, b) \\ b z(a, b) \end{pmatrix}, \text{ where } z(a, b) = \frac{b - 1 + \sqrt{(b-1)^2 + 4a}}{2a}.$$

We transform this (parameter dependent) fixed point into the origin. Let

$$\tilde{h}(x, a, b) = \begin{pmatrix} 1 + x_2 + (b-1)z(a, b) - a(x_1 + z(a, b))^2 \\ b x_1 \end{pmatrix},$$

then $\tilde{h}(0, a, b) = 0$ for all parameter $a > -\frac{(b-1)^2}{4}$.

To get a non-autonomous system, we define a sequence $a_{\mathbb{Z}}$, where $a_n \in [0.9, 1.9]$ is chosen randomly and consider the non-autonomous system

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}, \tag{38}$$

where f_n is defined as $f_n(x) := \tilde{h}(x, a_n, 0.3)$. By construction $f_n(0) = 0$ for all n, thus our assumptions **A1**, **A2** are fulfilled.

Note that we cannot transform orbits of (38) into orbits of (37), since these systems are not topologically equivalent. Here, we applied a transformation of the form

$$f_n = T_n^{-1} \circ h(\cdot, a_n, 0.3) \circ T_n.$$

Figure 1 shows an orbit segment of length $n_- = -20$, $n_+ = 20$, computed as a zero of our boundary operator (25), using projection boundary conditions. This orbit, of course, depends on the family $(f_n)_{n\in\mathbb{Z}}$ and therefore on the randomly chosen sequence $a_\mathbb{Z}$. For five different sequences, homoclinic orbits are computed in Figure 2.

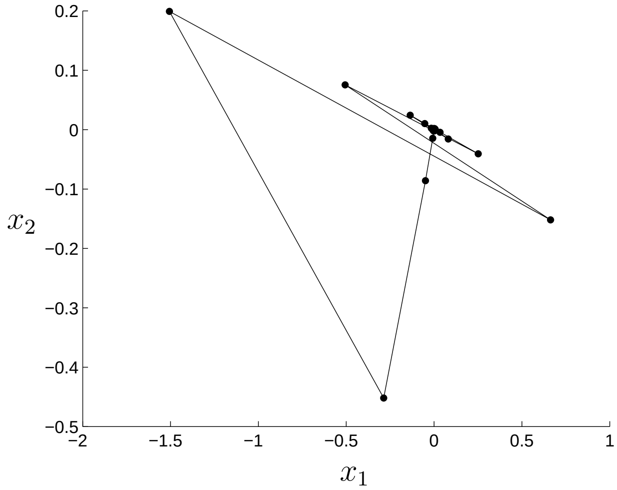

Figure 1: *Non-autonomous homoclinic orbit segment of length* $n_- = -20$, $n_+ = 20$ *of (38). The points of this orbit are connected with lines.*

For an illustration of the approximation error $d = \|\bar{x}_{|J} - x_J\|$, we compute a long orbit segment of length $n_- = -500$, $n_+ = 500$ as reference orbit. Note that an exact orbit is not known. By comparing this orbit with small orbit segments of length $-n_-, n_+ \in [1, 20]$ (computed for the same family $(f_n)_{n\in\mathbb{Z}}$), we get the diagram for the approximation error, shown in Figure 3.

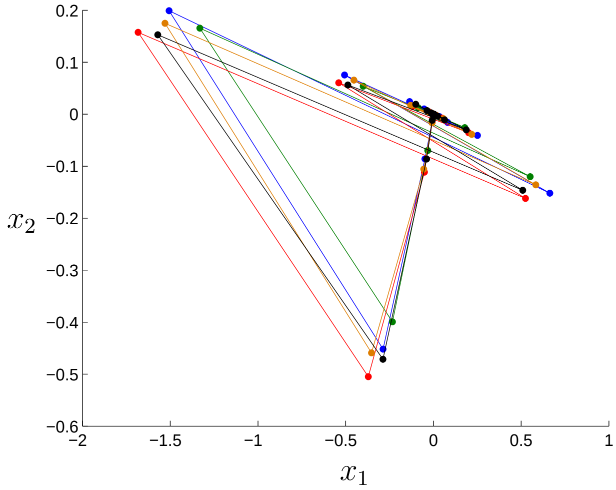

Figure 2: *Five non-autonomous homoclinic orbit segment of length* $n_- = -20$, $n_+ = 20$ *of (38) with different randomly chosen sequences* $a_{\mathbb{Z}}$.

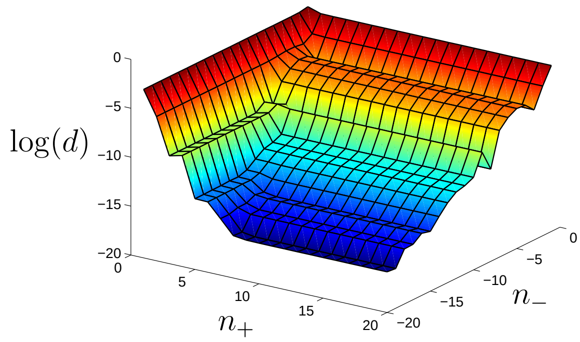

Figure 3: *Approximation error* $d = \|\bar{x}_{|J} - x_J\|$ *of small orbit segments of length* $-n_-, n_+ \in [1, 20]$, *plotted in a logarithmic scale.*

From Theorem 13 we get the error estimate

$$d = \|\bar{x}_{|J} - x_J\| \leq C\|b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \leq C\left(\|\bar{x}_{n_-}\|^{p_-} + \|\bar{x}_{n_+}\|^{p_+}\right).$$

In case of an autonomous system, it is well known that the projection boundary operator is at least of order $(2, 2)$. But in the non-autonomous context, we can only expect an order greater or equal to one, since we do not require the end point to lie in the linear approximation of the corresponding fiber bundle.

For approximating p_- and p_+ numerically, we fix $n_+ = 500$ and compute for $-n_- \in [1, 20]$ the expression $\frac{\log d}{\log \|\bar{x}_{n_-}\|}$, where $\bar{x}_{\mathbb{Z}}$ is our reference orbit of length

$n_- = -500$, $n_+ = 500$, cf. the left diagram in Figure 4. A similar computation for fixed $n_- = -500$, $n_+ \in [1, 20]$ is shown in the right diagram.

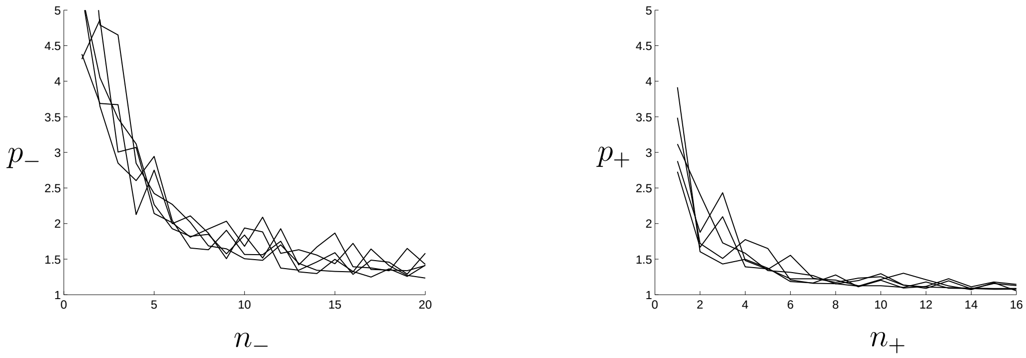

Figure 4: *Approximation of the order p_- (left) and p_+ (right) of the projection boundary operator. The diagrams show the results for five different sequences $a_{\mathbb{Z}}$.*

A Appendix

In this appendix, we state some well known results for exponential dichotomies from [12] and quote a Lipschitz inverse mapping theorem, cf. [17].

A.1 Exponential dichotomy

Definition 15 *A linear difference equation*

$$u_{n+1} = A_n u_n, \quad n \in \mathbb{Z}$$

*with invertible matrices $A_n \in \mathbb{R}^{k,k}$ and solution operator Φ has an **exponential dichotomy** with data $(K, \alpha, P_n^s, P_n^u)$ on $J \subset \mathbb{Z}$, if there exist two families of projectors P_n^s and $P_n^u = I - P_n^s$ and constants $K,\ \alpha > 0$, such that the following statements hold:*

$$P_n^s \Phi(n,m) = \Phi(n,m) P_m^s \quad \forall n, m \in J,$$

$$\left.\begin{array}{rcl} \|\Phi(n,m)P_m^s\| & \leq & K e^{-\alpha(n-m)} \\ \|\Phi(m,n)P_n^u\| & \leq & K \mathrm{e}^{-\alpha(n-m)} \end{array}\right. \quad \forall n \geq m,\ n, m \in J.$$

We introduce an important perturbation result for exponential dichotomies, frequently named as Roughness-Theorem, cf. [12, Proposition 2.10].

Theorem 16 *Assume that the difference equation*

$$u_{n+1} = A_n u_n, \quad A_n \in \mathbb{R}^{k,k} \textit{ invertible}, \quad \|A_n^{-1}\| \leq M\ \forall n \in J$$

with an interval $J \subseteq \mathbb{Z}$, possesses an exponential dichotomy with data $(K, \alpha, P_n^s, P_n^u)$. Suppose $0 < \delta < \alpha$ and $B_n \in \mathbb{R}^{k,k}$ satisfies for all $n \in J$

$$\begin{array}{rcl} \|B_n\| & < & M^{-1}, \\ 2K(1+\mathrm{e}^{-\alpha})(1-\mathrm{e}^{-\alpha})^{-1}\|B_n\| & \leq & 1, \\ 2K\mathrm{e}^{\alpha}(\mathrm{e}^{-\delta}+1)(\mathrm{e}^{\delta}-1)^{-1}\|B_n\| & \leq & 1. \end{array}$$

Then $A_n + B_n$ is invertible and the perturbed difference equation

$$u_{n+1} = (A_n + B_n)u_n$$

possesses an exponential dichotomy on J with data $\big(2K(1+\mathrm{e}^{\delta})(1-\mathrm{e}^{-\delta})^{-1}, \alpha - \delta, Q_n^s, Q_n^u\big)$, where $\operatorname{rank}(Q_n^s) = \operatorname{rank}(P_n^s)$ and

$$\|P_n^s - Q_n^s\| \leq 2K^2 \frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}} \sup_{m \in J} \|B_m\| \quad \textit{for all } n \in J. \tag{39}$$

An alternative representation of the range of the dichotomy-projector is introduced in the following lemma, cf. [12, Proposition 2.3].

Lemma 17 *Assume that the difference equation*

$$u_{n+1} = A_n u_n, \quad A_n \in \mathbb{R}^{k,k} \textit{ invertible},$$

possesses an exponential dichotomy on J with data $(K, \alpha, P_n^s, P_n^u)$.

Then it holds in case $J = \mathbb{Z}^+$ for each $n \in \mathbb{Z}^+$

$$\mathcal{R}(P_n^s) = \left\{ u \in \mathbb{R}^k : \sup_{m \geq n} \|\Phi(m,n)u\| < \infty \right\}.$$

In case $J = \mathbb{Z}^-$, we get for each $n \in \mathbb{Z}^-$

$$\mathcal{R}(P_n^u) = \left\{ u \in \mathbb{R}^k : \sup_{m \leq n} \|\Phi(m,n)u\| < \infty \right\}.$$

The following Lemma states a criterion for the existence of a unique bounded solution of an inhomogeneous difference equation, cf. [12, Lemma 2.7].

Lemma 18 *Assume that the difference equation*

$$u_{n+1} = A_n u_n, \quad A_n \textit{ invertible }, \quad n \in J, \quad J = \mathbb{Z}^- \textit{ or } J = \mathbb{Z}^+,$$

possesses an exponential dichotomy on J with data $(K, \alpha, P_n^s, P_n^u)$ and let $h_J \in (\mathbb{R}^k)^J$ be a bounded sequence. Then the inhomogeneous initial value problem

$$\begin{array}{rcl} u_{n+1} & = & A_n u_n + h_n, \quad n \in J \\ P_0^s u_0 & = & P_0^s \xi, \quad (\textit{if } J = \mathbb{Z}^+), \\ P_0^u u_0 & = & P_0^u \xi, \quad (\textit{if } J = \mathbb{Z}^-) \end{array}$$

has for each $\xi \in \mathbb{R}^k$ a unique bounded solution u_J, fulfilling

$$\|u_n\| \leq K\|\xi\| + K \frac{1+\mathrm{e}^{-\alpha}}{1-\mathrm{e}^{-\alpha}} \sup_{m \in J} \|h_m\|, \quad n \in J.$$

A.2 A Lipschitz inverse mapping theorem

The Lipschitz inverse mapping theorem (cf. Vainikko's Lemma [17]) is an essential tool for proving our approximation theorem.

Lemma 19 *Assume Y and Z are Banach spaces, $F \in C^1(Y, Z)$ and $F'(y_0)$ is for $y_0 \in Y$ a homeomorphism. Let $\kappa,\ \sigma,\ \delta > 0$ be three constants, such that the following estimates hold:*

$$\begin{aligned} \left\|F'(y) - F'(y_0)\right\| &\leq \kappa < \sigma \leq \frac{1}{\left\|F'(y_0)^{-1}\right\|} \quad \forall y \in B_\delta(y_0), && (40)\\ \left\|F(y_0)\right\| &\leq (\sigma - \kappa)\delta. && (41)\end{aligned}$$

Then F has a unique zero $\bar{y} \in B_\delta(y_0)$ and the estimates

$$\begin{aligned} \left\|F'(y)^{-1}\right\| &\leq \frac{1}{\sigma - \kappa} \quad \forall y \in B_\delta(y_0), && (42)\\ \|y_1 - y_2\| &\leq \frac{1}{\sigma - \kappa}\left\|F(y_1) - F(y_2)\right\| \quad \forall y_1,\ y_2 \in B_\delta(y_0) && (43)\end{aligned}$$

are fulfilled.

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