Numerical Fixed Grid Methods for Differential Inclusions

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Abstract

Numerical methods for initial value problems for differential inclusions usually require a discretization of time as well as of the set valued right hand side. In this paper, two numerical fixed grid methods for the approximation of the full solution set are proposed and analyzed. Convergence results are proved which show the combined influence of time and (phase) space discretization.

1 Introduction

In order to obtain a numerical scheme for the approximation of the set $S^F(x_0, [0,T])$ of solutions of the initial value problem

$$\dot{x}(t) \in F(x(t)) \text{ a.e. in } [0,T],\ x(0) = x_0, \tag{1}$$

it is necessary to discretize the problem appropriately. Unlike the classical ODE-case, a discretization of the time interval only is usually not sufficient and generically leads to an uncountable set of solutions. There is an extensive literature on error estimates for such semi-discretized systems, see e.g. [2, 3].

In our view there are two meaningful types of spatial discretization: either an approximation of the images of F by a finite set of points is generated and used to construct approximating trajectories, or the phase space is decomposed into small cells (usually cartesian boxes related to a fixed grid) and the cells visited by trajectories are recorded. A method of the first type is analyzed in [4], where convex combinations of extremal points of F-images are used. A method of the second type is employed in [7] to compute the reachable set of (1) with a prescribed accuracy.

A decomposition of the phase space into boxes has the nice effect that the state space of the numerical methods becomes discrete and even finite for finite times. In [5, 6, 10] it has been shown that methods from graph theory can be applied to the discretized flow in order to determine the control sets and the reachable sets of a system or to compute optimal feedback stabilization.

The influence of small so called realization-errors on Runge-Kutta schemes has been studied in [8], but the global setup of the estimates there is quite different from that presented here which makes a direct comparison of the results difficult.

The aim of this paper is to investigate and compare both types of discretization that are both based on box approximations: the first in velocity space of F-images and the second in phase space of x-variables. In fact, for the Euler scheme it turns out that both approaches transform into each other, which, however, cannot be expected for higher order schemes. In section 3 we discuss some implementational details and show applications to two examples: the normal form system for Hopf bifurcation and the Michaelis Menten system from reaction kinetics. This demonstrates that our method leads to fully discretized differential inclusions that simultaneously allows for a reasonable convergence analysis on finite time intervals.

2 Main Results

Let $0 = t_0 < \ldots < t_N = T$ be an equidistant grid with step-size $h = \frac{T}{N}$ in $[0, T]$, and let $\Delta_\rho = \rho\mathbb{Z}^d$ be a ρ-mesh in $\mathbb{R}^d$ equipped with the maximum norm $|\cdot|$. For any $x \in \Delta_\rho$, the set $B_{\frac{\rho}{2}}(x) = \{x' \in \mathbb{R}^d : |x - x'| \leq \frac{\rho}{2}\}$ will be referred to as the box with center x. Let $c_\rho : 2^{\mathbb{R}^d} \to 2^{\Delta_\rho}$ be the mapping given by

$$c_\rho(A) = \big(A + B_{\frac{\rho}{2}}(0)\big) \cap \Delta_\rho,\ \ A \in 2^{\mathbb{R}^d},$$

which maps a set A to the collection of the centers of those boxes which have a nontrivial intersection with A.

Let $S^F_{h,\rho}(x_0, [0, T])$ be the set of functions $\varphi : [0, T] \to \mathbb{R}^d$ which satisfy

$$\varphi(0) = x_0,\ \ \varphi(t_{k+1}) \in \varphi(t_k) + hc_\rho(F(\varphi(t_k))), \tag{2}$$

and

$$\varphi(t) = \frac{t_{k+1} - t}{t_{k+1} - t_k}\varphi(t_k) + \frac{t - t_k}{t_{k+1} - t_k}\varphi(t_{k+1})\ \forall t \in [t_k, t_{k+1}].$$

These functions can be interpreted as piecewise linear solutions of an Euler scheme, where F has been modified by an appropriate projection of its images to the mesh Δ_ρ.

Alternatively, consider the set $\tilde{S}^F_{h,\rho}(x_0,[0,T])$ of functions $\varphi:[0,T]\to\mathbb{R}^d$ which satisfy

$$\varphi(0)=x_0,\ \ \varphi(t_{k+1})\in\varphi(t_k)+c_\rho(hF(\varphi(t_k))), \tag{3}$$

and

$$\varphi(t)=\frac{t_{k+1}-t}{t_{k+1}-t_k}\varphi(t_k)+\frac{t-t_k}{t_{k+1}-t_k}\varphi(t_{k+1})\ \forall t\in[t_k,t_{k+1}].$$

These solutions are obtained by a successive application of the Euler method using the original multifunction F, and an appropriate projection of the Euler method's images to the mesh Δ_ρ. In contrast to the first approach, the projection happens in the phase space, such that

$$\varphi(t_k)\in x_0+\Delta_\rho\ \forall 0\le k\le N,\ \ \forall\varphi\in\tilde{S}^F_{h,\rho}(x_0,[0,T]).$$

Nevertheless the following relation holds

$$S^F_{h,\frac{\rho}{h}}(x_0,[0,T])=\tilde{S}^F_{h,\rho}(x_0,[0,T]), \tag{4}$$

the implications of which will be discussed in Remark 3.

Let $\mathcal{CC}(\mathbb{R}^d)$ denote the set of all compact convex subsets of $\mathbb{R}^d$ and let dist and dist_H be the unsymmetric and the symmetric Hausdorff distance, resprectively. The following theorems provide estimates for the accuracy of both approximations.

Theorem 1. *Let $F:\mathbb{R}^d\to\mathcal{CC}(\mathbb{R}^d)$ be Lipschitz continuous w.r.t.* dist_H *with Lipschitz constant $L>0$, and let $P>0$ such that $F(x)\subset B_P(0)\ \forall x\in\mathbb{R}^d$. Then the estimate*

$$\mathrm{dist}(S^F_{h,\rho}(x_0,[0,T]),S^F(x_0,[0,T]))\le(e^{LT}-1)(\frac{1}{2L}\rho+\frac{h}{2}\rho+Ph)$$

holds in $C([0,T],\mathbb{R}^d,||\cdot||_\infty)$.

Proof. Let $\varphi\in S^F_{h,\rho}(x_0,[0,T])$. Then $\dot\varphi(t)\in c_\rho(F(\varphi(t_k)))\ \forall t\in(t_k,t_{k+1})$, and

$$|\varphi(t)-\varphi(t_k)|\le\int_{t_k}^t|\dot\varphi(s)|ds\le Ph+\frac{h}{2}\rho.$$

Thus

$$\begin{aligned}\mathrm{dist}(\dot\varphi(t),F(\varphi(t)))&\le\mathrm{dist}_H(c_\rho(F(\varphi(t_k))),F(\varphi(t)))\\&\le\mathrm{dist}_H(F(\varphi(t_k)),F(\varphi(t)))+\frac{\rho}{2}\\&\le L|\varphi(t_k)-\varphi(t)|+\frac{\rho}{2}\\&\le LPh+L\frac{h}{2}\rho+\frac{\rho}{2}.\end{aligned}$$

By the Gronwall-Filippov-Wazewski theorem (Theorem 2.4.1 in [1]) there exists a $\psi \in S^F(x_0, [0,T])$ such that

$$\sup_{0\le t\le T} |\varphi(t)-\psi(t)| \le \int_0^T e^{L(T-s)}(LPh+L\frac{h}{2}\rho+\frac{\rho}{2})ds = (e^{LT}-1)(\frac{1}{2L}\rho+\frac{h}{2}\rho+Ph).$$

$\square$

Theorem 2. *Let $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be Lipschitz continuous w.r.t. dist_H with Lipschitz constant $L > 0$, and let $P > 0$ such that $F(x) \subset B_P(0)$ $\forall x \in \mathbb{R}^d$. Then the estimate*

$$\mathrm{dist}(S^F(x_0, [0,T]), S^F_{h,\rho}(x_0, [0,T])) \le (LPh + \frac{\rho}{2})Te^{LT} + 2Ph + \frac{\rho}{2}h.$$

holds in $C([0,T], \mathbb{R}^d, ||\cdot||_\infty)$.

Proof. Let $\psi \in S^F(x_0, [0,T])$. The aim is to construct a function $\varphi \in S^F_{h,\rho}(x_0, [0,T])$ close to ψ. Without loss of generality $\dot\psi(t) \in F(\psi(t))$ $\forall t \in [0,T]$. By Theorem I.6.13 in [11],

$$\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot\psi(s)ds \in \overline{\mathrm{co}}(\{\dot\psi(s) : s \in [t_k, t_{k+1}]\}),$$

where $\overline{\mathrm{co}}$ denotes the closure of the convex hull. Thus for every $\epsilon > 0$ there exist $\lambda_{\epsilon,0}, \dots, \lambda_{\epsilon,d} \in [0,1]$ with $\sum_{i=0}^d \lambda_{\epsilon,i} = 1$ and $s_{\epsilon,0}, \dots, s_{\epsilon,d} \in [t_k, t_{k+1}]$ such that

$$|\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot\psi(s)ds - \sum_{i=0}^d \lambda_{\epsilon,i}\dot\psi(s_{\epsilon,i})| < \epsilon.$$

Furthermore, for $0 \le i \le d$ there exist $\xi_{\epsilon,i} \in F(\varphi(t_k))$ such that

$$\begin{aligned} |\dot\psi(s_{\epsilon,i}) - \xi_{\epsilon,i}| &\le \mathrm{dist}_H(F(\psi(s_{\epsilon,i})), F(\varphi(t_k))) \\ &\le L|\psi(s_{\epsilon,i}) - \varphi(t_k)| \\ &\le L(|\psi(t_k) - \varphi(t_k)| + \int_{t_k}^{s_{\epsilon,i}} |\dot\psi(s)|ds) \\ &\le L|\psi(t_k) - \varphi(t_k)| + LPh. \end{aligned}$$

Because of the convexity of $F(\varphi(t_k))$, we have $\xi_\epsilon := \sum_{i=0}^d \lambda_{\epsilon,i}\xi_{\epsilon,i} \in F(\varphi(t_k))$,

$$|\sum_{i=0}^d \lambda_{\epsilon,i}\dot\psi(s_{\epsilon,i}) - \xi_\epsilon| \le L|\psi(t_k) - \varphi(t_k)| + LPh,$$

and

$$\begin{aligned}
& |\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - \xi_\epsilon| \\
\leq\ & |\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - \sum_{i=0}^{d} \lambda_{\epsilon,i}\dot{\psi}(s_{\epsilon,i})| + |\sum_{i=0}^{d} \lambda_{\epsilon,i}\dot{\psi}(s_{\epsilon,i}) - \xi_\epsilon| \\
\leq\ & \epsilon + L|\psi(t_k) - \varphi(t_k)| + LPh.
\end{aligned}$$

Let $\epsilon_n := \frac{1}{n}$. Since $F(\varphi(t_k))$ is compact, there exists a convergent subsequence $(\xi_{\epsilon_{n'}})$ of (ξ_{ϵ_n}) such that $\xi := \lim_{n'\to\infty} \xi_{\epsilon_{n'}} \in F(\varphi(t_k))$, and

$$|\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - \xi| \leq L|\psi(t_k) - \varphi(t_k)| + LPh.$$

Now take some $\hat{\xi} \in c_\rho(\xi)$ and define $\varphi(t_{k+1}) := \varphi(t_k) + h\hat{\xi}$. Then we obtain

$$\begin{aligned}
& |\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - \hat{\xi}| \\
\leq\ & |\frac{1}{h}\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - \xi| + |\xi - \hat{\xi}| \\
\leq\ & L|\psi(t_k) - \varphi(t_k)| + LPh + \frac{\rho}{2},
\end{aligned}$$

or

$$|\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - h\hat{\xi}| \leq Lh|\psi(t_k) - \varphi(t_k)| + LPh^2 + \frac{1}{2}\rho h.$$

Hence

$$\begin{aligned}
|\psi(t_{k+1}) - \varphi(t_{k+1})| &\leq |\psi(t_k) - \varphi(t_k)| + |\int_{t_k}^{t_{k+1}} \dot{\psi}(s)ds - h\hat{\xi}| \\
&\leq (1 + Lh)|\psi(t_k) - \varphi(t_k)| + LPh^2 + \frac{1}{2}\rho h.
\end{aligned}$$

By induction,

$$\begin{aligned}
|\psi(t_k) - \varphi(t_k)| &\leq (LPh^2 + \frac{1}{2}\rho h)\sum_{j=0}^{k-1}(1 + Lh)^k \\
&\leq (LPh^2 + \frac{1}{2}\rho h)\frac{T}{h}e^{LT} \\
&= (LPh + \frac{\rho}{2})Te^{LT}
\end{aligned}$$

for all $0 \leq k \leq N$. If $t \in (t_k, t_{k+1})$ for some $0 \leq k < N$, then

$$\begin{aligned} |\psi(t) - \varphi(t)| &\leq \int_{t_k}^{t} |\dot{\psi}(s)| ds + (t - t_k)|\hat{\xi}| + |\psi(t_k) - \varphi(t_k)| \\ &\leq 2Ph + \frac{\rho}{2} h + |\psi(t_k) - \varphi(t_k)|. \end{aligned}$$

Thus

$$|\psi(t) - \varphi(t)| \leq (LPh + \frac{\rho}{2}) T e^{LT} + 2Ph + \frac{\rho}{2} h \;\forall t \in [0, T].$$

□

Remark 3. *For the Euler scheme, the discretization of the velocity space (presented above) and the discretization of the phase space coincide in the following sense: If one chooses $\rho = \sigma h^\alpha$ with $\sigma > 0$, $\alpha > 0$, the equality*

$$S^F_{h,\sigma h^\alpha}(x_0, [0, T]) = \tilde{S}^F_{h,\sigma h^{1+\alpha}}(x_0, [0, T]) \tag{5}$$

can be obtained by the transformation

$$\begin{aligned} & x_n + h c_{\sigma h^\alpha}(F(x_h)) \\ = \;& x_n + h \sigma h^\alpha c_1(\frac{1}{\sigma h^\alpha} F(x_n)) \\ = \;& x_n + \sigma h^{1+\alpha} c_1(h \frac{1}{\sigma h^{1+\alpha}} F(x_n) \\ = \;& x_n + c_{\sigma h^{1+\alpha}}(h(F(x_n)) \end{aligned}$$

with $\rho = \sigma h^\alpha$, and the same error estimates as in theorems 1 and 2 are valid for the corresponding phase space discretization if ρ is chosen to be $\sigma h^{1+\alpha}$. For a proper balance, one should use $\rho = h$ in Theorems 1 and 2 in order to obtain $\mathcal{O}(h)$-convergence, and thus one should use $\rho = h^2$ for phase space discretizations.

Note that the relation above is not true for more complicated schemes. Consider for example the Heun method

$$x_{n+1} \in x_n + \frac{h}{2} F(x_n) + \frac{h}{2} F(x_n + hF(x_n)),$$

where a discretization in the velocity space

$$x_{n+1} \in x_n + \frac{h}{2} c_{\rho_1}(F(x_n)) + \frac{h}{2} c_{\rho_1}(F(x_n + h c_{\rho_1}(F(x_n))))$$

leads to two successive projections in the third term, while a discretization in the phase space

$$x_{n+1} \in x_n + c_{\rho_2}(\frac{h}{2} F(x_n) + \frac{h}{2} F(x_n + hF(x_n)))$$

needs only one projection.

The following Lemma gives some information on the error caused by the algorithm if only an approximation G of the right hand side F is available. This is useful for the implementation, because it might be difficult to represent F properly in a computer program. According to the theorems above it seems reasonable to allow an error δ which does not exceed LPh.

Lemma 4. *Let $F, G : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be Lipschitz continuous with Lipschitz constant L, and let $\delta > 0$ such that*

$$\mathrm{dist}_H(F(x), G(x)) \leq \delta$$

for all $x \in \mathbb{R}^d$. Then

$$\mathrm{dist}_H(S^F_{h,\rho}(x_0, [0,T]), S^G_{h,\rho}(x_0, [0,T])) \leq (\delta + \rho)Te^{LT}.$$

Proof. Since the problem is symmetric, it is sufficient to estimate

$$\mathrm{dist}(S^F_{h,\rho}(x_0, [0,T]), S^G_{h,\rho}(x_0, [0,T])).$$

Let $x(\cdot) \in S^F_{h,\rho}(x_0, [0,T])$ be given by

$$x(0) = x_0,\ x((n+1)h) = x(nh) + h\xi_n,\ \xi_n \in c_\rho(F(x(nh))).$$

Now construct a piecewise linear $y(\cdot) \in S^G_{h,\rho}(x_0, [0,T])$ close to x. For a given $y(nh)$, there exists a $\phi_n \in c_\rho(F(y(nh)))$ such that

$$\begin{aligned}
|\xi_n - \phi_n| &\leq \mathrm{dist}_H(c_\rho(F(x(nh))), c_\rho(G(y(nh)))) \\
&\leq \mathrm{dist}_H(F(x(nh)), G(y(nh))) + \rho \\
&\leq \mathrm{dist}_H(F(x(nh)), G(x(nh))) + \mathrm{dist}_H(G(x(nh)), G(y(nh))) + \rho \\
&\leq \delta + L|x(nh) - y(nh)| + \rho.
\end{aligned}$$

Hence

$$\begin{aligned}
|x((n+1)h) - y((n+1)h)| &\leq |x(nh) - y(nh)| + h|\xi_n - \phi_n| \\
&\leq (1 + Lh)|x(nh) - y(nh)| + (\delta + \rho)h,
\end{aligned}$$

and by induction,

$$|x(nh) - y(nh)| \leq (\delta + \rho)h \sum_{k=0}^{n-1} (1 + Lh)^k \leq (\delta + \rho)Te^{LT}.$$

Since trajectories are obtained via linear interpolation, this estimate holds not only for $\{nh : 0 \leq n \leq N\}$ but for all $t \in [0,T]$. $\square$

Under the same conditions as above the relation (4) implies the estimate

$$\mathrm{dist}_H(\tilde{S}^F_{h,\rho}(x_0, [0,T]), \tilde{S}^G_{h,\rho}(x_0, [0,T])) \leq (\delta + \frac{\rho}{h})Te^{LT}.$$

3 Implementation and Example

3.1 Approximation of all Trajectories

If one wants to approximate all trajectories, the data must be organized as a graph (as sketched in Figure 1) in order to save memory and computational costs. Once the graph is computed, one can simply read off the trajectories: Every path of length N starting with the initial value is an element of $S_{h,\rho}^{F}(x_0,[0,T])$ and vice versa. A similar but considerably larger graph is

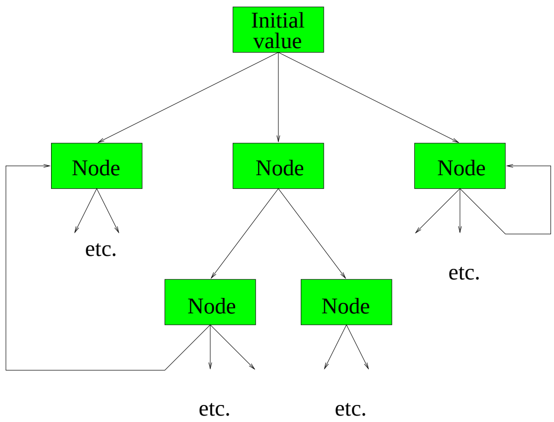

Figure 1: Trajectories of the differential inclusion stored as a graph

used in [5, 6, 10] for the approximation of control sets and optimal feedback stabilization.

3.2 Approximation of the Reachable Sets

In order to compute the reachable set $\mathcal{R}^F(x_0,T)$ of (1) at time T, one can use the methods presented above without storing the trajectories. Let the relevant region in the phase space be decomposed into boxes which can be marked either as 'true' or 'false'. The set of the centers of the boxes which are marked as 'true' in the n-th step of the algorithm will be denoted by $\mathcal{M}_n$.

At the beginning $\mathcal{M}_0 = \{x_0\}$, i.e. only the box with center x_0 is marked. Then our method is successively applied:

$$\mathcal{M}_{n+1} = \cup_{x\in\mathcal{M}_n} x + c_\rho(hF(x)).$$

The image $\mathcal{M}_n$ can either be stored as a list or as a matrix with entries 'true' and 'false'. The first technique is time consuming because of the maintainance of the tree, while the second technique needs a lot of memory

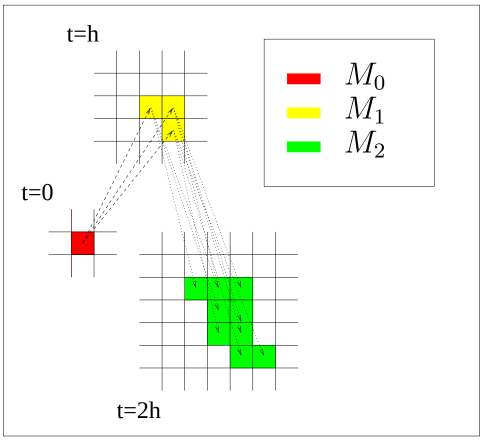

Figure 2: The evolution of the approximations of the reachable sets

because of the required fine spatial discretization. We propose to run a test routine from time to time which determines whether the first or the second approach is more efficient in that moment.

3.3 Example

3.3.1 A classical system from bifurcation analysis

We apply our method to the two-dimensional equation

$$\begin{aligned} \dot{x}_1 &= ax_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + ax_2 - x_2(x_1^2 + x_2^2) \end{aligned} \tag{6}$$

which is the normal form of the Hopf bifurcation. By a transformation to polar coordinates

$$\begin{aligned} \dot{\rho} &= \rho(a - \rho^2) \\ \dot{\varphi} &= 1 \end{aligned}$$

it becomes obvious that for $a > 0$ the origin is an unstable fixed point and that there is a stable limit cycle of radius $\sqrt{a}$.

We study the set valued system

$$\begin{aligned} \dot{x}_1 &\in [0.3, 0.6]x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &\in x_1 + [0.3, 0.6]x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{7}$$

As it is impossible to visualize the set of all solutions of the differential inclusion, we just show some randomly chosen ones in Figure 4.

Figure 5 allows more insight into the behaviour of the set of solutions as a whole. The pictures show the set $S^F_{0.1,0.01}(x_0, t)$ for $t = 0, \ldots, 15$. In order to

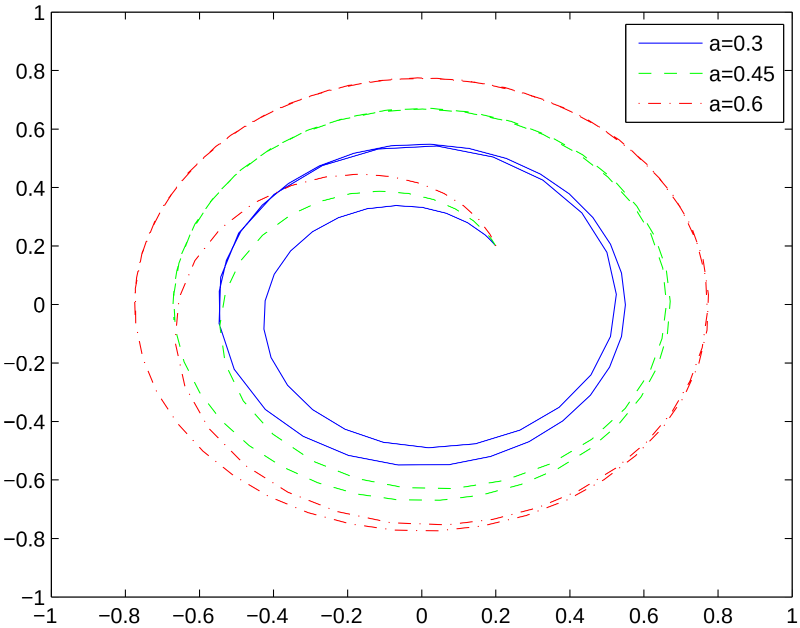

Figure 3: Typical deterministic trajectories of (6)

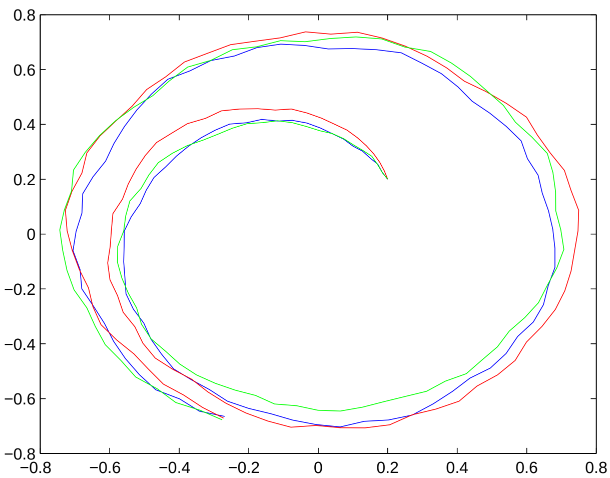

Figure 4: Some solutions of the differential inclusion (7)

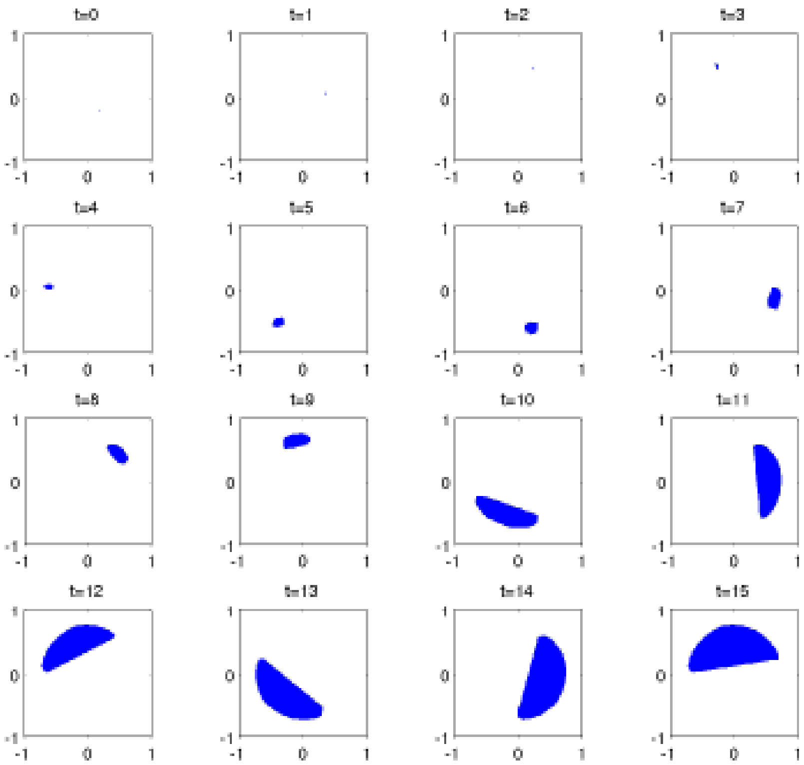

Figure 5: Approximations of the reachable sets of (7)

give an idea of the computational costs, we present some statistics in Figure 6. The graphs show how many boxes and how often these boxes have been visited by some trajectory. Note that the figures might vary depending on the algorithm which is used for the approximation. Nevertheless, it is clear that an increase of precision leads to an enourmous increase of the computational costs, which is a problem for list based algorithms in particular.

3.3.2 Reaction Kinetics: The Michaelis-Menten Model

The Michaelis-Menten model describes a simple biochemical process, during which an organic substrate molecule is changed into a product by the help of an enzyme. At first, the substrate molecule forms a complex with an enzyme. In a second step, the substrate molecule changes its shape and becomes the product, which leaves the complex in a third step.

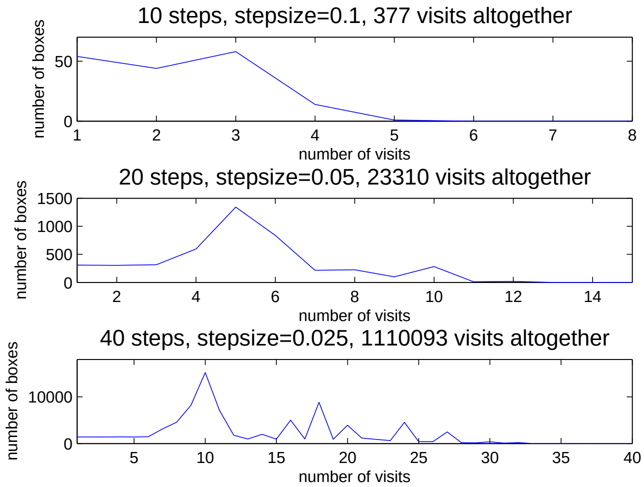

Figure 6: Distribution of hits in model (7)

The chemical equation

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} SE \overset{k_2}{\to} P + E.$$

can easily be expressed as a four dimensional system of differential equations which can be reduced to the system

$$\begin{aligned} \dot{s} &= -k_1 e_0 s + (k_1 s + k_{-1})c \\ \dot{c} &= k_1 e_0 s - (k_1 s + k_{-1} + k_2)c, \end{aligned} \tag{8}$$

where the k_i are rate constants, e_0 is the concentration of the enzyme, and s and c stand for the concentrations of the substrate and substrate-enzyme complex, respectively. For a more detailed discussion of this model see e.g. [9].

Figure 7 shows a phase portrait of the system for the parameter values $k_1 = 2.65$, $k_{-1} = 0.06$, $k_2 = 3.5$, and $e_0 = 0.1$. A set valued F is obtained from (8) by formally replacing some parameters by intervals (which for example may reflect incomplete knowledge of reaction constants). The set $S^F_{h,\rho}(x_0, t)$ of solutions of

$$\begin{aligned} \dot{s} &\in -[2.5, 2.8]e_0 s + ([2.5, 2.8]s + k_{-1})c \\ \dot{c} &\in [2.5, 2.8]e_0 s - ([2.5, 2.8]s + k_{-1} + [3.4, 3.6])c \end{aligned} \tag{9}$$

for $t = 0, \ldots, 11$ is displayed in Figure 8, where $h = \frac{1}{40}$ and $\rho = \frac{1}{1600}$. Again, we try to illustrate the computational costs by the statistics in Figure 9.

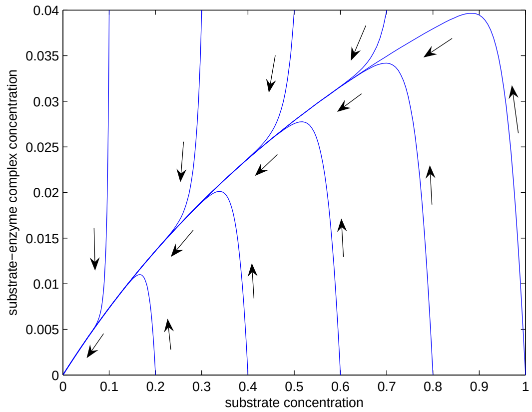

Figure 7: Phase portrait of the Michaelis-Menten system (8)

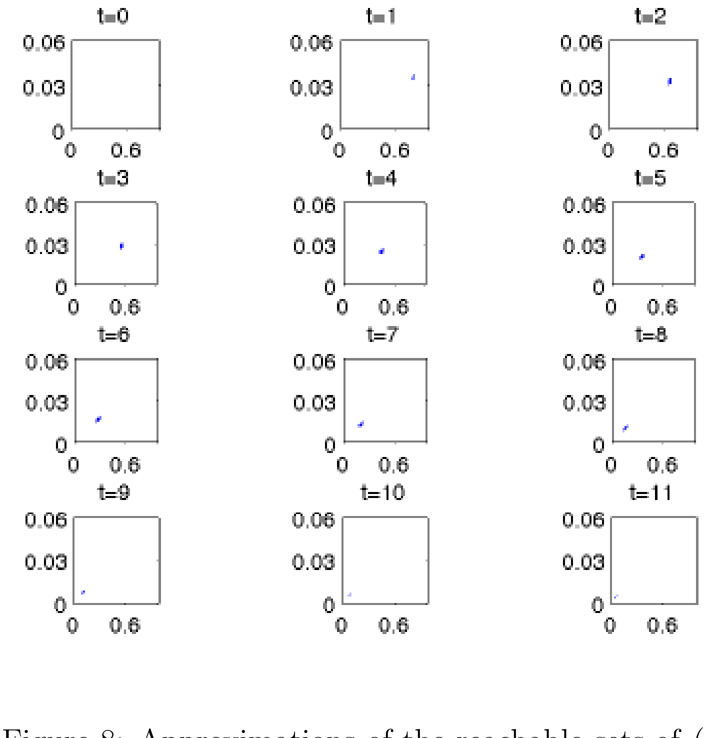

Figure 8: Approximations of the reachable sets of (9)

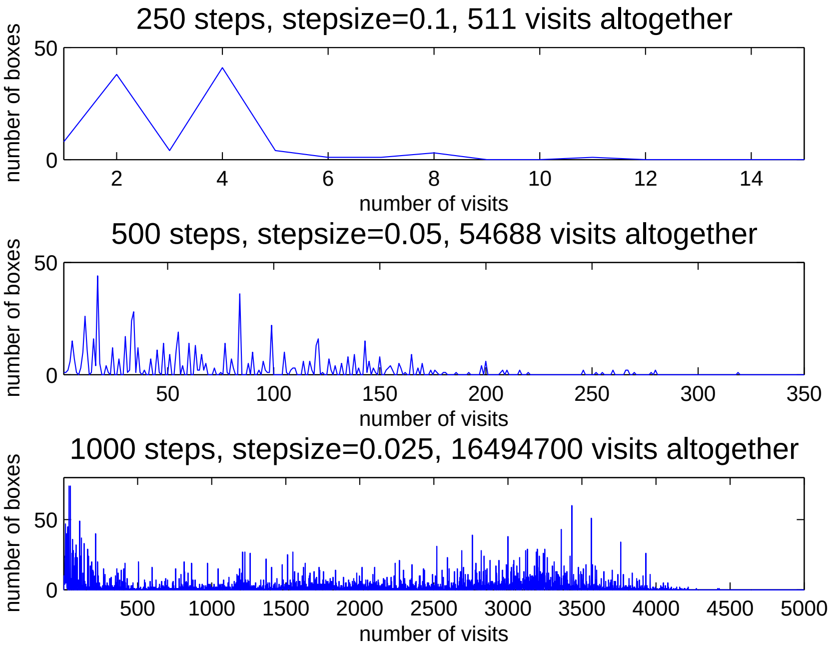

Figure 9: Distribution of hits in model (9)

4 Discussion

The quality of a numerical method is not only specified by its order of convergence, but also by its computational costs. For differential inclusions these costs depend on the behaviour of F in a very sensitive manner which makes a worst case analysis meaningless.

Theorems 1 and 2 indicate that reasonable convergence for a scheme with time step h can only be expected for grid spacings like $\rho = \mathcal{O}(h^\alpha)$ with $\sigma > 0$, $\alpha > 0$. The character of this result is independent of the underlying Euler scheme. The nature of the projection to a mesh is that of a round-off error which inevitably appears as an additive term in the final estimate.

In algorithms which discretize the images of F, the round-off error

$$\mathrm{dist}_H(F(x), c(F(x))) \le \rho$$

contributes an additive term of magnitude ρh to the local discretization error, because the latter is obtained by integration over an interval of lenght h. If the phase space is decomposed into boxes, the round-off error of magnitude

ρ constitutes an additive term of the local discretization error by definition, causing an additive term $\frac{\rho}{h}$ to appear in the error of convergence.

Hence we draw the conclusion that any fixed grid method with order of convergence $\mathcal{O}(h^p)$ needs a spatial discretization $\rho = \sigma h^p$ in the velocity space or $\rho = \sigma h^{p+1}$ in the phase space respectively. This finding does not encourage the development of higher order schemes, because for a given set $A \subset \mathbb{R}^d$, the set $c(A) \subset \Delta_\rho$ for suitable ρ is a very cheap approximation w.r.t. the Hausdorff distance for a desired accuracy. In most cases it will be considerably cheaper than the approximation of A by convex combinations of its extremal points as proposed in [4].

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