Homoclinic Trajectories of Non-Autonomous Maps

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Abstract

For non-autonomous difference equations of the form

$$x_{n+1} = f(x_n, \lambda_n), \quad n \in \mathbb{Z}$$

we consider homoclinic trajectories. These are pairs of trajectories that converge in both time directions towards each other. We derive a numerical method to approximate such homoclinic trajectories in two steps. In the first step one of the infinite trajectories is approximated by a finite segment and precise error estimates are given. In a subsequent step, a second trajectory that is homoclinic to the first one is computed as follows. We transform the original system into a topologically equivalent form, having zero as an n-independent fixed point. Then, the techniques, developed in Hüls (2006) apply and we gain an approximation of a non-autonomous homoclinic orbit, converging towards the origin. Transforming back to the original coordinates leads to the desired homoclinic trajectories. The approximation method and the validity of the error estimates are illustrated by an example.

Keywords: Non-autonomous discrete time dynamical systems, Homoclinic trajectories, Numerical approximation, Error analysis.

1 Introduction

For autonomous systems it is well known that the dynamics in a neighborhood of a homoclinic orbit is chaotic, see Smale (1967). Therefore, homoclinic orbits were

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analyzed in various studies, cf. Palis & Takens (1993) for an historical overview. Approximation results are of particular importance, see for example Beyn (1990) for continuous time systems as well as the current version of the bifurcation toolbox MATCONT Dhooge et al. (2003) for an implementation. For discrete time systems, we refer to Beyn & Hüls (2004), Beyn et al. (2004), Beyn & Kleinkauf (1997), Hüls (2005).

In several realistic applications from physics or mathematical biology, the limitation to autonomous systems is too restrictive. These models require the development of non-autonomous tools. In the preliminary article Hüls (2006) the non-autonomous difference equation

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z} \tag{1}$$

is considered. It is assumed that $f_n \in \mathcal{C}^\infty(\mathbb{R}^k, \mathbb{R}^k)$ is a diffeomorphism for all $n \in \mathbb{Z}$, having zero as an n-independent fixed point, i.e. $f_n(0) = 0$ for all $n \in \mathbb{Z}$. With respect to this fixed point, a homoclinic orbit is computed numerically in Hüls (2006). Note that the points of a homoclinic orbit lie in the intersection of the corresponding stable and unstable fiber bundles of the fixed point 0. These fiber bundles are the non-autonomous equivalent of the invariant manifolds in autonomous systems, cf. Hirsch et al. (1977), Pötzsche & Siegmund (2004).

More precisely, a homoclinic orbit $\bar{x}_\mathbb{Z} = (\bar{x}_n)_{n\in\mathbb{Z}}$ is a solution of (1), fulfilling $\lim_{n\to\pm\infty} \bar{x}_n = 0$. The proposed method for computing a finite approximation on some interval $J = [n_-, n_+] \cap \mathbb{Z}$, requires to solve the boundary value problem

$$0_\mathbb{Z} = \Gamma_J(y_J) := \Big(\big(y_{n+1} - f_n(y_n)\big)_{n=n_-,\dots,n_+-1}, b(y_{n_-}, y_{n_+}) \Big), \tag{2}$$

with an appropriately chosen boundary operator $b \in \mathcal{C}^1(\mathbb{R}^{2k}, \mathbb{R}^k)$, restricting the end points $y_{n_\pm}$, for example, to the unstable and stable subspace of the fixed matrix $Df_0(0)$, respectively, cf. Hüls (2006). Under reasonable assumptions, the boundary value problem (2) possesses a unique solution in a sufficiently small neighborhood of the exact solution.

In this paper, we push these ideas one step further by skipping the assumption that an n-independent fixed point exists. Then, the only candidate for the role of the fixed point $\bar{\xi} = 0$ from the previous setup, is a bounded trajectory $\bar{\xi}_\mathbb{Z}$ of (1), cf. Langa et al. (2002). Thus a homoclinic orbit $\bar{x}_\mathbb{Z}$ is a trajectory, converging in both time directions towards $\bar{\xi}_\mathbb{Z}$, i.e.

$$\lim_{n\to\pm\infty} \|\bar{x}_n - \bar{\xi}_n\| = 0. \tag{3}$$

On the other hand $\bar{\xi}_\mathbb{Z}$ is also a homoclinic orbit w.r.t. $\bar{x}_\mathbb{Z}$. Due to this symmetry, we call two trajectories homoclinic if they satisfy (3).

Systems of the form (1) are typically generated by parameter dependent maps, where the parameter varies in time. Therefore, we consider parameter-dependent systems of the form

$$x_{n+1} = f(x_n, \lambda_n), \quad n \in \mathbb{Z}, \tag{4}$$

where $\lambda_{\mathbb{Z}}$ denotes some sequence of parameter values. In this paper, we analyze the following problems:

(1) Determine a bounded solution $\bar{\xi}_{\mathbb{Z}}$ of (4), given the sequence $\bar{\lambda}_{\mathbb{Z}}$.

(2) Determine an orbit $\bar{x}_{\mathbb{Z}}$, homoclinic to $\bar{\xi}_{\mathbb{Z}}$.

Note that both trajectories are generally not known explicitly.

In Section 2, we first introduce our basic assumptions and prove dichotomy results for the variational equation. Then we derive an algorithm for the numerical approximation of the bounded trajectory $\bar{\xi}_{\mathbb{Z}}$. For the computations, we solve the boundary value problem $\Gamma_J(\xi_J) = 0$ on some interval $J = [n_-, n_+]$, using periodic boundary conditions. Doing so, the error at the outer points $\xi_{n_\pm}$ is quite large, since the boundary condition is not accurate and in addition each point $\bar{\xi}_n$ of the exact orbit depends on all parameter values $\bar{\lambda}_{\mathbb{Z}}$, cf. Figure 1. Fortunately, this influence decreases exponentially fast toward the middle of the interval, see Theorem 4. By taking only the inner points, cf. Theorem 5, we gain an approximation that is accurate up to any given accuracy. We state the corresponding algorithm in Section 2.3.

In Section 3, an algorithm for the approximation of a second trajectory $\bar{x}_{\mathbb{Z}}$ that is homoclinic to $\bar{\xi}_{\mathbb{Z}}$, is introduced. The idea is to consider the topologically equivalent system

$$y_{n+1} = f(y_n + \bar{\xi}_n) - \bar{\xi}_{n+1}, \quad n \in \mathbb{Z}$$

and apply the techniques from Hüls (2006) in order to approximate a homoclinic orbit w.r.t. the fixed point 0. Transforming back to the original coordinates, we finally obtain an approximation of the homoclinic trajectory $\bar{x}_{\mathbb{Z}}$.

For an illustration, we consider Hénon's map in Section 4. One of its parameters is chosen at random and we get a non-autonomous system of the form (4). We especially indicate that the approach gives us high accuracy approximations of bounded trajectories. Furthermore, homoclinic trajectories are computed numerically.

2 Approximation of bounded trajectories

Consider the non-autonomous difference equation

$$x_{n+1} = f_n(x_n), \quad n \in \mathbb{Z}. \tag{5}$$

In Hüls (2006) the existence of a fixed point of f_n is assumed that does not depend on n and approximation results for homoclinic orbits w.r.t. this fixed point are introduced.

In this paper, we consider a more general setup in which an n-independent fixed point does not exist. The only replacement for a fixed point is a complete trajectory, which is a solution $\xi_{\mathbb{Z}}$ of (5), cf. Langa et al. (2002). Thus, the non-autonomous analog of a homoclinic orbit $\bar{x}_{\mathbb{Z}}$, converging in both time directions towards a fixed point $\bar{\xi}$, is a trajectory $\bar{x}_{\mathbb{Z}}$ that converges towards another trajectory $\bar{\xi}_{\mathbb{Z}}$. These trajectories are called homoclinic. Note that if a trajectory $\bar{x}_{\mathbb{Z}}$ is homoclinic to $\bar{\xi}_{\mathbb{Z}}$ i.e.

$$\lim_{n\to\pm\infty} \|\bar{x}_n - \bar{\xi}_n\| = 0, \tag{6}$$

then $\bar{\xi}_{\mathbb{Z}}$ is also homoclinic to $\bar{x}_{\mathbb{Z}}$.

Definition 1 *Let $\bar{\xi}_{\mathbb{Z}}$ and $\bar{x}_{\mathbb{Z}}$ be two solutions of (5). These trajectories are* ***homoclinic*** *to each other, if (6) holds.*

Non-autonomous difference equations of the form (5) occur in several applications in form of parameter dependent maps, in which the parameter varies in time. Therefore, we restrict ourselves to the non-autonomous difference equation

$$x_{n+1} = f(x_n, \lambda_n), \quad n \in \mathbb{Z}, \tag{7}$$

where $\lambda_{\mathbb{Z}} = (\lambda_n)_{n\in\mathbb{Z}}$ is a bounded sequence. Throughout this paper, we address questions of the following type. Assume $\lambda_{\mathbb{Z}}$ and therefore the non-autonomous family $f_n = f(\cdot, \lambda_n)$, $n \in \mathbb{Z}$ is given. Can one approximate bounded or homoclinic trajectories of (7) with high accuracy?

First, we impose the following assumptions on f.

A1 $f \in \mathcal{C}^\infty(\mathbb{R}^k \times \mathbb{R}, \mathbb{R}^k)$ and $f(\cdot, \lambda)$ is a diffeomorphism for all $\lambda \in \mathbb{R}$.

A2 There exists a sequence $\bar{\lambda}_{\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ such that (7) possesses the bounded solution $\bar{\xi}_{\mathbb{Z}}$.

A3 The variational equation

$$u_{n+1} = D_x f(\bar{\xi}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z}$$

possesses an exponential dichotomy on $\mathbb{Z}$, cf. Definition 9.

Let $J = [n_-, n_+] \cap \mathbb{Z}$ be a discrete interval, where the cases $n_- = -\infty$ and $n_+ = \infty$ are included. We define the space of bounded sequences on J w.r.t. $\|\cdot\|$ by

$$X_J := \left\{ u_J = (u_n)_{n\in J} \in (\mathbb{R}^k)^J : \sup_{n\in J} \|u_n\| < \infty \right\}$$

and denote by 0_J the zero element in X_J.

For a given sequence $\lambda_{\mathbb{Z}}$, an orbit $x_{\mathbb{Z}} \in X_{\mathbb{Z}}$, i.e. a solution of (7), is a zero of the operator $\Gamma : X_{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}} \to X_{\mathbb{Z}}$, defined as

$$\Gamma(\xi_{\mathbb{Z}}, \lambda_{\mathbb{Z}}) := \big(\xi_{n+1} - f(\xi_n, \lambda_n)\big)_{n\in\mathbb{Z}}. \tag{8}$$

Let $\bar{\lambda}_{\mathbb{Z}}$ be the sequence from assumption **A2**. In general, the bounded solution $\bar{\xi}_{\mathbb{Z}}$ is not known explicitly. Even worse, the sequence $\bar{\xi}_{\mathbb{Z}}$ is not convergent. The main task, we consider in this section, is to compute an approximation of this bounded trajectory. First, we prove that a bounded trajectory also exists in some neighborhood of $\bar{\xi}_{\mathbb{Z}}$, if the parameter sequence $\lambda_{\mathbb{Z}}$ varies slightly around $\bar{\lambda}_{\mathbb{Z}}$.

Lemma 2 *Assume **A1**–**A3**. Then there exist two neighborhoods $U(\bar{\lambda}_{\mathbb{Z}})$ and $V(\bar{\xi}_{\mathbb{Z}})$, such that*

$$\Gamma(\xi_{\mathbb{Z}}, \lambda_{\mathbb{Z}}) = 0_{\mathbb{Z}} \tag{9}$$

has for all $\lambda_{\mathbb{Z}} \in U(\bar{\lambda}_{\mathbb{Z}})$ a unique solution $\xi_{\mathbb{Z}} \in V(\bar{\xi}_{\mathbb{Z}})$.

Proof: Since $\Gamma(\bar{\xi}_{\mathbb{Z}}, \bar{\lambda}_{\mathbb{Z}}) = 0_{\mathbb{Z}}$, the assertion follows from the implicit function theorem, if $D_x\Gamma(\bar{\xi}_{\mathbb{Z}}, \bar{\lambda}_{\mathbb{Z}})$ is invertible.

Note that $u_{\mathbb{Z}}$ is a solution of $D_x\Gamma(\bar{\xi}_{\mathbb{Z}}, \bar{\lambda}_{\mathbb{Z}})u_{\mathbb{Z}} = 0$ if and only if

$$u_{n+1} = D_x f(\bar{\xi}_n, \bar{\lambda}_n)u_n, \quad \text{for all } n \in \mathbb{Z}. \tag{10}$$

By assumption **A3**, equation (10) possesses an exponential dichotomy on $\mathbb{Z}$. Therefore, $u_{\mathbb{Z}} = 0_{\mathbb{Z}}$ is the only bounded solution of (10).

■

Let $\lambda_{\mathbb{Z}} \in U(\bar{\lambda}_{\mathbb{Z}})$ and denote by $\xi_{\mathbb{Z}}$ the unique bounded solution of (7) in $V(\bar{\xi}_{\mathbb{Z}})$. The next lemma shows that the variational equation

$$u_{n+1} = D_x f(\xi_n, \lambda_n)u_n, \quad n \in \mathbb{Z} \tag{11}$$

possesses an exponential dichotomy on $\mathbb{Z}$.

Lemma 3 *Assume **A1**–**A3**. Then a neighborhood V of $\bar{\xi}_{\mathbb{Z}}$ exists, such that the difference equation*

$$u_{n+1} = D_x f(\varrho_n, \lambda_n)u_n, \quad n \in \mathbb{Z}$$

possesses for $\varrho_{\mathbb{Z}} \in V$, $\lambda_{\mathbb{Z}} \in U(\bar{\lambda}_{\mathbb{Z}})$ an exponential dichotomy on $\mathbb{Z}$. The dichotomy constants do not depend on the specific sequence $\varrho_{\mathbb{Z}}$.

Proof: Due to assumption **A3**, the difference equation (10) has an exponential dichotomy on $\mathbb{Z}$. An application of the Roughness-Theorem 10 guarantees the existence of an exponential dichotomy of the perturbed equation

$$u_{n+1} = \left(D_x f(\bar{\xi}_n, \bar{\lambda}_n) + \left[D_x f(\varrho_n, \lambda_n) - D_x f(\bar{\xi}_n, \bar{\lambda}_n)\right]\right) u_n, \quad n \in \mathbb{Z}$$

if

$$\|D_x f(\varrho_n, \lambda_n) - D_x f(\bar{\xi}_n, \bar{\lambda}_n)\| \le \beta \tag{12}$$

holds, where β is specified in Theorem 10. But the inequality (12) is satisfied if V is chosen sufficiently small.

■

For sufficiently small $\|\lambda_{\mathbb{Z}} - \bar{\lambda}_{\mathbb{Z}}\|$, it holds that $\xi_{\mathbb{Z}} \in V$, and consequently, (11) possesses an exponential dichotomy.

In the following, we denote by V the sufficiently small convex neighborhood of $\bar{\xi}_\mathbb{Z}$ from Lemma 3. Let $\lambda_\mathbb{Z} \in U(\bar{\lambda}_\mathbb{Z})$ and denote by $\xi_\mathbb{Z}(\lambda_\mathbb{Z})$ the unique solution of (9), cf. Lemma 2. We choose U such that $\xi_\mathbb{Z}(\lambda_\mathbb{Z}) \in V$ holds for all $\lambda_\mathbb{Z} \in U$.

For the numerical approximation of the bounded trajectory $\bar{\xi}_\mathbb{Z}$ we introduce an algorithm and derive error estimates in the three steps. First, we prove in Section 2.1 that the difference between two solutions of (7) for different sequences $\lambda_\mathbb{Z}$, $\mu_\mathbb{Z}$ that coincide on some interval J, decreases exponentially fast towards the middle. Then, an approximation result for bounded trajectories is introduced in Section 2.2, assuming that the parameter sequences is convergent, i.e. $\lim_{n\to\pm\infty} \lambda_n = \lambda$. Combining these results, we gain an approximation for arbitrary trajectories in Section 2.3.

2.1 Bounded trajectories with varying tails

Assume that the sequence $\bar{\lambda}_\mathbb{Z}$ is given, cf. assumption **A2**. For computing a finite approximation z_J of the bounded trajectory $\bar{\xi}_\mathbb{Z}$, we introduce the boundary value problem, cf. Hüls (2006)

$$\Gamma_J(z_J, \bar{\lambda}_J) := \Big(\big(z_{n+1} - f(z_n, \bar{\lambda}_n)\big)_{n\in\tilde{J}}, b(z_{n_-}, z_{n_+}) \Big) = 0_J, \tag{13}$$

where $J = [n_-, n_+]$ and $\tilde{J} = [n_-, n_+ - 1]$ are finite intervals. Obviously, the finite middle part of the sequence $\bar{\xi}_\mathbb{Z}$, denoted by $\bar{\xi}_J$, depends on $\bar{\lambda}_J$ but also on the parameters $\bar{\lambda}_n$, $n \notin J$. On the other hand, the finite approximations, i.e. the solutions of the boundary value problem (13), coincide for all sequences $\bar{\mu}_\mathbb{Z}$ and $\bar{\lambda}_\mathbb{Z}$ fulfilling $\bar{\mu}_n = \bar{\lambda}_n$ for $n \in J$. Thus, no matter what boundary operator we choose, we will have a relatively large approximation error at the boundary. For numerical calculations we choose periodic boundary conditions

$$b_{\text{per}}(x, y) := (x - y). \tag{14}$$

Fortunately, the influence of the outer points decreases exponentially fast towards the middle of the interval J as one can see from the following theorem.

Theorem 4 *Assume* ***A1**–**A3**. Let J be a finite interval and U, V are given as stated above. Choose $\lambda_\mathbb{Z}$, $\mu_\mathbb{Z} \in U$ such that $\lambda_n = \mu_n$ for $n \in J$. Denote by $\xi_\mathbb{Z}$, $\zeta_\mathbb{Z} \in V$ the bounded solutions w.r.t. $\lambda_\mathbb{Z}$ and $\mu_\mathbb{Z}$, respectively, cf. Lemma 2. Then there exist two constants C, $\alpha > 0$ that do not depend on $\lambda_\mathbb{Z}$ and $\mu_\mathbb{Z}$, such that*

$$\|\xi_n - \zeta_n\| \le C \left(\mathrm{e}^{-\alpha(n-n_-)} + \mathrm{e}^{-\alpha(n_+-n)} \right) \tag{15}$$

holds for all $n \in J$.

Proof: Due to our assumptions it holds that

$$\xi_{n+1} = f(\xi_n, \lambda_n) \quad \text{and} \quad \zeta_{n+1} = f(\zeta_n, \mu_n), \quad n \in \mathbb{Z}.$$

Let $d_{\mathbb{Z}} = \zeta_{\mathbb{Z}} - \xi_{\mathbb{Z}}$ and $h_{\mathbb{Z}} = \mu_{\mathbb{Z}} - \lambda_{\mathbb{Z}}$. Then $d_{\mathbb{Z}}$ is a solution of the following difference equation

$$\begin{aligned} d_{n+1} &= f(\xi_n + d_n, \lambda_n + h_n) - f(\xi_n, \lambda_n) \\ &= f(\xi_n + d_n, \lambda_n) + \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau \ h_n - f(\xi_n, \lambda_n) \\ &= f(\xi_n, \lambda_n) + \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau \ d_n \\ &\quad + \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau \ h_n - f(\xi_n, \lambda_n) \\ &= \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau \ d_n + \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau \ h_n. \end{aligned}$$

The homogeneous difference equation

$$u_{n+1} = D_x f(\bar{\xi}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z}$$

possesses by assumption **A3** an exponential dichotomy on $\mathbb{Z}$. Let $\beta > 0$ as demanded in the Roughness-Theorem 10. From the construction of U and V and (12), we get

$$\sup_{n\in\mathbb{Z}} \left\| \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n) - D_x f(\bar{\xi}_n, \bar{\lambda}_n)\mathrm{d}\tau \right\| \le \sup_{n\in\mathbb{Z}} \int_0^1 \beta \mathrm{d}\tau = \beta.$$

Applying the Roughness-Theorem 10, we get an exponential dichotomy on $\mathbb{Z}$ of the difference equation

$$u_{n+1} = A_n u_n, \quad A_n = \int_0^1 D_x f(\xi_n + \tau d_n, \lambda_n)\mathrm{d}\tau, \quad n \in \mathbb{Z}. \tag{16}$$

Let $(K, \alpha, P_n^s, P_n^u)$ be the corresponding dichotomy data and denote the solution operator of (16) by Φ, i.e. $u_n = \Phi(n, m)u_m$ for all $n, \ m \in \mathbb{Z}$.

Now, consider the inhomogeneous difference equation

$$u_{n+1} = A_n u_n + r_n, \quad r_n = \int_0^1 D_\lambda f(\xi_n + d_n, \lambda_n + \tau h_n)\mathrm{d}\tau \ h_n, \quad n \in \mathbb{Z}. \tag{17}$$

The unique bounded solution of (17) on $\mathbb{Z}$ is

$$u_n = \sum_{m\in\mathbb{Z}} G(n, m+1) r_m,$$

cf. Palmer (1988), where G is Green's function, defined as

$$G(n,m) = \begin{cases} \Phi(n,m)P_m^s, & n \ge m, \\ -\Phi(n,m)P_m^u, & n < m. \end{cases} \tag{18}$$

Since (16) possesses an exponential dichotomy the following estimates hold

$$\|G(n,m)\| = \|\Phi(n,m)P_m^s\| \le K\mathrm{e}^{-\alpha(n-m)}, \quad \text{for } n \ge m, \tag{19}$$
$$\|G(n,m)\| = \|\Phi(n,m)P_m^u\| \le K\mathrm{e}^{-\alpha(m-n)}, \quad \text{for } n < m. \tag{20}$$

Note that due to our assumptions $\|r_n\|$ is bounded from above by some constant R for all $n \in \mathbb{Z}$ and $r_n = 0$ for $n \in J = [n_-, n_+]$, since $\lambda_n = \mu_n$ for $n \in J$.

For $n \in J$ we derive an estimate of $\|u_n\|$

$$\begin{aligned}
\|u_n\| &\le \sum_{m\in\mathbb{Z}} \|G(n,m+1)r_m\| \\
&= \sum_{m=-\infty}^{n_- -1} \|G(n,m+1)r_m\| + \sum_{m=n_+ +1}^{\infty} \|G(n,m+1)r_m\| \\
&= \sum_{m=-\infty}^{n_- -1} \|\Phi(n,m+1)P_{m+1}^s r_m\| + \sum_{m=n_+ +1}^{\infty} \|\Phi(n,m+1)P_{m+1}^u r_m\| \qquad (21)\\
&\le \sum_{m=-\infty}^{n_- -1} RK\mathrm{e}^{-\alpha(n-m-1)} + \sum_{m=n_+ +1}^{\infty} RK\mathrm{e}^{-\alpha(m+1-n)} \\
&= RK\left(\sum_{m=-\infty}^{0} \mathrm{e}^{-\alpha(n-m-n_-)} + \sum_{m=0}^{\infty} \mathrm{e}^{-\alpha(m+n_+ +2-n)}\right) \\
&= \frac{RK}{1-\mathrm{e}^{-\alpha}}\left(\mathrm{e}^{-\alpha(n-n_-)} + \mathrm{e}^{-\alpha(n_+ -n+2)}\right).
\end{aligned}$$

By construction, $d_{\mathbb{Z}}$ is the bounded solution of (17), thus the estimate

$$\|d_n\| \le C\left(\mathrm{e}^{-\alpha(n-n_-)} + \mathrm{e}^{-\alpha(n_+ -n)}\right)$$

holds for all $n \in J$ with the constant $C = \frac{RK}{1-\mathrm{e}^{-\alpha}}$.

■

2.2 Approximation of bounded trajectories with constant tails

From the previous section, we know that for two given sequences $\bar{\lambda}_{\mathbb{Z}}$, $\bar{\mu}_{\mathbb{Z}}$ that coincide on the interval J, the corresponding solutions $\bar{\xi}_{\mathbb{Z}}$, $\bar{\zeta}_{\mathbb{Z}}$ of (7) are exponentially close in the middle of J. On the other hand, the solution of the boundary value problem (13) does not depend on $\bar{\lambda}_n$ for $n \notin J$.

In this section, we prove an approximation theorem for bounded trajectories $\bar{\zeta}_{\mathbb{Z}}$ in case the parameter sequences $\bar{\mu}_{\mathbb{Z}}$ is convergent. We impose the following assumptions.

A4 A sequence $\bar{\mu}_{\mathbb{Z}} \in U$ with corresponding solution $\bar{\zeta}_{\mathbb{Z}} \in V$ of (7) and $\bar{\mu} \in \mathbb{R}, \bar{\zeta} \in \mathbb{R}^k$ exist, such that

$$\lim_{n \to +\infty} \bar{\mu}_n = \lim_{n \to -\infty} \bar{\mu}_n =: \bar{\mu} \quad \text{and} \quad \lim_{n \to +\infty} \bar{\zeta}_n = \lim_{n \to -\infty} \bar{\zeta}_n =: \bar{\zeta}. \tag{22}$$

Note that if $\bar{\lambda}_{\mathbb{Z}}$ and $\bar{\xi}_{\mathbb{Z}}$, introduced in assumption **A2**, do not vary to much as $n \to \pm\infty$, sequences $\bar{\mu}_{\mathbb{Z}} \in U$, $\bar{\zeta}_{\mathbb{Z}} \in V$ exist that satisfy (22). On the other hand, the condition **A4** is naturally fulfilled in several applications. Let $\bar{\zeta}$ be a hyperbolic fixed point of $f(\cdot, \bar{\mu})$. For the constant sequence $\bar{\mu}_{\mathbb{Z}}$ defined as $\bar{\mu}_n = \bar{\mu}$, the constant trajectory $\bar{\zeta}_{\mathbb{Z}}$ ($\bar{\zeta}_n = \bar{\zeta}$) obviously is a solution of (7). By Lemma 2 a generally non-constant bounded trajectory $\bar{\xi}_{\mathbb{Z}}$ exist for $\bar{\lambda}_{\mathbb{Z}} \in U(\bar{\mu}_{\mathbb{Z}})$, fulfilling **A2**, see the example in Section 4.

We assume that $\bar{\mu}_{\mathbb{Z}}$ is given as in assumption **A4** and approximate the bounded trajectory $\bar{\zeta}_{\mathbb{Z}}$ by a finite orbit segment.

Theorem 5 *Assume* ***A1**–**A4**. Then constants δ, N, $C > 0$ exist, such that the approximating system $\Gamma_J(z_J, \bar{\mu}_J) = 0$, cf. (13), with periodic boundary conditions (14), possesses a unique solution*

$$z_J \in B_\delta(\bar{\zeta}_J) \quad \textit{for } J = [n_-, n_+], \ -n_-, \ n_+ \geq N.$$

The approximation error can be estimated as

$$\|\bar{\zeta}_J - z_J\| \leq C \|\bar{\zeta}_{n_-} - \bar{\zeta}_{n_+}\|. \tag{23}$$

Proof: First, we show that $D_1\Gamma_J(\bar{\zeta}_J, \bar{\mu}_J)$ has for sufficiently large intervals J a uniformly bounded inverse.

Let $(y_{\tilde{J}}, r) \in X_{\tilde{J}} \times \mathbb{R}^k$. Then the inhomogeneous equation $D_1\Gamma_J(\bar{\zeta}_J, \bar{\mu}_J)u_J = (y_{\tilde{J}}, r)$ is equivalent to

$$u_{n+1} - D_x f(\bar{\zeta}_n, \bar{\mu}_n) u_n = y_n, \quad n \in \tilde{J}, \tag{24}$$

$$u_{n_-} - u_{n_+} = r. \tag{25}$$

Let Φ be the solution operator of the homogeneous equation

$$u_{n+1} = D_x f(\bar{\zeta}_n, \bar{\mu}_n) u_n, \quad n \in \mathbb{Z}.$$

This difference equation possesses according to Lemma 3 an exponential dichotomy on $\mathbb{Z}$. Therefore, any solution of (24) is of the form

$$u_n = \Phi(n, 0)v + \sum_{m \in \tilde{J}} G(n, m+1) y_m, \quad n \in J,$$

with some $v \in \mathbb{R}^k$. Here, G denotes Green's function, introduced in (18). We introduce the following decomposition of v

$$v = \Phi(0, n_-) V_{n_-}^{-1} v_- + \Phi(0, n_+) W_{n_+}^{-1} v_+, \quad v_- \in \mathcal{R}(P^s), \ v_+ \in \mathcal{R}(P^u),$$

where P^s, P^u denote the constant dichotomy projectors of $u_{n+1} = D_x f(\bar{\zeta}, \bar{\mu}) u_n$, $n \in \mathbb{Z}$ and

$$\begin{aligned} V_{n_-} &:= I + P^s - P^s_{n_-} : \mathcal{R}(P^s_{n_-}) \to \mathcal{R}(P^s), \\ W_{n_+} &:= I + P^u - P^u_{n_+} : \mathcal{R}(P^u_{n_+}) \to \mathcal{R}(P^u). \end{aligned}$$

Since $P^{s,u}_n \to P^{s,u}$ as $n \to \pm\infty$, V_{n_-} and W_{n_+} have a uniformly bounded inverse for sufficiently large $-n_-, n_+$. In this notation, (25) reads

$$V_{n_-}^{-1} v_- + \Phi(n_-, n_+) W_{n_+}^{-1} v_+ - \Phi(n_+, n_-) V_{n_-}^{-1} v_- - W_{n_+}^{-1} v_+ = R, \qquad (26)$$

where $R = \sum_{m \in \tilde{J}} G(n_+, m+1) y_m - \sum_{m \in \tilde{J}} G(n_-, m+1) y_m + r$. Employing the dichotomy estimates, the second and third term in (26) converge exponentially fast to 0 as $n_\pm \to \pm\infty$ and it follows that (26) has a unique solution for sufficiently large $-n_-$, n_+. From (19) and (20), we get with some generic constant $C > 0$ the uniform estimate $\|R\| \le C\left(\|y_{\tilde{J}}\| + \|r\|\right)$ and therefore $\|v_\pm\| \le C\left(\|y_{\tilde{J}}\| + \|r\|\right)$. Furthermore, using the dichotomy estimates we get

$$\begin{aligned} \|\Phi(n,0)v\| &\le \|\Phi(n, n_-) P^s_{n_-}\| \|V_{n_-}^{-1} v_-\| + \|\Phi(n, n_+) P^u_{n_+}\| \|W_{n_+}^{-1} v_+\| \\ &\le C\left(\|v_-\| + \|v_+\|\right). \end{aligned}$$

Thus, a J-independent constant σ exists, such that

$$\|u_J\| \le \sigma^{-1}\left(\|y_{\tilde{J}}\| + \|r\|\right),$$

and consequently $\|D_1 \Gamma_J(\bar{\zeta}_J, \bar{\mu}_J)^{-1}\| \le \sigma^{-1}$.

The remaining part of the proof is an application of Lemma 11 with the setting

$$Y = (X_J, \|\cdot\|), \quad Z = \left(X_{\tilde{J}} \times \mathbb{R}^k, \|\cdot\| + \|\cdot\|\right), \quad F = \Gamma_J(\cdot, \bar{\mu}_J), \quad y_0 = \bar{\zeta}_J.$$

We show that assumption (41) of Lemma 11 is fulfilled. By assumption **A1** there exists a $\delta > 0$ such that

$$\|D_1 \Gamma_J(z_J, \bar{\mu}_J) - D_1 \Gamma_J(\bar{\zeta}_J, \bar{\mu}_J)\| \le \sup_{n \in \tilde{J}} \|D_x f(z_n, \bar{\mu}_n) - D_x f(\bar{\zeta}_n, \bar{\mu}_n)\| \le \frac{\sigma}{2}$$

for $z_J \in B_\delta(\bar{\zeta}_J)$.

Assumption (42) can also be verified with $\kappa = \frac{\sigma}{2}$:

$$\|\Gamma_J(\bar{\zeta}_J, \bar{\mu}_J)\| = \left\| \left(\bar{\zeta}_{n+1} - f(\bar{\zeta}_n, \bar{\mu}_n)\right)_{n \in \tilde{J}} \right\| + \|b(\bar{\zeta}_{n_-}, \bar{\zeta}_{n_+})\| = \|\bar{\zeta}_{n_-} - \bar{\zeta}_{n_+}\| \le \frac{\sigma}{2}\delta$$

holds due to assumption **A4** for sufficiently large $-n_-$, n_+.

By Lemma 11 a unique solution z_J of $\Gamma_J(z_J, \bar{\mu}_J) = 0$ exists in $B_\delta(\bar{\zeta}_J)$ for J sufficiently large, and an estimate of the approximation error follows from (44):

$$\|\bar{\zeta}_J - z_J\| \le \frac{1}{\sigma - \kappa} \|\Gamma_J(\bar{\zeta}_J, \bar{\mu}_J) - \Gamma_J(z_J, \bar{\mu}_J)\| = \frac{2}{\sigma} \|\bar{\zeta}_{n_-} - \bar{\zeta}_{n_+}\|.$$

■

2.3 Approximation of bounded trajectories with varying tails

Let $\bar{\lambda}_{\mathbb{Z}}$ be the given sequence from assumption **A2**. Denote by $J = [n_-, n_+]$ a finite interval. Combining the previous results, we show that one obtains an approximation of the trajectory $\bar{\xi}_{\mathbb{Z}} \in V$ on the finite interval J that is accurate up to any given accuracy Δ. The main idea is to compute an approximation of the orbit on a longer interval. Since the approximation errors occur at the boundary of this interval, we only take the accurate middle part. Formally, we carry out these computations in two steps.

In the first step, we define for an interval $\bar{J}$ the sequence $\bar{\mu}^{\bar{J}}_{\mathbb{Z}}$ by

$$\bar{\mu}^{\bar{J}}_n = \begin{cases} \bar{\lambda}_n, & \text{for } n \in [\bar{n}_-, \bar{n}_+], \\ \bar{\mu}, & \text{for } n \notin [\bar{n}_-, \bar{n}_+], \end{cases}$$

where $\bar{\mu}$ is defined as in **A4**. Denote by $\bar{\zeta}_{\mathbb{Z}} = \bar{\zeta}_{\mathbb{Z}}(\bar{J})$ the unique bounded solution of

$$\zeta_{n+1} = f(\zeta_n, \bar{\mu}^{\bar{J}}_n), \quad n \in \mathbb{Z}$$

in V (Lemma 2). For sufficiently large intervals $\bar{J}$, a non-empty interval $J \subset \bar{J}$ can be chosen due to Theorem 4, such that

$$\|\bar{\xi}_n - \bar{\zeta}_n\| \leq \frac{\Delta}{2} \quad \text{holds for all } n \in J. \tag{27}$$

In the second step, we compute a finite approximation of $\bar{\zeta}_{\mathbb{Z}}$ by solving (13). Due to Theorem 5, an interval $\hat{J} \supset J$ exists, such that the following error estimate holds, cf. (23):

$$\|\bar{\zeta}_{\hat{J}} - z_{\hat{J}}\| \leq \frac{\Delta}{2}. \tag{28}$$

Combining the results (27) and (28), we get for $n \in J$

$$\|\bar{\xi}_n - z_n\| \leq \|\bar{\xi}_n - \bar{\zeta}_n\| + \|\bar{\zeta}_n - z_n\| \leq \frac{\Delta}{2} + \frac{\Delta}{2} = \Delta.$$

Thus, the middle part z_J is a finite approximation of $\bar{\xi}_{\mathbb{Z}}$ on the interval J with accuracy Δ.

This algorithm is illustrated in Section 4.1 by an example.

3 Homoclinic trajectories

Applying the approach introduce in the previous section, we gain for a given parameter sequence $\bar{\lambda}_{\mathbb{Z}}$ a finite approximation of the bounded trajectory $\bar{\xi}_{\mathbb{Z}}$ of (7). In the following, we derive an algorithm for computing a second trajectory $\bar{x}_{\mathbb{Z}}$ that is homoclinic to the first one, see Definition 1. First, we assume existence as well as transversality.

A5 Let $\bar{\lambda}_{\mathbb{Z}}$ as in **A2**. For this parameter sequence a solution $\bar{x}_{\mathbb{Z}}$ of

$$x_{n+1} = f(x_n, \bar{\lambda}_n), \quad n \in \mathbb{Z}$$

exists, that is homoclinic to $\bar{\xi}_{\mathbb{Z}}$ and non-trivial, i.e. $\bar{x}_{\mathbb{Z}} \neq \bar{\xi}_{\mathbb{Z}}$.

A6 The trajectory $\bar{x}_{\mathbb{Z}}$ is transversal, i.e.

$$u_{n+1} = D_x f(\bar{x}_n, \bar{\lambda}_n) u_n,\ n \in \mathbb{Z} \text{ for } u_{\mathbb{Z}} \in X_{\mathbb{Z}} \quad \Longleftrightarrow \quad u_{\mathbb{Z}} = 0.$$

First, we prove that the exponential dichotomy of the variational equation

$$u_{n+1} = D_x f(\bar{\xi}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z} \tag{29}$$

implies exponential dichotomies on $\mathbb{Z}^-$ and on $\mathbb{Z}^+$ of the difference equation

$$u_{n+1} = D_x f(\bar{x}_n, \bar{\lambda}_n) u_n, \quad n \in \mathbb{Z}. \tag{30}$$

Lemma 6 *Assume **A1**–**A5**. Then the difference equation (30) possesses exponential dichotomies on $\mathbb{Z}^-$ and on $\mathbb{Z}^+$.*

Proof: Since (29) possesses an exponential dichotomy on $\mathbb{Z}$ and $\|\bar{x}_n - \bar{\xi}_n\| \to 0$ as $n \to \pm\infty$, there exists an $N > 0$, such that

$$\left\| D_x f(\bar{x}_n, \bar{\lambda}_n) - D_x f(\bar{\xi}_n, \bar{\lambda}_n) \right\| < \beta, \quad \text{for all } |n| \geq N,$$

where β is the bound for the additive perturbation in Theorem 10. Thus, (30) possesses exponential dichotomies on $(-\infty, -N]$ and on $[N, \infty)$ which can be extended to $\mathbb{Z}^-$ and $\mathbb{Z}^+$, respectively.

■

After these preparations, we introduce techniques for the numerical approximation of the second trajectory $\bar{x}_{\mathbb{Z}}$ which is homoclinic to the first one. The main idea is to transform the system (7) into a topologically equivalent form, having zero as an n-independent fixed point. Then we apply an approach for approximating homoclinic orbits w.r.t. a constant fixed point to the transformed system. The corresponding algorithm is introduced in Hüls (2006).

First, a topologically equivalent system is introduced, having a constant fixed point. We refer to Aulbach & Wanner (2003), where the notion of topological equivalence for non-autonomous systems is introduced.

Lemma 7 *Assume **A1**–**A5**. Let $T_n(y) := y + \bar{\xi}_n$ and*

$$g_n(y) := T_{n+1}^{-1} \circ f(\cdot, \bar{\lambda}_n) \circ T_n(y).$$

Then the two difference equations (7) and

$$y_{n+1} = g_n(y_n), \quad n \in \mathbb{Z} \tag{31}$$

are topologically equivalent and $0_{\mathbb{Z}}$ is a solution of (31).

Proof: For proving topological equivalence, we show that orbits of (31) transform into orbits of (7) and vice versa.

Let $y_{\mathbb{Z}}$ be an orbit of (31). Then

$$x_n := T_n(y_n), \quad n \in \mathbb{Z}$$

is a solution of (7), since

$$\begin{aligned} x_{n+1} &= T_{n+1}(y_{n+1}) = T_{n+1} \circ T_{n+1}^{-1} \circ f(\cdot, \bar{\lambda}_n) \circ T_n(y_n) \\ &= f(\cdot, \bar{\lambda}_n) \circ T_n(y_n) = f(x_n, \bar{\lambda}_n). \end{aligned}$$

Furthermore,

$$\begin{aligned} g_n(0) &= T_{n+1}^{-1} \circ f(\cdot, \bar{\lambda}_n) \circ T_n(0) \\ &= T_{n+1}^{-1} \circ f(\bar{\xi}_n, \bar{\lambda}_n) = T_{n+1}^{-1}(\bar{\xi}_{n+1}) = 0, \quad \text{for all } n \in \mathbb{Z}. \end{aligned}$$

■

As a consequence, the task of computing a second trajectory of (7) that is homoclinic to $\bar{\xi}_{\mathbb{Z}}$ is equivalent to the computation of a homoclinic orbit $\bar{y}_{\mathbb{Z}}$ of (31) w.r.t. the fixed point 0.

In the second case, we obtain a finite approximation of the homoclinic orbit $\bar{y}_{\mathbb{Z}}$ on the interval J by solving

$$\Gamma_J(y_J) = \big((y_{n+1} - g_n(y_n))_{n \in \tilde{J}}, b(y_{n_-}, y_{n_+})\big) = 0_J, \tag{32}$$

where the boundary operator $b \in \mathcal{C}^1(\mathbb{R}^{2k}, \mathbb{R}^k)$ is chosen such that the end points $y_{n_\pm}$ are restricted to the unstable and stable subspace of the constant matrix $Dg_0(0)$, cf. Hüls (2006). Formally, we define the projection boundary operator as

$$b(x, y) := \begin{pmatrix} Y_s^T x \\ Y_u^T y \end{pmatrix}, \quad x, y \in \mathbb{R}^k, \tag{33}$$

where the columns of Y_s and Y_u form an orthogonal basis of the stable and unstable subspace of $Dg_0(0)^T$. Note that the stable and unstable subspace of $Dg_0(0)^T$ is orthogonal to the unstable and stable subspace of $Dg_0(0)$, respectively. We do not construct a boundary operator, restricting the end points to the linearizations of the corresponding unstable and stable fiber bundles. These fiber bundles are generally not known explicitly, cf. Hirsch et al. (1977) for the case of normally hyperbolic manifolds. Furthermore, the effort to approximate these fiber bundles numerically is not justified by a slightly better rate of convergence of the above described algorithm, see Hüls (2006).

By assumption **A5**, $\bar{x}_{\mathbb{Z}}$ and $\bar{\xi}_{\mathbb{Z}}$ are homoclinic trajectories, and $\bar{y}_{\mathbb{Z}}$ defined as

$$\bar{y}_n = T_n^{-1}(\bar{x}_n) = \bar{x}_n - \bar{\xi}_n, \quad n \in \mathbb{Z}$$

is a homoclinic orbit of (31) w.r.t. the fixed point 0. Furthermore, the variational equations (30) and

$$u_{n+1} = Dg_n(\bar{y}_n)u_n, \quad n \in \mathbb{Z} \tag{34}$$

coincide. Consequently, (34) possesses an exponential dichotomy with the same data as (30), and the transversality assumption **A6** holds for the transformed system, too. Geometrically, the transversality assumption **A6** states that the corresponding stable and unstable fiber bundles of the fixed point 0 intersect transversally for the transformed system (31), cf. (Hüls 2006, Lemma 3.7). The following theorem, cf. (Hüls 2006, Theorem 4.2), applies and guarantees (local) existence of a solution of (32) and therefore (local) well-posedness of our approach.

Theorem 8 *Assume **A1**–**A6**. There exist constants δ, N, $C > 0$, such that the approximating system $\Gamma_J(y_J) = 0$ possesses a unique solution*

$$y_J \in B_\delta(\bar{y}_{|J}) \quad \textit{for all } J = [n_-, n_+],$$

where $-n_-$, $n_+ \geq N$. The approximation error can be estimated as

$$\|\bar{y}_{|J} - y_J\| \leq C\|b(\bar{y}_{n_-}, \bar{y}_{n_+})\|. \tag{35}$$

Transforming y_J back to the original coordinates, we get a finite approximation

$$x_n := T_n(y_n) = y_n + \bar{\xi}_n, \quad n \in J$$

of $\bar{x}_{\mathbb{Z}}$, fulfilling the same error estimate (35).

4 Example

For an illustration of our approach, we approximate homoclinic trajectories for the well known Hénon-map

$$x \mapsto h(x, \lambda, b) = \begin{pmatrix} 1 + x_2 - \lambda x_1^2 \\ bx_1 \end{pmatrix}, \tag{36}$$

cf. Mira (1987), Devaney (1989), Hale & Koçak (1991). Fix the parameters $b = 0.3$ and let $\hat{\lambda} = 1.5$. Hénon's map possesses for $\lambda > -\frac{(b-1)^2}{4}$ the fixed point

$$\xi(\lambda, b) = \begin{pmatrix} z(\lambda, b) \\ bz(\lambda, b) \end{pmatrix}, \text{ where } z(\lambda, b) = \frac{b - 1 + \sqrt{(b-1)^2 + 4\lambda}}{2\lambda}. \tag{37}$$

The matrix

$$A = D_x h(\xi, \hat{\lambda}, b) = \begin{pmatrix} -b + 1 - \sqrt{(b-1)^2 + 4\hat{\lambda}} & 1 \\ b & 0 \end{pmatrix}$$

has the eigenvalues $\sigma_s \approx 0.15$ and $\sigma_u \approx -1.998$ and consequently this matrix is hyperbolic. For fixed parameters $\hat{\lambda} = 1.5$, $b = 0.3$, a transversal homoclinic orbit $x_{\mathbb{Z}}$ w.r.t. the fixed point $\xi(\hat{\lambda}, b)$ exists, cf. Beyn et al. (2004). In the language of this paper, $x_{\mathbb{Z}}$ and $\xi_{\mathbb{Z}}$, where $\xi_n = \xi(\hat{\lambda}, b)$ for all $n \in \mathbb{Z}$, are two homoclinic trajectories. When $\lambda_{\mathbb{Z}}$ varies in a sufficiently small neighborhood $U(\hat{\lambda}_{\mathbb{Z}})$, we obtain two, generally non-constant, homoclinic trajectories $\xi_{\mathbb{Z}}(\lambda_{\mathbb{Z}})$ and $x_{\mathbb{Z}}(\lambda_{\mathbb{Z}})$. Let $\bar{\lambda}_{\mathbb{Z}} \in U(\hat{\lambda}_{\mathbb{Z}})$ and denote by $\bar{\xi}_{\mathbb{Z}}$ and $\bar{x}_{\mathbb{Z}}$ the corresponding homoclinic trajectories, then our assumptions **A2**-**A6** are satisfied. For the forthcoming numerical computations, we choose the interval $I = [1, 2]$ and take a sequence $\lambda_{\mathbb{Z}} \in I^{\mathbb{Z}}$ at random.

4.1 Approximation of the bounded trajectory

First, we approximate the bounded trajectory $\bar{\xi}_{\mathbb{Z}}$ on the finite interval $J = [n_-, n_+]$. To this end, we compute a longer orbit segment $\xi_{\bar{J}}$ on $\bar{J} = [\bar{n}_-, \bar{n}_+]$ and take only the accurate middle part ξ_J as suggested in Section 2.3.

For a first illustration, we take two randomly chosen sequences on the interval $[-40, 40]$ that coincide in the middle interval $[-20, 20]$. The solutions of the boundary value problem (13), (14) are shown in Figure 1.

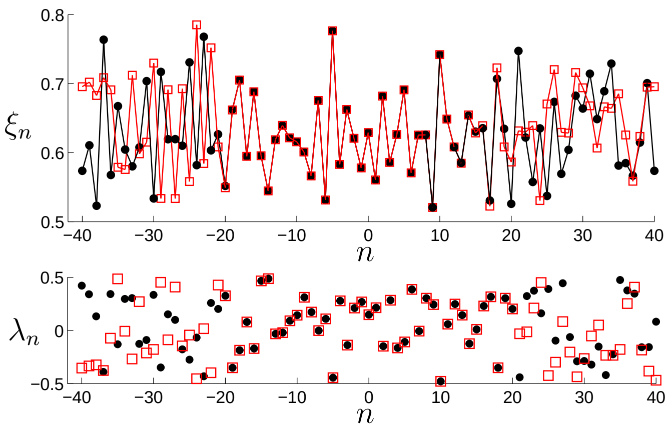

Figure 1: *For two randomly chosen parameter sequences with the same middle part (lower diagram), the upper picture shows the corresponding solutions of (13), (14).*

The choice of $\bar{n}_\pm$ is guided by Theorem 4. We choose $\bar{n}_\pm$ such that the difference between two solutions with different tails is of the order Δ, i.e.

$$e^{-\alpha_-(n_- - \bar{n}_-)} + e^{-\alpha_+(\bar{n}_+ - n_+)} = \mathcal{O}(\Delta).$$

Here $\alpha_\pm$ denote the dichotomy constants w.r.t. the stable and unstable direction and $\Delta = 10^{-16}$ is the precision of the machine. Let

$$\bar{n}_- = \left\lfloor n_- - \frac{\log \Delta}{\alpha_-} \right\rfloor \quad \text{and} \quad \bar{n}_+ = \left\lceil n_+ - \frac{\log \Delta}{\alpha_+} \right\rceil. \tag{38}$$

As a guess of $\alpha_\pm$, we take into account the weakest rates in the stable and unstable directions and define $\alpha_- = \log|\sigma_s|$ and $\alpha_+ = \log|\sigma_u|$, cf. (21).

For testing the validity of this ansatz, let $\bar{J} = [-100, 100]$, $\tilde{J} = [-150, 150]$ and choose a sequence $\lambda_{\tilde{J}} \in I^{\tilde{J}}$ at random. Then, a second sequence $\mu_{\tilde{J}}$ is defined, such that $\mu_{\bar{J}} = \lambda_{\bar{J}}$ holds.

With respect to the parameter sequences $\lambda_{\tilde{J}}$, $\mu_{\tilde{J}}$, we compute the associated bounded trajectories $\xi_{\tilde{J}}$, $\zeta_{\tilde{J}}$, respectively, using Newton's method for solving the non-linear systems. As an initial guess $x_{\tilde{J}}$, we take the fixed points $x_n = \xi(\lambda_n, b)$ for $n \in \tilde{J}$, cf. (37). For an illustration, $\|\xi_n - \zeta_n\|$ is plotted over n in a logarithmic scale. In Figure 2, these computations are performed for 10 sequences $\mu_{\tilde{J}}$, having different, randomly chosen tails. Define $J = [n_-, n_+]$, where $n_\pm$ are given in (38). We expect that the influence of parameter values outside the interval $\bar{J}$ is of magnitude $\mathcal{O}(\Delta)$. For an illustration, two lines are drawn, connecting the points $\left(\bar{n}_-, \frac{1}{2}\right)$ with (n_-, Δ) and $\left(\bar{n}_+, \frac{1}{2}\right)$ with (n_+, Δ). As one can see from Figure 2, these results are quite accurate.

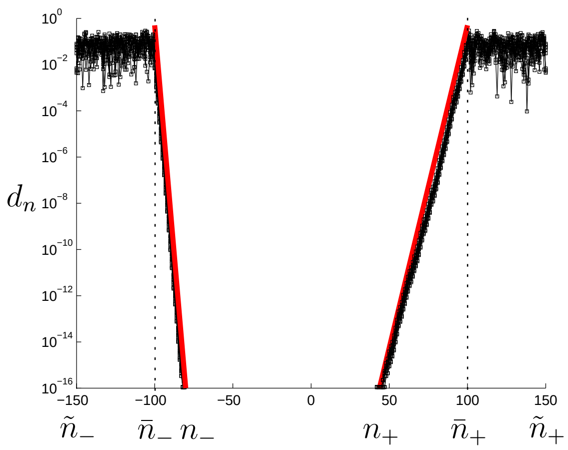

Figure 2: *Difference $d_n = \|\xi_n - \zeta_n\|$ between two solutions of (7). $\xi_{\tilde{J}}$ is computed w.r.t. the reference parameter sequence $\lambda_{\tilde{J}}$, and $\zeta_{\tilde{J}}$ is a solution w.r.t. the sequences $\mu_{\tilde{J}}$, where $\lambda_{\tilde{J}}$ and $\mu_{\tilde{J}}$ coincide on $\bar{J} = [\bar{n}_-, \bar{n}_+]$. The results for 10 different $\mu_{\tilde{J}}$ are shown. The red lines indicate the predicted differences.*

For a randomly chosen sequence $\lambda_{\mathbb{Z}} \in I^{\mathbb{Z}}$, we illustrate the numerical approximation of a homoclinic trajectory of length $n_- = -20$, $n_+ = 20$. To this end, we compute $\bar{n}_\pm$ as $\bar{n}_- = -40$, $\bar{n}_+ = 74$ using (38), and solve the boundary value problem (7) on the interval $\bar{J} = [\bar{n}_-, \bar{n}_+]$ as described in Section 4.1, using periodic boundary conditions. Figure 3 shows the solution (left) and the accurate middle part (right).

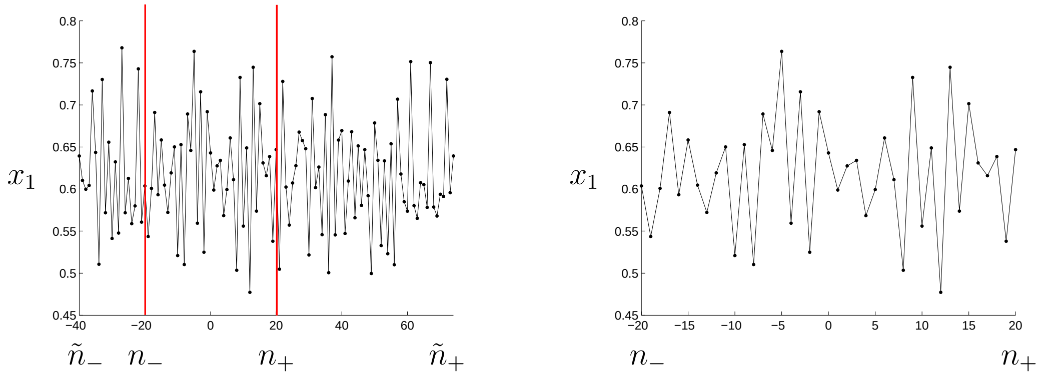

Figure 3: *Approximation of the bounded trajectory $\xi_{\mathbb{Z}}$ (left) and the accurate middle part (right), projected onto the (n, x_1)-plane.*

4.2 Approximation of a second homoclinic trajectory

In the next step, a homoclinic orbit y_J of the transformed system

$$y_{n+1} = h(y_n + \xi_n, \lambda_n, b) - \xi_{n+1}, \quad n \in J \tag{39}$$

is computed w.r.t. the fixed point 0, see Figure 4 (left). In the right picture, the distance to the fixed point $\|y_n\|$ is given in a logarithmic scale, thus one can see the exponentially fast convergence of the orbit towards the fixed point 0.

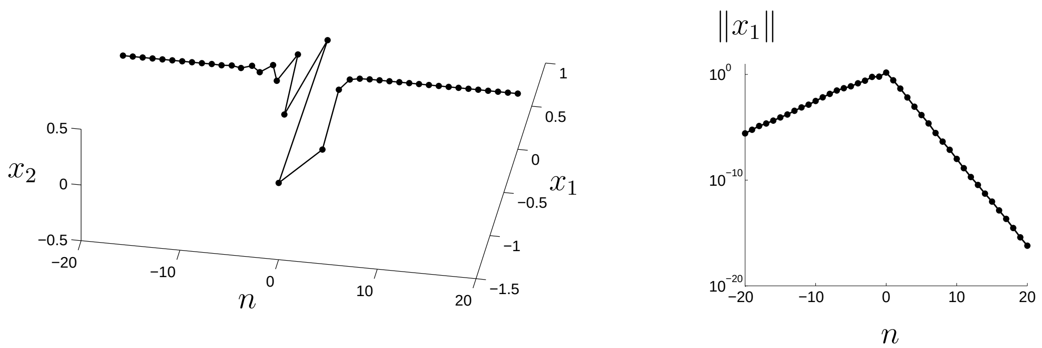

Figure 4: *Homoclinic orbit of the transformed system (39) (left). The right diagram illustrates the exponential rate of convergence of the orbit towards the fixed point 0.*

Transforming the orbit y_J back to the original coordinates we obtain an approximation of a second trajectory x_J, where $x_n = y_n + \xi_n$ for $n \in J$, that is homoclinic to ξ_J. The two homoclinic trajectories x_J (in black) and ξ_J (in red) are shown in Figure 5.

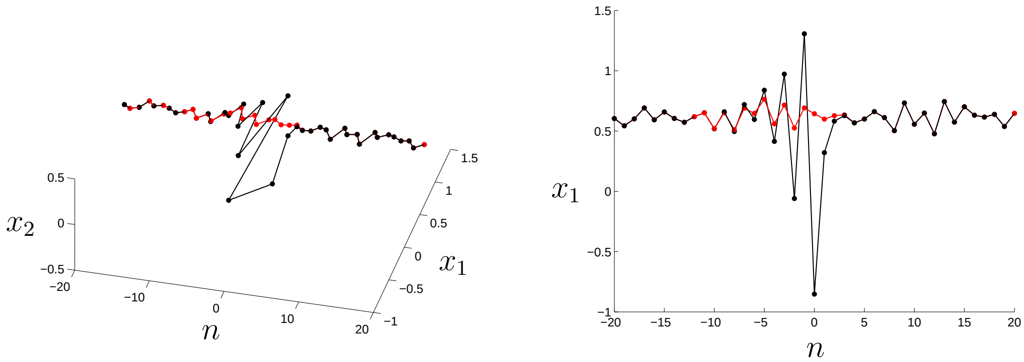

Figure 5: *Two homoclinic trajectories x_J (in black) and ξ_J (in red). In the right figure, a projection onto the (n, x_1) plane is given.*

A Exponential dichotomy

In this appendix, we state some well known results for exponential dichotomies from Palmer (1988).

Definition 9 *A linear difference equation*

$$u_{n+1} = A_n u_n, \quad n \in \mathbb{Z}$$

*with invertible matrices $A_n \in \mathbb{R}^{k,k}$ and solution operator Φ has an **exponential dichotomy** with data $(K, \alpha, P_n^s, P_n^u)$ on $J \subset \mathbb{Z}$, if there exist two families of projectors P_n^s and $P_n^u = I - P_n^s$ and constants $K,\ \alpha > 0$, such that the following statements hold:*

$$P_n^s \Phi(n,m) = \Phi(n,m) P_m^s \quad \forall n, m \in J,$$

$$\left.\begin{array}{rcl} \|\Phi(n,m)P_m^s\| & \leq & Ke^{-\alpha(n-m)} \\ \|\Phi(m,n)P_n^u\| & \leq & Ke^{-\alpha(n-m)} \end{array}\right. \quad \forall n \geq m,\ n, m \in J.$$

We introduce an important perturbation result for exponential dichotomies, frequently named as Roughness-Theorem, cf. (Palmer 1988, Proposition 2.10).

Theorem 10 *Assume that the difference equation*

$$u_{n+1} = A_n u_n, \quad A_n \in \mathbb{R}^{k,k} \text{ invertible}, \quad \|A_n^{-1}\| \leq M \ \forall n \in J$$

with an interval $J \subseteq \mathbb{Z}$, possesses an exponential dichotomy with data $(K, \alpha, P_n^s, P_n^u)$. *Suppose* $0 < \delta < \alpha$ *and* $B_n \in \mathbb{R}^{k,k}$ *satisfies* $\|B_n\| \leq \beta$ *for all* $n \in J$, *where*

$$\begin{aligned} \beta &< M^{-1}, \\ 2K(1 + \mathrm{e}^{-\alpha})(1 - \mathrm{e}^{-\alpha})^{-1}\beta &\leq 1, \\ 2K\mathrm{e}^{\alpha}(\mathrm{e}^{-\delta} + 1)(\mathrm{e}^{\delta} - 1)^{-1}\beta &\leq 1. \end{aligned}$$

Then $A_n + B_n$ *is invertible and the perturbed difference equation*

$$u_{n+1} = (A_n + B_n)u_n$$

possesses an exponential dichotomy on J *with data* $\left(2K(1 + \mathrm{e}^{\delta})(1 - \mathrm{e}^{-\delta})^{-1}, \alpha - \delta, Q_n^s, Q_n^u\right)$, *where* $\operatorname{rank}(Q_n^s) = \operatorname{rank}(P_n^s)$ *and*

$$\|P_n^s - Q_n^s\| \leq 2K^2 \frac{1 + \mathrm{e}^{-\alpha}}{1 - \mathrm{e}^{-\alpha}} \sup_{m \in J} \|B_m\| \quad \textit{for all } n \in J. \tag{40}$$

B A Lipschitz inverse mapping theorem

We apply a quantitative version of the Lipschitz inverse mapping theorem, cf. Irwin (2001), for proving our approximation theorem.

Lemma 11 *Assume* Y *and* Z *are Banach spaces,* $F \in C^1(Y, Z)$ *and* $F'(y_0)$ *is for* $y_0 \in Y$ *a homeomorphism. Let* κ, σ, $\delta > 0$ *be three constants, such that the following estimates hold:*

$$\left\|F'(y) - F'(y_0)\right\| \leq \kappa < \sigma \leq \frac{1}{\left\|F'(y_0)^{-1}\right\|} \quad \forall y \in B_\delta(y_0), \tag{41}$$

$$\left\|F(y_0)\right\| \leq (\sigma - \kappa)\delta. \tag{42}$$

Then F *has a unique zero* $\bar{y} \in B_\delta(y_0)$ *and the following inequalities are satisfied*

$$\left\|F'(y)^{-1}\right\| \leq \frac{1}{\sigma - \kappa} \quad \forall y \in B_\delta(y_0), \tag{43}$$

$$\|y_1 - y_2\| \leq \frac{1}{\sigma - \kappa}\left\|F(y_1) - F(y_2)\right\| \quad \forall y_1,\ y_2 \in B_\delta(y_0). \tag{44}$$

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