Shadowing for discrete approximations of abstract parabolic equations

Wolf-Jürgen Beyn [1*] **and Sergey Piskarev** [2**]
[1] Department of Mathematics, Bielefeld University,
P.O. Box 100131, 33501 Bielefeld, Germany
beyn@mathematik.uni-bielefeld.de
[2] Scientific Research Computer Center, Moscow State University,
Vorobjevy Gory, Moscow 119899, Russia
s_piskarev@yahoo.com

February 5, 2007

Abstract

This paper is devoted to the numerical analysis of abstract semilinear parabolic problems $u'(t) = Au(t) + f(u(t)), u(0) = u^0$, in some general Banach space E. We prove a shadowing Theorem that compares solutions of the continuous problem with those of a semidiscrete approximation (time stays continuous) in the neighborhood of a hyperbolic equilibrium. We allow rather general discretization schemes following the theory of discrete approximations developed by F. Stummel, R.D. Grigorieff and G. Vainikko. We use a compactness principle to show that the decomposition of the flow into growing and decaying solutions persists for this general type of approximation. The main assumptions of our results are naturally satisfied for operators with compact resolvents and can be verified for finite element as well as finite difference methods. In this way we obtain a unified approach to shadowing results derived e.g. in the finite element context ([19, 20, 21]).

1991 Mathematics Subject Classification: 65J, 65M, 65N, 35J, 35K
Keywords and phrases: Theory of shadowing, abstract parabolic problems, analytic C_0-semigroups, hyperbolic equilibrium point, semidiscretization, discretization in space, fractional powers of operators, compact convergence of resolvents.

$*$ Corresponding author W.-J. Beyn, Department of Mathematics, University of Bielefeld, P.O. Box 100131, 33501 Bielefeld, Germany.

$**$ Research was supported by grant Russian Foundation for Basic Research 04-01-00026, by grant from DAAD and by grant SFB 701 "Spectral Structures and Topological Methods in Mathematics", Bielefeld University.

1 Introduction

Classical shadowing results state that pseudo-trajectories of a finite dimensional dynamical system can be 'shadowed' by true trajectories provided the system has some kind of hyperbolicity. This is usually stated in a more quantitative form as follows. For any given $\varepsilon > 0$ there exists some $\delta > 0$ such that for any pseudo-trajectory, that allows jumps of size δ at successive time instances, there exists a true trajectory within an ε-neighborhood of the pseudo-trajectory uniformly in time. Such statements hold for both time continuous and time discrete dynamical systems within or in the neighborhood of (suitably defined) hyperbolic sets. We refer to the monographs [23] and [24] for an excellent account of various shadowing results.

When numerical approximations are to be included into such an approach one realizes that the concept of a pseudo-trajectory needs a considerable extension for a shadowing principle to be still valid. For example, discretizing an autonomous ODE by a one-step method leads to a mapping depending on step-size and shadowing now means approximation of a discrete time orbit by a true continuous trajectory or vice versa. A result of this type was derived in [7] near stationary hyperbolic points and for more general hyperbolic situations in [12].

Shadowing results for numerical approximations of time-dependent partial differential equations usually involve both time and space discretization, i.e. the continuous trajectory in an infinite dimensional space should be shadowed by a discrete time trajectory in a finite dimensional space and vice versa.

Such a result was derived in [21] for a finite element method combined with the backward Euler discretization in time when applied to a nonlinear reaction diffusion system in the neighborhood of stationary hyperbolic solution. These results extended earlier work on semidiscretizations with finite elements by the same authos [20]. Shadowing results for semi-discretizations in time were shown earlier in [1] near hyperbolic stationary states and, more recently, for a linear but nonautonomous setting in [22].

The purpose of this paper is to study shadowing properties of rather general spatial discretizations of a nonlinear evolution equation in some Banach space

$$u'(t) = Au(t) + f(u(t)), t \geq 0 \quad u(0) = u^0 \in E, \tag{1.1}$$

where A is a closed operator that generates an analytic semigroup $\exp(tA)$ on E. For the discretization in space we use the theory of discrete approximations as developed in [13],[28],[32],[33],[34]. As is known for stationary problems this theory provides a unified framework for handling such diverse approximations as (conforming and nonconforming) finite elements methods, finite difference methods (see [32]) and perturbations of domains ([4],[29]).

We consider a family of discretized problems indexed by $n \in \mathbb{N}$

$$u_n'(t) = A_n u_n(t) + f(u_n(t)),\ t \geq 0, \quad u_n(0) = u_n^0 \in E_n, \tag{1.2}$$

where the spaces are related by discretization maps $p_n : E \mapsto E_n$ and the closed operators A_n satisfy certain compatibility requirements with respect to the continuous operator A.

Our main results (see Theorems 4.4 and 4.6) show that mild solutions near a hyperbolic equilibrium of the system (1.1) can be shadowed on arbitrary large time intervals by corresponding mild solutions of the system (1.2) and vice versa. While our approach still follows the general idea of constructing shadowing trajectories from boundary value problems as in [1],[7],[20],[21], we encounter several difficulties that must be resolved when working in the general framework:

(i) When using the standard interpolation spaces E^α, E_n^α (see e.g. [15]) for the operators $f : E^\alpha \mapsto E$, $f_n : E_n^\alpha \mapsto E_n$, $\exp(tA) : E \mapsto E^\alpha$, $\exp(tA_n) : E_n \mapsto E_n^\alpha$ it becomes necessary to construct discretization maps $p_n^\alpha : E^\alpha \mapsto E_n^\alpha$ that inherit properties of p_n.

(ii) The discretization maps p_n, p_n^α need to be adapted to the hyperbolic splitting of the linear operators obtained by linearizing about the hyperbolic equilibria.

(iii) While the theory of discrete approximations allows to control eigenvalues of finite multiplicity in a bounded domain it is necessary to assume resolvent estimates for A_n in a sectorial region of the complex plane (condition (B_1) in Theorem 3.5).

(iv) When shadowing solutions of (1.2) by those of (1.1) one needs to approximate elements from E_n by those of E which is not obvious since we avoid the use of interpolation operators.

Our main tool to solve these problems will be compactness properties of resolvents as well as of initial values of trajectories to be shadowed. In section 2 and 3 we collect the main technical tools for proving the shadowing theorems in section 4. The application to finite element and finite difference approximations is discussed in section 5. In particular, it turns out that several of the issues raised above are resolved in a natural way and that we retrieve some shadowing results from [20],[21].

We have limited the current paper to the simplest case where shadowing is possible in the framework of discrete approximations. We expect that the approach can be extended substantially to cover systems with more general hyperbolic structures (see [8] for the case of attractors) to derive error bounds for stable and unstable manifolds as well as shadowing estimates with weak singularities for noncompact initial values.

2 Preliminaries

Let $B(E)$ denote the Banach algebra of all bounded linear operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. For $B \in \mathcal{C}(E)$ let $\sigma(B)$ be its spectrum and $\rho(B)$ be its resolvent set. In a Banach space E we consider the following inhomogeneous Cauchy problem:

$$\begin{aligned} &u'(t) = Au(t) + g(t),\ t \in [0,T], \\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

where $A \in \mathcal{C}(E)$ generates a C_0-semigroup and $g(\cdot)$ is a function from $[0,T]$ into E. The problem (2.1) can be considered in various function spaces. The most popular spaces for which well-posedness can be shown are $C([0,T];E), C_0^\alpha([0,T];E)$, and $L^p([0,T];E)$ spaces (see [5, 35]).

In general one considers a mild or so-called generalized solution of (2.1), i.e. the function

$$u(t) = \exp(tA)u^0 + \int_0^t \exp((t-s)A)g(s)ds,\ t \geq 0. \tag{2.2}$$

Now we proceed to the semilinear autonomous parabolic problem

$$\begin{aligned} &u'(t) = Au(t) + f(u(t)),\ t \geq 0, \\ &u(0) = u^0 \in E, \end{aligned} \tag{2.3}$$

where the function $f(\cdot) : E \to E$ is locally Lipschitz, bounded and continuously Fréchet differentiable. It is well known that under these assumptions the mild solution of (2.3) exists on a maximal interval. Moreover, writing the solution as $u(t) = T(t)u^0$, we obtain a nonlinear semigroup $T(\cdot)$ on E that satisfies the variation of constants formula

$$T(t)u^0 = \exp(tA)u^0 + \int_0^t \exp((t-s)A)f(T(s)u^0)ds,\ t \geq 0. \tag{2.4}$$

The equilibria of (2.3) are solutions $u \in D(A)$ of the equation

$$Au + f(u) = 0. \tag{2.5}$$

Definition 2.1. *A solution u^* of (2.5) is called hyperbolic if $\sigma(A + f'(u^*)) \cap i\mathbb{R} = \emptyset$.*

In case A has a compact resolvent we can conclude that any hyperbolic solution is isolated. Moreover, if all solutions of (2.5) are hyperbolic, then there is an odd number of them [8].

One should note that even in case of (2.2) and analyticity of the C_0-semigroup $\exp(\cdot A)$ the function $u(\cdot)$ is not necessary differentiable if $g(\cdot) \in C([0,T];E)$, i.e. (2.1) is not classically well-posed in $C([0,T];E)$ for a general Banach space E. However, the problem (2.1) is classically well-posed in $C([0,T];E^\alpha)$, (see [9, 10, 27]), where $E^\alpha = (E, D(A))_\alpha$ is a suitable interpolation space. Moreover, if one has in mind applications like the space $E = L^p(\Omega)$ and needs Fréchet differentiability of the function $f(\cdot) : L^2(\Omega) \to L^2(\Omega)$, then the function $f(\cdot)$ must be linear, cf. [2]. To cover more general nonlinearities one would like to work with the weaker assumption that $f(\cdot)$ maps $H_0^1(\Omega)$ into $L^2(\Omega)$ and is differentiable. The difficulties caused by these facts can be resolved by considering the problem (2.3) in a Banach space $E^\alpha, 0 \leq \alpha < 1$, and assume that $f(\cdot) : E^\alpha \to E$ is Fréchet differentiable with derivative at the equilibrium $f'(u^*) \in B(E^\alpha, E)$. In case of $E = L^2(\Omega)$ and $A = \Delta$ one normally has $E^{1/2} = H_0^1(\Omega)$ for the $\alpha = 1/2$ case.

In the following let $A : D(A) \subseteq E \to E$ be a closed linear operator, such that

$$\|(\lambda I - A)^{-1}\|_{B(E)} \leq \frac{M}{1 + |\lambda|} \text{ for any } \mathrm{Re}\lambda \geq 0. \tag{2.6}$$

In such a situation we have $\theta(A) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\} < 0$. Let $(-A)^\alpha$, $\alpha \in \mathbb{R}^+$, denote the fractional power operators (see [15, 18]) associated to A and let $E^\alpha := D((-A)^\alpha)$ be the corresponding spaces endowed with the graph norm $\|x\|_{E^\alpha} = \|(-A)^\alpha x\|_E$. Let $\mathcal{U}_{E^\alpha}(0;\rho)$ denote the ball with center 0 and radius $\rho > 0$ in E^α space.

For some $0 < \alpha \leq 1$ consider the semilinear equation (2.3) in the space E^α

$$\begin{aligned} &u'(t) = Au(t) + f(u(t)),\ t \geq 0, \\ &u(0) = u^0 \in E^\alpha, \end{aligned} \tag{2.7}$$

where $f(\cdot) : E^\alpha \subseteq E \to E$ satisfies the following condition:

(F1) For some $\rho > 0$ the function $f : \mathcal{U}_{E^\alpha}(u^*;\rho) \mapsto E$ is continuously Fréchet differentiable and for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|f'(w) - f'(z)\|_{B(E^\alpha,E)} \le \epsilon$ for all $w, z \in \mathcal{U}_{E^\alpha}(u^*;\rho)$ with $\|w - z\|_{E^\alpha} \le \delta$.

Here and in what follows u^* always denotes a hyperbolic equilibrium of (2.3).

By the change of variables $v(\cdot) = u(\cdot) - u^*$ problem (2.7) may be written in the form

$$v'(t) = A_{u^*}v(t) + F_{u^*}(v(t)),\ v(0) = v^0,\ t \ge 0, \tag{2.8}$$

where $v^0 = u^0 - u^*$ and

$$A_{u^*} = A + f'(u^*), F_{u^*}(w) = f(w + u^*) - f(u^*) - f'(u^*)w \ \text{ for } \ \|w\|_{E^\alpha} \le \rho. \tag{2.9}$$

Note that $F_{u^*}(w) = f(w + u^*) - f(u^*) - f'(u^*)w$ is of order $o(\|w\|_{E^\alpha})$ and that the operator $A_{u^*} = A + f'(u^*)$ generates an analytic C_0-semigroup since $f'(u^*) \in B(E^\alpha, E)$.

We assume that the part σ^+ of the spectrum of $A+f'(u^*)$ which is located strictly to the right of the imaginary axis consists of a finite number of eigenvalues with finite multiplicity. This assumption is satisfied, for instance, if the resolvent of the operator A is compact. In case of a hyperbolic point u^* there is no spectrum of A_{u^*} on $i\mathbb{R}$. Let $U(\sigma^+) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ be an open connected neighborhood of σ^+ with a closed rectifiable curve $\partial U(\sigma^+)$ as boundary. We decompose E^α using the Riesz projection

$$P(\sigma^+) := P(\sigma^+, A_{u^*}) := \frac{1}{2\pi i}\int_{\partial U(\sigma^+)} \Big(\zeta I - A_{u^*}\Big)^{-1} d\zeta \tag{2.10}$$

defined by σ^+. Due to this definition and analyticity of the C_0-semigroup $\exp(tA_{u^*})$ we have positive constants $M_1, \beta > 0$, such that (cf. [15])

$$\|\exp(tA_{u^*})v\|_{E^\alpha} \le \begin{cases} M_1 e^{-\beta t}\|v\|_{E^\alpha}, & t \ge 0, \quad v \in (I - P(\sigma^+))E^\alpha, \\ M_1 e^{\beta t}\|v\|_{E^\alpha}, & t \le 0, \quad v \in P(\sigma^+)E^\alpha. \end{cases} \tag{2.11}$$

Without loss of generality we can adapt the norm in E^α such that

$$\|v\|_{E^\alpha} = \max(\|P(\sigma+)v\|_{E^\alpha}, \|(I - P(\sigma+))v\|_{E^\alpha}). \tag{2.12}$$

If v^0 is close to 0, i.e. say $v^0 \in \mathcal{U}_{E^\alpha}(0;\rho)$, then the mild solution $v(t;v^0)$ of (2.8) can stay in the ball $\mathcal{U}_{E^\alpha}(0;\rho)$ for some time. We will recognize such a solution as a solution of a boundary value problem where the stable part is prescribed at the beginning and the unstable part at the end. More precisely, for any two $v^-, v^+ \in \mathcal{U}_{E^\alpha}(0;\rho)$ and for any $0 < T \le \infty$ we consider the boundary value problem

$$\begin{cases} v'(t) = A_{u^*}v(t) + F_{u^*}(v(t)), \quad 0 \le t \le T, \\ (I - P(\sigma^+))v(0) = (I - P(\sigma^+))v^-,\ P(\sigma^+)v(T) = P(\sigma^+)v^+. \end{cases} \tag{2.13}$$

In case $T = \infty$ the second boundary condition is empty and the differential equation holds on $[0,\infty)$. A mild solution of problem (2.13) satisfies the integral equation

$$v(t) = \exp((t-T)A_{u^*})P(\sigma^+)v^+ \ + \ \exp(tA_{u^*})(I - P(\sigma^+))v^- + \tag{2.14}$$

$$+\int_0^T \Gamma_T(t,s) F_{u^*}(v(s))ds,\ 0 \le t \le T,$$

where we define the Green's function Γ_T by

$$\Gamma_T(t,s) = \begin{cases} \exp((t-s)A_{u^*})(I - P(\sigma^+)), & 0 \le s \le t \le T, \\ \exp((t-s)A_{u^*})P(\sigma^+), & 0 \le t < s \le T. \end{cases} \tag{2.15}$$

Note that (2.11) implies

$$\|\Gamma_T(t,s)z\|_{E^\alpha} \le M_1 \frac{e^{-\beta|t-s|}}{|t-s|^\alpha} \|z\|_E. \tag{2.16}$$

Again, in case $T = \infty$ we set the term involving v^+ in (2.14) equal to zero. Existence and uniqueness of solutions to (2.14) is established by the following Proposition.

Proposition 2.2. *Let A and f satisfy the conditions above, in particular, let $(F1)$ be satisfied. Then there exists $\hat\rho > 0$ such that for any $0 < \hat\rho_2 \le \hat\rho$ we find a $0 < \hat\rho_1 \le \hat\rho_2$ with the property that equation (2.14) has a unique solution $v(\cdot) = v(u^+, u^-, \cdot) \in C([0,T]; \mathcal{U}_{E^\alpha}(0;\hat\rho_2))$ for all $v^\pm \in \mathcal{U}_{E^\alpha}(0,\hat\rho_1)$ and all $0 < T \le \infty$. If $T = \infty$, then $\|v(t)\|_{E^\alpha} \to 0$ as $t \to \infty$.*

Proof. We apply Lemma 6.1 with the setting $Y = Z = C([0,T]; E^\alpha)$ which is to be understood as the space of continuous and bounded functions in case $T = \infty$. We further set $y_0 = 0, F(v) = v - G(u^-, u^+; v)$, where the operator $G(v^-, v^+; v)$ is defined by the right hand side of (2.14). First note that $F'(0) = I - G'_v(u^-, u^+; 0) = I$, so that we can take $\sigma = 1$ in Lemma 6.1. For any two $v, w \in \mathcal{U}_Y(0, \hat\rho_2)$ we have by (2.16) and $(F1)$

$$\|\Big(G'_v(v^-, v^+; v) - G'_v(v^-, v^+; w)\Big)u\|_Z \le$$

$$\le \sup_{0\le t\le T} \|\int_0^T \Gamma_T(t,s)\Big(F'_{u^*,v}(v(s)) - F'_{u^*,v}(w(s))\Big)u(s)ds\|_{E^\alpha} \le$$

$$\le M_1 \sup_{0\le t\le T} \int_0^T \frac{e^{-\beta|t-s|}}{|t-s|^\alpha} \|\Big(f'(v(s)+u^*) - f'(w(s)+u^*)\Big)u(s)\|_E ds \le$$

$$\le M_1 \frac{1}{\beta}\|u\|_Y \sup_{v_1,v_2 \in \mathcal{U}_{E^\alpha}(0,\hat\rho_2)} \|f'(v_1) - f'(v_2)\|_{B(E^\alpha;E)} \le \frac{1}{2}\|u\|_Y \tag{2.17}$$

for $\hat\rho_2$ sufficiently small. Finally choose $\hat\rho_1 = \frac{\hat\rho_2}{4M_1}$ and obtain

$$\|F(0)\|_{E^\alpha} \le \sup_{0\le t\le T} \Big(M_1 e^{-\beta t}\|(I - P(\sigma^+))u^-\|_{E^\alpha} + M_1 e^{\beta(t-T)}\|P(\sigma^+)u^+\|_{E^\alpha}\Big) \le$$

$$\le M_1 2\hat\rho_1 \le \frac{1}{2}\hat\rho_2.$$

Consider now the case $T = \infty$. where we write (2.14) as

$$v(t) = \exp(tA_{u^*})(I - P(\sigma^*))v(0) + \tag{2.18}$$

$$+\int_0^t \exp((t-s)A_{u^*})(I-P(\sigma^+))F_{u^*}(v(s))ds+\int_t^\infty \exp((t-s)A_{u^*})P(\sigma^+)F_{u^*}(v(s))ds,$$

Now, we argue as above with the space $C([0,\infty);E^\alpha)$ replaced by $C_0([0,\infty);E^\alpha) = \{u \in C([0,\infty);E^\alpha) : \|u(t)\|_{E^\alpha} \to 0 \text{ as } t \to \infty\}$. Note that $G(u^-,u^+,\cdot)$ maps this space into itself since $\|v(t)\|_{E^\alpha} \to 0$ as $t \to \infty$ implies

$$\|F_{u^*}(v(t))\|_E \to 0 \text{ as } t \to \infty. \tag{2.19}$$

The operator $G(v^-;v)$ defined on the right of (2.18) is continuous in both arguments and maps the space $C_0([0;\infty);E^\alpha))$ into itself. Indeed, for $t \geq T$ we have

$$\|(-A)^\alpha G(v^-;v)(t)\|_E \leq M_1 e^{-t\beta}\|(I-P(\sigma^+))v^-\|_{E^\alpha}+ \tag{2.20}$$

$$+M_1 e^{-\beta(t-T)}\int_0^T \frac{e^{-\beta(T-s)}}{|t-s|^\alpha}\|F_{u^*}(v(s))\|_E\,ds+\int_T^\infty M_1\frac{e^{-\beta|t-s|}}{|s-t|^\alpha}\|F_{u^*}(v(s))\|_E\,ds.$$

Given $\epsilon > 0$ we first take T so large that the first term and the second integral are below $\epsilon/3$ for all $t \geq T$ then we choose t large so that the first integral is below $\epsilon/3$.

So there is a unique solution of the equation $v(\cdot) = G(v^-;v)$ in $C_0([0;\infty);E^\alpha)$ and the result follows by uniqueness. ■

3 Discretization of operators and semigroups

In the papers [13, 28, 31, 32, 33, 36] a general framework was developed that allows to analyze convergence properties of numerical discretizations in a unifying way. This approach is able to cover such diverse approximations as (conforming and nonconforming) finite elements, finite differences, collocation methods and perturbation of domains. Our paper aims at showing that one can derive shadowing properties within this framework.

3.1 General approximation scheme

We first describe the general approximation scheme as developed in [13, 28, 32, 33, 34]. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of bounded linear operators $p_n : E \to E_n, p_n \in B(E,E_n), n \in \mathbb{N} = \{1,2,\cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E. \tag{3.1}$$

Definition 3.1. *The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$. We write this as $x_n \xrightarrow{\mathcal{P}} x$.*

Definition 3.2. *The sequence of bounded linear operators $B_n \in B(E_n), n \in \mathbb{N}$, is called $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We then write $B_n \xrightarrow{\mathcal{PP}} B$.*

The simplest case is $E_n = E$ and $p_n = I$ for each $n \in \mathbb{N}$, where I is the identity on E. Then Definition 3.1 leads to the traditional pointwise convergence of bounded linear operators which we denote by $B_n \to B$. For various notions related to $\mathcal{P}-$convergence and for several applications we refer to [28],[32],[33]. An elementary consequence of Definition 3.2 is the following (see [28], [32])

Lemma 3.3. *Let B_n, B be as above. Then $B_n \xrightarrow{\mathcal{PP}} B$ is equivalent to boundedness of $\|B_n\|$ and the condition $B_n p_n x \xrightarrow{\mathcal{P}} Bx \; \forall x \in E$. If this holds then for any compact set $K \subset E$ we have*

$$\sup_{x \in K} \|B_n p_n x - p_n Bx\| \to 0 \text{ as } n \to \infty. \tag{3.2}$$

Proof. For the first statement we refer to [28], [32]. The proof of (3.2) is by contradiction. Assume $\|B_n p_n x^n - p_n B x^n\| \geq \varepsilon > 0$ for some sequence $x^n \in K, n \in \mathbb{N}$, and some $\varepsilon > 0$. Then take a subsequence $x^n, n \in \mathbb{N}' \subset \mathbb{N}$, with $x^n \to x$ for some $x \in K$ and find a contradiction from

$$\|B_n p_n x^n - p_n B x^n\| \leq \|B_n\| \|p_n(x^n - x)\| + \|B_n p_n x - p_n Bx\| + \|p_n B(x - x^n)\| \to 0$$

as $n \in \mathbb{N}'$. ■

For unbounded operators that occur as infinitesimal generators of PDE's the notion of *compatibility* turns out to be useful.

Definition 3.4. *The sequence of closed linear operators $A_n \in \mathcal{C}(E_n), n \in \mathbb{N}$, is called compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax$. We write (A_n, A) are compatible.*

For analytic C_0-semigroups the following ABC Theorem holds, see [25].

Theorem 3.5. *Let the operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).*

- (A) *Compatibility. There exists a $\lambda \in \rho(A) \cap \bigcap_{n \in \mathbb{N}} \rho(A_n)$ such that the resolvents converge $(\lambda I - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;*
- (B_1) *Stability. There are constants $M_2 \geq 1$ and $\omega_2 \in \mathbb{R}$ such that*

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|}, \; Re\lambda > \omega_2, n \in \mathbb{N};$$

- (C_1) *Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$ we have*

$$\max_{\eta \in \Sigma(\theta,\mu)} \| \exp(\eta A_n) u_n^0 - p_n \exp(\eta A) u^0 \| \to 0 \text{ as } n \to \infty \text{ whenever } u_n^0 \xrightarrow{\mathcal{P}} u^0.$$

In (C_1) we denote by $\Sigma(\theta, \mu) = \{z \in \mathbb{C} : |z| \leq \mu, |arg(z)| \leq \theta\}$ the sector of angle 2θ and radius μ.

As a simple corollary we obtain uniform convergence on compact sets (the proof follows in the same way as (3.2))

Corollary 3.6. *Under the assumptions* (A) *and* (B_1) *of Theorem 3.5 we have for any compact set* $K \subset E$

$$\max_{u^0 \in K} \max_{\eta \in \Sigma(\theta,\mu)} \|(\exp(\eta A_n)p_n - p_n \exp(\eta A))u^0\| \to 0 \quad as \quad n \to \infty. \tag{3.3}$$

The semidiscrete approximation of (2.1) are the following Cauchy problems in Banach spaces E_n:

$$\begin{aligned} u_n'(t) &= A_n u_n(t) + g_n(t),\ t \in [0,T], \\ u_n(0) &= u_n^0, \end{aligned} \tag{3.4}$$

with operators A_n which generate C_0-semigroups, A_n and A are compatible, $u_n^0 \xrightarrow{\mathcal{P}} u^0$ and $g_n(\cdot) \xrightarrow{\mathcal{P}} g(\cdot)$ in an appropriate sense. It is natural to assume for a typical semidiscretization that conditions like (A) and (B_1) are satisfied.

Definition 3.7. *The region of stability* $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \rho(A_n)$ *for almost all* n *and such that the sequence* $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n\in\mathbb{N}}$ *is bounded. The region of convergence* $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \Delta_s(\{A_n\})$ *and such that the sequence of operators* $\{(\lambda I_n - A_n)^{-1}\}_{n\in\mathbb{N}}$ *is* $\mathcal{PP}$*-convergent to some operator* $S(\lambda) \in B(E)$.

Definition 3.8. *A sequence of operators* $\{L_n\}$, $L_n \in \mathcal{C}(E_n)$, *is called regularly compatible with an operator* $L \in \mathcal{C}(E)$ *if the following conditions hold*

(*i*) (L_n, L) *are compatible,*

(*ii*) *for any bounded sequence* $\|x_n\|_{E_n} = O(1)$ *such that* $x_n \in D(L_n)$ *and* $\{L_n x_n\}$ *is* $\mathcal{P}$*-compact, it follows that* $\{x_n\}$ *is* $\mathcal{P}$*-compact;*

(*iii*) *the* $\mathcal{P}$*-convergence of* $\{x_n\}$ *to some element* x *and convergence of* $\{L_n x_n\}$ *to some element* y *as* $n \to \infty$ *imply that* $x \in D(L)$ *and* $Lx = y$.

Definition 3.9. *The region of regularity* $\Delta_r = \Delta_r(\{A_n\}, A)$, *is defined as the set of all* $\lambda \in \mathbb{C}$ *such that* $(L_n(\lambda), L(\lambda))$ *are regularly compatible, where* $L_n(\lambda) = \lambda I_n - A_n$ *and* $L(\lambda) = \lambda I - A$.

The relationships between these regions are given by the following result, see [34].

Proposition 3.10. *Suppose that* $\Delta_c \neq \emptyset$ *and* $\mathcal{N}(S(\lambda)) = \{0\}$ *for at least one point* $\lambda \in \Delta_c$. *Then* (A_n, A) *are compatible and*

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r.$$

Note that $\Delta_c \neq \emptyset$ and $\mathcal{N}(S(\lambda)) = \{0\}$ imply $S(\lambda) = (\lambda I - A)^{-1}$.

Let $\Lambda \subseteq \mathbb{C}$ be some open connected set. For an isolated point $\lambda \in \sigma(A)$ we denote the corresponding generalized eigenspace by $\mathcal{W}(\lambda; A) = Q(\lambda)E$, where $Q(\lambda) = \frac{1}{2\pi i} \int\limits_{|\zeta-\lambda|=\delta} (\zeta I - A)^{-1} d\zeta$ and δ is small enough so that there are no points of $\sigma(A)$ in the disc $\{\zeta : |\zeta - \lambda| \leq \delta\}$ different from λ. The isolated point $\lambda \in \sigma(A)$ is called a

Riesz point of A if $\lambda I - A$ is a Fredholm operator of index zero and $Q(\lambda)$ is of finite rank. In a similar way we define the invariant subspace

$$\mathcal{W}(\lambda, \delta; A_n) = \bigoplus_{\substack{\lambda_n \in \sigma(A_n), \\ |\lambda_n - \lambda| < \delta}} \mathcal{W}(\lambda_n, A_n),$$

where $\mathcal{W}(\lambda, \delta; A_n) = Q_n(\lambda) E_n$, and the projector $Q_n(\lambda)$ is given by

$$Q_n(\lambda) = \frac{1}{2\pi i} \int_{|\zeta - \lambda| = \delta} (\zeta I_n - A_n)^{-1} d\zeta.$$

The following theorem gives rather complete information about the approximation of the spectrum, see [32].

Theorem 3.11. *Assume that $L_n(\lambda) = \lambda I_n - A_n$ and $L(\lambda) = \lambda I - A$ are Fredholm operators of index zero for all $\lambda \in \Lambda$. Suppose that $L_n(\lambda) \to L(\lambda)$ regularly for any $\lambda \in \Lambda$ and $\rho(B) \cap \Lambda \neq \emptyset$. Then the following assertions hold*

- (*i*) *for any $\lambda_0 \in \sigma(A) \cap \Lambda$, there exists a sequence $\{\lambda_n\}$, $\lambda_n \in \sigma(A_n)$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda_0$ as $n \to \infty$;*
- (*ii*) *if for some sequence $\{\lambda_n\}, \lambda_n \in \sigma(A_n)$, $n \in \mathbb{N}$, one has $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to \infty$, then $\lambda_0 \in \sigma(A)$;*
- (*iii*) *for any $x \in \mathcal{W}(\lambda_0, A)$, there exists a sequence $\{x_n\}$, $x_n \in \mathcal{W}(\lambda_0, \delta; A_n)$, $n \in \mathbb{N}$, such that $x_n \to x$ as $n \to \infty$;*
- (*iv*) *there exists $n_0 \in \mathbb{N}$ such that* $\dim \mathcal{W}(\lambda_0, \delta; A_n) = \dim \mathcal{W}(\lambda_0, A)$ *for all $n \geq n_0$;*
- (*v*) *any sequence $\{x_n\}, x_n \in \mathcal{W}(\lambda_0, \delta; A_n)$, $n \in \mathbb{N}$, with $\|x_n\|_{E_n} = 1$ is $\mathcal{P}$-compact and any limit point of this sequence belongs to $\mathcal{W}(\lambda_0, A)$.*

It is shown in [34] that this theorem holds for closed operators as well.

Remark 3.12. *A Riesz point $\lambda_0 \in \sigma(A)$ is called strongly stable in Kato's sense if* $\dim \mathcal{W}(\lambda_0, \delta; A_n) \leq \dim \mathcal{W}(\lambda_0, A)$ *for all $n \geq n_0$. It was shown in [34] that a Riesz point $\lambda_0 \in \sigma(A)$ is strongly stable in Kato's sense iff $\lambda_0 \in \Lambda \cap \Delta_r \cap \sigma(A)$.*

In the case of operators which have compact resolvent it is natural to consider approximate operators which 'preserve' the property of compactness.

Definition 3.13. *A sequence of operators $\{B_n\}$, $B_n : E_n \to E_n$, $n \in \mathbb{N}$, converges compactly to an operator $B : E \to E$ if $B_n \xrightarrow{\mathcal{PP}} B$ and the following compactness condition holds:*

$$\|x_n\|_{E_n} = O(1) \Longrightarrow \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

Definition 3.14. *The region of compact convergence of resolvents, $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.*

The region of compact convergence Δ_{cc} can be characterized as follows, see [26].

Theorem 3.15. *Assume that $\Delta_{cc} \neq \emptyset$. Then for any $\zeta \in \Delta_s$ the following implication holds*:

$$\|x_n\|_{E_n} = O(1) \ \& \ \|(\zeta I_n - A_n)x_n\|_{E_n} = O(1) \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact.} \tag{3.5}$$

Conversely, if for some $\zeta \in \Delta_c \cap \rho(A)$ implication (3.5) holds, then $\Delta_{cc} \neq \emptyset$.

The condition $\Delta_{cc} \neq \emptyset$ has many consequences, for example [8]

$$\Delta_{cc} \neq \emptyset \Longrightarrow (-A_n)^{-\alpha} \xrightarrow{\mathcal{PP}} (-A)^{-\alpha} \text{ compactly for all } 0 < \alpha \leq 1. \tag{3.6}$$

One can compare the conditiion $\Delta_{cc} \neq \emptyset$ with regular compatibility (ii) of Definition 3.8 and see that $\Delta_{cc} \neq \emptyset$ implies regular compatibility. Moreover, by [14] we have

$$\Delta_{cc} \neq \emptyset \Longrightarrow \Delta_{cc} = \Delta_c \cap \rho(A) \quad \text{and} \quad \Delta_r = \mathbb{C}. \tag{3.7}$$

Lemma 3.16. *Assume that $\Delta_{cc} \neq \emptyset$. Let Λ be a compact subset of $\rho(A)$. Then there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$ and*

$$\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - A_n)^{-1}\| < \infty. \tag{3.8}$$

Proof. First select $\mu \in \Delta_{cc}$ and prove that there is a $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$. Suppose contrary to our claim that we have a subsequence $v_n \in D(A_n), \|v_n\| = 1, n \in \mathbb{N}'$ and $\lambda_n \in \Lambda$ such that $\lambda_n v_n - A_n v_n = 0$. Then $\lambda_n \to \lambda \in \Lambda$ for some subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and from the boundedness of $\mu v_n - A_n v_n = (\mu - \lambda_n)v_n$ we obtain from Theorem 3.15 that $v_n \xrightarrow{\mathcal{P}} v \in E, \|v\| = 1$ for some subsequence $\mathbb{N}''' \subset \mathbb{N}''$. Since $\Delta_r = \mathbb{C}$ by (3.7) and $(\lambda I_n - A_n)v_n \xrightarrow{\mathcal{P}} 0$ Definition 3.8 leads to $v \in D(A), (\lambda I - A)v = 0$, which contradicts $\lambda \in \rho(A)$.

To prove (3.8) consider Hilbert's identity

$$(\lambda I_n - A_n)^{-1} - (\mu I_n - A_n)^{-1} = (\mu - \lambda)(\lambda I_n - A_n)^{-1}(\mu I_n - A_n)^{-1}$$

for any $\lambda \in \rho(A_n)$. One can write

$$(\lambda I_n - A_n)^{-1} = (I_n - (\mu - \lambda)(\mu I_n - A_n)^{-1})^{-1}(\mu I_n - A_n)^{-1},$$

for $\lambda \in \Lambda$. Since $(\mu - \lambda)(\mu I_n - A_n)^{-1}$ converges compactly to $(\mu - \lambda)(\mu I - A)^{-1}$ and $I_n \xrightarrow{\mathcal{PP}} I$ stably one gets that $I_n - (\mu - \lambda)(\mu I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} I - (\mu - \lambda)(\mu I - A)^{-1}$ regularly which implies that

$$\|(I_n - (\mu - \lambda)(\mu I_n - A_n)^{-1})^{-1}\| \leq C \text{ for any } \lambda \in \Lambda.$$

The Lemma is proved. ■

3.2 Estimates for the linear case

The following Lemma gives the crucial linear estimate for our main Theorem 4.4.

Lemma 3.17. *Assume that $\Delta_{cc} \neq \emptyset$ and let the conditions (2.6), (B_1) be satisfied. Then there exist constants $\tilde{M} \geq 1, \tilde{\omega} > 0$ such that for all $0 \leq \alpha \leq 1$*

$$\| \exp(tA_n) \|_{B(E_n, E_n^\alpha)} \leq \tilde{M} t^{-\alpha} e^{-\tilde{\omega} t} \, \textit{for all} \, t > 0. \tag{3.9}$$

Proof. We introduce the sector $\Sigma(\pi - \phi, \omega) = \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| < \pi - \phi\}$ for $\omega \in \mathbb{R}$ and $0 < \phi < \frac{\pi}{2}$. From condition (B_1) we obtain suitable $\omega_3 \in \mathbb{R}$, $n_\omega \in \mathbb{N}$ and $0 < \phi < \frac{\pi}{2}$ such that

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_3}{|\lambda - \omega_3|} \text{ for all } \lambda \in \Sigma(\pi - \phi, \omega_3) \text{ and } n \geq n_\omega. \tag{3.10}$$

From (2.6) it follows that $Re\sigma(A) < \omega'$ for some $\omega' < 0$ and moreover $\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - \omega'|}$ for $Re\lambda > \omega'$. Now let $0 < \tilde{\omega} < |\omega'|$ and consider the triangular region $\Lambda = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq -\tilde{\omega} \text{ and } \lambda \notin \Sigma(\pi - \phi, \omega_3)\}$, see Figure 1.

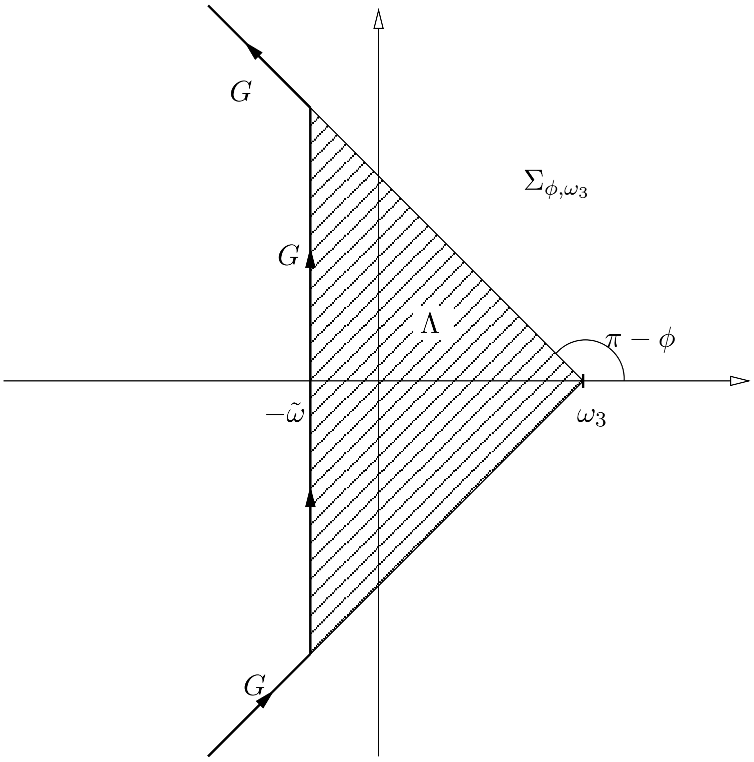

Figure 1: Figure 1

Then Λ is compact and $\Lambda \subset \rho(A)$. By Lemma 3.16 there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$ and $\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - A_n)^{-1}\| < \infty$. With the contour G composed of $\{\lambda : Re\lambda = -\tilde{\omega}\}$ and part of the boundary $\partial\Sigma(\pi - \phi, \omega_3)$ (see Figure 1) we have the representation

$$\exp(tA_n) = \frac{1}{2\pi i} \int_G e^{\lambda t}(\lambda I_n - A_n)^{-1} d\lambda.$$

Using the estimate (3.10) of resolvents $\|(\lambda I_n - A_n)^{-1}\|$ and the uniform bound on G one gets the estimate (3.9) for $t > 0$, cf. (2.16). ■

Next we introduce the operators

$$p_n^\alpha = (-A_n)^{-\alpha} p_n (-A)^\alpha \in B(E^\alpha, E_n^\alpha) \tag{3.11}$$

and show that they have the property (3.1), but for the spaces E^α, E_n^α. We then write $x_n \xrightarrow{\mathcal{P}^\alpha} x$ iff $\|x_n - p_n^\alpha x\|_{E_n^\alpha} \to 0$ as $n \to \infty$. Obviously we have
$\|x_n - p_n^\alpha x\|_{E_n^\alpha} = \|(-A_n)^\alpha x_n - p_n(-A)^\alpha x\|$ and

$$\|p_n^\alpha x\|_{E_n^\alpha} = \|p_n(-A)^\alpha x\|_{E_n} \to \|(-A)^\alpha x\|_{E_n} = \|x\|_{E^\alpha} \text{ as } n \to \infty$$

for any $x \in D((-A)^\alpha)$ so that (3.1) is satisfied.

For the nonlinear result we need a theorem on uniform convergence for linear inhomogeneous problems with compact data.

Theorem 3.18. *Let $A : D(A) \subset E \to E$ be a closed operator satisfying (2.6) (in particular, A generates an exponentially decreasing semigroup). For the approximate system (3.4) assume $\Delta_{cc} \neq \emptyset$ and let conditions (A) and (B_1) of Theorem 3.5 hold. Let $K_1 \subset E^\alpha$ and $K_2 \subset E$ be compact sets. Then for any $\varepsilon > 0$ there exist $n_1 = n_1(\varepsilon) \in \mathbb{N}$ and $\delta = \delta(\varepsilon) > 0$ such that the following property holds for all $0 < T \le \infty$. For any solution $u(t)$ of (2.1) with $u(0) = u^0 \in K_1$ and $g(\cdot) \in C([0,T]; K_2)$ and for any solution $u_n(t)$ of (3.4) with $u_n(0) = p_n^\alpha u^0$ and $g_n(\cdot) \in C([0,T]; E_n)$ with $\|g_n(t) - p_n g(t)\|_{E_n} \le \delta$ for $t \in [0,T], n \ge n_1$ we have the estimate*

$$\|u_n(t) - p_n^\alpha u(t)\|_{E_n^\alpha} \le \epsilon \text{ for all } n \ge n_1,\ 0 \le t \le T. \tag{3.12}$$

Proof. Let $u(t), u(0) = u^0$ and $u_n(t; u_n^\alpha), u_n^\alpha = p_n^\alpha u^0$ denote the mild solution of (2.1) and (3.4), respectively and let g, g_n be as in the Theorem. Then the following holds

$$u(t; u^0) = \exp(tA)u^0 + \int_0^t \exp((t-s)A) g(s) ds,\ t \in [0,T],$$

$$u_n(t; u_n^\alpha) = \exp(tA_n)u_n^\alpha + \int_0^t \exp((t-s)A_n) g_n(s) ds,\ t \in [0,T].$$

By Lemma 3.17 we have $\|\exp(tA_n)\| \le Me^{-\omega t}$ for $\omega > 0, t \ge 0$. Therefore for any $\epsilon > 0$ there is T_ϵ such that

$$\|\Big(\exp(tA_n)p_n^\alpha - p_n^\alpha \exp(tA)\Big)u^0\|_{E_n^\alpha} \le Me^{-\omega t}\|u^0\|_{E^\alpha} \le \epsilon,\ t \ge T_\epsilon, \tag{3.13}$$

and by Corollary 3.6 there is $n(T_\epsilon)$ such that

$$\|\,(-A_n)^\alpha \Big(\exp(tA_n)p_n^\alpha - p_n^\alpha \exp(tA)\Big)u^0\|_{E_n} \le \epsilon,\ t \in [0, T_\epsilon] \text{ as } n \ge n(T_\epsilon). \tag{3.14}$$

We note that in case $T_\epsilon > T$ we merely need a compactness argument. A similar remark applies to the following estimates of integrals. For the ease of presentation we extend $g(t)$ and $g_n(t)$ as constants for $t \geq T$ and argue in the following for $0 \leq t < \infty$.

Consider the E_n^α-norm of the difference of integrals, i.e.

$$(-A_n)^\alpha \int_0^t \Big(\exp((t-s)A_n) g_n(s) - p_n^\alpha \exp((t-s)A) g(s) \Big) ds = \tag{3.15}$$

$$= (-A_n)^\alpha \int_0^t \exp((t-s)A_n) \Big(g_n(s) - p_n g(s) \Big) ds$$

$$+ (-A_n)^\alpha \int_0^t \Big(\exp((t-s)A_n) p_n - p_n^\alpha \exp((t-s)A) \Big) g(s)\, ds.$$

The first term can be estimated in two parts

$$(-A_n)^\alpha \Big(\int_0^{t_1} + \int_{t_1}^t \Big) = (-A_n)^\alpha \int_0^{t_1} \exp(sA_n) \Big(g_n(t-s) - p_n g(t-s) \Big) ds$$

$$+ (-A_n)^\alpha \int_0^{t-t_1} \exp((t_1+\eta)A_n) \Big(g_n(t - t_1 - \eta) - p_n g(t - t_1 - \eta) \Big) d\eta.$$

By (3.9) the second part can be made smaller than ε uniformly in n by taking t_1 sufficiently large. Then the first part with finite t_1 can be made small by the majorant term $\|g_n(t) - p_n g(t)\|_{E_n} \leq \delta$ as $n \geq n_1$. The second term in (3.15) for any $0 < t_1 \leq t$ can be re-written in the same way as

$$(-A_n)^\alpha \int_0^t \Big(\exp(\eta A_n) p_n - p_n^\alpha \exp(\eta A) \Big) g(t-\eta) d\eta =$$

$$(-A_n)^\alpha \int_0^{t_1} \Big(\exp(\eta A_n) p_n - p_n^\alpha \exp(\eta A) \Big) g(t-\eta) d\eta$$

$$+ (-A_n)^\alpha \int_0^{t-t_1} \Big(\exp((t_1+\eta)A_n) p_n - p_n^\alpha \exp((t_1+\eta)A) \Big) g(t - t_1 - \eta) d\eta.$$

Again, we first choose t_1 to make the second term small uniformly in n. Then by Corollary 3.6 the first term converges to 0 for finite t_1 since $g(\xi)$ is in the compact set K_2. The second term can be decomposed into two parts:

$$(-A_n)^\alpha \exp(t_1 A_n) \int_0^{t-t_1} \Big(\exp(\eta A_n) p_n - p_n^\alpha \exp(\eta A) \Big) g(t - t_1 + \eta) d\eta$$

and

$$(-A_n)^\alpha \Big(\exp(t_1 A_n) p_n^\alpha - p_n^\alpha \exp(t_1 A) \Big) \int_0^{t-t_1} \exp(\eta A) g(t - t_1 + \eta) d\eta.$$

Both parts can be made small by the choice of t_1, using (3.9), (2.16) and a uniform bound on the integrals. ■

3.3 Estimates for the nonlinear case

Consider in the Banach spaces E_n^α the family of Cauchy problems

$$\begin{aligned} &u_n'(t) = A_n u_n(t) + f_n(u_n(t)),\ t \geq 0, \\ &u_n(0) = u_n^0 \in E_n^\alpha, \end{aligned} \tag{3.16}$$

where $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$ and the operators (A_n, A) are compatible. We assume the nonlinear maps $f_n(\cdot) : E_n^\alpha \to E_n$ to have the following properties

(F2) The mappings f_n are continuously differentiable in $\mathcal{U}_{E_n^\alpha}(p_n^\alpha u^*, \rho)$ and whenever $x_n \in \mathcal{U}_{E_n^\alpha}(p_n^\alpha u^*, \rho)$ and $x_n \xrightarrow{\mathcal{P}^\alpha} x$ then $f_n(x_n) \xrightarrow{\mathcal{P}} f(x)$ and $f_n'(x_n) \xrightarrow{\mathcal{P}^\alpha \mathcal{P}} f'(x)$.

(F3) For any $\epsilon > 0$ there is $\delta > 0$ such that $\|f_n'(w_n) - f_n'(z_n)\|_{B(E_n^\alpha, E_n)} \leq \epsilon$ as $\|w_n - z_n\|_{E_n^\alpha} \leq \delta$ for all $w_n, z_n \in \mathcal{U}_{E_n^\alpha}(p_n^\alpha u^*; \rho)$.

Under the above assumptions the mild solution of (3.16) exists on a maximal interval $[0, \tau)$ in $\mathcal{U}_{E_n^\alpha}(p_n^\alpha u^*, \rho)$ (see [15, 35]) and we denote it by $u_n(\cdot) = T_n(\cdot) u_n^0 : \mathbb{R}^+ \to E_n$. The nonlinear semigroup $T_n(\cdot)$ satisfies the variation of constants formula

$$T_n(t) u_n^0 = \exp(tA_n) u_n^0 + \int_0^t \exp((t-s)A_n) f_n(T_n(s) u_n^0) ds,\ t \in [0, \tau). \tag{3.17}$$

Similar to (2.5) we consider the family of nonlinear problems

$$A_n u_n + f_n(u_n) = 0 \tag{3.18}$$

and define $\mathcal{E}_n = \{u_n^* \in D(A_n) : A_n u_n^* + f_n(u_n^*) = 0\}$. The following result is taken from [8].

Proposition 3.19. *Assume that the problem (2.5) has a hyperbolic solution u^* and let conditions (F2),(F3) hold. Then, there exist $n^* \in \mathbb{N}$ and $\rho^* > 0$, so that equation (3.18) has a unique solution $u_n^* \in D(A_n) \cap \mathcal{U}_{E_n^\alpha}(p_n^\alpha u^*, \rho^*)$ for each $n \geq n^*$. Moreover, u_n^* is hyperbolic and satisfies $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \to 0 \quad$ as $\ n \to \infty$.*

Proof. Define the operators $\mathcal{M}(w) = Iw + A^{-1} f(w)$, $\mathcal{M}_n(w_n) = I_n w_n + A_n^{-1} f_n(w_n)$. The derivative $\mathcal{M}'(w) = I + A^{-1} f'(w)$ is an operator from E^α to E^α, since A^{-1} maps E into $D(A) \subset E^\alpha$. From (F3) we obtain $\mathcal{M}_n(v_n) \xrightarrow{\mathcal{P}^\alpha \mathcal{P}^\alpha} \mathcal{M}(v)$ as $v_n \xrightarrow{\mathcal{P}^\alpha} v$ and $\|\mathcal{M}_n'(w_n + p_n^\alpha u^*) - \mathcal{M}_n'(p_n^\alpha u^*)\|_{E_n^\alpha} \leq \rho$ if $\|w_n\|_{E_n^\alpha} \leq \delta$ with $\rho = \rho(\delta) \to 0$ as $\delta \to 0$ uniformly in n. From condition (F2) it is also clear that $\mathcal{M}_n'(p_n u^*) \xrightarrow{\mathcal{P}^\alpha \mathcal{P}^\alpha} \mathcal{M}'(u^*)$ regularly, $\mathcal{M}_n'(p_n^\alpha u^*)$ are Fredholm of index 0 and $\mathcal{N}(A + f'(u^*)) = \{0\}$. Now Theorem 2 from [33] applies and yields the result. Hyperbolicity of u_n^* will follow from the spectral considerations below, cf. (3.21). ■

Remark 3.20. *If $\Delta_{cc} \neq \emptyset$ and we do not assume continuity of $\mathcal{M}_n'(\cdot)$ uniformly in n one can still get the existence of solutions u_n^* of equation (3.18) and convergence $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$, but uniqueness may fail.*

From now on we consider a hyperbolic point u^* and the corresponding fixed points $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$ from Proposition 3.19. Near the equilibrium u_n^* we set $u_n(t) = u_n^* + v_n(t)$, then equation (3.16) reads

$$v_n'(t) = A_{u_n^*,n} v_n(t) + F_{u_n^*,n}(v_n(t)),\ v_n(0) = v_n^0,\ t \geq 0, \tag{3.19}$$

where

$$A_{u_n^*,n} = A_n + f_n'(u_n^*),\ F_{u_n^*,n}(w_n) = f_n(v_n(t) + u_n^*) - f_n(u_n^*) - f_n'(u_n^*) w_n. \tag{3.20}$$

We decompose E_n^α using the spectral projection

$$P_n(\sigma_n^+) := P_n(\sigma_n^+, A_{u_n^*,n}) := \frac{1}{2\pi i} \int_{\partial U(\sigma_n^+)} \Big(\zeta I_n - A_{u_n^*,n}\Big)^{-1} d\zeta, \tag{3.21}$$

where $\partial U(\sigma_n^+)$ is the boundary of the region $\{\lambda \in \mathbb{C} : Re\lambda \geq 0, \lambda \notin \Sigma(\pi - \phi, \omega_2)\}$ with some $0 < \theta < \frac{\pi}{2}$ and ω_2 given by (B_1) for the operator $A_{u_n^*,n}$, compare Lemma 3.17. Note that the part of $\partial U(\sigma_n^+)$ on $i\mathbb{R}$ does not intersect $\sigma(A_{u_n^*,n})$ due to Lemma 3.16. In particular, this implies that the fixed points u_n^* are hyperbolic.

By σ_n^+ we denote the part of $\sigma(A_{u_n^*,n})$ that is inside the contour. From the representations (2.10) and (3.21) we obtain

$$P_n(\sigma_n^+) \xrightarrow{\mathcal{PP}} P(\sigma^+) \text{ compactly as } n \to \infty. \tag{3.22}$$

In order to see this one first modifies the contour in (2.10) so that it coincides with that in (3.21), then one uses the convergence $(\zeta I_n - A_{u_n^*,n})^{-1} \xrightarrow{\mathcal{PP}} (\zeta I - A_{u^*})^{-1}$ for $\zeta \in \partial U(\sigma^+)$ (see Proposition 3.10, and (3.7)) and the fact that the convergence $(\zeta I_n - A_{u_n^*,n})^{-1} p_n x \xrightarrow{\mathcal{P}} (\zeta I - A_{u^*})^{-1} x$ is uniform for $\zeta \in \partial U(\sigma^+)$. Moreover, the condition $\Delta_{cc} \neq \emptyset$ implies compact convergence of projectors and, therefore, by (3.7) and Theorem 3.11 $\dim P_n(\sigma_n^+) = \dim P(\sigma^+)$ for $n \geq n_0$.

Applying Lemma 3.17 to the semigroup $\exp(tA_{u_n^*,n})$ we obtain constants $M_2, \tilde\beta > 0$, such that

$$\|\exp(tA_{u_n^*,n}) v_n\|_{E_n^\alpha} \leq \begin{cases} M_2 \frac{e^{-\tilde\beta t}}{|t|^\alpha} \|v_n\|_{E_n}, & t \geq 0, \quad v_n \in (I_n - P_n(\sigma_n^+))E_n, \\ M_2 \frac{e^{\tilde\beta t}}{|t|^\alpha} \|v_n\|_{E_n}, & t \leq 0, \quad v_n \in P_n(\sigma_n^+)E_n. \end{cases} \tag{3.23}$$

Similar to (2.13) we consider for $v_n^-, v_n^+ \in \mathcal{U}_{E_n^\alpha}(0;\rho)$ the boundary value problem

$$v_n'(t) = A_{u_n^*,n} v_n(t) + F_{u_n^*,n}(v_n(t)),\ 0 \leq t \leq T, \tag{3.24}$$

$$(I_n - P_n(\sigma_n^+)) v_n(0) = (I_n - P_n(\sigma_n^+)) v_n^-, \quad P_n(\sigma_n^+) v_n(T) = P_n(\sigma_n^+) v_n^+,$$

where the case $T = \infty$ is included in the usual way. Using condition $(F2)$ one finds that the mild solution of problem (3.24) satisfies for $0 \leq t \leq T$ the equation

$$v_n(t) = \exp((t-T)A_{u_n^*,n}) P_n(\sigma_n^+) v_n^+ + \exp(tA_{u_n^*,n})(I_n - P_n(\sigma_n^+)) v_n^- + \tag{3.25}$$

$$+ \int_0^T \Gamma_{T,n}(t,s) F_{u_n^*,n}(v_n(s)) ds,$$

where the Green's function is

$$\Gamma_{T,n}(t,s) = \begin{cases} \exp((t-s)A_{u^*,n})(I_n - P_n(\sigma_n^+)), & 0 \le s \le t \le T, \\ \exp((t-s)A_{u^*,n})P_n(\sigma_n^+), & 0 \le t < s \le T. \end{cases} \tag{3.26}$$

In case $T = \infty$ the v_n^+ term in (3.25) vanishes. The analog of Proposition 2.2 is the following result.

Proposition 3.21. *Let the above assumptions on A, A_n and conditions $(F2)$, $(F3)$ be satisfied. Then there exists $\tilde{\rho} > 0$ such that for any $0 < \tilde{\rho}_2 \le \tilde{\rho}$ we find $0 < \tilde{\rho}_1 \le \tilde{\rho}_2$ with the property that equation (3.25) has a unique solution $v_n(\cdot) = v_n(v_n^-, v_n^+, \cdot) \in C([0,T]; \mathcal{U}_{E_n^\alpha}(0; \tilde{\rho}_2))$ for all $v_n^-, v_n^+ \in \mathcal{U}_{E_n^\alpha}(0; \tilde{\rho}_1)$ and for all $0 < T \le \infty$.*

Proof. We repeat the proof of Proposition 2.2 for the space of continuous bounded functions $C([0,T]; E_n^\alpha)$ with the operators $A_{u_n^*,n}, F_{u_n^*,n}(\cdot), P_n(\sigma_n^+)$ and $G_n(v_n^-, v_n^+; \cdot)$ defined by the right-hand side of (3.25). Note that $(F3)$ guarantees that the estimates in (3.23) can be done with constants independent of n and, therefore, Lemma 6.1 applies with uniform data. Moreover, the estimates (2.19) and (2.20) hold uniformly in n. From (6.7) we find a constant $C^* > 0$ such that for any two $v_n, w_n \in C([0,T], \mathcal{U}_{E_n^\alpha}(0; \tilde{\rho}_2))$ we have for $\|\cdot\| = \|\cdot\|_{C([0,T];E_n^\alpha)}$ the estimate

$$\|v_n - w_n\| \le C^* \|v_n - G_n(v_n^-, v_n^+; v_n) - (w_n - G_n(v_n^-, v_n^+; w_n))\|. \tag{3.27}$$

■

4 Shadowing for a general discretization scheme

4.1 Adapted discretization maps

In addition to p_n, p_n^α we need discretizing maps that are adapted to the hyperbolic splitting. First note that the spectral projectors $P = P(\sigma^+)$ and $P_n = P_n(\sigma_n^+)$ are finite dimensional and satisfy $P_n \xrightarrow{\mathcal{PP}} P$ compactly, see (3.22). Then define the discretizing maps $\tilde{p}_n : E \mapsto E_n$ by

$$\tilde{p}_n = P_n p_n P + (I_n - P_n) p_n (I - P) \tag{4.1}$$

and $\tilde{p}_n^\alpha : E^\alpha \mapsto E_n^\alpha$ by

$$\tilde{p}_n^\alpha x = \begin{cases} (A_{u_n^*,n})^{-\alpha} P_n p_n (A_{u^*})^\alpha P x, & x \in P E^\alpha, \\ (-A_{u_n^*,n})^{-\alpha} (I_n - P_n) p_n (-A_{u^*})^\alpha (I - P) x, & x \in (I - P) E^\alpha. \end{cases} \tag{4.2}$$

Note that the spectra of $A_{u^*} = A + f'(u^*)$ and $A_{u_n^*,n} = A_n + f_n'(u_n^*)$ are partitioned in such a way that the fractional powers of the operators are well defined.

Proposition 4.1. *The system $\{\tilde{p}_n\}$ is equivalent to the system $\{p_n\}$ on E and the system $\{\tilde{p}_n^\alpha\}$ is equivalent to the system $\{p_n^\alpha\}$ on E^α. In particular, we have bounds*

$$\|\tilde{p}_n\|_{B(E,E_n)} \le \tilde{C}, \quad \|\tilde{p}_n^\alpha\|_{B(E^\alpha,E_n^\alpha)} \le \tilde{C}_\alpha \quad \textit{for all } n \in \mathbb{N} \tag{4.3}$$

and $\sup_{x \in K_2} \|(p_n - \tilde{p}_n)x\| \to 0$, $\sup_{x \in K_1} \|(p_n^\alpha - \tilde{p}_n^\alpha)x\| \to 0$ as $n \to \infty$ for compact sets $K_2 \subset E, K_1 \subset E^\alpha$.

Proof. From the equality

$$(\tilde{p}_n - p_n)x = 2(P_n p_n - p_n P)Px - (P_n p_n - p_n P)x,$$

we find that the system $\{\tilde{p}_n\}$ is equivalent to $\{p_n\}$ on E. In the α-case the following holds on $(I-P)E^\alpha$

$$\begin{aligned}
p_n^\alpha - \tilde{p}_n^\alpha &= (-A_n)^{-\alpha}(p_n - \tilde{p}_n)(-A)^\alpha + (-A_n)^{-\alpha}\tilde{p}_n(-A)^\alpha - (-A_{u_n^*,n})^{-\alpha}\tilde{p}_n(-A_{u^*})^\alpha \\
&= (-A_n)^{-\alpha}(p_n - \tilde{p}_n)(-A)^\alpha + ((-A_n)^{-\alpha} - (-A_{u_n^*,n})^{-\alpha})\tilde{p}_n(-A)^\alpha \\
&\quad + (-A_{u_n^*,n})^{-\alpha}\tilde{p}_n((-A)^\alpha - (-A_{u^*})^\alpha) \\
&= (-A_n)^{-\alpha}(p_n - \tilde{p}_n)(-A)^\alpha + \Big((-A_n)^{-\alpha} - (-A_{u_n^*,n})^{-\alpha}\Big)\tilde{p}_n(-A)^\alpha \\
&\quad + (-A_{u_n^*,n})^{-\alpha}\tilde{p}_n((-A)^\alpha - (-A_{u^*})^\alpha) \\
&= (-A_n)^{-\alpha}(p_n - \tilde{p}_n)(-A)^\alpha + (-A_{u_n^*,n})^{-\alpha}\Big((-A_{u_n^*,n})^\alpha(-A_n)^{-\alpha} - I_n\Big)\tilde{p}_n(-A)^\alpha \\
&\quad - (-A_{u_n^*,n})^{-\alpha}\tilde{p}_n((-A_{u^*})^\alpha(-A)^{-\alpha} - I)(-A)^\alpha \\
&= (-A_n)^{-\alpha}(p_n - \tilde{p}_n)(-A)^\alpha(-A_{u_n^*,n})^{-\alpha} \\
&\quad + \Big(((-A_{u_n^*,n})^\alpha(-A_n)^{-\alpha} - I_n)\tilde{p}_n - \tilde{p}_n((-A_{u^*})^\alpha(-A)^{-\alpha} - I)\Big)(-A)^\alpha.
\end{aligned}$$

In the last step we show that $(-A_{u_n^*,n})^\alpha(-A_n)^{-\alpha} \xrightarrow{\mathcal{PP}} (-A_{u^*})^\alpha(-A)^{-\alpha}$. To this end following [15] we consider the formula

$$(-A_{u^*})^{-\alpha} - (-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty z^{-\alpha}\Big((zI + A_{u^*})^{-1} - (zI + A)^{-1}\Big)dz = \tag{4.4}$$

$$= \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty z^{-\alpha}\Big((zI + A_{u^*})^{-1}(f'(u^*)(-A)^{-\alpha})(-A)^\alpha(zI + A)^{-1}\Big)dz.$$

Since $\|(-A)^\alpha(zI + A)^{-1}\| = O(|z|^{\alpha-1})$ as $z \to \infty$ the integral will converge even if we apply $(-A_{u^*})^\alpha$. Therefore convergence $(-A_{u_n^*,n})^\alpha(-A_n)^{-\alpha} \xrightarrow{\mathcal{PP}} (-A_{u^*})^\alpha(-A)^{-\alpha}$ follows from $f_n'(u_n^*)A_n^{-\alpha} \xrightarrow{\mathcal{PP}} f'(u^*)A^{-\alpha}$ and Lebesgue's dominated convergence theorem. The uniform convergence on compact sets follows as in Lemma 3.3. ■

Remark 4.2. *It is easy to see that $(-A)^\alpha(-A_{u^*})^{-\alpha}$ is a bounded operator and $(-A_n)^\alpha(-A_{u_n^*,n})^{-\alpha} \xrightarrow{\mathcal{PP}} (-A)^\alpha(-A_{u^*})^{-\alpha}$. Indeed, $\|(-A)^\alpha(zI - A_{u^*})^{-1}\| = O(|z|^{\alpha-1})$ as $z \to \infty$ and therefore interchanging A and A_{u^*} in (4.4) one gets the statements on boundedness and convergence in the same way as in Proposition 4.1.*

Proposition 4.3. *Consider $0 < \alpha < \gamma \le 1$ and let $(F1) - (F3)$ be satisfied, then*

$$\sup_{w \in \mathcal{U}_{E^\gamma}(0;\rho)} \|F_{u_n^*,n}(\tilde{p}_n^\alpha w) - \tilde{p}_n F_{u^*}(w)\|_{E_n} \to 0 \text{ as } n \to \infty. \tag{4.5}$$

Proof. First note

$$\begin{aligned}
F_{u_n^*,n}(\tilde{p}_n^\alpha w) - \tilde{p}_n F_{u^*}(w) &= f_n(u_n^* + \tilde{p}_n^\alpha w) - \tilde{p}_n f(u^* + w) \\
&- \big(f_n(u_n^*) - \tilde{p}_n f(u^*) + f_n'(u_n^*)\tilde{p}_n^\alpha w - \tilde{p}_n f'(u^*)w\big),
\end{aligned}$$

and observe that by Proposition 4.1 we can replace the maps $\tilde{p}_n, \tilde{p}_n^\alpha$ by p_n, p_n^α. From Proposition 3.19 and (F2) we obtain $f(u_n^* + p_n^\alpha w) \xrightarrow{\mathcal{P}} f(u^* + w)$, $f_n(u_n^*) \xrightarrow{\mathcal{P}} f(u^*)$ and $f_n'(u_n^*) \xrightarrow{\mathcal{P}^\alpha\mathcal{P}} f'(u^*)$. Following the proof of (3.2) one then shows that the convergence is uniform on compact sets in E^α. ■

4.2 Shadowing with discretization in space variables

The first result approximates orbits of the general evolution equation (2.7) by appropriate orbits of the 'spatially discretized' system (3.16).

Theorem 4.4. *Let A be the generator of an exponentially decreasing analytic C_0-semigroup and consider $0 \le \alpha < \gamma < 1$. For the discretized system (3.16) assume that the linear parts satisfy $\Delta_{cc} \neq \emptyset$ and condition (B_1) and the nonlinear parts satisfy conditions $(F1), (F2), (F3)$. Then there exists $\rho_0 > 0$ with the following property. For any $\varepsilon_0 > 0$ there is an $n_0 = n_0(\varepsilon_0) \in \mathbb{N}$ such that for any mild solution $u(t)$ of (2.7) satisfying $u(t) \in \mathcal{U}_{E\gamma}(u^*, \rho_0), 0 \le t \le T$ for some $0 < T \le \infty$ there exist initial values $u_n^0 \in E_n^\alpha, n \ge n_0$ such that the mild solution $u_n(t; u_n^0)$ of (3.16) exists on $[0, T]$ and satisfies*

$$\sup_{0 \le t \le T} \|p_n^\alpha u(t) - u_n(t; u_n^0)\|_{E_n^\alpha} \le \varepsilon_0 \quad \forall n \ge n_0(\varepsilon). \tag{4.6}$$

Proof. We will collect the conditions on n_0 and ρ_0 during the proof. Let $u(t)$, $0 \le t \le T$, be a mild solution of (2.7) such that $u(t) \in \mathcal{U}_{E^\gamma}(u^*, \rho_0)$ for all $0 \le t \le T$. Then $v(t) := u(t) - u^* \in \mathcal{U}_{E^\gamma}(0, \rho_0)$ is a solution of (2.14) with

$$v^- = (I - P)v(0), \quad v^+ = Pv(0). \tag{4.7}$$

By the choice of norms (2.12) we have

$$\|v^-\|_{E^\gamma} \le \rho_0, \quad \|v^+\|_{E^\gamma} \le \rho_0. \tag{4.8}$$

We apply Proposition 2.2 with γ in place of α and with $\hat{\rho}_2 = \hat{\rho}$. We require $\rho_0 \le \hat{\rho}_1$ which implies $\rho_0 \le \hat{\rho}_2$ as well. Because of uniqueness in $C([0, T], \mathcal{U}_{E^\gamma}(0, \hat{\rho}_2))$ the solution $v(v^-, v^+, \cdot)$ from Proposition 2.2 satisfies $v(t) = v(v^-, v^+, t), 0 \le t \le T$. Next we define the discrete boundary values

$$v_n^- = \tilde{p}_n^\alpha v^-, \quad v_n^+ = \tilde{p}_n^\alpha v^+. \tag{4.9}$$

By the definition (4.2) we have $v_n^- = \tilde{p}_n^\alpha v^- = (I_n - P_n)\tilde{p}_n^\alpha v^-$ and $v_n^+ = \tilde{p}_n^\alpha v^+ = P_n \tilde{p}_n^\alpha v^+$ and from (4.3)

$$\|v_n^\pm\|_{E_n^\alpha} \le \tilde{C}_\alpha \|v^\pm\|_{E^\alpha} \le \tilde{C}_\alpha \rho_0. \tag{4.10}$$

Then we apply Proposition 3.21 with $\tilde{\rho}_2 = \tilde{\rho}$ and require $\tilde{C}_\alpha \rho_0 \le \tilde{\rho}_1(\tilde{\rho}_2)$. Taking the corresponding unique solutions of (3.25)

$$v_n(\cdot) = v_n(v_n^-, v_n^+)(\cdot) \in C([0, T], \mathcal{U}_{E_n^\alpha}(0, \tilde{\rho}_2)) \tag{4.11}$$

we claim that

$$u_n^0 = u_n^* + v_n(0), \quad u_n(t) = v_n(t) + u_n^* \tag{4.12}$$

satisfy the assertion of the present Theorem. We require $\tilde{C}_\alpha \rho_0 \le \tilde{\rho}_1$ so that we can apply (3.27) to $v_n(t)$ and $w_n(t) = \tilde{p}_n^\alpha v(t)$. We obtain

$$\sup_{0 \le \le T} \|v_n(t) - \tilde{p}_n^\alpha v(t)\|_{E_n^\alpha} \le C^* \sup_{0 \le t \le T} \|\eta_n^-(t) + \eta_n^+(t)\|_{E_n^\alpha}, \tag{4.13}$$

where the terms on the right-hand side are given by

$$
\begin{aligned}
\eta_n^-(t) &= \tilde{p}_n^\alpha \exp(tA_{u^*})(I-P)v^- - \exp(tA_{u_n^*,n})(I_n-P_n)v_n^- \\
&+ \tilde{p}_n^\alpha \int_0^t \exp((t-s)A_{u^*})(I-P)F_{u^*}(v(s))ds \\
&- \int_0^t \exp((t-s)A_{u_n^*,n})(I_n-P_n)F_{u_n^*,n}(\tilde{p}_n^\alpha v(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
\eta_n^+(t) &= \tilde{p}_n^\alpha \exp((t-T)A_{u^*})Pv^+ - \exp((t-T)A_{u_n^*,n})P_n v_n^+ \\
&+ \tilde{p}_n^\alpha \int_t^T \exp((t-s)A_{u^*})PF_{u^*}(v(s))ds \\
&- \int_t^T \exp((t-s)A_{u_n^*,n})P_n F_{u_n^*,n}(\tilde{p}_n^\alpha v(s))ds.
\end{aligned}
$$

We estimate η_n^+ by an application of Theorem 3.18 with the settings $\tilde{E}=(I-P)E$, $D(\tilde{A})=D(A)\cap\tilde{E}$, $\tilde{A}=A_{u^*}$, $g(t)=(I-P)F_{u^*}(v(t))$, $g_n(t)=(I_n-P_n)F_{u_n^*,n}(\tilde{p}_n^\alpha v(t))$, $K_1=\mathcal{U}_{E^\gamma}(0,\rho_0)$, $K_2=\{(I-P)F_{u^*}(w) : w\in\mathcal{U}_{E^\gamma}(0,\rho_0)\}$, $\tilde{\varepsilon}=\frac{\varepsilon_0}{4C^*}$, $u^0=v^-$, $\tilde{E}_n=(I_n-P_n)E_n$, $\tilde{E}_n^\alpha=(I_n-P_n)E_n^\alpha$, $u_n(0)=\tilde{p}_n^\alpha u(0)=\tilde{p}_n^\alpha v^-$. Note that by the continuity of F_{u^*} from E^α to E and the compact embedding of E^γ in E^α the set K_1 is compact in E^α and K_2 is compact in E. The estimate (3.12) applies for $n\ge n_3=\max(n_1(\tilde{\varepsilon}),n_2)$ where n_2 is chosen by Proposition 4.3 such that for $n\ge n_2$

$$
\begin{aligned}
\sup_{0\le t\le T}\|g_n(t)-\tilde{p}_n g(t)\|_{\tilde{E}_n} &\le \sup_{0\le t\le T}\|(I-P_n)\left(F_{u_n^*,n}(\tilde{p}_n^\alpha v(t))-p_nF_{u^*}(v(t))\right)\|_{E_n} \\
&+ \sup_{w\in K_2}\|(I-P_n)(p_nP-P_np_n)F_{u^*}(w)\|_{E_n} \\
&\le \delta=\delta(\tilde{\varepsilon}).
\end{aligned}
$$

Therefore we have $\|\eta_n^-\|_{E_n^\alpha}\le\tilde{\varepsilon}$ and by an analogous reasoning the same estimate for $\|\eta_n^+\|_{E_n^\alpha}$ and $n\ge n_4$. Finally, by (4.13) and Proposition 3.19 we obtain for some $n_5\ge n_4$ and all $0\le t\le T, n\ge n_5$

$$
\|u_n(t)-p_n^\alpha u(t)\|_{E_n^\alpha}\le\|v_n(t)-p_n^\alpha v(t)\|_{E_n^\alpha}+\|u_n^*-p_n^\alpha u^*\|_{E_n^\alpha}\le\frac{\varepsilon_0}{2}+\frac{\varepsilon_0}{2}=\varepsilon_0.
$$

■

In the following Lemma we approximate vectors of a compact sequence in the discrete spaces by discretizations of continuous elements.

Lemma 4.5. *Let $0\le\alpha<\gamma<1$ and let $\{v_n^0\}$ be a bounded sequence in E_n^γ. Then for any $\varepsilon>0$ there is a number $n_0(\varepsilon)$ such that $\inf_{v\in E^\alpha}\|v_n^0-\tilde{p}_n^\alpha v\|_{E_n^\alpha}\le\varepsilon$ for $n\ge n_0(\varepsilon)$. In addition, if $0\le\alpha\le\beta<\gamma<1$ and the sequence $\{v_n^0\}$ lies in $P_nE_n^\gamma$ and satisfies $\|v_n^0\|_{E_n^\gamma}\le b$ then one can find $n_1(\varepsilon)$ and a constant $\hat{C}>0$ such that*

$$
\inf_{v\in PE^\beta,\|v\|_{E^\beta}\le\hat{C}b}\|v_n^0-p_n^\alpha v\|_{E_n^\alpha}\le\varepsilon\quad\text{for all } n\ge n_1(\varepsilon). \tag{4.14}
$$

If $v_n^0\in(I_n-P_n)E_n^\gamma$ instead of $v_n^0\in P_nE_n^\gamma$ then (4.14) holds with the infimum taken over $v\in(I-P)E^\beta$, $\|v\|_{E^\beta}\le\hat{C}b$.

Proof. Proposition 4.1 shows that it is sufficient to prove the assertion with p_n^α in place of $\tilde{p}_n^\alpha$. Assume that the statement is not true. Then there exists a sequence $\{v_n^0\}$, an $\varepsilon > 0$ and a subsequence $\mathbb{N}' \subseteq \mathbb{N}$ such that $\|(-A_n)^\gamma v_n^0\| \le b$ and $\|v_n^0 - p_n^\alpha v\|_{E_n^\alpha} > \varepsilon$ for all $n \in \mathbb{N}', v \in E^\alpha$. First of all, the sequence $(-A)_n^{-\alpha}(-A)_n^\alpha v_n^0 = v_n^0$ is $\mathcal{P}$-compact by (3.6) and therefore there are $\mathbb{N}'' \subseteq \mathbb{N}'$ and $\bar{v} \in E$ such that $v_n^0 \xrightarrow{\mathcal{P}} \bar{v}$ as $n \in \mathbb{N}''$. Since $0 \le \alpha < \gamma < 1$ and $(-A_n)^{\alpha-\gamma}(-A_n)^\gamma v_n^0 = (-A_n)^\alpha v_n^0$ we obtain in a similar way from (3.6) that $(-A_n)^\alpha v_n^0$ is $\mathcal{P}$-compact. Thus there is $\mathbb{N}''' \subseteq \mathbb{N}''$ such that $(-A_n)^\alpha v_n^0 \xrightarrow{\mathcal{P}} z \in E$ as $n \in \mathbb{N}'''$ and we conclude $(-A_n)^{-\alpha}(-A_n)^\alpha v_n^0 \xrightarrow{\mathcal{P}} (-A)^{-\alpha} z$ as $n \in \mathbb{N}'''$. On the other hand $v_n^0 \xrightarrow{\mathcal{P}} \bar{v}$ as $n \in \mathbb{N}''$ which implies $\bar{v} = (-A)^{-\alpha} z \in E^\alpha$. Finally we have $\|v_n^0 - p_n^\alpha \bar{v}\|_{E_n^\alpha} = \|(-A_n)^\alpha v_n^0 - p_n z\|_{E_n} \to 0$ as $n \in \mathbb{N}'''$ for some $\bar{v} \in E^\alpha$ which is a contradiction. To prove the second assertion we extend the previous argument. First by (3.6) and Lemma 3.3 we have for every $0 \le \lambda < 1$ a constant $C_{[\lambda]} > 0$ such that

$$\|(-A)^{-\lambda}\|_{B(E,E)}, \|(-A_n)^{-\lambda}\|_{B(E_n,E_n)} \le C_{[\lambda]} \quad \text{for all } n \in \mathbb{N}. \tag{4.15}$$

Define $\hat{C} = C_{[\gamma-\beta]} + 1$. In the proof above we can arrange that $(-A_n)^\beta v_n^0 \xrightarrow{\mathcal{P}} y \in E$ for $n \in \mathbb{N}'''$ and then obtain $\bar{v} = (-A)^{-\beta} y \in E^\beta$. Using $v_n^0 \in P_n E_n^\gamma$ and the convergence of projectors (3.22) we find

$$0 = (I_n - P_n) v_n^0 \xrightarrow{\mathcal{P}} \bar{v} - P\bar{v},$$

hence $\bar{v} \in PE^\beta$. Moreover for $n \in \mathbb{N}'''$ large

$$\|\bar{v}\|_{E^\beta} = \|y\|_E \le \|(-A_n)^{\beta-\gamma}(-A_n)^\gamma v_n^0\|_{E_n} + b \le (C_{[\gamma-\beta]} + 1)b = \hat{C}b.$$

This contradicts $\|v_n^0 - p_n^\alpha v\| > \varepsilon$ for all $v \in PE^\beta$ with $\|v\|_{E^\beta} \le \hat{C}b$. ■

This Lemma will be used for constructing the appropriate boundary data for the following inverse shadowing result.

Theorem 4.6. *Let the assumptions of Theorem 4.4 hold. Then there exists $\rho_0 > 0$ with the following property. For any $\varepsilon_0 > 0$ there is an $n_0 = n_0(\varepsilon_0) \in \mathbb{N}$ such that for any mild solution $u_n(t)$, $n \ge n_0$ of (3.16) satisfying $u_n(t) \in \mathcal{U}_{E_n^\gamma}(u_n^*, \rho_0), 0 \le t \le T$ for some $0 < T \le \infty$ there exist initial values $u^{n,0} \in E^\alpha, n \ge n_0$, such that the mild solution $u(t; u^{n,0})$ of (2.7) exists on $[0,T]$ and satisfies*

$$\sup_{0 \le t \le T} \|u_n(t) - p_n^\alpha u(t; u^{n,0})\|_{E_n^\alpha} \le \varepsilon_0 \quad \forall n \ge n_0(\varepsilon_0). \tag{4.16}$$

Proof. As in Theorem 4.4 we take some $\varepsilon_0 > 0$ and list the conditions on n_0 and ρ_0 during the proof. Consider a mild solution $u_n(t), 0 \le t \le T$ of (3.16) that lies in $\mathcal{U}_{E_n^\gamma}(u_n^*, \rho_0)$ and define

$$v_n(t) = u_n(t) - u_n^*, \quad v_n^- = (I_n - P_n) v_n(0), \quad v_n^+ = P_n v_n(T). \tag{4.17}$$

By the uniform boundedness of projectors we have for some $C_b \ge 1$

$$\|v_n^-\|_{E_n^\gamma} \le C_b \rho_0, \quad \|v_n^+\|_{E_n^\gamma} \le C_b \rho_0. \tag{4.18}$$

We apply Proposition 3.21 to the values $v_n^{\pm}$ with γ instead of α and $\tilde{\rho}_2 = \tilde{\rho}$. We require $C_b \rho_0 \le \tilde{\rho}_1$ so that $v_n(t) = v_n(v_n^-, v_n^+, t), 0 \le t \le T$ holds by the uniqueness of solutions in $C([0,T], \mathcal{U}_{E_n^\gamma}(0, \tilde{\rho}))$. Now take $\alpha < \beta < \gamma$ and use Lemma 4.5 to construct boundary values $v^{n,-} \in (I-P)E^\beta, v^{n,+} \in PE^\beta, n \ge n_1(\varepsilon_0)$ such that

$$\|v_n^- - \tilde{p}_n^\alpha v^{n,-}\|_{E_n^\alpha} + \|v_n^+ - \tilde{p}_n^\alpha v^{n,+}\|_{E_n^\alpha} \le \frac{\varepsilon_0}{16 M_2 C^*}, \quad \|v^{n,\pm}\|_{E^\beta} \le \hat{C} \rho_0, \tag{4.19}$$

see (3.23),(3.27). In the next step we apply Proposition 2.2 with boundary values $v^{n,\pm}$ and β instead of α. Choose $\hat{\rho}_2$ such that (cf. (4.15),(4.3))

$$\tilde{C}_\alpha C_{[\beta-\alpha]} \hat{\rho}_2 \le \tilde{\rho}_2,$$

require $(C_{[\gamma-\alpha]} + \hat{C})\rho_0 \le \tilde{\rho}_2$ and denote the unique solution in $C([0,T]; \mathcal{U}_{E^\beta}(0; \hat{\rho}_2))$ by $v^n(t), 0 \le t \le T$. We will show that

$$u^{n,0} = v^n(0) + u^*, \quad u(t; u^{n,0}) = v^n(t) + u^*, \ 0 \le t \le T, \tag{4.20}$$

satisfies (4.6). For this purpose we insert $v_n(\cdot)$ and $w_n(\cdot) := \tilde{p}_n^\alpha v^n(\cdot)$ into (3.27). This inequality is valid since $\|v_n(t)\|_{E_n^\alpha} \le C_{[\gamma-\alpha]} \rho_0$,

$$\|w_n(t)\|_{E_n^\alpha} \le \tilde{C}_\alpha \|v^n(t)\|_{E^\alpha} \le \tilde{C}_\alpha C_{[\beta-\alpha]} \|v^n(t)\|_{E^\beta} \le \tilde{C}_\alpha C_{[\beta-\alpha]} \hat{\rho}_2 \le \tilde{\rho}_2$$

and $\|v_n^\pm\|_{E_n^\alpha} \le C_{[\gamma-\alpha]} \|v_n^\pm\|_{E_n^\gamma} \le C_{[\gamma-\alpha]} C_b \rho_0 \le \tilde{\rho}_1$.

We obtain the estimate

$$\begin{aligned} \|v_n - w_n\|_{C([0,T];E_n^\alpha)} &\le C^* \|w_n - G_n(v_n^-, v_n^+, w_n)\|_{C([0,T];E_n^\alpha)} && (4.21) \\ &\le C^* \sup_{0 \le t \le T} \|\eta_n^-(t) + \eta_n^+(t) + \varphi_n^-(t) + \varphi_n^+(t)\|_{E_n^\alpha}, && (4.22) \end{aligned}$$

where the terms on the right-hand side are given by

$$\begin{aligned} \eta_n^-(t) &= \tilde{p}_n^\alpha \exp(tA_{u^*})(I-P)v^{n,-} - \exp(tA_{u_n^*,n})(I_n - P_n)\tilde{p}_n^\alpha v^{n,-} \\ &+ \tilde{p}_n^\alpha \int_0^t \exp((t-s)A_{u^*})(I-P)F_{u^*}(v^n(s))ds \\ &- \int_0^t \exp((t-s)A_{u_n^*,n})(I_n - P_n)F_{u_n^*,n}(\tilde{p}_n^\alpha v^n(s))ds, \\ \eta_n^+(t) &= \tilde{p}_n^\alpha \exp((t-T)A_{u^*})Pv^{n,+} - \exp((t-T)A_{u_n^*,n})P_n \tilde{p}_n^\alpha v^{n,+} \\ &+ \tilde{p}_n^\alpha \int_t^T \exp((t-s)A_{u^*})PF_{u^*}(v^n(s))ds \\ &- \int_t^T \exp((t-s)A_{u_n^*,n})P_n F_{u_n^*,n}(\tilde{p}_n^\alpha v^n(s))ds, \\ \varphi_n^-(t) &= \exp(tA_{u_n^*,n})(I_n - P_n)(\tilde{p}_n^\alpha v^{n,-} - v_n^-), \\ \varphi_n^+(t) &= \exp((t-T)A_{u_n^*,n})P_n(\tilde{p}_n^\alpha v^{n,+} - v_n^+). \end{aligned}$$

We obtain the estimate $\|\eta_n^\pm\| \le \frac{\varepsilon_0}{8C^*}$ by Theorem 3.18 in almost the same way as in Theorem 4.4, cf. (4.13), the main difference being that $v(s)$ is replaced by $v^n(s)$

and the compact sets are given by $K_2 = \{(I-P)F_{u^*}(w) : w \in \mathcal{U}_{E^\beta}(0,\hat{\rho}_2)\}$ and $K_1 = \mathcal{U}_{E^\beta}(0,\hat{C}\rho_0)$, cf. (4.19).

For the second terms we have from Lemma 3.17 and (4.19)

$$\|\varphi_n^-(t) + \varphi_n^+(t)\| \leq M_2 e^{-\tilde{\beta}t} \frac{\varepsilon_0}{8M_2C^*} \leq \frac{\varepsilon_0}{8C^*}.$$

Using this in (4.21) and Proposition 3.19 we finally have for n large

$$\|u_n(t) - p_n^\alpha u(t,u^{n,0})\|_{E_n^\alpha} \leq \|v_n(t) - p_n^\alpha v^n(t)\|_{E_n^\alpha} + \|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \leq \varepsilon_0.$$

■

5 Applications

In this section we show how the assumptions of the Theorems 4.4 and 4.6 can be satisfied for finite element and finite difference methods.

Example 5.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded smooth domain. Consider the second order strongly elliptic operator*

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x)u_{x_ix_j}(x) + \sum_{j=1}^d b_j(x)u_{x_j}(x) + c(x)u(x), \tag{5.1}$$

where the coefficientes a_{ij}, b_j, c are smooth bounded functions. Consider the associated parabolic problem

$$u_t(t,x) = Lu(t,x) + f(u(t,x)), \quad t>0,\ x \in \Omega, \tag{5.2}$$

$$u(t,x) = 0, \quad t>0,\ x \in \partial\Omega,\ u(0,x) = u^0(x) \in H_0^1(\Omega).$$

Let $E = L^2(\Omega)$ and define the operator $A : D(A) \subset E \to E$ by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = Lu$ for all $u \in D(A)$. It is well known that A generates an analytic and compact C_0-semigroup $\{\exp(tA) : t \geq 0\}$. Assume that $c(x)$ is chosen such that the spectrum of A is located to the left of the imaginary axis. Then, we can define the fractional powers $(-A)^\alpha$ of $-A$ as before. It is well known that $E^1 = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $E^{1/2} = H_0^1(\Omega)$.

As to the nonlinear term $f(\cdot)$, it is known (see [3, 19]) that under some growth conditions, the problem (5.2) is locally well-posed in $E^{1/2}$ and the operator-function $f(\cdot)$ is Frechet differentiable as a function from E^α to E. For example, these assumptions are as follows, cf. [19]: the scalar function $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$, $x \in \Omega$, is in $C^2(\mathbb{R},\mathbb{R})$ and one has

$$|f_\xi^{(l)}(x,\xi)| \leq C(1 + |\xi|^{\delta+1-l}) \text{ for any } \xi \in \mathbb{R}, x \in \Omega,$$

where $l = 1,2$, and $\delta = 2$ if $d = 3$ and $\delta \in [1,\infty)$ if $d = 2$. Then one can show (see [19]) that

$$\|f'(u) - f'(v)\|_{B(E^{1/2},E)} \leq C(\rho)\|u - v\|_{E^{1/2}} \tag{5.3}$$

and

$$\|f(u) - f(v) - f'(w)(u-v)\|_{E^{\frac{k-1}{2}}} \leq C(\rho)(\|u-w\|_{E^{1/2}} + \|v-w\|_{E^{1/2}})\|u-v\|_{E^{k/2}} \tag{5.4}$$

for $k = 1, 2$, *and any* $v, w, u \in \{z \in E : \|z - u^*\|_{E^{1/2}} \le \rho\}$. *The inequalities (5.3)-(5.4) imply condition* $(F1)$.

With the operator A *and the function* $f(\cdot)$ *defined in this way the problem (2.3) is well posed and has all properties we need for our main Theorems.*

Moreover, the approximation problems (3.16) also have the required properties when defined by the finite element method.

Indeed, let A, E *and* $E^{1/2}$ *be as before. It is well known (see [17]) that* A *is in one-to-one correspondence with a sesquilinear form* $a : E^{1/2} \times E^{1/2} \to \mathbb{C}$ *such that*

$$|a(u,v)| \le c_1 \|u\|_{E^{1/2}} \|v\|_{E^{1/2}}, \quad u, v \in E^{1/2},$$

$$\operatorname{Re} a(u,u) \ge c_2 \|u\|_{E^{1/2}}, \quad u \in E^{1/2},$$

$$a(u,v) = \langle -Au, v\rangle, \quad u \in D(A),\ v \in E^{1/2}.$$

Consider a convex polygon $\Omega \subset \mathbb{R}^2$ *and a regular triangulation where the triangles have maximum diameter* h. *Denote by* S_h *the space of functions in* $E^{1/2}$ *that are linear in each element and vanish on the boundary.*

Then S_h *is a family of finite dimensional subspaces of* $H_0^1(\Omega)$ *with the standard approximation property (see[30])*

$$\inf_{\chi \in S_h} \left(\|v - \chi\|_E + h\|v - \chi\|_{E^{1/2}} \right) \le Ch^2 \|v\|_{E^1} \ \text{ for } \ v \in H^2(\Omega) \cap H_0^1(\Omega).$$

We denote by $P_h u$ *the projection of* $u \in E$ *onto* $S_h = E_h^{1/2}$ *with respect to the* $L^2(\Omega)$ *inner product. These operators play the role of the connecting mappings* $\{p_h\}$. *In this framework, the finite element approximation* $A_h : S_h \to S_h$ *of* A *is defined by*

$$\langle -A_h \phi_h, \psi_h \rangle = a(\phi_h, \psi_h), \quad \phi_h, \psi_h \in E_h^{1/2}.$$

In other words, A_h *is the operator associated with the sesquilinear form* $a_h(\cdot,\cdot)$ *which is the restriction of* $a(\cdot,\cdot)$ *to* $E_h^{1/2} \times E_h^{1/2}$. *In this setting one can prove [11] that there exists a constant* C *and an acute angle* θ *such that for* $u \in E$ *and* $\theta \le |\arg z| \le \pi$ *we have*

$$\|(zI - A)^{-1}u - (zI_h - A_h)^{-1} P_h u\|_E \le Ch^2 \|u\|_E.$$

This estimate shows $\mathcal{P}$*-convergence with uniform convergence of resolvents. Since our resolvent* $(\lambda I - A)^{-1}$ *is compact for some* λ, *then the above inequality yields (with* $\mu(\cdot)$ *being the measure of noncompactness)*

$$\mu((zI_h - A_h)^{-1}x_h) \le \mu((zI - A)^{-1}x_h) + \overline{\lim}_{h\to 0} \|(zI - A)^{-1}x_h - (zI_h - A_h)^{-1}x_h\|_E = 0$$

and therefore compact convergence of resolvents as $h \to 0$. *In this way, our basic assumption* $\Delta_{cc} \ne \emptyset$ *can be verified.*

Finally with $f_h(v_h) = p_h f(v_h)$ *for* $v_h \in E_h^{1/2}$ *and* $F_{u_h^*,h}(v_h) = f_h(v_h - u_h^*) - f_h(u_h^*) - f_h'(u_h^*)v_h$ *the problems (3.18),(3.19) are well-defined. The estimate (5.4) shows* $\|F_{u^*}(v(t))\|_E \le c(\rho)\|v(t)\|^2_{E^{1/2}}$ *and, moreover, one has* $F'_{u^*}(0) = 0$ *and condition* $(F3)$ *in the form* $\|F_{u_h^*,h}(v_h)\|_{E_n} \le \tilde{c}(\rho)\|v_h\|^2_{E_h^{1/2}}$, *since* $\|p_h\|$ *is uniformly bounded.*

Example 5.2. *Resolvent estimates such as* (B_1) *were proved for finite element and finite difference methods for example in* [5, 6]. *Without going into details we show how to get compact convergence of resolvents for the finite difference method. Consider, for example, in the space* $E = L^2(0,1)$ *the operator* A *defined by*

$$Av(x) = \frac{d^2v(x)}{dx^2} \text{ with } D(A) = \{v(\cdot) \in H_0^1(0,1) \cap H^2(0,1) : v(0) = v(1) = 0\}.$$

We choose step-sizes $h = \frac{1}{n}$ *and approximate* A *by the operators*

$$A_n u_n = \bar{\partial}_h \, \partial_h u_n = \{\frac{1}{h^2}(u_{n,(k+1)h} - 2u_{n,kh} + u_{n,(k-1)h})\}_{k=1}^{n-1}, \tag{5.5}$$

where $u_{n,\cdot} \in E_n = L_h^2(0,1) = D(A_n) = \{\{u_{n,kh}\}_{k=1}^{n-1} \in \mathbb{R}^{n-1}\}$. *Note that we set* $u_{n,0} = u_{n,nh} = 0$ *in (5.5). The discretization maps* p_n *are given by (cf. [32])*

$$(p_n u)_{kh} = \frac{1}{h} \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x)dx, \quad u \in E. \tag{5.6}$$

With $\langle u_{n,\cdot}, v_{n,\cdot}\rangle_{E_n} = h \sum_{k=1}^{n-1} u_{n,kh} v_{n,kh}$ *the summation by parts formula reads (using* $u_{n,0} = u_{n,nh} = 0$ *again)*

$$-\langle \bar{\partial}_h \, \partial_h u_n, u_n\rangle_{E_n} = \langle \partial_h u_n, \partial_h u_n\rangle_{E_n} = \|\partial_h u_n\|_{E_n}^2 = h \sum_{k=1}^{n} (\frac{u_{n,kh} - u_{n,(k-1)h}}{h})^2.$$

This implies $\|\partial_h u_n\|_{E_n}^2 \le \|A_n u_n\|_{E_n} \|u_n\|_{E_n}$. *Hence for any bounded sequence* $\{u_n\}$ *such that* $\{A_n u_n\}$ *is also bounded in* E_n *the last inequality shows that* $\|u_n\|_{E_n^{1/2}}$ *is bounded. This implies that* $\{u_n\}$ *is* $\mathcal{P}$*-compact. Now we can use Theorem* 3.15 *to get compact convergence of resolvents.*

6 Appendix

We use the following quantitative Lipschitz inverse mapping theorem, cf. [16], [32].

Lemma 6.1. *Assume* Y *and* Z *are Banach spaces,* $F \in C^1(Y,Z)$ *and* $F'(y_0)$ *is a homeomorphism for some* $y_0 \in Y$. *Let* $k, \sigma, \delta > 0$ *be three constants, such that the following estimates hold:*

$$\|F'(y) - F'(y_0)\| \le k < \sigma \le \frac{1}{\|F'(y_0)^{-1}\|} \quad \text{for any} \quad y \in \mathcal{U}_Y(y_0;\delta),$$
$$\|F(y_0)\| \le (\sigma - k)\delta.$$

Then F *has a unique zero* $\bar{y} \in \mathcal{U}_Y(y_0;\delta)$ *and the following inequalities are satisfied*

$$\|F'(y)^{-1}\| \le \frac{1}{\sigma - k} \quad \text{for all} \quad y \in \mathcal{U}_Y(y_0;\delta),$$

$$\|y_1 - y_2\| \le \frac{1}{\sigma - k} \|F(y_1) - F(y_2)\| \quad \text{for any} \quad y_1, y_2 \in \mathcal{U}_Y(y_0;\delta). \tag{6.7}$$

References

[1] *F. Alouges, A. Debussche.* On the qualitative behavior of the orbits of a parabolic partial differential equation and its discretizatin in the neighborhood of a hyperbolic fixed point. Numer. Funct. Anal. Optim. 12 (1991), 253-269.

[2] *A. Ambrosetti, G. Prodi.* A primer of nonlinear analysis. Corrected reprint of the 1993 original. Cambridge Studies in Advanced Mathematics, 34. Cambridge University Press, Cambridge, 1995.

[3] *J. M. Arrieta, A. N. Carvalho, A. Rodriguez-Bernal.* Parabolic problems with nonlinear boundary conditions and critical nonlinearities. J. Differential Equations 156 (1999), no. 2, 376–406.

[4] *J. M. Arrieta, A. N. Carvalho, G. Lozada-Cruz.* Dynamics in dumbbell domains I. Continuity of the set of equilibria. J. Diff. Equ. 231, (2006), 551-597.

[5] *A. Ashyralyev and P. E. Sobolevskii.* Well-Posedness of Parabolic Difference Equations, Operator Theory Advances and Applications, Birkhäuser Verlag, Basel, Boston, Berlin, vol.69(1994).

[6] *N. Yu. Bakaev, V. Thomee, L. B. Wahlbin.* Maximum-norm estimates for resolvents of elliptic finite element operators. Math. Comp. 2003. **72**, N 244. 1597–1610.

[7] *W.-J. Beyn.* On the numerical approximation of phase portraits near stationary points. SIAM J. Numer. Anal. 24 (1987), no. 5, 1095–1113.

[8] *A. N. Carvalho, S. Piskarev.* A general approximation scheme for attractors of abstract parabolic problems. Numer. Funct. Anal. Optim. 27 (2006), no. 7-8, 785 - 829.

[9] *G. Da Prato, P. Grisvard.* Sommes d'opérateus liéaires et équations différentielles opérationnelles. J. Math. Pures Appl. (9)**54**(1975), no. 3, 305-387.

[10] *Da Prato, G. and Grisvard, P.* Équations d'évolution abstraites non linus é aires de type parabolique. C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 9, A709-A711.

[11] *H. Fujita and A. Mizutani.* On the finite element method for parabolic equations. I. Approximation of holomorphic semigroups. *J. Math. Soc. Japan*, **28**, No. 4, 749–771 (1976).

[12] *B. M. Garay.* On structural stability of ordinary differential equations with respect to discretization methods. Numer. Math. 72 (1996), no. 4, 449–479.

[13] *R.D. Grigorieff.* Diskrete Approximation von Eigenwertproblemen. II. Konvergenzordnung. Numer. Math. 1975. **24**, N 5. 415–433.

[14] *D. Guidetti, B. Karasozen and S. Piskarev.* Approximation of abstract differential equations. J. Math. Sci. (N. Y.) 122 (2004), no. 2, 3013–3054.

[15] *D. Henry.* Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin, 1981.

[16] *M.C. Irwin.* Smooth dynamical systems. Advanced Series in Nonlinear Dynamics, 17. World Scientific Publishing Co., Inc., River Edge, NJ, (2001).

[17] *T. Kato.* Perturbation Theory for Linear Operators, Classics in Mathematics. Springer-Verlag, Berlin (1995).

[18] *S. G. Krein.* Linear differential equations in Banach space. American Mathematical Society, Providence, R.I., 1971. Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 29.

[19] *S. Larsson.* Numerical analysis of semilinear parabolic problems. (English) Ainsworth, Mark (ed.) et al., The graduate student's guide to numerical analysis '98. Lecture notes from the 8th EPSRC summer school in numerical analysis. Leicester, GB, July 5-17, 1998. Berlin: Springer. Springer Ser. Comput. Math. 26, 83-117 (1999).

[20] *S. Larsson, J. M. Sanz-Serna.* The behavior of finite element solutions of semilinear parabolic problems near stationary points. *SIAM J. Numer. Anal.*, 1994, **31**, No. 4, 1000–1018.

[21] *S. Larsson, J.-M. Sanz-Serna.* A shadowing result with applications to finite element approximation of reaction-diffusion equations. Math. Comp. 68 (1999), no. 225, 55–72.

[22] *A. Ostermann, C. Palencia.* Shadowing for nonautonomous parabolic problems with applications to long-time error bounds. SIAM J. Numer. Anal. 37 (2000), 1399-1419.

[23] *K. Palmer.* Shadowing in dynamical systems. Theory and applications. Mathematics and its Applications, 501. Kluwer Academic Publishers, Dordrecht, 2000.

[24] *S. Yu. Pilyugin.* Shadowing in dynamical systems. Lecture Notes in Mathematics, 1706. Springer-Verlag, Berlin, 1999. xviii+271 pp.

[25] *S. Piskarev.* On approximation of holomorphic semigroups. Tartu Riikl. Ul. Toimetised, 1979, N 492, 3–23.

[26] *S. Piskarev.* Differential equations in Banach space and it's approximation. Moscow, Moscow State University Publish House (in Russian), 2005.

[27] *P.E. Sobolevskii.* Some properties of the solutions of differential equations in fractional spaces. in: Trudy Nauchn. -Issled. Inst. Mat. Voronezh. Gos. Univ. No. 74(1975), 68-76. (Russian).

[28] *F. Stummel.* Diskrete Konvergenz linearer Operatoren. III. In Linear operators and approximation (Proc. Conf., Oberwolfach, 1971), pages 196–216. Internat. Ser. Numer. Math., Vol. 20. Birkhäuser, Basel, 1972.

[29] *F. Stummel.* Perturbation of domains in elliptic boundary value problems. pp. 110-136 in Springer Lecture Notes in Math., 503, 1976.

[30] *V. Thomée.* Finite element methods for parabolic problems – some steps in the evolution, In: *Finite element methods (Jyväskylä, 1993)*, Vol. 164 of *Lecture Notes in Pure and Appl. Math.*, Dekker, New York (1994), pp. 433–442.

[31] *H. Trotter.* Approximation of semigroups of operatiors. Pacific J. Math. 1958. **8**. 887–919.

[32] *G. Vainikko.* Funktionalanalysis der Diskretisierungsmethoden. Leipzig. B. G. Teubner Verlag, 1976. Mit Englischen und Russischen Zusammenfassungen, Teubner-Texte zur Mathematik.

[33] *G. Vainikko.* Approximative methods for nonlinear equations (two approaches to the convergence problem) Nonlinear Anal. 1978. **2**. 647–687.

[34] *G. Vainikko, S. Piskarev.* Regularly compatible operators, *Izv. Vuzov, Mat.*, **10**, 25–36 (1977).

[35] *V. V. Vasil'ev, S.I. Piskarev.* Differential equations in Banach spaces. II. Theory of cosine operator functions. J. Math. Sci. (N. Y.) 122 (2004), no. 2, 3055–3174.

[36] *T. Ushijima.* Approximation theory for semigroups of linear operators and its appliations to approximations of wave equations. Japan J. Math. 1975. **1**, N 1. 185–224.