A regularity theorem for differential inclusions with Lipschitz continuous right hand sides

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March 22, 2007

Abstract

By definition, solutions of differential inclusions are absolutely continuous functions with L^1 derivatives. We prove that at least for Lipschitz continuous right hand sides the continuously differentiable solutions are dense in the set of all solutions with respect to the supremum norm, and an application is presented.

Key words. Differential inclusions

AMS(MOS) subject classifications. 34A60

1 Introduction

Differential inclusions are used in a variety of fields ranging from dynamical systems with discontinuous right hand sides [7, 8] to control theory [2, 9]. One reason why they are a powerful concept is that their solutions are only required to be absolutely continuous and not necessarily differentiable, so that their behavioural repertoire is superior to that of the solutions of ordinary differential equations. The price one has to pay for this flexibility is that the derivatives of the solutions are merely spooky L^1 functions instead of nice continuous curves. This work is supposed to ease this problem for differential inclusions with Lipschitz continuous right hand side, at least up to an arbitrarily small epsilon.

As an application we prove that the T-flow of a differential inclusion with a suitable dissipative right hand side is a contraction.

2 The result

Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$. For a set $A \subset \mathbb{R}^d$, we set

$$||A|| := \sup_{a \in A} |a|.$$

Let $\mathcal{CC}(\mathbb{R}^d)$ denote the set of all compact convex subsets of $\mathbb{R}^d$ and let dist and dist_H be the nonsymmetric and the symmetric Hausdorff distance, resprectively. The convex hull of a set A will be denoted by $co(A)$.

The following lemma formalizes a simple geometric principle:

Lemma 1. *Let $F_i \subset \mathbb{R}^d$ $\forall i \in I$ and $G \subset \mathbb{R}^d$ be closed and convex. Then*

$$\text{dist}(co(\cup_{i\in I} F_i), G) \leq \sup_{i\in I} \text{dist}(F_i, G), \tag{1}$$

$$\text{dist}(G, co(\cup_{i\in I} F_i)) \leq \inf_{i\in I} \text{dist}(G, F_i) \tag{2}$$

and

$$\text{dist}_H(co(\cup_{i\in I} F_i), G) \leq \sup_{i\in I} \text{dist}_H(F_i, G). \tag{3}$$

Proof. Let $f \in co(\cup_{i\in I} F_i)$. There exist $\lambda_0, \ldots, \lambda_d \in [0,1]$ and $f_0, \ldots, f_d \in \cup_{i\in I} F_i$, $f_j \in F_{i_j}$ such that

$$f = \sum_{j=0}^{d} \lambda_j f_j, \qquad \sum_{j=0}^{d} \lambda_j = 1.$$

Let $g_j \in G$ be such that

$$|f_j - g_j| = \text{dist}(f_j, G) \leq \text{dist}(F_{i_j}, G).$$

Then $g := \sum_{j=0}^{d} \lambda_j g_j \in G$, and

$$\begin{aligned} |f - g| &\leq \sum_{j=0}^{d} \lambda_j |f_j - g_j| \\ &\leq \sum_{j=0}^{d} \lambda_j \, \text{dist}(F_{i_j}, G) \\ &\leq \sup_{i\in I} \text{dist}(F_i, G), \end{aligned}$$

which shows (1). Inequality (2) follows from

$$\begin{aligned}\operatorname{dist}(G, co(\cup_{i\in I} F_i)) &\le \operatorname{dist}(G, \cup_{i\in I} F_i)\\ &\le \inf_{i\in I} \operatorname{dist}(G, \cup_{i\in I} F_i) \qquad (4)\\ &\le \sup_{i\in I} \operatorname{dist}(G, \cup_{i\in I} F_i),\end{aligned}$$

and (1) and (4) give (3). □

Now we formulate our main result:

Theorem 2. *Let $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be Lipschitz continuous. Then the continuously differentiable solutions of the initial value problem*

$$\dot{x}(t) \in F(x(t)) \ a.e. \ in \ [0,T], \quad x(0) = x_0 \qquad (5)$$

are dense in the set of all solutions with respect to the maximum norm.

Proof. For every solution x of (5) there exists an à-priori bound: Let $z(s) \in F(0)$ such that $|\dot{x}(s) - z(s)| = \operatorname{dist}(\dot{x}(s), F(0))$. Then

$$\begin{aligned}|x(t) - x(0)| \le \int_0^t |\dot{x}(s)| ds&\\ &\le \int_0^t |\dot{x}(s) - z(s)| + |z(s)| ds\\ &\le \int_0^t \operatorname{dist}_H(F(x(s)), F(x(0))) + ||F(x(0))|| ds\\ &\le t||F(x(0))|| + \int_0^t L|x(s) - x(0)| ds,\end{aligned}$$

and by the Gronwall lemma,

$$\begin{aligned}|x(t) - x(0)| &\le t||F(x(0))|| + \int_0^t s||F(x(0))|| L e^{L(t-s)} ds\\ &= t||F(x(0))|| + \frac{1}{L}||F(x(0))||(e^{Lt} - Lt - 1)\\ &= \frac{1}{L}||F(x(0))||(e^{Lt} - 1).\end{aligned}$$

In particular,

$$\begin{aligned}
|x(t+\eta)-x(t)| &\leq \frac{1}{L}||F(x(t))||(e^{L\eta}-1)\\
&\leq \frac{1}{L}\big[||F(x(0))||+\mathrm{dist}_H(F(x(t)),F(x(0)))\big](e^{L\eta}-1)\\
&\leq \frac{1}{L}\big[||F(x(0))||+L|x(t)-x(0)|\big](e^{L\eta}-1)\\
&\leq \frac{1}{L}\big[||F(x(0))||+||F(x(0))||(e^{Lt}-1)\big](e^{L\eta}-1)\\
&\leq \underbrace{\frac{1}{L}||F(x(0))||e^{Lt}}_{=:C_1}(e^{L\eta}-1). \qquad (6)
\end{aligned}$$

Now we construct a regular approximation x_δ of x. Without loss of generality we can assume that

$$\dot{x}(t)\in F(x(t))\ \forall t\in[0,T]$$

as a function. We formally continue it as a function $\dot{x}\in L^1(\mathbb{R},\mathbb{R}^d)$ by setting

$$\dot{x}(t):=\begin{cases} 0, & t>T+1\\ \dot{x}(T), & T<t\leq T+1\\ \dot{x}(t), & 0<t\leq T\\ \dot{x}(0), & -1<t\leq 0\\ 0, & t<-1\end{cases}.$$

For given $\delta>0$, there exists a function $\varphi_\delta\in C_0^\infty(\mathbb{R},\mathbb{R})$ satisfying $\mathrm{supp}(\varphi_\delta)\subset[-\delta,\delta]$ and $\int_{\mathbb{R}}\varphi_\delta(\tau)d\tau=1$, such that

$$y_\delta(s):=\int_{\mathbb{R}}\varphi_\delta(\tau)\dot{x}(s-\tau)d\tau$$

is a function $y_\delta\in C^\infty(\mathbb{R},\mathbb{R}^d)$ with

$$||y_\delta-\dot{x}||_{L^1}\leq\delta.$$

Hence $x_\delta\in C^\infty([0,T],\mathbb{R}^d)$ given by

$$x_\delta(t):=x(0)+\int_0^t y_\delta(s)ds$$

satisfies

$$||x_\delta-x||_\infty\leq\delta.$$

Consider the time dependent mapping

$$\tilde{F} : \mathbb{R} \times \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d), \tilde{F}(t,x) := F(x) - y_\delta(t).$$

Since y_δ is continuous, $\tilde{F}$ is continuous w.r.t the Hausdorff metric. By theorem 1.7.1 in [1], the minimal selection $(t,x) \mapsto m(t,x)$ of $\tilde{F}$ is also continuous. Obviously

$$|m(t,x)| = \mathrm{dist}(y_\delta(t), F(x)),$$

and

$$y_\delta(t) + m(t,x) \in F(x) \ \forall t \in [0,T], \ \forall x \in \mathbb{R}^d.$$

By the Cauchy-Peano theorem, the initial value problem

$$\dot{a}_\delta(t) = y_\delta(t) + m(t, a_\delta(t)), \quad a_\delta(0) = x(0) \tag{7}$$

admits a solution a_δ on a maximal subinterval $J \subset [0,T]$ with $0 \in J$. For $t \in J$ one obtains

$$\begin{aligned} |x_\delta(t) - a_\delta(t)| &\le \int_0^t |y_\delta(s) - (y_\delta(s) + m(s, a_\delta(s)))| ds \\ &= \int_0^t |m(s, a_\delta(s))| ds \\ &= \int_0^t \mathrm{dist}(y_\delta(s), F(a_\delta(s))) ds. \end{aligned}$$

According to theorem 1.6.13 in [9],

$$y_\delta(s) \in \overline{co}\{\dot{x}(\tau) : \tau \in s - \mathrm{supp}(\varphi_\delta)\}. \tag{8}$$

Hence

$$\begin{aligned} |x_\delta(t) - a_\delta(t)| &\le \int_0^t \mathrm{dist}(\overline{co}\{\cup_{s-\mathrm{supp}(\varphi_\delta)} F(x(\tau))\}, F(a_\delta(s))) ds \\ &\overset{\text{Lemma 1}}{\le} \int_0^t \sup_{\tau \in [s-\delta, s+\delta]} \mathrm{dist}(F(x(\tau)), F(a_\delta(s))) ds \\ &\le \int_0^t \sup_{\tau \in [s-\delta, s+\delta]} L|x(\tau) - a_\delta(s)| ds \\ &\overset{(6)}{\le} \int_0^t L(|x(s) - a_\delta(s)| + C_1(e^{L\delta} - 1)) ds \\ &\le \int_0^t L|x_\delta(s) - a_\delta(s)| + \underbrace{L(\delta + C_1(e^{L\delta} - 1))}_{=:C_2(\delta)} ds. \end{aligned}$$

The Gronwall lemma yields

$$\begin{aligned}
|x_\delta(t) - a_\delta(t)| &\leq C_2(\delta)t + \int_0^t C_2(\delta) s L e^{L(t-s)} ds \\
&= C_2(\delta)t + \frac{1}{L} C_2(\delta)(e^{Lt} - Lt - 1) \\
&= \frac{1}{L} C_2(\delta) t (e^{Lt} - 1),
\end{aligned}$$

and thus

$$\begin{aligned}
|x(t) - a_\delta(t)| &\leq \delta + \frac{1}{L} C_2(\delta) t (e^{Lt} - 1) \\
&\leq \delta + \frac{1}{L} C_2(\delta) T (e^{LT} - 1). \qquad (9)
\end{aligned}$$

In particular, a_δ is bounded on J. Hence $J = [0, T]$, and

$$\begin{aligned}
||x - a_\delta||_\infty &\leq \delta + \frac{1}{L} C_2(\delta) T (e^{LT} - 1) \qquad (10) \\
&\longrightarrow 0 \text{ as } \delta \to 0.
\end{aligned}$$

□

Note that the techniques of our proof take full advantage of the Lipschitz continuity of the right hand side and hence cannot be modified easily to obtain a more general result.

3 An application

Now consider the T-flow of a differential inclusion

$$\dot{x}(t) \in F(x(t)) \text{ a.e. in } [0, T] \qquad (11)$$

with the following property:

Definition 3. *A set valued mapping $F : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ satisfies the relaxed one-sided Lipschitz condition (ROSL) with a constant $\mu \in \mathbb{R}$ if for any $x', x'' \in \mathbb{R}^d$ and $y' \in F(x')$ there exists a $y'' \in F(x'')$ such that*

$$\langle y'' - y', x'' - x' \rangle \leq \mu |x'' - x'|^2.$$

Set valued mappings which satisfy the ROSL condition or related dissipativity concepts have been thoroughly investigated in [4], [5], and [6]. The techniques displayed in those works suggest that the following theorem should also be correct under weaker assumptions. The aim of the following proof is merely to illustrate the usefulness of theorem 2.

Theorem 4. *Let $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be Lipschitz continuous with Lipschitz constant $L > 0$ and ROSL with constant $\mu < 0$. Then the T-flow of the differential inclusion* (11) *is a contraction w.r.t. the Hausdorff metric.*

Proof. Let x be any solution of (11). Without loss of generality we can assume that $x(0) = 0$. Let $v \in \mathbb{R}^d$ be given. By Theorem 2, for every $\epsilon > 0$ there exists a solution $x_\epsilon \in C^1([0,T], \mathbb{R}^d)$ of (11) such that $x_\epsilon(0) = 0$ and $||x_\epsilon - x||_\infty < \epsilon$.

The mapping

$$G(t,x) := \{y \in \mathbb{R}^d : \langle y - \dot{x}_\epsilon(t), x - x_\epsilon(t)\rangle \leq -\mu|x - x_\epsilon(t)|^2\} \tag{12}$$

has a closed graph, because x_ϵ, $\dot{x}_\epsilon$, and the inner product are continuous. As F is ROSL, the right hand side of the time dependent differential inclusion

$$\dot{y}(t) \in F(y(t)) \cap G(t, y(t)) \tag{13}$$

is nonempty, and it is obviously convex and compact. By Theorem 1.1.1 in [1], it is upper semicontinuous in (t,x). Hence there exists a solution $x_{v,\epsilon}$ of (13) with $x_{v,\epsilon}(0) = v$ according to Theorem 5.1 in [3].

Now

$$\begin{aligned} \frac{d}{dt}|x_\epsilon(t) - x_{v,\epsilon}(t)|^2 &= 2\langle \dot{x}_\epsilon(t) - \dot{x}_{v,\epsilon}(t), x_\epsilon(t) - x_{v,\epsilon}(t)\rangle \\ &\leq -2\mu|x_\epsilon(t) - x_{v,\epsilon}(t)|^2 \end{aligned}$$

implies that

$$|x_\epsilon(T) - x_{v,\epsilon}(T)| \leq |x_\epsilon(0) - x_{v,\epsilon}(0)|e^{-\mu T} = e^{-\mu T}|v|, \tag{14}$$

and

$$|x(T) - x_{v,\epsilon}(T)| \leq \epsilon + e^{-\mu T}|v|. \tag{15}$$

Since this estimate holds for every $\epsilon > 0$, $e^{-\mu T}$ is a Lipschitz constant for the T-flow w.r.t. the Hausdorff distance. □

Note that x_ϵ being an exact solution and not just a smooth approximate trajectory is crucial for the right hand side in (13) to be nonempty.

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