On r-periodic orbits of k-periodic maps

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Abstract

In this paper, we analyze r-periodic orbits of k-periodic difference equations, i.e.

$$x_{n+1} = F_n(x_n), \quad F_n = F_{n \bmod k}, \quad x_n = x_{n \bmod r}, \quad n \in \mathbb{Z}$$

and their stability. This notion was introduced in [7]. We discuss that, depending on the values of r and k, such orbits generically only occur in finite dimensional systems that depend on sufficiently many parameters, i.e. they have a large codimension in the sense of bifurcation theory. As an example, we consider the periodically forced Beverton-Holt model, for which explicit formulas for the globally attracting periodic orbit, having the minimal period $k = r$, can be derived. When r factors k the Beverton-Holt model with two time-variant parameters is an example that can be studied explicitly and that exhibits globally attracting r-periodic orbits. For arbitrarily chosen periods r and k, we develop an algorithm for the numerical approximation of an r-periodic orbit and of the associated parameter set, for which this orbit exists. The algorithm is applied to the generalized Beverton-Holt and the two-dimensional stiletto model.

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1 Introduction

Non-autonomous difference equations on a metric space Y

$$x_{n+1} = F_n(x_n), \quad n \in \mathbb{Z}, \quad x_n \in Y \tag{1}$$

occur in several applications, for example, from mathematical biology, when the law, transforming one state into the next, depends on time. Various approaches for the analysis of non-autonomous systems have been developed. One well known technique is the skew-product flow mechanism, cf. [19], by which non-autonomous systems are transformed back to the autonomous world.

The analysis simplifies even further, when environmental influences are periodic, and the corresponding models are periodically forced with some period k, i.e. $F_n = F_{n \bmod k}$. In this case it often suffices to consider the autonomous system generated by $\Psi(k, 0)$, where $\Psi(n, m)$ is the solution operator of (1), transferring state m to state n (see equation (5)).

Important structures in dynamical systems are fixed points and periodic orbits. In [7] the more general concept of r-periodic orbits for k-periodic maps is introduced, see also [1, 8, 9]. These orbits satisfy the condition

$$x_{n+1} = F_n(x_n), \quad F_n = F_{n \bmod k}, \quad x_n = x_{n \bmod r}, \quad n \in \mathbb{Z}. \tag{2}$$

Note that due to periodicity, it suffices to consider (2) for $n = 0, \ldots, \operatorname{lcm}(k, r) - 1$. We introduce two notions of stability for r-periodic orbits in Section 2 and prove their equivalence.

In several applications, F_n is generated by a parameter-dependent map $f(\cdot, \cdot)$, i.e. $F_n = f(\cdot, \Lambda_n)$ where $(\Lambda_n)_{n\in\mathbb{Z}}$ is a k-periodic sequence of parameters.

As an example, we take the famous Beverton-Holt map from population biology, cf. [2]

$$F_n(x) := \frac{\mu K_n x}{K_n + (\mu - 1)x}. \tag{3}$$

This function transforms into a model-map, introduced in [13, 15] that is useful in the analysis of non-autonomous bifurcations, since the orbits and the solution operator of this map can be given explicitly. Applying these results to the Beverton-Holt model, one gets explicit formulas for the solution operator. We choose a sequence $(K_n)_{n\in\mathbb{Z}}$ with minimal period k and gain an explicit representation of a k-periodic orbit that is stable and globally attracting in $\mathbb{R}^+$, hence unique. In this way, we obtain an alternative proof of the first Cushing-Henson conjecture, see [3, 4, 8, 9, 16, 21]. As a consequence, r-periodic orbits of k-periodic Beverton-Holt maps cannot occur in case $r < k$.

We show, however, that r-periodic orbits with r a non-trivial factor of k, do occur in the Beverton-Holt model if the second parameter μ_n is also time variant. In this general setup it is again possible to derive explicit formulas for the solution operator and for the stable and globally attracting periodic orbit.

Our main goal is the numerical computation of r-periodic orbits of k-periodic maps.

In Section 4, we consider parameterized systems (as in (3)) and introduce the codimension of an r-periodic orbit. This is the number of parameters, necessary to find such an orbit. Then we define an operator the zeroes of which yield an r-periodic orbit and the associated parameter set. For the numerical computation, we then use Newton's method. We discuss several cases where one can expect the period of the computed orbit to be minimal and the computed solution to be regular (which guarantees local convergence of Newton's method). For instance, both properties hold in a generic sense if r is a factor of k or k is a factor of r. In all other cases, orbits of period $\gcd(k, r)$ may occur as singular solutions.

We apply the algorithm to compute 4-periodic orbits for the generalized 8-periodic Beverton-Holt map and for the two-dimensional stiletto map. The first case requires 4 and the second 8 free parameters. Finally, we consider a parameterized 3-periodic system with a 2-periodic orbit.

2 r-periodic orbits and their stability

Consider the non-autonomous difference equation

$$x_{n+1} = F_n(x_n), \quad n \in \mathbb{Z}, \tag{4}$$

and denote by Ψ its solution operator, defined as

$$\Psi(n,m)(x) := \begin{cases} F_{n-1} \circ \ldots \circ F_m(x), & \text{for } n > m, \\ x, & \text{for } n = m, \\ F_n^{-1} \circ \ldots \circ F_{m-1}^{-1}(x), & \text{for } n < m. \end{cases} \tag{5}$$

We assume that

A1 $F_n : Y \to Y$, $n \in \mathbb{Z}$ is a k-periodic family of homeomorphism on a metric space (Y, d), i.e. $F_n = F_{n \bmod k}$ for all $n \in \mathbb{Z}$.

Definition 1 *An r-tuple $X = (x_0, \ldots, x_{r-1})$, $x_i \in \mathbb{R}^\ell$ is called **r-periodic orbit of a k-periodic family** $(F_n)_{n\in\mathbb{Z}}$, if*

$$\Psi(n,m)(x_{m \bmod r}) = x_{n \bmod r} \quad \text{for all } n, m \in \mathbb{Z}. \tag{6}$$

Note that condition (6) may be equivalently written as

$$F_{(i+nr) \bmod k}(x_i) = x_{(i+1) \bmod r}, \quad \text{for } i = 0, \ldots, r-1 \text{ and all } n \in \mathbb{Z}. \tag{7}$$

r-periodic orbits of k-periodic maps are called geometric r-cycles in [7].

In the following, we assume that an r-periodic orbit exists.

A2 $X = (x_0, \dots, x_{r-1})$ is an r-periodic orbit of (4).

For r-periodic orbits, we introduce notions of stability, applying the classical definition for sets, cf. [22], to $\{x_0, \dots, x_{r-1}\}$. We use the Hausdorff semi-distance for sets $A, B \subset Y$ defined as $\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$.

Definition 2 *Assume **A1**. An r-periodic orbit $X = (x_0, \dots, x_{r-1})$ is*

- ***stable*** *if for any neighborhood U of X there exist neighborhoods V_i of x_i, $i = 0, \dots, r-1$, such that for all $i = 0, \dots, r-1$ and any $x \in V_i$.*

$$\Psi(n, i)(x) \in U \quad \textit{holds for all } n \geq i,$$

- ***attracting*** *if neighborhoods V_i of x_i exist, such that for all $i = 0, \dots, r-1$ and any $x \in V_i$*

$$\lim_{n \to \infty} \mathrm{dist}(\Psi(n, i)(x), \{x_0, \dots, x_{r-1}\}) = 0, \tag{8}$$

- ***globally attracting*** *if (8) holds true for all $x \in \mathbb{R}^\ell$.*

- ***asymptotically stable*** *if X is stable and attracting.*

In case $r = k = 1$, this is the classical definition of an (asymptotically) stable fixed point of an autonomous map. In the general case, x_0 is a fixed point of $G := \Psi(s, 0)$, where $s = \mathrm{lcm}(r, k)$, and one can define alternatively that the r-periodic orbit X is (asymptotically) stable, if x_0 is an (asymptotically) stable fixed point of G, cf. [1, 6]. We find it instructive, to prove that these notions of stability are equivalent.

Lemma 3 *Assume **A1** and **A2**. Then $X = (x_0, \dots, x_{r-1})$ is (asymptotically) stable if and only if x_0 is an (asymptotically) stable fixed point of $\Psi(s, 0)$, where $s = \mathrm{lcm}(r, k)$.*

Proof: We only prove one direction by showing that if x_0 is an (asymptotically) stable fixed point of G, then X is an (asymptotically) stable r-periodic orbit. One can show the converse, using similar arguments. The main part of the proof lies in the construction of suitable neighborhoods, cf. [18].

Let x_0 be a stable fixed point of G and let U be a neighborhood of X. Due to our assumptions **A1**, **A2**, the functions $\Psi(j, 0)$ are continuous for all $j = 0, \dots, s$ and $\Psi(j, 0)(x_0) = x_{j \bmod r}$. Therefore a neighborhood U_0 of x_0 exists, such that

$$\Psi(j, 0)(U_0) \subset U, \quad \text{for all } j = 0, \dots, s-1. \tag{9}$$

Since x_0 is a stable fixed point of G, there is a neighborhood V_0 of x_0, such that

$$G^n(V_0) \subset U_0 \quad \text{for all } n \geq 0. \tag{10}$$

We define recursively the neighborhoods

$$V_{s-1} := F_{s-1}^{-1}(V_0) \cap U, \quad \text{and} \quad V_i := F_i^{-1}(V_{i+1}) \cap U \text{ for } i = s-2, \dots, 1. \tag{11}$$

Obviously, V_i is a neighborhood of $x_{i \bmod r}$ for $i = 0, \dots, s-1$, and it holds for $i = 1, \dots, s-1$ that

$$x \in V_i \quad \Rightarrow \quad \Psi(s,i)(x) \in V_0 \subset U. \tag{12}$$

Stability of the orbit follows if we show

$$\Psi(m,j)(V_j) \subset U \quad \text{for all } j = 0, \dots, s-1 \text{ and } m \geq j. \tag{13}$$

In case $m < s$, the assertion follows directly from (9) and (12) and $V_0 \subset U_0$. For $j \in \{0, \dots, s-1\}$ and $m \geq s$ we choose n such that $0 \leq m - ns < s$ holds and get for $x \in V_j$ using **A1**

$$\begin{aligned} \Psi(m,j)(x) &= \Psi(m,ns) \circ \Psi(ns,s) \circ \Psi(s,j)(x) \\ &= \Psi(m \bmod s, 0) \circ G^n \circ \Psi(s,j)(x). \end{aligned}$$

Applying (12), (10) and finally (9) we obtain (13).

Now we prove attraction of X. Let $\bar{x}_0$ be an attracting fixed point of G. Choose a sufficiently small neighborhood V_0 such that $\lim_{n\to\infty} G^n(x) = \bar{x}_0$ holds for all $x \in V_0$ and construct neighborhood around $\bar{x}_n$ as in (11). For $i = 0, \dots, s-1$ and $x \in V_i$ it follows for all $n \in \mathbb{N}$

$$\begin{aligned} \Psi(n,i)(x) &= \Psi\left(n, \left\lfloor \frac{n}{s} \right\rfloor s\right) \circ \Psi\left(\left\lfloor \frac{n}{s} \right\rfloor s, s\right) \circ \Psi(s,i)(x) \\ &= \Psi(n \bmod s, 0) \circ G^{\left\lfloor \frac{n}{s} \right\rfloor} \circ \Psi(s,i)(x). \end{aligned}$$

Using (12) we get

$$\lim_{n\to\infty} d\big(\Psi(n,i)(x), \bar{x}_{n \bmod r}\big) = 0.$$

■

3 The periodically forced Beverton-Holt model

We consider r-periodic orbits in models originating from population biology. A classical one is the Beverton-Holt model, which describes the density of a population in a fluctuating environment, cf. [2].

For a sequence $K_{\mathbb{N}} = (K_n)_{n\in\mathbb{N}}$ of positive numbers and $\mu > 1$ the Beverton-Holt model is defined as

$$x_{n+1} = g_n(x_n, \mu), \; n \in \mathbb{N}, \quad \text{where } g_n(x,\mu) := \frac{\mu K_n x}{K_n + (\mu - 1)x}. \tag{14}$$

The parameter K_n models the carrying capacity of the environment at time n and μ is the inherent growth rate of the population.

We choose a k-periodic sequence K_n such that we get a periodically forced model of the form (4) that satisfies assumption **A1**. Since population densities are positive, we restrict $g_n(\cdot, \mu) : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Note that the origin is an unstable fixed point of $g_n(\cdot, \mu)$.

This model has been studied in various papers and in 2002 the Cushing-Henson conjecture was published in [4], saying that if the sequence $K_{\mathbb{N}}$ is k-periodic, then a k-periodic orbit $(\bar{x}_n)_{n\in\mathbb{N}}$ that is asymptotically stable on $\mathbb{R}^+$ exists. Furthermore, is holds that

$$\frac{1}{k}\sum_{i=0}^{k-1} \bar{x}_i < \frac{1}{k}\sum_{i=0}^{k-1} K_i.$$

Proofs or parts of proofs for these results can be found, for example, in [16, 7, 9, 8, 10, 21].

Here, we give a short proof for the existence of a stable and globally attracting periodic orbit with minimal period k. In [15], one of the authors introduced a model map for the explicit study of non-autonomous bifurcations. This model map has the remarkable property that its solution operator can be given explicitly. In a special case, this map transforms into the Beverton-Holt map (14) and consequently we get explicit formulas for the orbits of the periodically forced Beverton-Holt map, too.

We note that a proof of the first conjecture can already be read off from the article [3], see also [16]. The proof in [3] is based on an explicit representation of the fixed point of $\Psi(k, 0)$ in the more general setup in which $\mu = \mu_n$ is also time-variant. We analyze this general form in Section 4.1.1.

In our proof, we provide an explicit representation of the period orbit, which to our knowledge cannot be found in the literature. Furthermore, we put special emphasis on the minimality of the period of the orbit, cf. [9, Section 6].

Proposition 4 *Let $K_{\mathbb{N}}$ be a sequence of positive numbers with minimal period k, and let $\mu > 1$. For $n \geq m$ and $x > 0$ the solution operator Ψ of (14) has the explicit form*

$$\Psi(n, m)(x, \mu) = \frac{\mu^{n-m} x}{1 + x \sum_{i=m}^{n-1} \frac{\mu-1}{K_i} \mu^{i-m}}. \tag{15}$$

Let

$$\bar{x}_n := \frac{\mu^k - 1}{\sum_{i=n}^{n+k-1} \frac{\mu-1}{K_i} \mu^{i-n}}, \quad \textit{for } n \in \mathbb{N}. \tag{16}$$

Then $\bar{X} := (\bar{x}_0, \ldots, \bar{x}_{k-1})$ is a globally attracting and stable periodic orbit of (14) on $\mathbb{R}^+$ with minimal period k.

Proof: The explicit representation of the solution operator follows from [15] and can be directly verified by induction.

For all $x \in \mathbb{R}^+$ and any $n, p \in \mathbb{N}$ it follows from (15) and the k-periodicity of $K_\mathbb{N}$

$$\begin{aligned}
\Psi(n+k,n)^p(x,\mu) &= \Psi(n+pk,n)(x,\mu) = \frac{\mu^{pk}x}{1+(\mu-1)x\sum_{i=n}^{n+pk-1}\frac{\mu^{i-n}}{K_i}} \\
&= \frac{\mu^{pk}x}{1+(\mu-1)x\sum_{i=n}^{n+k-1}\frac{\mu^{i-n}}{K_i}\sum_{j=0}^{p-1}\mu^{jk}} \\
&= \frac{\mu^{pk}}{\frac{1}{x}+(\mu-1)\frac{\mu^{pk}-1}{\mu^k-1}\sum_{i=n}^{n+k-1}\frac{\mu^{i-n}}{K_i}} \\
&= \frac{\mu^k-1}{\frac{\mu^k-1}{\mu^{pk}x}+(\mu-1)\frac{(\mu^{pk}-1)}{\mu^{pk}}\sum_{i=n}^{n+k-1}\frac{\mu^{i-n}}{K_i}}.
\end{aligned}$$

In the limit $p \to \infty$, we get

$$\lim_{p\to\infty}\Psi(n+pk,n)(x,\mu) = \bar{x}_n, \tag{17}$$

where this limit does not depend on the point $x \in \mathbb{R}^+$. The sequence $\bar{x}_\mathbb{N}$ is a solution of (14), since due to the periodicity of $K_\mathbb{N}$ we obtain for any $n, m \in \mathbb{N}$ and $x \in \mathbb{R}^+$

$$\begin{aligned}
\Psi(m,n)(\bar{x}_n,\mu) &= \Psi(m,n)\lim_{p\to\infty}\Psi(n+pk,n)(x,\mu) \\
&= \lim_{p\to\infty}\Psi(m+pk,n+pk)\circ\Psi(n+pk,m)\circ\Psi(m,n)(x,\mu) \\
&= \lim_{p\to\infty}\Psi(m+pk,m)\big(\Psi(m,n)(x,\mu)\big) = \bar{x}_m.
\end{aligned}$$

Furthermore, it holds that

$$\bar{x}_{n+k} = \frac{\mu^k-1}{\sum_{i=n+k}^{n+2k-1}\frac{\mu-1}{K_i}\mu^{i-n-k}} = \frac{\mu^k-1}{\sum_{i=n}^{n+k-1}\frac{\mu-1}{K_{i+k}}\mu^{i-n}} = \bar{x}_n,$$

and as a consequence, $\bar{x}_\mathbb{N}$ is a k-periodic solution of (14) that is globally attracting on $\mathbb{R}^+$.

For any $n \in \mathbb{N}$ and $0 < c_1 \le \bar{x}_n \le c_2$ it holds that the convergence in (17) is uniform on the interval $[c_1, c_2]$, since $\Psi(i,j)(c_1,\mu) \le \Psi(i,j)(c_2,\mu)$ for all $i \ge j$, and as a consequence, $\bar{X}$ is stable.

In the remaining part of the proof, we show that the period k is minimal. In case $\mu\bar{x}_n - \bar{x}_{n+1} \neq 0$ we can rewrite the difference equation

$$\bar{x}_{n+1} = g_n(\bar{x}_n, \mu), \quad n \in \mathbb{N}$$

with respect to K_n and get

$$K_n = \frac{(\mu - 1)\bar{x}_n \bar{x}_{n+1}}{\mu\bar{x}_n - \bar{x}_{n+1}}, \quad n \in \mathbb{N}. \tag{18}$$

Note that

$$\begin{aligned}
\mu\bar{x}_n - \bar{x}_{n+1} &= \frac{(\mu^k - 1)\left(\sum_{i=n+1}^{n+k} \frac{\mu-1}{K_i}\mu^{i-n} - \sum_{i=n}^{n+k-1} \frac{\mu-1}{K_i}\mu^{i-n}\right)}{\left(\sum_{i=n}^{n+k-1} \frac{\mu-1}{K_i}\mu^{i-n}\right)\left(\sum_{i=n+1}^{n+k} \frac{\mu-1}{K_i}\mu^{i-n-1}\right)} \\
&= \frac{(\mu^k - 1)\frac{\mu-1}{K_n}(\mu^k - \mu^0)}{\left(\sum_{i=n}^{n+k-1} \frac{\mu-1}{K_i}\mu^{i-n}\right)\left(\sum_{i=n+1}^{n+k} \frac{\mu-1}{K_i}\mu^{i-n-1}\right)} \neq 0
\end{aligned}$$

for $\mu > 1$.

It follows from (18) that if $x_{\mathbb{N}}$ is r-periodic with $r < k$, then $K_{\mathbb{N}}$ is also r-periodic. This is a contradiction, since the period k of the sequence $K_{\mathbb{N}}$ is assumed to be minimal.

■

As a consequence, the precise answer to the first Cushing-Henson conjecture, introduced in [4], is: There exists a stable and globally attracting solution with minimal period k which has the explicit form (16).

4 Genericity and numerical approximation

As we have seen in Section 3, r-periodic orbits occur in the periodically forced Beverton-Holt system with minimal period k only in case $r = k$. On the other hand, one can easily construct examples of k-periodic maps, exhibiting r-periodic orbits for arbitrarily chosen k and r, cf. the constructions in [7, 1]. In the following the phase space is $Y = \mathbb{R}^\ell$.

Our focus is on r-periodic orbits of k-periodic systems that occur in a natural way through parameter variations from single functions $f : \mathbb{R}^\ell \times \mathbb{R}^{q+1} \to \mathbb{R}^\ell$, depending on $q + 1$ parameters, i.e. the non-autonomous difference equation (4) is generated by

$$F_n = f(\cdot, \Lambda_n), \quad n \in \mathbb{Z}, \tag{19}$$

where $\Lambda_n \in \mathbb{R}^{q+1}$ is assumed to be k-periodic.

We introduce an algorithm that computes simultaneously an r-periodic orbit as well as the parameter set for which this orbit exists.

It turns out that, depending on the values of r, k and the dimension ℓ of the space, many parameters are needed to find these objects in generic systems. For our purposes it is useful to define the codimension of an r-periodic orbit for a k-periodic system in $\mathbb{R}^\ell$ as

$$\operatorname{codim}(k, r, \ell) := (\operatorname{lcm}(k, r) - r)\ell. \tag{20}$$

This definition will be motivated below.

4.1 The case r factors k

Let r and k be two natural numbers such that r is a divisor of k and let $p = k/r$. This is the only case in which a globally attracting r-periodic orbit with $r \leq k$ can exist, cf. [7, Theorem 3.4].

To guarantee minimality of the period k of the sequence $F_{\mathbb{Z}}$, we separate one parameter

$$\Lambda_n = (\lambda_n, \mu_n), \quad \lambda_n \in \mathbb{R}^q,\ \mu_n \in \mathbb{R}$$

and choose a non-constant k-periodic sequence $\mu_{\mathbb{Z}}$. Typically, the influence of the parameter μ is such that the following assumption is satisfied.

A3 Let $\mu_{\mathbb{Z}}$ be a k-periodic sequence of parameters, and let k be the minimal period of the sequence $(f(\cdot, \cdot, \mu_n))_{n \in \mathbb{Z}}$.

Using the notions from above, the condition for r-periodic orbits given in (7) is equivalent to

$$f(x_i, \Lambda_{i+nr}) = x_{(i+1) \bmod r}, \quad i = 0, \ldots, r-1,\ n = 0, \ldots, p-1. \tag{21}$$

Note that due to assumption **A3**, none of the equations in (21) can be omitted.

System (21) consists of $k\ell$ equations with $r\ell$ unknown variables x_i, thus $(k-r)\ell$ extra parameters are needed in order to find an r-periodic orbit of a k-periodic map. This means that the map f must depend on sufficiently many parameters, i.e. $q \geq \frac{(k-r)\ell}{k}$.

Remark 5 *A more precise meaning to this counting is given by the notion of codimension in (algebraic) singularity theory (cf. contact equivalence in [12]) which is the number of parameters, necessary to construct a universal unfolding. In particular, a zero $\bar{x} \in \mathbb{R}^m$ of some smooth function $G : \mathbb{R}^m \to \mathbb{R}^j$, $j \geq m$ has codimension $j - m$ provided* $\operatorname{rank}(G_x(\bar{x})) = m$. *Applying this to $X = (x_0, \ldots, x_{r-1}) \in \mathbb{R}^{r\ell}$ and*

$$G(X) = \left(F_{i+nr}(x_i) - x_{(i+1) \bmod r},\ i = 0, \ldots, r-1,\ n = 0, \ldots, p-1\right) \in \mathbb{R}^{k\ell}$$

leads to the codimension $(k-r)\ell$, provided $G_x(X)$ has rank $r\ell$. This coincides with (20), since r is a divisor of k. The rank condition holds if and only if the matrices

$$\prod_{i=0}^{r-1} \left(F_{i+nr}\right)_x (x_i), \quad n = 0, \ldots, p-1$$

do not have a common eigenvector with corresponding eigenvalue 1.

We note that the notion of codimension in dynamical bifurcation theory is much more involved, see [17].

For the numerical computations it is convenient to assume $\lambda_n \in \mathbb{R}^\ell$ for $n = 0, \dots, k-1$ so that $q \geq \frac{(k-r)\ell}{k}$ is automatically satisfied.

We fix the parameters $\lambda_0, \dots, \lambda_{r-1}$ and determine the remaining $(k-r)\ell$ parameters $\lambda_r, \dots, \lambda_{k-1}$ by computing a zero of the operator $\Gamma_r : \mathbb{R}^{k\ell} \to \mathbb{R}^{k\ell}$, defined as

$$\Gamma_r \begin{pmatrix} x_0 \\ \vdots \\ x_{r-2} \\ x_{r-1} \\ \lambda_r \\ \vdots \\ \lambda_{k-1} \end{pmatrix} = \begin{pmatrix} f(x_0, \lambda_0, \mu_0) - x_1 \\ \vdots \\ f(x_{r-2}, \lambda_{r-2}, \mu_{r-2}) - x_{r-1} \\ f(x_{r-1}, \lambda_{r-1}, \mu_{r-1}) - x_0 \\ f(x_0, \lambda_r, \mu_r) - x_1 \\ \vdots \\ f(x_{(k-1) \bmod r}, \lambda_{k-1}, \mu_{k-1}) - x_0 \end{pmatrix}. \tag{22}$$

The derivative of Γ_r turns out to have lower block diagonal structure which leads to the following characterization of well-posedness.

Lemma 6 *Let the parameters $\bar{\lambda}_0, \dots, \bar{\lambda}_{r-1} \in \mathbb{R}^\ell$ be given.*

Then the vector $(\bar{x}_0, \dots, \bar{x}_r, \bar{\lambda}_r, \dots, \bar{\lambda}_{k-1})^T \in \mathbb{R}^{k\ell}$ is a solution of (22) if and only if $\bar{X} = (\bar{x}_0, \dots, \bar{x}_{r-1})$ is an r-periodic orbit of the k-periodic map (19) with parameters $\Lambda_n = (\bar{\lambda}_n, \mu_n)$. Moreover, this solution is a regular zero of Γ_r if and only if the matrices $I - \prod_{i=0}^{r-1} f_x(\bar{x}_i, \bar{\lambda}_i, \mu_i)$ and $\prod_{i=r}^{k-1} f_\lambda(\bar{x}_{i \bmod r}, \bar{\lambda}_i, \mu_i)$ are non-singular.

Lemma 6 gives sufficient conditions under which the problem of finding a zero of Γ_r is well posed. But these do not guarantee minimality of the period r, in general. However, we expect the period of the zeroes of Γ_r to be r in a generic sense. Assume, for example, that the solution has period ϱ where ϱ is a non-trivial factor of r. Then such an orbit has the codimension

$$\operatorname{codim}(k, \varrho, \ell) = (k - \varrho)\ell > (k - r)\ell = \operatorname{codim}(k, r, \ell)$$

and needs a corresponding number of parameters for its stable computation. Since the numerically computed solution has in general the lowest possible codimension, a ϱ-periodic solution will typically not occur.

4.1.1 The Beverton-Holt model

For an illustration, we revisit the Beverton-Holt model, introduced in Section 3. We compute in case $k = 8$ a 4-periodic orbit. From Proposition 4 we know that for constant parameter μ, such an orbit cannot exist. However, as we will see, such orbits exists for the more general Beverton-Holt map

$$g(x, K_n, \mu_n) := \frac{\mu_n K_n x}{K_n + (\mu_n - 1)x}. \tag{23}$$

For this map, explicit formulas for the solution operator and the periodic orbit can be found. We introduce the analog of Proposition 4, see also [3]. Here it is important to mention that the period k of the orbit $\bar{X}$ is in general not minimal.

Proposition 7 *Let $K_{\mathbb{N}}$ and $\mu_{\mathbb{N}}$ be two sequences with minimal period k, such that $K_n > 0$ and $\mu_n > 1$ for all $n \in \mathbb{N}$. For $n \geq m$ and $x > 0$ the solution operator Ψ of (14) has the explicit form*

$$\Psi(n,m)(x) = \frac{x \prod_{i=m}^{n-1} \mu_i}{1 + x \sum_{i=m}^{n-1} \frac{\mu_i - 1}{K_i} \prod_{j=m}^{i-1} \mu_j}.$$

Let

$$\bar{x}_n := \frac{\left(\prod_{i=0}^{k-1} \mu_i\right) - 1}{\sum_{i=n}^{n+k-1} \frac{\mu_i - 1}{K_i} \prod_{j=n}^{i-1} \mu_j}, \quad \text{for } n \in \mathbb{N}.$$

Then $\bar{X} := (\bar{x}_0, \ldots, \bar{x}_{k-1})$ is a globally attracting and stable periodic orbit of (23) on $\mathbb{R}^+$.

The proof follows along the lines of Proposition 4 and will be omitted.

After this modification, we have sufficiently many parameters, to set up the system (22). More precisely, we fix $\mu_0, \ldots, \mu_7$ such that $g(\cdot, \cdot, \mu_n) \neq g(\cdot, \cdot, \mu_m)$ for $n \neq m$ and choose the values $K_0, \ldots, K_3$. Then we solve

$$\Gamma_4(\bar{x}_0, \ldots, \bar{x}_3, K_4, \ldots, K_7)^T = 0$$

numerically, using Newton's method. For the values $K_n = 0.2(n+1)$ for $n = 0, \ldots, r-1$ and $\mu_n = 1.1 + 0.1n$ for $n = 0, \ldots, k-1$ the corresponding orbit $x_{\mathbb{N}}$ and the sequence of parameters $K_{\mathbb{N}}$ is displayed in Figure 1.

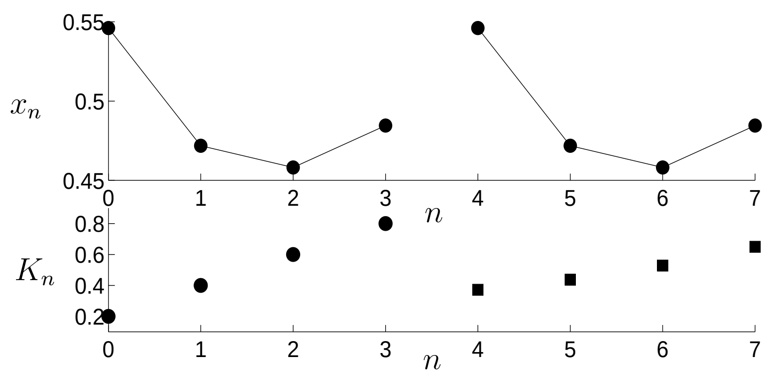

Figure 1: *A 4-periodic orbit of the 8-periodic Beverton-Holt model (upper picture). The lower diagram displays the corresponding sequence K_n of parameters. A circle represents a fixed and a box a computed parameter.*

As one can see from this figure, the period 4 of the orbit is indeed minimal.

4.1.2 The Stiletto map

As a second example, we consider a non-autonomous version of the two-dimensional stiletto map

$$f(x, \lambda_n^1, \lambda_n^2, \mu_n) := \begin{pmatrix} \left(x_1 + \frac{1}{\lambda_n^1}\right) e^{\lambda_n^1(1-x_1)-1} - \frac{1}{\lambda_n^1} + x_2 \\ \lambda_n^2 x_1 - \mu_n x_2 \end{pmatrix},$$

see [20, 11, 14]. This function generalizes Ricker's equation, analogous to the way in which the Hénon map generalizes the one-dimensional logistic map.

Also in this example we choose $k = 8$ and compute a 4-periodic orbit. To this end, we fix the parameters $\mu_0, \ldots, \mu_7$ and $\lambda_0, \ldots, \lambda_3 \in \mathbb{R}^2$, and by solving (22) we compute a 4-periodic orbit as well as the 8 parameters $\lambda_4, \ldots, \lambda_7 \in \mathbb{R}^2$ for which this orbit exists. For the numerical computation, we set $\mu_i = 0.02(i + 1)$ for $i = 0, \ldots, k - 1$ and $\lambda_i = \big(0.1(i + 1), -0.3 - 0.1i\big)$ for $i = 0, \ldots, r - 1$. The corresponding solution of (22) is given in Figure 2. As one can see, the period 4 of

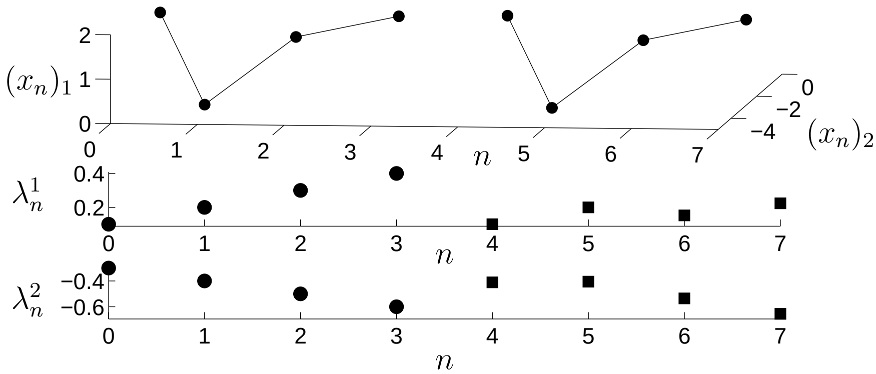

Figure 2: 4-*periodic orbit of the* 8-*periodic stiletto map (upper picture). The lower diagrams display the corresponding sequence* λ_n^1 *and* λ_n^2 *of parameters. A circle represents a fixed and a box a computed parameter.*

the orbit in this example is minimal. In Section 4.3 we point out that this orbit is indeed attracting.

4.2 The general case

We generalize our approach (22) for arbitrarily chosen natural numbers r and k. Here both cases $r \leq k$ and $r \geq k$ are permitted. Let $s = \operatorname{lcm}(k, r)$ and assume that the smooth function f depends on sufficiently many parameters $\Lambda \in \mathbb{R}^{q+1}$, where $q \geq \left\lceil \frac{s-r}{k} \ell \right\rceil$. To guarantee minimality of the period of the functions, see **A3**, we separate one parameter $\Lambda_n = (\lambda_n, \mu_n) \in \mathbb{R}^q \times \mathbb{R}$.

Let $\Delta = (\lambda_0, \dots, \lambda_{k-1})^T$. We obtain an r-periodic orbit by solving (7) for $i = 0, \dots, r-1$ and $n = 0, \dots, \frac{s}{r}$. For the numerical realization, we compute a zero of the operator Γ_r, defined as

$$\Gamma_r \begin{pmatrix} x_0 \\ \vdots \\ x_{r-2} \\ x_{r-1} \\ \nu_0 \\ \vdots \\ \nu_{s-r-1} \end{pmatrix} = \begin{pmatrix} f(x_0, \lambda_0, \mu_0) - x_1 \\ \vdots \\ f(x_{r-2}, \lambda_{(r-2) \bmod k}, \mu_{(r-2) \bmod k}) - x_{r-1} \\ f(x_{r-1}, \lambda_{(r-1) \bmod k}, \mu_{(r-1) \bmod k}) - x_0 \\ f(x_0, \lambda_{r \bmod k}, \mu_{r \bmod k}) - x_1 \\ \vdots \\ f(x_{(s-1) \bmod r}, \lambda_{(s-1) \bmod k}, \mu_{(s-1) \bmod k}) - x_0 \end{pmatrix}, \tag{24}$$

where the j-th component of ν_i is defined as

$$(\nu_i)_j = \Delta_{i\ell+j}, \quad \text{for } i = 0, \dots, s-r-1,\ j = 1, \dots, \ell.$$

Since none of the equations in (24) can be omitted, we need $(s-r)\ell$ free parameters for the computation of an r-periodic orbit. Depending on r and k these orbits may have an extremely high codimension.

Remark 8 *This motivates the definition of the codimension from (20). Applying the formalism, introduced in Remark 5 to* $X = (x_0, \dots, x_{r-1})$, $p = \frac{\mathrm{lcm}(k,r)}{r}$ *and*

$$G(X) = \left(F_{(i+nr) \bmod k}(x_i) - x_{(i+1) \bmod r},\ {\scriptstyle i=0,\dots,r-1,\ n=0,\dots,p-1}\right) \in \mathbb{R}^{\mathrm{lcm}(k,r)\ell}$$

leads to the codimension $(\mathrm{lcm}(k,r) - r)\ell$*, provided* $G_x(X)$ *has rank* $r\ell$*, see (20).*

When solving $\Gamma_r = 0$, we do not insist on the minimality of the period r. Indeed it may happen that an orbit of period ϱ shows up, where ϱ is a factor of r. In numerical computations, we will typically find the solution with the lowest codimension. For a more precise analysis of the codimension, let r_1 be the greatest common divisor of r and k, i.e.

$$r = r_1 r_2, \quad k = r_1 k_1, \quad r_2,\ k_1 \text{ relative prime.}$$

Assuming that ϱ is a factor of r, we analyze the codimension of the corresponding ϱ-periodic orbit. Note that a ϱ-periodic orbit is also an r-periodic orbit. Let ϱ_1 be the greatest common divisor of ϱ and r_1, i.e.

$$\varrho = \varrho_1 \varrho_2, \quad r_1 = \varrho_1 r_3, \quad \varrho_2,\ r_3 \text{ relative prime.}$$

In the following lemma, we prove that, given r and k, sub-periodic orbits of length $r_1 = \gcd(k,r)$ have minimal codimension.

Lemma 9 *Let* ϱ *be a factor of* r*. Then*

$$\mathrm{codim}(k, \gcd(k,r), \ell) \leq \mathrm{codim}(k, \varrho, \ell).$$

Equality holds if and only if $\varrho = \gcd(k,r)$ *or if both* r *and* ϱ *are multiples of* k*.*

Proof: It holds

$$\begin{aligned}
\operatorname{codim}(k,\varrho,\ell) &= (\operatorname{lcm}(k,\varrho)-\varrho)\ell \\
&= (k_1\varrho_1 r_3\varrho_2-\varrho_1\varrho_2)\ell = \varrho_1\varrho_2(k_1r_3-1)\ell \\
&\geq \varrho_1\varrho_2 r_3(k_1-1)\ell = r_1\varrho_2(k_1-1)\ell \\
&\geq r_1(k_1-1)\ell = (k_1r_1-r_1)\ell = (\operatorname{lcm}(k,r_1)-r_1)\ell \\
&= \operatorname{codim}(k,\gcd(k,r),\ell).
\end{aligned}$$

The second assertion follows since equality holds if and only if $r_3 = 1$ and ($k_1 = 1$ or $\varrho_2 = 1$).

■

As a consequence of Lemma 9, the solutions of $\Gamma_r = 0$ generically will be orbits of period $r_1 = \gcd(k,r)$. The only exception is $r = r_2k$, $k = k_1$ which implies $\operatorname{codim}(k,r,\ell) = 0$.

Suppose we have found a solution $X = (x_0,\ldots,x_{r-1}) \in \mathbb{R}^{r\ell}$ of $\Gamma_r = 0$ that has period $r_1 = \gcd(k,r) < r,k$. We argue that we then expect the Jacobian of Γ_r to be singular. Let $\nu = (\nu_0,\ldots,\nu_{s-r-1}) \in \mathbb{R}^{(s-r)\ell}$ be the parameter set such that $\Gamma_r(X,\nu) = 0$. Since X has period r_1 the vectors $\tilde{X} = (x_0,\ldots,x_{r_1-1}) \in \mathbb{R}^{r_1\ell}$ and $\tilde{\nu} = (\nu_0,\ldots,\nu_{k-r_1-1})$, (note that $k = \operatorname{lcm}(r_1,k)$) solve the $k\ell$-dimensional system

$$\Gamma_{r_1}(\tilde{X},\tilde{\nu},\hat{\nu}) = 0, \tag{25}$$

where Γ_{r_1} is defined as in (24) with (r_1,k) instead of (r,s) and where

$$\hat{\nu} = (\nu_{k-r_1},\ldots,\nu_{s-r-1}) \in \mathbb{R}^{(s-r-(k-r_1))\ell}, \quad s-r > k-r_1$$

collects the remaining parameters. In view of Lemma 6 we expect the Jacobian $\frac{\partial\Gamma_{r_1}}{\partial(\tilde{X},\tilde{\nu})}$ to be non-singular. By the implicit function theorem, equation (25) has an $(s-r-k+r_1)\ell$-dimensional manifold of solutions $(\tilde{X}(\hat{\nu}),\tilde{\nu}(\hat{\nu}),\hat{\nu})$ which also solves the original equation $\Gamma_r = 0$. Therefore, the rank of the Jacobian at these solutions drops at least by $(s-r-k+r_1)\ell$.

However, the existence of such manifolds of singular solutions of (24) may well coexist with further isolated and regular solutions, see the following section.

4.2.1 The case $r < k$

If r is a factor of k, then $\gcd(k,r) = r$, and the period of the solution of (24) is indeed minimal in a generic sense, cf. the examples in the Sections 4.1.1 and 4.1.2.

We tested the general approach numerically, in case $r = 2$, $k = 3$ for the Beverton-Holt model, and the resulting 2-periodic orbit is indeed the fixed point 0 (an $r_1 = 1$-periodic orbit). Furthermore, the Jacobian of Γ_2 at this fixed point is singular and Newton's method is ill-conditioned.

In case $r = 2$, $k = 3$ we discuss an example, where it is possible to compute 2-periodic orbits. Consider the following system

$$\begin{aligned} F_0(x, \lambda_0, \lambda_1) &= -\lambda_0 x + x^3 + \lambda_1, \\ F_1(x, \lambda_2, \lambda_3) &= -2x + \lambda_2 x^5 + \lambda_3, \\ F_2(x, \lambda_4, \lambda_5) &= \lambda_4 \sin(-\lambda_5 x). \end{aligned}$$

A 2-periodic orbit of a 3-periodic map has the codimension $\mathrm{lcm}(3,2) - 2 = 4$ and as a consequence, we can fix two parameters $\lambda_4 = 1$, $\lambda_5 = \frac{\pi}{2}$.

Note that this system fits into our framework (19) by taking the map

$$f(x, \lambda, \mu) = \frac{1}{2}(2-\mu)(1-\mu)F_0(x, \lambda) + \mu(2-\mu)F_1(x, \lambda) + \frac{1}{2}\mu(\mu-1)F_2(x, \lambda)$$

with $\mu = (0,\ 1,\ 2)^T$ and observing $\dim(\lambda) \geq \left\lceil \frac{\mathrm{lcm}(3,2)-2}{3} \right\rceil = 2$.

When solving $\Gamma_2 = 0$ numerically, we get, depending on the initial values, with equal probability either one of the two-periodic orbits $(-1, 1)$ and $(1, -1)$ and the corresponding parameters $\lambda_0 = 2$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0$ or Newton's method does not converge.

From Lemma 9, we know that the sub-periodic orbit with the lowest codimension in this example is a fixed point. The fixed point $x_0 = x_1 = 0$ with arbitrary values of λ_0, λ_2 leads to singular solutions of equation $\Gamma_2 = 0$ (see the discussion above). Nevertheless Newton's method finds easily the 2-periodic orbit with the higher codimension. This is caused by a separation of the space $\mathbb{R}^2 \times \mathbb{R}^4$ in this example. In order to understand this more clearly, we write (24) in a permuted form

$$\begin{pmatrix} x_0 \\ x_1 \\ \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \mapsto \begin{pmatrix} F_2(x_0) - x_1 \\ F_2(x_1) - x_0 \\ F_0(x_0, \lambda_0, \lambda_1) - x_1 \\ F_0(x_1, \lambda_0, \lambda_1) - x_0 \\ F_1(x_0, \lambda_2, \lambda_3) - x_1 \\ F_1(x_1, \lambda_2, \lambda_3) - x_0 \end{pmatrix}.$$

Now, one can decouple the computation of x_0, x_1 and λ_0, λ_1 and λ_2, λ_3. In this example, F_2 possesses the unique fixed point 0 and the 2-periodic orbits $(-1, 1)$ and $(1, -1)$. A fixed point and a 2-periodic orbit of a single map have the same codimension 0, cf. Section 4.2.2, and in fact, Newton's method converges locally to both solutions. Thus, depending on the initial values, both objects arise in this example with equal probability. The stability of the 2-periodic orbit is analyzed in Section 4.3

4.2.2 The case $r \geq k$

If k is not a factor of r we gain generally an orbit with minimal period $\gcd(k, r)$, as discussed previously. More interesting is the case, in which k is a factor of r.

Then $(\mathrm{lcm}(k,r)-r)\ell = 0$ parameters are needed and the orbits are of codimension 0. For an illustration, we revisit the stiletto map, introduced in Section 4.1.2. Let $F_n = f(\cdot, \lambda_n^1, \lambda_n^2, \mu_n)$. In case $k=3$ we compute a fixed point of

$$G := F_2 \circ F_1 \circ F_0$$

for the parameters

$$\lambda^1 = \begin{pmatrix} 0.6 \\ 0.2 \\ 0.3 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} -0.3 \\ -0.4 \\ -0.5 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.02 \\ 0.04 \\ 0.06 \end{pmatrix}.$$

A fixed point of G corresponds to a 3-periodic orbit of the 3-periodic family $(F_n)_{n\in\mathbb{N}}$. We continue this fixed point w.r.t. the parameter λ_0^1, using the bifurcation-toolbox MATCONT, see [5]. As one can see in Figure 3 various period doubling bifurcations, denoted by PD, see [17], occur that result in 6- and 12-periodic orbits of $(F_n)_{n\in\mathbb{N}}$. Note that periodic orbits with period $n \cdot k$ are of codimension 0.

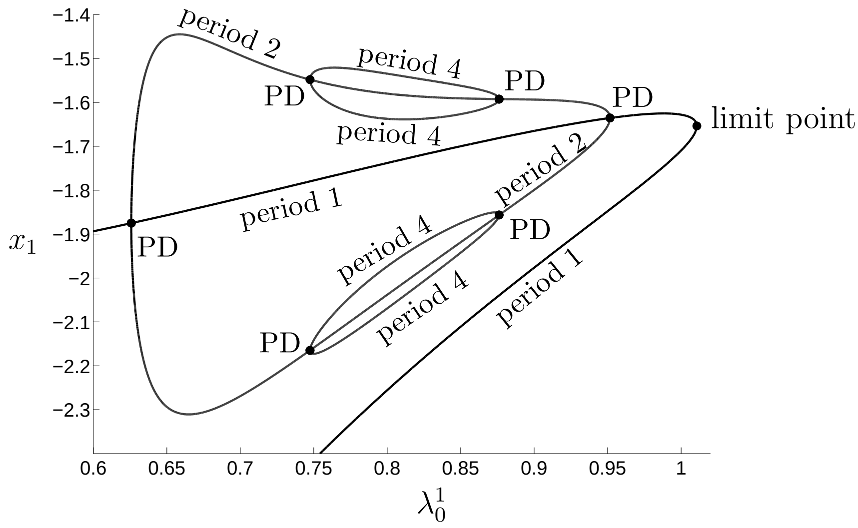

Figure 3: *Bifurcation diagram of fixed points and periodic orbits of G w.r.t. the parameter λ_0^1.*

4.3 Stability

Local attraction of an r-periodic orbit for a k-periodic map can be tested easily, applying the linearized stability principle, combined with Lemma 3. Let $s = \mathrm{lcm}(k,r)$, then a zero $\bar{x}_0, \ldots, \bar{x}_{r-1}$ of (24) is asymptotically stable, if

$$\begin{aligned} D\Psi(s,0)(\bar{x}_0) &= D\left(F_{(s-1) \bmod k} \circ \cdots \circ F_0\right)(\bar{x}_0) \\ &= DF_{(s-1) \bmod k}(\bar{x}_{(k-1) \bmod r}) \cdot \ldots \cdot DF_0(\bar{x}_0) \end{aligned} \tag{26}$$

has only eigenvalues inside the unique circle.

If r is a factor of k it may even be that the orbit is asymptotically stable and globally attracting, cf. [7].

We test condition (26) for the examples from Section 4.1.1 and 4.1.2.

For the generalized Beverton-Holt model, the orbit, computed in Section 4.1.1 is asymptotically stable, since (26) has the eigenvalue 0.056678. Furthermore, we know from Proposition 7, that this orbit is also globally attracting. We illustrate this result in Figure 4, where the orbits of several initial points are plotted in a logarithmic scale. As one can see, each orbit converges after a few steps towards $\bar{x}_n$.

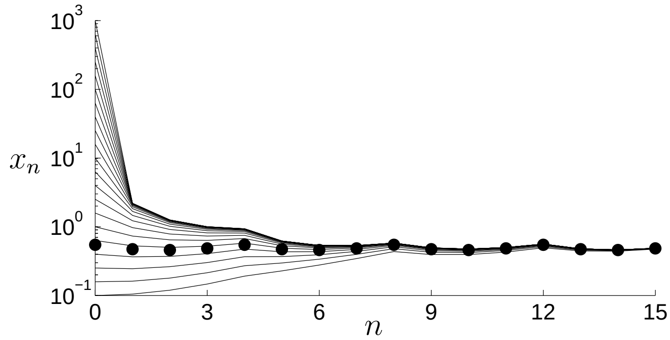

Figure 4: *A stable and globally attracting* 4*-periodic orbit of the* 8*-periodic Beverton-Holt model (black dots), cf. Figure 1, and the orbits for the initial points* $x_0 = 10^{-1+i(0.2)}$, $i = 0, \ldots, 20$.

For the stiletto map, equation (26) possesses for the parameter setup from Section 4.1.2 the eigenvalues -0.07313 and -0.00772 and as a consequence, the computed 4-periodic orbit is asymptotically stable. For an illustration of the attraction of the 4-periodic orbit, the corresponding orbits of several initial points from a small neighborhood of $\bar{x}_0$ are plotted in Figure 5 (left). The right picture of Figure 5 displays the domain of attraction of the fixed point $\bar{x}_0$ of $\Psi(k, 0)$ ($\bar{x}_0$ is marked by a white point). As a result, the 4-periodic orbit is not globally attracting.

Finally, we revisit the example from Section 4.2.1 that exhibits 2-periodic orbits of a 3-periodic map. The fixed points -1 and 1 of $\Psi(6, 0)$ which correspond to the 2-periodic orbits $(-1, 1)$ and $(1, -1)$, respectively, possess the eigenvalue $8.3261 \cdot 10^{-32}$ and consequently, these orbits are asymptotically stable. The fixed point 0 possesses the eigenvalue 39.47842 and is therefore unstable.

In this example, it is possible to study global attraction. Since $F_2(x) = \sin(-\frac{\pi}{2}x)$ maps all points to $[-1, 1]$ we get for any initial value $x_0 \in \mathbb{R}$

$$\lim_{n\to\infty} \Psi(6, 0)^n(x_0) = \begin{cases} 1 & \text{if } \Psi(3, 0)(x_0) \in [-1, 0), \\ 0 & \text{if } \Psi(3, 0)(x_0) = 0, \\ -1 & \text{if } \Psi(3, 0)(x_0) \in (0, 1]. \end{cases}$$

This result is illustrated in Figure 6.

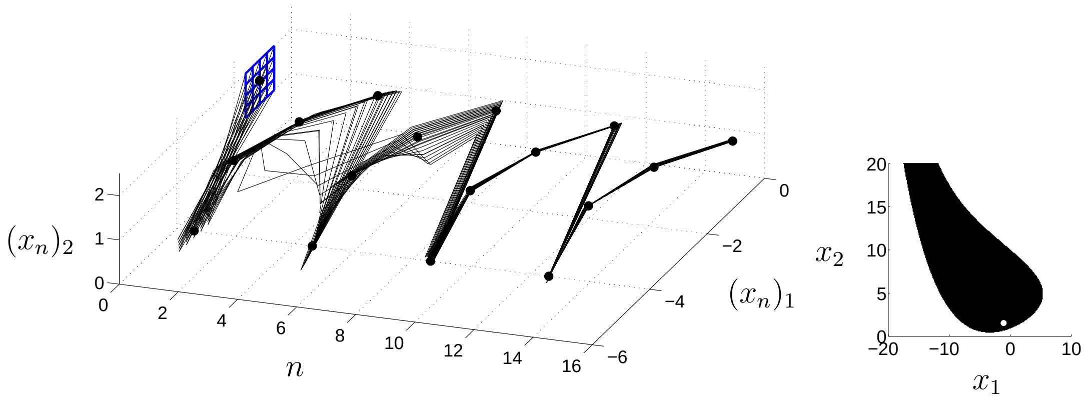

Figure 5: *In the left diagram the* 4*-periodic orbit (black dots), cf. Figure* 2*, of the* 8*-periodic stiletto map are shown, together with the orbits of points chosen from a* 5×5 *initial lattice. The right diagram, shows the domain of attraction of the fixed point* $\bar{x}_0$ *(white point) of* $\Psi(k, 0)$.

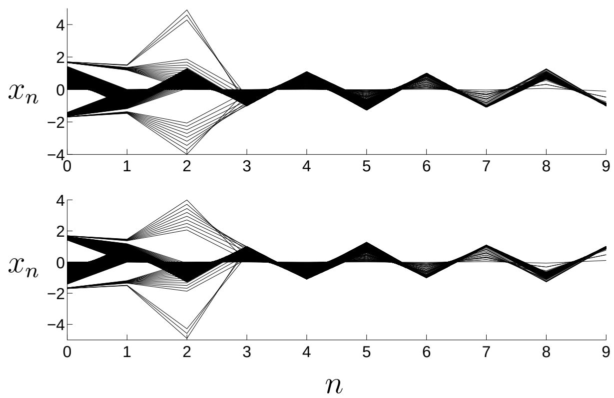

Figure 6: *The upper picture shows the orbits for* $x_0 \in [-1.7, 1.7]$ *that converge towards the* 2*-periodic orbit* $(1, -1)$*, while in the lower diagram, those orbits are displayed that converge towards* $(-1, 1)$.

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