# Shadowing and inverse shadowing in set-valued dynamical systems. Contractive case

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**Abstract**

We obtain several results on shadowing and inverse shadowing for set-valued dynamical systems that have a contractive property. Applications to $T$-flows of differential inclusions are discussed.

**Key words.** set-valued dynamical systems, shadowing, inverse shadowing, differential inclusions
**AMS(MOS) subject classifications.** 54C60, 37C50, 34A60

## 1 Introduction and Basic Notation

The shadowing property for classical dynamical systems (with discrete or continuous time) is now well-studied (see, for example, the monographs [11, 10]). This property means that, near approximate trajectories, there exist exact trajectories of the system considered. If a dynamical system has the shadowing property, then, for example, results of numerical modeling reflect the global structure of trajectories of the system.

Another type of shadowing properties is related to the following question: Given a (large enough) family of approximate trajectories, can we find, for a chosen exact trajectory, a close approximate trajectory from the given family? The corresponding property is called the inverse shadowing property. It was introduced in [5] and studied intensively by various authors (see, for example, [12, 9]).

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Let us also mention that from results of W.-J. Beyn [3] it follows that an autonomous system of differential equations has an analog of the inverse shadowing property in a neighbourhood of a hyperbolic rest point with respect to pseudotrajectories generated by numerical one-step methods.

In this paper, we study the shadowing and inverse shadowing properties for set-valued dynamical systems. One of the main objects to which our results are applicable are $T$-flows of differential inclusions (very important, for example, in control theory [4, 14]).

We introduce a contractive property (stability condition) for set-valued dynamical systems and show that any system satisfying this condition has both shadowing and inverse shadowing properties (and the distance between the corresponding exact and approximate trajectories is estimated linearly in terms of the error).

Let us pass to basic notation. A set-valued dynamical system on a metric space $(M, \text{dist})$ is determined by a set-valued mapping $F : M \to 2^M \setminus \{\emptyset\}$ and its iterates. In what follows, we identify the mapping $F$ and the corresponding dynamical system.

A sequence $\eta = \{p_k\}$ is a trajectory of the system $F$ if

$$p_{k+1} \in F(p_k) \text{ for any } k \in \mathbb{Z}. \tag{1}$$

A sequence $\xi = \{x_k\}$ is called a d-pseudotrajectory of $F$ if an error of size $d > 0$ is allowed in every step, i.e., if

$$\text{dist}(x_{k+1}, F(x_k)) \leq d \quad \text{for any } k \in \mathbb{Z}. \tag{2}$$

The distance between two subsets $A$ and $B$ of $\mathbb{R}^m$ is measured by the deviation

$$\text{dev}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|$$

or by the Hausdorff distance

$$\text{dist}_H(A, B) = \max\{\text{dev}(A, B), \text{dev}(B, A)\}.$$

If $A$ and $B$ are compact sets, there exists a (possibly not unique) vector $\text{Dev}(A, B) \in \mathbb{R}^m$ such that $\text{Dev}(A, B) = b - a$ for some $a \in A$ and $b \in B$ with $|b - a| = \text{dev}(A, B)$. Let us note that if $A$ is a point and $B$ is a convex set, then the vector $\text{Dev}(A, B)$ is defined uniquely. In addition, if $B(t)$ is a continuous (w.r.t. $\text{dist}_H$) family of convex sets, then the vector-function $\text{Dev}(A, B(t))$ is continuous in $t$ as well (cf. Theorem 1.7.1 of [2]).

The collection of compact subsets of $R^m$ will be denoted by $\mathcal{C}(\mathbb{R}^m)$, while the class of compact and convex subsets of $R^m$ will be denoted by $\mathcal{CC}(\mathbb{R}^m)$.

As usual, for a sequence $\eta = \{\eta_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$,

$$||\eta||_\infty = \sup_{k\in\mathbb{Z}} |\eta_k|.$$

In section 2, we establish several shadowing results (for convex-valued dynamical systems and for dynamical systems with sufficiently large "continuous convex kernels"). In section 3, similar inverse shadowing results are proved.

# 2 Shadowing

We begin with a result on shadowing for set-valued dynamical systems.

**Theorem 1.** *Let $F : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ be a set-valued mapping for which there exist numbers $a > 0$ and $\lambda \in (0, 1)$ such that $F$ satisfies the following stability condition:*

$$\mathrm{dist}_H(F(x), F(x + v)) \le \lambda|v| \quad \text{for any } x \in \mathbb{R}^m \quad \text{and } |v| \le a. \tag{3}$$

*If $\xi = \{x_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ is a $d$-pseudotrajectory for some $d < (1 - \lambda)a$, then there exists a solution $\eta \in (\mathbb{R}^m)^{\mathbb{Z}}$ of (1) such that*

$$||\xi - \eta||_\infty \le \frac{d}{1 - \lambda}. \tag{4}$$

*Proof.* Define the sets $H_d := \{v \in \mathbb{R}^n : |v| \le \frac{d}{1-\lambda}\}$ and $H_d^\infty := (H_d)^{\mathbb{Z}}$. Then $H_d \subset \mathbb{R}^n$ is compact w.r.t the Euclidean topology and $H_d^\infty \subset (\mathbb{R}^m)^{\mathbb{Z}}$ is compact w.r.t. the Tikhonov topology.

Take some $V = \{v_k\} \in H_d^\infty$ and define a sequence $W = \{w_k\}$ by

$$w_{k+1} = \mathrm{Dev}(x_{k+1}, F(x_k + v_k)).$$

Such a sequence $W$ is unique since the sets $F(x_k + v_k)$ are convex.

Condition (3) implies that the mapping $F$ is continuous w.r.t. $\mathrm{dist}_H$. Hence, the mapping $v_k \mapsto w_{k+1}$ is continuous by Theorem 1.7.1. of [2] mentioned in the introduction. Furthermore,

$$\begin{aligned} |w_{k+1}| &\le \mathrm{dist}(x_{k+1}, F(x_k)) + \mathrm{dist}(F(x_k), F(x_k + v_k)) \\ &\le d + \lambda|v_k| \le d + \lambda\frac{d}{1-\lambda} = \frac{d}{1-\lambda}. \end{aligned}$$

Hence, $W \in H_d^\infty$. Thus, the operator $\sigma$ defined by $\sigma(V) = W$ maps the compact convex set $H_d^\infty$ into itself.

Since the $(k+1)$th element of $\sigma(V)$ depends on the $k$ th element of $V$ only, the operator $\sigma$ is continuous w.r.t. the Tikhonov topology. By the Tikhonov-Schauder fixed point theorem, there is a sequence $V = \{v_k\} \in H_d^\infty$ such that $\sigma(V) = V$. Thus,

$$x_{k+1} + v_{k+1} = x_{k+1} + \mathrm{Dev}(x_{k+1}, F(x_k + v_k)) \in F(x_k + v_k),$$

and the trajectory $\eta = \{p_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ given by $p_k := x_k + v_k$ is a solution of (1) with

$$||\eta - \xi||_\infty = ||V||_\infty \leq \frac{d}{1-\lambda}.$$

□

**Remark 2.** *Let us note that the problem of shadowing for set-valued dynamical systems has been considered by E. Akin (see Appendix 11 of [1]). In [1], the author defines hyperbolic sets for set-valued dynamical systems. According to the definition of [1], on a hyperbolic set, a pseudotrajectory with a small error is shadowed by a* unique *real trajectory.*

*The following example shows that, in contrast to the case studied by Akin, in the conditions of Theorem 1 the shadowing trajectory is not necessarily unique.*

*Consider the plane* $\mathbb{R}^2$ *with coordinates* $x = (x_1, x_2)$ *and the segment* $I = \{x : x_1 = 0,\ 0 \leq x_2 \leq 1\}$.

*Define a set-valued dynamical system generated by the constant mapping* $F(x) = I$, $x \in \mathbb{R}^2$. *Obviously, the mapping* $F$ *satisfies the stability condition of Theorem 1 with any* $a > 0$ *and* $\lambda \in (0, 1)$.

*It is easy to see that the conclusion of Theorem 1 holds in the following form: If* $\xi$ *is a d-pseudotrajectory with any* $d > 0$, *then there exists an exact trajectory* $\eta$ *such that*

$$||\xi - \eta||_\infty \leq d \tag{5}$$

*(indeed, it is enough to repeat the proof of Theorem 1 taking into account the obvious inequality* $|w_{k+1}| \leq d$*).*

*Let us show that the corresponding shadowing trajectory is not necessarily unique. Indeed, fix* $d > 0$ *and consider the sequence* $\xi = \{z_k\}$ *with* $z_k = (d(1 - 2^{-|k|}), 0)$. *Clearly,* $\xi$ *is a d-pseudotrajectory of* $F$ *(and at the same time,* $\xi$ *is not a* $\delta$*-pseudotrajectory with* $\delta < d$*).*

*A sequence* $\eta = \{p_k\}$ *is a trajectory of* $F$ *if and only if* $p_k \in I$ *for all* $k$. *Note that we cannot find an exact trajectory* $\eta$ *for which an analog of* (5) *holds with a smaller constant on the right.*

*Now it is easy to understand that inequality* (5) *holds for any sequence* $\eta = \{r_k\}$, *where* $r_k$ *belongs to the vertical side of the triangle with vertices* $(0, 0)$, $z_k$, *and* $(0, d\sqrt{1 - (1 - 2^{-|k|})^2})$.

The following theorem states that a shadowing result can still be obtained if $F$ is not convex-valued itself but contains a sufficiently large "continuous convex kernel" $G$.

**Theorem 3.** *Let $F : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ be a set-valued mapping for which there exist numbers $a > 0$ and $\lambda \in (0, 1)$ such that $F$ satisfies the stability condition* (3).

*Fix $d < \frac{1-\lambda}{2}a$ and assume that there exists a set-valued mapping $G : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ which is continuous w.r.t. $\mathrm{dist}_H$ and satisfies the following conditions:*

$$G(x) \subset F(x) \text{ and } \mathrm{dist}_H(F(x), G(x)) < d \text{ for any } x \in \mathbb{R}^m.$$

*If $\xi = \{x_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ is a $d$-pseudotrajectory of $F$, then there exists a solution $\eta \in (\mathbb{R}^m)^{\mathbb{Z}}$ of* (1) *such that*

$$||\xi - \eta||_\infty \le \frac{2d}{1-\lambda}. \tag{6}$$

*Proof.* The proof follows the same lines as that of Theorem 1. Define the sets $H_d := \{v \in \mathbb{R}^n : |v| \le \frac{2d}{1-\lambda}\}$ and $H_d^\infty := (H_d)^{\mathbb{Z}}$.

Take some $V = \{v_k\} \in H_d^\infty$ and define a sequence $W = \{w_k\}$ by

$$w_{k+1} = \mathrm{Dev}(x_{k+1}, G(x_k + v_k)).$$

Since the sets $G(x_k+v_k)$ are convex, the above relations define an operator $\sigma$ by $\sigma(V) = W$. This operator is continuous w.r.t. the Tikhonov topology.

The inequalities

$$|w_{k+1}| \le \mathrm{dist}(x_{k+1}, F(x_k)) + \mathrm{dist}(F(x_k), F(x_k + v_k)) +$$

$$+ \mathrm{dist}(F(x_k + v_k), G(x_k + v_k)) \le d + \lambda|v_k| + d \ \le \ 2d + \lambda\frac{2d}{1-\lambda} \ = \ \frac{2d}{1-\lambda}$$

imply that if $V \in H_d^\infty$, then $W \in H_d^\infty$.

By the Tikhonov-Schauder fixed point theorem, there is a sequence $V = \{v_k\} \in H_d^\infty$ such that $\sigma(V) = V$. Thus,

$$x_{k+1} + v_{k+1} = x_{k+1} + \mathrm{Dev}(x_{k+1}, G(x_k + v_k)) \in F(x_k + v_k),$$

and the trajectory $\eta = \{p_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ given by $p_k := x_k + v_k$ is a solution of (1) with

$$||\eta - \xi||_\infty = ||V||_\infty \le \frac{2d}{1-\lambda}.$$

□

**Remark 4.** *At first glance it seems that, for a continuous mapping* $F : \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ *with connected images and with the property that for every* $x \in \mathbb{R}^m$ *there exists a convex set* $\tilde{F}(x) \subset F(x)$ *such that* $\text{dist}_H(\tilde{F}(x), F(x)) < \vartheta$*, there must exist a "continuous convex kernel"* $G : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ *such that* $G(x) \subset F(x)$ *and* $\text{dist}_H(G(x), F(x)) < \vartheta$*. Simple counterexamples show that this is not the case.*

The aim of the following reasoning is to prove a shadowing result for the (usually not convex) $T$-flow of a differential inclusion. Consider a differential inclusion

$$\dot{x}(t) \in F(x(t)) \text{ almost everywhere.} \tag{7}$$

We fix a point $x_0$ and denote by

$$S^F(x_0, [0,T]) := \{\varphi \in AC([0,T], \mathbb{R}^d) : \varphi \text{ is a solution of } (7),\ \varphi(0) = x_0\}$$

the set of all solutions of the differential inclusion with the initial condition $x(0) = x_0$, where $AC([0,T], \mathbb{R}^d)$ is the space of all absolutely continuous functions. For $0 \le t \le T$ let

$$S^F(x_0, t) := \{\varphi(t) : \varphi \in S^F(x_0, [0,T])\}$$

be the reachable set at time $t$ and set $B_P(0) = \{x \in \mathbb{R}^m : |x| \le P\}$.

**Lemma 5.** *Let* $F : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ *be Lipschitz continuous and let* $F(x) \subset B_P(0)$ *for some* $P > 0$ *and all* $x \in \mathbb{R}^m$*. For every* $T > 0$ *there exists a Lipschitz continuous selection of* $S^F(\cdot, T)$*.*

*Proof.* According to Theorem 1.9.1 of [2], there exists a Lipschitz continuous selection $f : \mathbb{R}^m \to \mathbb{R}^m$ of $F$; let $L > 0$ be a Lipschitz constant of $f$. If $\varphi(t, x_0)$ is the unique solution of

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0,\ 0 \le t \le T,$$

then $\varphi(t, x_0) \in S^F(x_0, [0,T])$ and $\varphi(T, x_0) \in S^F(x_0, T)$. The inequalities

$$\begin{aligned} |\varphi(T, x_0) - \varphi(T, x_1)| &\le |x_0 - x_1| + \int_0^T |f(\varphi(t, x_0)) - f(\varphi(t, x_1))| dt \le \\ &\le |x_0 - x_1| + L \int_0^T |\varphi(t, x_0) - \varphi(t, x_1)| dt \end{aligned}$$

and the Gronwall lemma imply that

$$|\varphi(T, x_0) - \varphi(T, x_1)| \le |x_0 - x_1| e^{LT},$$

and $\varphi(T, \cdot)$ is a Lipschitz continuous selection. □

Moreover, every solution $x(t)$ of the differential inclusion (7) with $x(0) = x_0$ satisfies the inequality

$$|x(T) - x(0)| \leq \int_0^T |\dot{x}(s)| ds \leq PT,$$

and

$$\text{diam}(S^F(x_0, T)) \leq 2PT. \tag{8}$$

**Theorem 6.** *Assume that the mapping $F$ in the differential inclusion* (7) *satisfies the conditions of Lemma 5. Assume, in addition, that there exist numbers $a > 0$ and $\lambda \in (0, 1)$ such that the $T$-flow $S^F(\cdot, T)$ satisfies the stability condition*

$$\text{dist}_H(S^F(x, T), S^F(x + v, T)) \leq \lambda |v| \quad \textit{for any } x \in \mathbb{R}^m \quad \textit{and } |v| \leq a. \tag{9}$$

*If*

$$T \leq \frac{d}{2P} \quad \textit{and} \quad d < \frac{1 - \lambda}{2} a,$$

*then for any $d$-pseudotrajectory $\xi = \{x_k \in \mathbb{R}^m\}$ of the set-valued dynamical system*

$$z_{n+1} \in S^F(z_n, T) \tag{10}$$

*there exists a solution $\eta$ of* (10) *such that*

$$||\xi - \eta||_\infty \leq \frac{2d}{1 - \lambda}.$$

*Proof.* Lemma 5 guarantees the existence of a Lipschitz continuous selection $g$ of $S^F(\cdot, T)$. By (8),

$$\text{dist}_H(g(x), S^F(x, T)) \leq 2PT \leq d,$$

and Theorem 3 applies. □

The stability condition (9) is a reasonable generalization of the classical concept of a contraction. Consider the T-flow of (7) with the following property:

**Definition 7.** *A set-valued mapping $F : \mathbb{R}^d \to 2^{\mathbb{R}^d}$ satisfies the relaxed one-sided Lipschitz condition (ROSL) with a constant $\mu \in \mathbb{R}$ if for any $x', x'' \in \mathbb{R}^d$ and $y' \in F(x')$ there exists a $y'' \in F(x'')$ such that*

$$\langle y'' - y', x'' - x' \rangle \leq \mu |x'' - x'|^2.$$

Set-valued mappings that satisfy the ROSL condition or related dissipativity concepts have been thoroughly investigated in [6], [7], and [8]. The results displayed there indicate that, if $\mu < 0$, then the $T$-flow of such a mapping is a contraction in the sense of the stability condition (9). Since this fact is not explicitly mentioned there, we would like to refer to a short proof in [13] in order to show that our stability condition is valid for a reasonably large class of differential inclusions.

# 3 Inverse Shadowing

The next two theorems provide some results on inverse shadowing in the stable case considered above.

**Theorem 8.** *Let $\eta = \{p_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ be a trajectory of a set-valued dynamical system generated by a mapping $F : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$.*

*Assume that there exist numbers $a > 0$ and $\lambda \in (0, 1)$ such that $F$ satisfies the following stability condition:*

$$\mathrm{dist}_H(F(p_k), F(p_k + v)) \le \lambda|v| \quad \text{for any } k \in \mathbb{Z} \quad \text{and} \quad |v| \le a \tag{11}$$

*at the trajectory $\eta$.*

*If $\Phi_k : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ is a family of mappings such that any $\Phi_k$ is continuous w.r.t. $\mathrm{dist}_H$ and $d$-close to $F$ in the $a$-neighborhood of $\eta$, i.e.,*

$$\mathrm{dist}_H(F(p_k + v), \Phi_k(p_k + v)) < d \quad \text{for any } k \in \mathbb{Z} \quad \text{and} \quad |v| \le a, \tag{12}$$

*where $d < (1 - \lambda)a$, then there exists a solution $\xi = \{x_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ of the inclusions*

$$x_{k+1} \in \Phi_k(x_k), \quad k \in \mathbb{Z}, \tag{13}$$

*such that*

$$||\eta - \xi||_\infty \le \frac{d}{1 - \lambda}.$$

*Proof.* Let $H_d$ and $H_d^\infty$ be as in the proof of Theorem 1. Take some $V = \{v_k\} \in H_d^\infty$ and define a sequence $W = \{w_k\}$ by

$$w_{k+1} = \mathrm{Dev}(p_{k+1}, \Phi_k(p_k + v_k)).$$

By our conditions on the mappings $\Phi_k$, the operator defined by $\sigma(V) := W$ is continuous w.r.t. the Tikhonov topology. Furthermore,

$$\begin{aligned} |w_{k+1}| &\le \mathrm{dist}(p_{k+1}, F(p_k)) + \mathrm{dist}_H(F(p_k), F(p_k + v_k)) + \\ &+ \mathrm{dist}_H(F(p_k + v_k), \Phi_k(p_k + v_k)) \le \\ &\le \lambda|v_k| + d \le \frac{d}{1 - \lambda}; \end{aligned}$$

hence, $W \in H_d^\infty$.

By the Tikhonov-Schauder theorem, there exists a sequence $V = \{v_k\} \in H_d^\infty$ such that $\sigma(V) = V$. Thus,

$$p_{k+1} + v_{k+1} = p_{k+1} + \mathrm{Dev}(p_{k+1}, \Phi_k(p_k + v_k)) \in \Phi_k(p_k + v_k),$$

and $\xi = \{x_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ given by $x_k := p_k + v_k$ is a solution of (13) such that

$$||\eta - \xi||_\infty = ||V||_\infty \leq \frac{d}{1-\lambda}.$$

□

**Remark 9.** *Note again that the $T$-flow of the differential inclusion* (7) *satisfies the stability condition* (11) *if its right hand side is ROSL with a constant $\mu < 0$. Hence, every trajectory of the $T$-flow is shadowed by an approximate trajectory.*

Similarly to the case of shadowing, Theorem 8 can be generalized to mappings which contain sufficiently large "continuous convex kernels".

**Theorem 10.** *Let $\eta = \{p_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ be a trajectory of a set-valued dynamical system generated by a mapping $F : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ that satisfies the stability condition* (11) *at the trajectory $\eta$.*

*Let $\{\Phi_k\}$ be a family of mappings $\Phi_k : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \{\emptyset\}$ that are $d$-close to $F$ in the $a$-neighborhood of $\eta$, i.e., conditions* (12) *are satisfied, where $d < (1-\lambda)a$.*

*Finally, let $\Gamma_k : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ be another family of mappings such that any mapping $\Gamma_k$ is continuous w.r.t.* $\mathrm{dist}_H$ *and satisfies the following assumptions:*

$$\Gamma_k(x) \subset \Phi_k(x) \ \text{ and } \ \mathrm{dist}_H(\Gamma_k(x), \Phi_k(x)) < d.$$

*Then there exists a solution $\xi = \{x_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ of* (13) *such that*

$$||\eta - \xi||_\infty \leq \frac{2d}{1-\lambda}.$$

*Proof.* Now we take the same $H_d$ and $H_d^\infty$ as in the proof of Theorem 3. Take $V = \{v_k\} \in H_d^\infty$ and define a sequence $W = \{w_k\}$ by

$$w_{k+1} = \mathrm{Dev}(p_{k+1}, \Gamma_k(p_k + v_k)).$$

Again, the operator defined by $\sigma(V) := W$ is continuous w.r.t. the Tikhonov topology. Furthermore,

$$\begin{aligned} |w_{k+1}| \ &\leq \ \mathrm{dist}(p_{k+1}, F(p_k)) + \mathrm{dist}_H(F(p_k), F(p_k + v_k)) + \\ &+ \ \mathrm{dist}_H(F(p_k + v_k), \Phi_k(p_k + v_k)) + \mathrm{dist}_H(\Phi_k(p_k + v_k), \Gamma_k(p_k + v_k)) \leq \\ &\leq \ \lambda |v_k| + 2d \ \leq \ \frac{2d}{1-\lambda}, \end{aligned}$$

hence $W \in H_d^\infty$.

By the Tikhonov-Schauder theorem, there exists a sequence $V = \{v_k\} \in H_d^\infty$ such that $\sigma(V) = V$. Thus,

$$p_{k+1} + v_{k+1} = p_{k+1} + \mathrm{Dev}(p_{k+1}, \Phi_k(p_k + v_k)) \in \Phi_k(p_k + v_k),$$

and $\xi = \{x_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$ given by $x_k := p_k + v_k$ is a solution of (13) such that

$$||\eta - \xi||_\infty = ||V||_\infty \leq \frac{2d}{1-\lambda}.$$

□

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