A hybrid method for calculating Lyapunov exponents

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Abstract

In this paper we propose a numerical method for computing all Lyapunov coefficients of a discrete time dynamical system by spatial integration. The method extends an approach of Aston and Dellnitz (1999) who use a box approximation of an underlying ergodic measure and compute the first Lyapunov exponent from a spatial average of the norms of the Jacobian for the iterated map. In the hybrid method proposed here, we combine this approach with classical QR-oriented methods by integrating suitable R-factors with respect to the invariant measure. In this way we obtain approximate values for all Lyapunov exponents. Assuming somewhat stronger conditions than those of Oseledec' multiplicative theorem, these values satisfy an error expansion that allows to accelerate convergence through extrapolation.

1   Introduction

Numerical methods for computing Lyapunov exponents of a dynamical system usually fall into two categories.

In the first category one assumes that a linear variable coefficient system is given for which Lyapunov exponents are defined in the classical way as *time averages.* For example, given a discrete time system in $\mathbb{R}^d$

$$x_{n+1} = A_n x_n, n = 0, 1, \dots, \quad \text{where } A_n \in \text{GL}(\mathbb{R}^d) \tag{1.1}$$

one defines Lyapunov exponents in direction $v \in \mathbb{R}^d$ by

$$\lambda(v) = \limsup_{n \to \infty} \frac{1}{n} \ln \|A_n \cdots A_0 v\|. \tag{1.2}$$

There are at most d such Lyapunov exponents and under certain regularity assumptions one can replace $\limsup$ by $\lim$ (see [22], [28], [31]). Then Lyapunov exponents can be computed numerically in a stable way by the discrete QR-method, which uses QR decompositions of the matrices A_n see e.g. [12], [14], [15], [20], [16]. For the continuous time analog $\dot{x} = A(t)x$ there is an extensive literature on QR-like methods and their error analysis (see [7],[6], [13], [14],[15]).

This approach is normally applied to linear systems (1.1) that derive from linearizations $A_n = Dg(y_n)$ about a single trajectory $\{y_n\}_{n \in \mathbb{N}}$ of a nonlinear system

$$y_{n+1} = g(y_n), \quad n = 0, 1, \ldots. \tag{1.3}$$

In doing so one tacitly assumes that the trajectory $\{y_n\}_{n \in \mathbb{N}}$ is typical in the sense of some invariant ergodic measure and, therefore, it seems justified to consider a specific trajectory or just a few of them. Due to the statistical nature of trajectories one usually has to compute very long trajectories and it seems difficult to devise theory-based stopping criteria.

Methods of the second category use *spatial averages* and build on numerical approximations of some invariant ergodic measure. In fact, as the multiplicative ergodic theorem of Oseledec shows, the general definition of Lyapunov exponents for the nonlinear system in some domain of the phase space depends on the choice of invariant measure, see e.g. [4], [23], [31]. Methods of the second category were proposed by Froyland and co-authors [17],[18] and in a series of papers by Aston and Dellnitz[1], [2],[3]. In the latter approach the authors approximate the first Lyapunov exponent by the integral

$$a_n = \frac{1}{n} \int \ln \|Dg^n(x)\| \, d\mu(x) \tag{1.4}$$

where μ is an invariant ergodic measure and Dg^n denotes the derivative of the n-th iterate g^n. The measure μ is computed approximately by set valued numerical methods as developed by Dellnitz and co-authors (see [8],[11] and the software GAIO discussed therein). In the vector method of [3] the integrand in (1.4) is replaced by $\ln \|Dg^n(x)v\|$ with some vector v.

The hybrid method proposed in this paper combines ideas from both categories and approximates the j-th Lyapunov exponent by

$$a_n^j = \frac{1}{n} \int \ln \left(R_{jj}(Dg^n(x)) \right) d\mu(x), \qquad (1.5)$$

where $R_{jj}(A)$ denote the j-th diagonal element of the R-factor in the unique QR-decomposition of a nonsingular matrix A. Note that in case $j = 1$ this corresponds to taking v as the first Cartesian basis vector in the vector method from [3].

Following ideas from [1],[2] it is shown in [27] that, under certain regularity assumptions, the sequence a_n^j has an error expansion of the form

$$a_n^j = \lambda_j + \frac{C_j}{n} + o\left(\frac{e^{-\theta_j n}}{n} \right) \quad \text{for} \quad j = 1, \cdots, d. \qquad (1.6)$$

A proof of this result including a discussion of the underlying assumptions will be published in a companion paper.

In applications it turns out that the number of $n-$iterations, needed for a_n^j to converge, is by orders of magnitude smaller than those needed for (1.2). And it can be even further reduced by extrapolating on the basis of (1.6). Of course, the price to be paid lies in the approximation of the invariant measure and in the multitude of short trajectories to be computed for any box in the support of the approximate measure.

In section 2 we briefly review the discrete QR-method for trajectories as well as the spatial integration method from [1],[2]. Then our hybrid method is set up and supported by a convergence theorem. Moreover, some implementational details are discussed along with a first example. Finally, in sections 4 and 5 we demonstrate by a few examples the efficiency of extrapolation based on (1.6) and we discuss the balance of errors caused by varying the number of iterations n and the resolution of the invariant measure.

2 A review of known methods

We consider a discrete dynamical system on a d-dimensional smooth submanifold M of some $\mathbb{R}^k$ generated by a C^1-diffeomorphism

$$g : M \mapsto M \quad . \qquad (2.1)$$

For any $x \in M$ and v in the tangent space $T_x M$ the Lyapunov exponent in direction v is defined through

$$\lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \ln \|Dg^n(x)v\|, \qquad (2.2)$$

provided the limit exists.

In the following we will always assume that we have some ergodic probability measure μ on the Borel σ-algebra of M. Moreover, we assume that the functions $\max(0, \ln \|Dg(x)\|)$ and $\max(0, \ln \|Dg^{-1}(x)\|)$ are μ-integrable. Then by the theorem of Oseledec (see [22], [28]) there exists a Borel set $M_\mu \subset M$ of full measure, invariant under g such that for all $x \in M_\mu$ the limit in (2.2) exists and is independent of x. Moreover, there is a measurable decomposition $T_x M = \bigoplus_{i=1}^s W^i(x)$ for some $s \le d$ and there are numbers $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_s$ such that the following holds for $j = 1, \ldots, s$

$$\tilde{\lambda}_j = \lambda(x, v) \quad \text{for all } x \in M_\mu \text{ and } v \in \bigoplus_{i=j}^s W^i(x) \setminus \bigoplus_{i=j+1}^s W^i(x). \qquad (2.3)$$

Counting the $\tilde{\lambda}_j$ values according to their multiplicities we obtain the Lyapunov exponents $\lambda_1 \le \ldots \le \lambda_d$. .

2.1 Discrete QR-method

The common discrete QR method for Lyapunov exponents is based on semi-trajectories $\{g^n(x)\}_{n \in \mathbb{N}}$ (cf. [12], [20] or [26]). This method is particularly simple and allows approximate calculation of all Lyapunov exponents that belong to the linear system (1.1) with $A_n = Dg(g^n(x))$.

Let $A = Q(A)R(A)$ be the *unique* QR-decomposition of a nonsingular matrix $A \in \mathbb{R}^{d \times d}$, i.e.

- $Q(A) \in \mathbb{R}^{d \times d}$ is orthogonal (unitary);

- $R(A) \in \mathbb{R}^{d \times d}$ is an upper triangular matrix with positive diagonal entries.

This decomposition may be obtained through the (modified) Gram-Schmidt process, see [21]. From $\|Av\| = \|R(A)v\|$ for all $v \in \mathbb{R}^d$ we have

$$\lim_{n \to \infty} \frac{1}{n} \ln \|Dg^n(x)v\| = \lim_{n \to \infty} \frac{1}{n} \ln \|R\left(Dg^n(x)\right)v\|.$$

Thus it is sufficient to consider the time evolution of $R\left(Dg^n(x)\right)$

More information about the relation between the Lyapunov exponents of g and the R-factor of the linearization $R\left(Dg^n(x)\right)$ is provided under the Oseledec conditions by the following theorem (cf. [28], [22], [23]) .

Theorem 1. [1] *Let $\lambda_1, \cdots, \lambda_d$ denote the Lyapunov exponents of the system including multiplicity. For any $x \in M_\mu$ there exists a permutation π_x such that*

$$\lambda_{\pi_x(j)} = \lim_{n \to \infty} \frac{1}{n} \ln R_{jj}\left(Dg^n(x)\right) \tag{2.4}$$

for $j = 1, \cdots, d$.

For the practical computation one proceeds as follows:

Take any nonsingular matrix $Z_0 \in \mathbb{R}^{d \times d}$ (e.g. $Z_0 = I_d$) and define the sequence $\{Z_n\}_{n \in \mathbb{N}_0}$ via

$$Z_{n+1} := Dg(g^n(x))\, Q(Z_n)\,, \quad n \in \mathbb{N}_0\,,$$

where $Q(Z_n)$ is the unique Q-factor in the decomposition

$$Z_n = Q(Z_n)R(Z_n), \quad n \in \mathbb{N}_0.$$

From the decomposition $Dg^n(x) = Q\left(Dg^n(x)\right) R\left(Dg^n(x)\right)$ one obtains by induction (see [12], [20])

$$R\left(Dg^n(x)\right) = \prod_{i=n}^{1} R(Z_i) \quad \text{and} \quad Q\left(Dg^n(x)\right) = Q(Z_n)$$

for $n \in \mathbb{N}_0$, By Theorem 1 we then find

$$\lambda_{\pi_x(j)} = \lim_{n \to \infty} \frac{1}{n} \ln \prod_{i=n}^{1} R_{jj}(Z_i) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln R_{jj}(Z_i) \tag{2.5}$$

for some permutation π_x.

Remark. *If we are only interested in a few (usually the largest) Lyapunov exponents, we may take $Z_0 \in \mathbb{R}^{d,k}, k < d$ and use the reduced QR-decomposition (again with modified Gram Schmidt) instead. Then the overall effort will be reduced to $\mathcal{O}(k^2 d)$ in each step (see [21]).*

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[1]This theorem is part of Lyapunov's theorem [24] adapted to discrete dynamical systems.

2.2 Spatio-temporal method of Aston & Dellnitz.

In [1] and [2] Aston and Dellnitz proposed to compute the largest Lyapunov exponent λ_1 as the limit of a sequence of spatial integrals. The formula

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{n} \int \ln \|Dg^n(x)\| \, d\mu(x)$$

suggests to approximate λ_1 by the sequence $\{a_n\}_{n \in \mathbb{N}}$ given by

$$a_n = \frac{1}{n} \int \ln \|Dg^n(x)\| \, d\mu(x). \tag{2.6}$$

In order to evaluate these integrals one needs a good approximation of the underlying invariant ergodic measure so that quadrature errors, introduced by this approximation, are balanced by the value of n in (2.6), see section 4 for a more detailed discussion.

For the approximation of an invariant measure the algorithm in [2] uses the software GAIO[2] based on set valued numerical methods as documented in [8]. Note that there is also theoretical evidence that this algorithm selects an SRB-measure (if it exists) from the set of all invariant ergodic measures, see [10]. For our purpose it is sufficient to summarize the main ingredients as follows.

- Find a covering of the (relative) global attractor of the system in some initial box Q by the subdivision algorithm [9]. The initial box is successively subdivided into finer boxes, while discarding those which contain no parts of the attractor. Let $\{B_j\}_{j=1}^N$ denote the final covering of the attractor that satisfies $g(\bigcup_{j=1}^N B_j) \subset Q \cap \bigcup_{j=1}^N B_j$.

- Use the box covering to approximate the Perron-Frobenius operator by an $N \times N$ column-stochastic matrix P given by

$$P_{i,j} = \frac{m(B_j \cap g^{-1}(B_i))}{m(B_j)} \qquad i,j \in \{1, \cdots, N\},$$

where m denotes Lebesgue measure. Compute the Frobenius eigenvector $(\tilde{\mu}_j)_{j=1}^N \in \mathbb{R}^N, \tilde{\mu}_j \geq 0$ of P that belongs to the eigenvalue $+1$ and normalize it such that $\sum_{j=1}^N \tilde{\mu}_j = 1$. Then an invariant

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[2]see http://www-math.upb.de/$\sim$ agdellnitz/Software/gaio.html for detailed information concerning GAIO.

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measure μ supported on the attractor is approximated by the pre-measure $\tilde{\mu}$ defined on the ring generated by the sets $\{B_n\}_{n=1}^N$ through $\tilde{\mu}(B_j) = \tilde{\mu}_j, j = 1, \ldots, N$.

Note that, in the second step, one assumes $+1$ to be a simple eigenvalue of P. Further eigenvalues on the unit circle or near 1 may be used to specify the dynamics on the attractor in more detail, see [11], [8].

3 The hybrid method

The hybrid method proposed in this paper combines the discrete QR-method with spatial integration. It is based on the following result.

Theorem 2. *Let* $\lambda_1 \geq \ldots \geq \lambda_d$ *denote the Lyapunov exponents of the system (2.1) including multiplicities. Further assume that*

$$\lambda_j = \lim_{n \to \infty} \frac{1}{n} \ln R_{jj}(Dg^n(x)) \quad \mu \ a.e., \quad j = 1, \cdots, d, \tag{3.1}$$

then the following holds

$$\lambda_j = \lim_{n \to \infty} \frac{1}{n} \int \ln R_{jj}(Dg^n(x)) d\mu(x) \quad j = 1, \cdots, d. \tag{3.2}$$

Proof. It is sufficient to show that Lebesgue's dominated convergence theorem applies. For $j = 1, \ldots, d$ and $x \in M$ we have

$$\frac{1}{n} \ln R_{jj}(Dg^n(x)) \leq \frac{1}{n} \ln ||R(Dg^n(x))|| = \frac{1}{n} \ln ||Dg^n(x)||$$

$$\leq \frac{1}{n} \ln \prod_{\nu=0}^{n-1} ||Dg(g^\nu(x))|| \leq \frac{1}{n} \sum_{\nu=0}^{n-1} \max(0, \ln ||Dg(g^\nu(x))||),$$

where the integral of the right-hand side is

$$\frac{1}{n} \sum_{\nu=0}^{n-1} \int_M \max(0, \ln ||Dg(g^\nu(x)||) d\mu(x) = \int_M \max(0, \ln ||Dg(x)||) d\mu(x) < \infty.$$

Similarly one has

$$-\frac{1}{n} \ln R_{jj}(Dg^n(x)) \leq \frac{1}{n} \sum_{\nu=0}^{n-1} \max(0, \ln ||Dg(g^\nu(x))^{-1}||)$$

where the right-hand side has a uniformly bounded integral. $\square$

In [27] a sufficient condition is given that allows to conclude (3.1) from (2.4), i.e. the permutation of Liapunov exponents is trivial. One expects this condition to hold in a generic sense.

Theorem 2 suggests to use the sequence of integrals

$$a_n^j = \frac{1}{n} \int \ln \left(R_{jj}(Dg^n(x)) \right) \, d\mu(x) \tag{3.3}$$

for computing the Lyapunov exponent λ_j.

Remark. *Note that in case $j = 1$ we have $R_{11}(Dg^n(x)) = ||Dg^n(x)e^1||$ where $e^1 = (1, 0, \ldots, 0)^T$. Then the hybrid method coincides with taking $v = e^1$ in the vector method proposed in [3].*

More precisely, our algorithm for approximating the first ℓ Lyapunov exponents $\lambda_1, \cdots, \lambda_\ell$ proceeds as follows:

- As in section 2.2 compute via GAIO a box covering $\{B_i^k\}_{i=1}^{N_k}$ of the attractor and the Frobenius eigenvector $\tilde{\mu}^k \in \mathbb{R}^{N_k}$ of the transfer matrix P_k. Here k denotes the number of rga-steps needed by GAIO to compute the covering by N_k boxes ('rga' stands for 'relative global attractor', and rga-steps form a special case of the general subdivision algorithm, [9],[8]).

- Choose a representative $x_i^k \in B_i^k$ for $i = 1, \cdots, N_k$, for example the center of the box.

- For each point x_i^k, $i = 1, \cdots, N_k$, use the QR-method from section 2.1 to compute the values $R_{jj}(Dg^n(x_i^k))$ for $n = 1, \ldots, T$ and $j = 1, \cdots, \ell$, where T denotes the number of time steps.

- Compute approximation $\tilde{a}_n^j(k)$ of a_n^j after k rga-steps for $j = 1, \cdots, \ell$ according to

$$\tilde{a}_n^j(k) = \frac{1}{n} \sum_{i=1}^{N_k} \tilde{\mu}_i^k \ln R_{jj}(Dg^n(x_i^k)) \quad n = 1, \cdots, T. \tag{3.4}$$

Example 1. For a first illustration we compare the hybrid method with the norm method from [1],[2] for the Hénon system

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \quad g(x, y) = \left(1 - ax^2 + y \,, \, bx \right) \tag{3.5}$$

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with the classical values $a = 1.4, b = 0.3$. The values $\lambda_1 = 0.4191$ and $\lambda_2 = -1.6231$, computed by the discrete QR-method with an extremely high number of iterations ($n = 10^7$), were taken as 'exact' values for comparison. We compare the values $\tilde{a}_n^1(k)$ from (3.4) with $\tilde{a}_n(k)$ obtained from the norm method (cf. (2.6))

$$\tilde{a}_n(k) = \frac{1}{n} \sum_{j=1}^{N_k} \ln \left\| Dg^n(x_j^k) \right\| \tilde{\mu}_j^k, \quad \text{for} \quad n = 1, \cdots, T$$

Figure 1 shows both sequences for a fixed number of $k = 24$ rga-steps $\{\tilde{a}_n(k)\}$. Both sequences exhibit smooth convergence behavior after very few steps, with the hybrid method giving substantially smaller errors. Then Figure 2 shows that the same type of convergence behavior can be observed for the hybrid method in case of the second Lyapunov exponent.
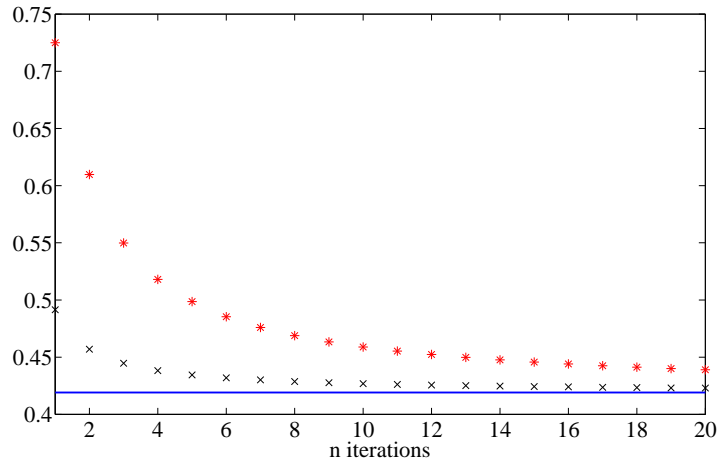

Figure 1: Hénon-map: comparison of approximations $\tilde{a}_n(24)$ (norm method, symbol $*$) and $\tilde{a}_n^1(24)$ (hybrid method, symbol $\times$) for the first Lyapunov exponent λ_1 (solid line).
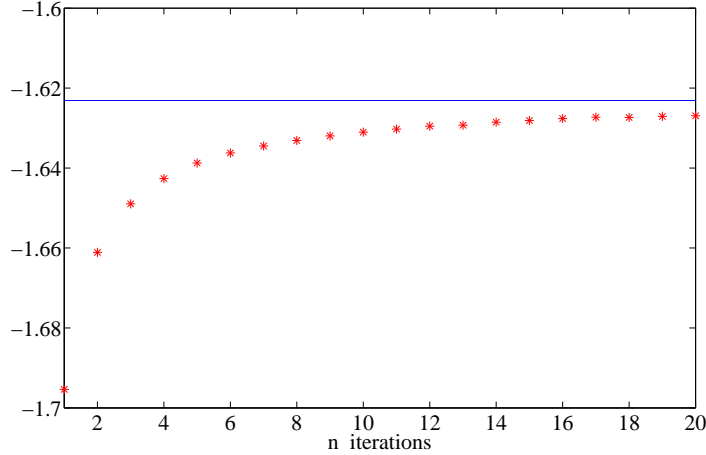
Figure 2: Hénon-map: approximation of the second Lyapunov exponent $\tilde{a}_n^2(24)$ (hybrid method).

4 Error expansion, extrapolation, and applications

4.1 Different types of error expansion

In [27] it is shown that the sequences $\{a_n^j\}_{n\in\mathbb{N}}$, $j = 1, \cdots, d$ satisfy an error expansion of the form

$$a_n^j = \lambda_j + \frac{C_j}{n} + o\left(\frac{1}{n}\right) \quad \text{for} \quad j = 1, \cdots, d. \tag{4.1}$$

Such a result holds if the Oseledec spaces (see (2.3)) are sufficiently separated and their products do not collapse in volume (see [27]). If, in addition, one requires hyperbolicity by assuming a gap between the Lyapunov exponents λ_j and λ_{j-1}, λ_{j+1}, then one can prove the following stronger result (cf. [27])

$$a_n^j = \lambda_j + \frac{C_j}{n} + o\left(\frac{e^{-\theta_j n}}{n}\right) \quad \text{for} \quad j = 1, \cdots, d \tag{4.2}$$

for some constants $C_j \in \mathbb{R}$ and $\theta_j > 0$ independent of n. Assuming that the Oseledec spaces in (2.3) are simple, i.e. $W^i(x) = \text{span}\{w_i(x)\}, \|w_i(x)\| = 1$, one finds the following expression (see [27])

$$C_j = \int_M \ln \left| \frac{P_{j+1}(x)G_j(x)}{P_j(x)G_{j-1}(x)} \right| d\mu(x). \tag{4.3}$$

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Here $P_j(x) = \det(W(j:d, j:d)(x))$ denotes the trailing principal minors of the Oseledec matrix $W(x) = \mathrm{col}(w_1(x), \ldots, w_d(x)) \in \mathbb{R}^{d \times d}$ and $G_j(x) = \sqrt{\det(W(1:d, 1:j)(x)^T W(1:d, 1:j)(x))}$ denotes the Gramian volume of the parallelepiped generated by the first j Oseledec vectors (cf. [19]).

Based on these estimates we investigate the following extrapolation rules proposed in [1], [2]

$$b_n^j = (n+1)a_{n+1}^j - na_n^j, \quad n = 1, 2, 3, \cdots \tag{4.4}$$

and

$$B_n^j = 2a_{2^n}^j - a_{2^{n-1}}^j, \quad n = 1, 2, 3, \cdots . \tag{4.5}$$

Note that the first expansion (4.1) implies

$$b_n^j = \lambda_j + o(1) \quad \text{for} \quad j = 1, \cdots, d,$$

which is no improvement over (4.1). However, expansion (4.2) leads to

$$b_n^j = \lambda_j + o(e^{-\theta_j n}) \quad j = 1, \ldots, d, \tag{4.6}$$

which is considerably better than (4.2). For the sequence $\{B_n^j\}_{n \in \mathbb{N}}$ from (4.5) we obtain the estimates

$$B_n^j = \lambda_j + \begin{cases} o(2^{-n}) & \text{if (4.1) holds} \\ o\left(2^{-n} e^{-\theta_j 2^n}\right) & \text{if (4.2) holds} \end{cases} \quad \text{for } j = 1, \ldots, d. \tag{4.7}$$

In both cases we have an improvement over $a_{2^n}^j = \lambda_j + O(2^{-n})$.

4.2   Numerical errors

It turns out that, in addition to the error caused by the choice of the index n, numerical errors involved in the approximation of a_n^j are crucial. Let us assume that after k rga-steps we have computed an approximation

$$\tilde{a}_n^j(k) = a_n^j + \varepsilon_n^j(k) \quad \text{for } n = 1, 2, 3, \cdots, \tag{4.8}$$

where the $\varepsilon_n^j(k)$ are due to approximation of the measure and the underlying attractor as well as to quadrature errors in (3.4) and the choice of representatives $x_i^k \in B_i^k, i = 1, \cdots, N_k$. For the extrapolated values

$$\tilde{b}_n^j(k) = (n+1)\tilde{a}_{n+1}^j(k) - n\tilde{a}_n^j(k), \quad n = 1, 2, 3, \cdots \tag{4.9}$$

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we find

$$\tilde{b}_n^j(k) = b_j^n + (n+1)\varepsilon_{n+1}^j(k) - n\varepsilon_n^j(k) \qquad j = 1, \cdots, d. \qquad (4.10)$$

This is potentially dangerous if the $\varepsilon_n^j(k)$ are uncorrelated (see the numerical experiments below).

Assuming the stronger expansion (4.2) yields the estimate

$$\left| \tilde{b}_n^j(k) - \lambda_j \right| \le (n+1) \left| \varepsilon_{n+1}^j(k) \right| + n \left| \varepsilon_n^j(k) \right| + Ce^{-\theta_j n}. \qquad (4.11)$$

Let us further assume that $\Delta_k = \sup \left\{ \varepsilon_n^j(k) \, | \, n \in \mathbb{N}, j \in \{1, \cdots, d\} \right\}$ and $\theta = \min\{\theta_1, \cdots, \theta_d\}$ are moderate constants. Then (4.11) implies

$$\left| \tilde{b}_n^j(k) - \lambda_j \right| \le (2n+1)\Delta_k + Ce^{-\theta n} =: f(n).$$

The right hand side becomes minimal at the integer closest to

$$n_{min} = -\frac{1}{\theta} \ln \left(\frac{2\Delta_k}{C\theta} \right), \qquad (4.12)$$

and the optimal error term is

$$f_{min} = f(n_{min}) = \frac{2\Delta_k}{\theta} \left[1 - \ln \left(\frac{2\Delta_k}{C\theta} \right) \right] + \Delta_k. \qquad (4.13)$$

Since C and θ are endogenous quantities, the only way to reduce the errors is via a better approximation of the measure. Note that n_{min} from (4.12) becomes positive only if Δ_k is sufficiently small. Even then, it may be reasonable to compute just the first few elements of the extrapolated sequence $\left\{ \tilde{b}_n^j(k) \right\}_{n \in \mathbb{N}}$, $j = 1, \cdots, d$ (see the example below). Similar results are obtained for the second type of extrapolation

$$\tilde{B}_n^j(k) = B_j^n + 2\varepsilon_{2^n}^j(k) - \varepsilon_{2^{n-1}}^j(k) \qquad j = 1, \cdots, d. \qquad (4.14)$$

Example 2. Hénon revisited

Figure 3 shows a plot of both approximations $\tilde{a}_n^1(k)$ and $\tilde{b}_n^1(k)$ in case of the Hénon map. Clearly, for k large extrapolation is superior to the simple approximation, but for large n it may destroy accuracy completely.

Figure 3: Hénon-map: values $\tilde{a}_n^1(k)$ from the hybrid method (left) and its extrapolated values $\tilde{b}_n^1(k)$ (right) for $k = 13, \ldots, 30$ rga-steps of GAIO
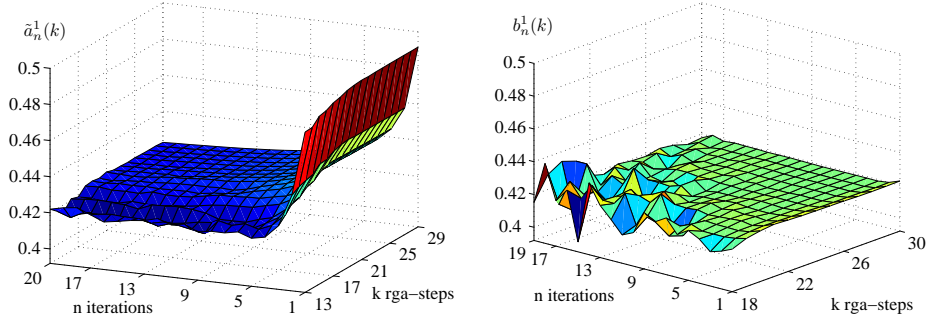
Remark. *In order to eliminate further errors from the measure approximation it is desirable to have a principle error term for the dependence on k, such as*

$$\varepsilon_n^j(k) \sim C_n(\gamma_n)^k \quad \text{for } n \in \mathbb{N} \text{ and } k = k_0, k_0 + 1, \cdots. \tag{4.15}$$

Formula (4.15) is motivated by two facts. First, it was shown in [9] that the Hausdorff distance of the relative global attractor A_Q and its box approximation $\bigcup_{i=1}^{N_k} B_i^k$ can be estimated after k rga-steps by

$$\text{dist}_H \left(A_Q, \bigcup_{i=1}^{N_k} B_i^k \right) \leq C\gamma^k,$$

where γ depends on the subdivision procedure and on the hyperbolicity constants of the attractor. Second, when integrals of smooth functions are replaced by Riemann sums over boxes of length h then the quadrature error behaves like $O(h)$. Since k rga-steps create box lengths of the form $h_k = \frac{h_1}{2^{k-1}}$ it is reasonable to assume that quadrature errors behave like $O(2^{-k})$.

However, numerical tests of relation (4.15) and of extrapolations based on (4.15) did not lead to convincing results. Therefore, extrapolation with increasing resolution of the measure remains as an open problem.

Example 3. The Lorenz system
We consider the time-1-map $g(\cdot) = \Phi(1, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$, where Φ denotes the

Table 1: Number N_k of boxes after k rga-steps.

k	13	15	17	19	21	23	25	27
N_k	432	1117	2698	6902	18237	44450	114156	303113

flow of the Lorenz system

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_1 x_3 - x_2 \qquad \text{where} \quad \sigma,\ \beta,\ \rho \in \mathbb{R}_+ . \qquad (4.16) \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \end{aligned}$$

We use the parameter values $\sigma = 16, \beta = 4, \rho = 40$ for which the Lyapunov exponents are

$$\lambda_1 = 1.368,\ \lambda_2 = 0.0,\ \lambda_3 = -22.368.$$

The first one is calculated with high accuracy by a QR-method, the second one follows from the flow property and the third one is obtained through $\lambda_3 = -(\sigma + 1 + \beta) - \lambda_1$ (see e.g. [16]). Note that the Lyapunov exponents of the flow $\lambda(x_0, v) = \lim_{n \to \infty} \frac{1}{t} \ln \|D_x \Phi(t, x_0) v\|$ coincide with those of the map if the limit exists. A fourth order Runge-Kutta method is used with step-size 0.02 for evaluating the map and its derivative via the variational equation.

The approximation of the global attractor (cf. [29], [30]) starts with the initial box $B_0 = [-35, 35] \times [-45, 45] \times [-10, 80]$. Then Table 1 shows the number of boxes used by GAIO for k rga-steps. The box covering and approximate measures are similar to [11], but note that our choice of parameters differs from [11] and that the map $\Phi(0.2, \cdot)$ is considered there which gives 5 times larger Lyapunov exponents.

Figure 4 displays the dependence of the sequences $\{\tilde{a}_n^j(k)\}_{n=1}^{10}$, $j = 1, 2, 3$ on the number k which increases with accuracy of the measure. We observe that the convergence behavior is smoother for the second and the third Lyapunov exponent while the first one needs more accuracy of the underlying measure. Figure 5 shows that extrapolation based on (4.4) is quite successful in all three cases. This may be taken as an indication that the expansion (4.2) is really valid, and so are perhaps the assumptions made in the proof.

Figure 4: The Lorenz system : Dependence of approximate values $\tilde{a}_n^j(k)$ for $\lambda_j, j = 1, 2, 3$ on numbers of iterations (n) and rga-steps (k).
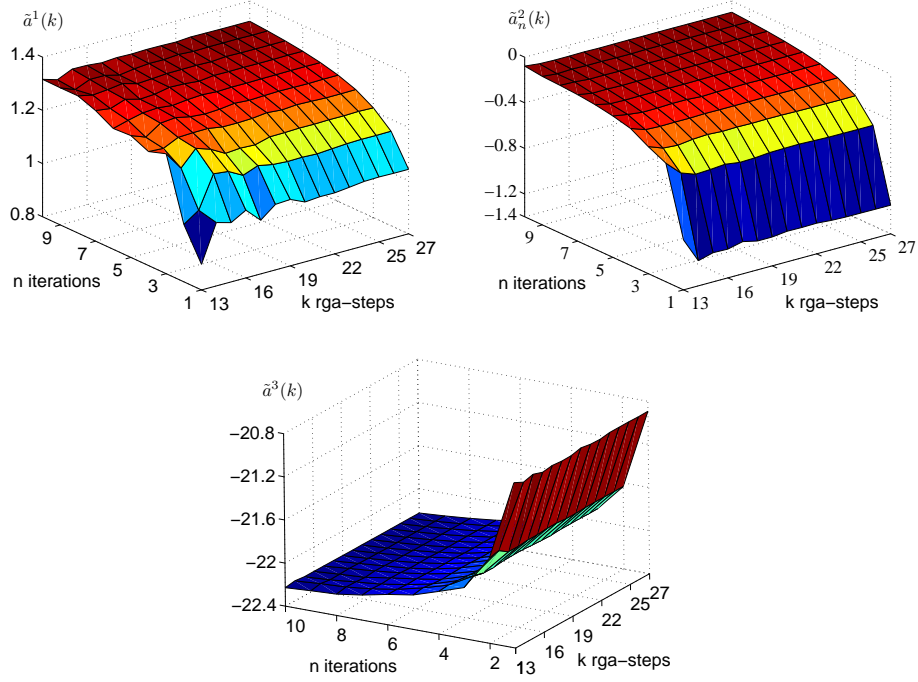
As an additional confirmation, we compute approximations of the principal error constants C_j from (4.2), (4.3) via

$$C_{j,n}(k) = \left| \tilde{a}_n^j(k) - \lambda_j \right| n, \quad j = 1, 2, 3. \tag{4.17}$$

For the case $k = 27$, Figure 6 shows the convergence of these approximation with growing n and it also indicates that one has $0 < C_1 < C_2 < C_3$.

Figure 5: The Lorenz system : Convergence of approximate values $\tilde{a}_n^j(25)$ (symbol $*$) and extrapolated values $\tilde{b}_n^j(25)$ (symbol $\circ$) for $j = 1, 2, 3$ after $k = 25$ rga steps.
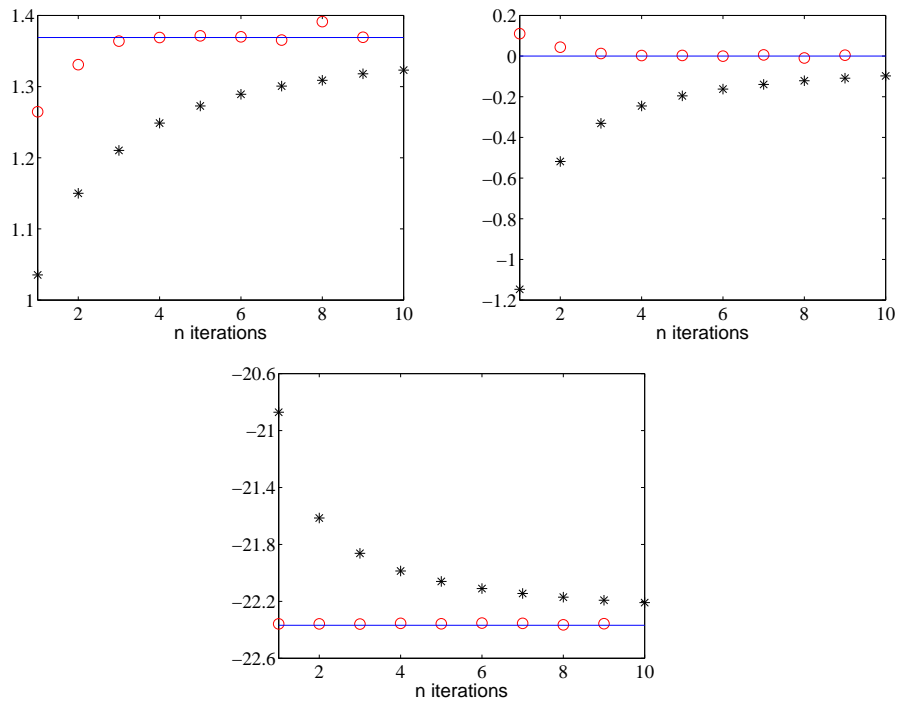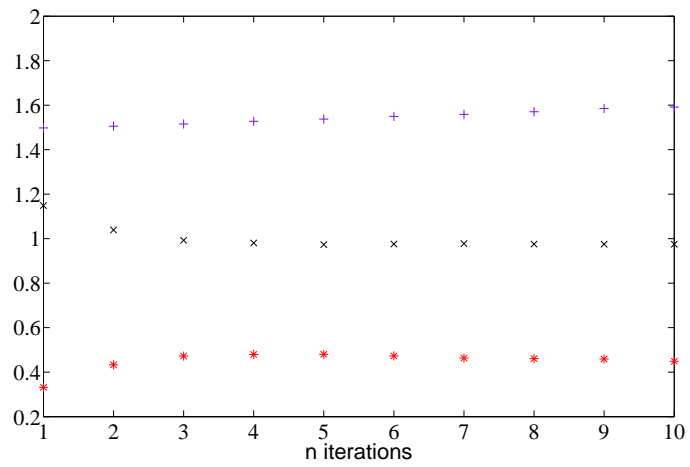
Figure 6: The Lorenz system : Estimates $C_{j,n}(27)$ of principal error constants C_j from (4.2).

5 Bekryaev's system

In [5] Bekryaev derived a model for atmospheric circulation and simplified it to the following 6-dimensional system. Some analytical and numerical studies of this system were undertaken by Lundstroem [25] (note that the printed version in [25] contains a wrong sign in the first bracket of the third line).

$$
\begin{aligned}
\dot{x}_1 &= -BU_T - Prx_1 + (2.2x_3 - Be)\, x_2, \\
&\quad + (U_U - x_3)\frac{x_5}{50} + \left(A + B + \frac{x_2}{50}\right)x_6, \\
\dot{x}_2 &= Bex_1 - Prx_2 - 2.2x_1x_3 + (x_3 - U_U)\frac{x_4}{50} + \left(C - \frac{x_1}{50}\right)x_6, \\
\dot{x}_3 &= -PPr\, x_3 - \left(\frac{A+B}{P} + \frac{x_2}{160}\right)x_4 + \left(\frac{x_1}{160} - \frac{C}{P}\right)x_5, \qquad (5.18) \\
\dot{x}_4 &= Q_F - U_Tx_2 - x_4 + (U_U - x_3)\, x_5 + x_2x_6, \\
\dot{x}_5 &= Q_Y + U_Tx_1 + (x_3 - U_U)\, x_4 - x_5 - x_1x_6, \\
\dot{x}_6 &= -x_2x_4 + x_1x_5 - Px_6.
\end{aligned}
$$

For the parameters $A, B, Be, C, P, Pr, Q_F, Q_Y, U_T, U_U$ we took the following values from [25]

$$
A = 2, \ B = 0, \ Be = -8.5242, \ C = 0, \ P = 3.2, \ Pr = 1,
$$
$$
Q_F = -2500, \quad Q_Y = 0, \quad U_T = 620.15, \quad U_U = 42.467.
$$

Note that extrapolation for the hybrid method works as efficiently as for the Lorenz system (see Figure 7) except that the principal error constants are somewhat larger and the convergence is slightly slower. Moreover, the extrapolated values (see Table 2) agree with those obtained by the classical $QR-$ algorithm (not shown). However, there is a noticeable difference for the $3. - 5.$ Lyapunov exponent when compared with the values given in [25] (in [25] Lyapunov exponents are calculated by the method from [7] applied to a single trajectory).

Table 2: The Bekryaev system: values of $\tilde{b}_n^j(30)$ for $n = 4, 5$ and $j = 1, \ldots, 6$ after 30 rga-steps.

	j=1	j=2	j=3	j=4	j=5	j=6
$n = 4$	2.9632	0.0198	-1.3481	-2.4455	-3.7273	-6.8990
$n = 5$	3.0049	0.0113	-1.3467	-2.4468	-3.7270	-6.9405

Figure 7: The Bekryaev system: Convergence of approximate values $\tilde{a}_n^j(30)$ (symbol $*$) and extrapolated values $\tilde{b}_n^j(30)$ (symbol$\circ$) for $j = 1, 2, 3, 4, 5, 6$ after $k = 30$ rga steps.
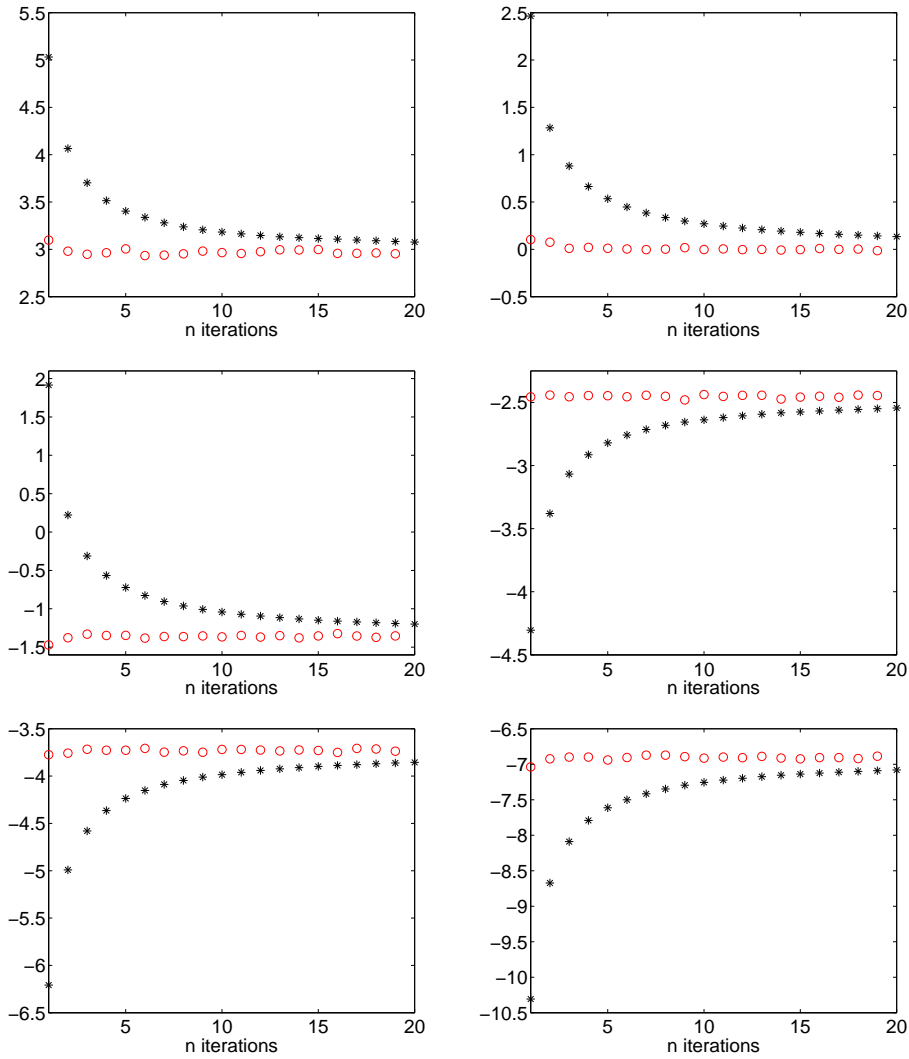
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