Phase conditions, symmetries, and PDE continuation

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Abstract

Integral phase conditions were first suggested by E.J. Doedel as an efficient tool for computing periodic orbits in dynamical systems. In general, phase conditions help in eliminating continuous symmetries as well as in reducing the effort for adaptive meshes during continuation. In this paper we discuss the usefulness of phase conditions for the numerical analysis of finite- and infinite-dimensional dynamical systems that have continuous symmetries. The general approach (called the *freezing method*) will be presented in an abstract framework for evolution equations that are equivariant with respect to the action of a (not necessarily compact) Lie group. We show particular applications of phase conditions to periodic, heteroclinic and homoclinic orbits in ODEs, to relative equilibria and relative periodic orbits in PDEs as well as to time integration of equivariant PDEs.

1 Introduction

There is a long tradition in using continuous symmetries for the analysis of differential equations; see for example the monographs [10, 21, 33]. In general, such symmetries are expressed as the equivariance of the differential operator with respect to the action of a Lie group. Solutions of the differential equation then come in group orbits and this has interesting consequences, for example inherent symmetries of solutions or symmetry breaking bifurcations. In the theory of equivariant systems one usually tries to reduce the differential equation to the so called orbit space the elements of which are equivalence classes created by applying the group action to a single point in phase space. After factoring out the group action in this way one applies specific results on existence and uniqueness of solutions, on bifurcations, or on asymptotic stability.

Contrary to the situation in theory the use of continuous equivariances for efficient numerical computations seems to be rather rare. An early exception is E.J. Doedel's integral phase condition [13] for the computation of periodic orbits in autonomous ODEs.

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It is a typical example to show that a judicious use of symmetry (in this case equivariance with respect to time shifts) can enhance rather than hamper efficiency of a numerical method. Namely, less effort for mesh adaptation is needed and larger continuation steps are possible.

The ODE example also shows another paradigm of numerical bifurcation analysis. While theory prefers to *reduce* problems, e.g by Lyapunov Schmidt or center manifold reduction, it seems advantageous to rather *extend* the problem for numerical purposes (e.g. by choosing unfolding parameters) and adding extra constraints (e.g. normalizing conditions for eigenvectors). In this way one can keep as much structure as possible from the original problem and simultaneously use the normalizing conditions to optimize the conditioning of the extended problem.

For equivariant evolution equations this means that one should have an extra parameter (here an element in the associated Lie algebra) that determines the position on the group orbit, and impose further constraints or phase conditions such that the point in phase space (e.g. the spatial profile in case of a PDE) varies as little as possible. Such an approach was developed in [34] and [7] and will be the main topic of this paper.

After reviewing in Section 2 phase conditions that eliminate the time shift in ODEs, we set up in Section 3 the general freezing method within an abstract framework. We then apply our method to the computation of various spatio-temporal patterns such as traveling and modulated waves in one, spiral waves in two and scroll waves in three space dimensions. For problems in one space dimension we will also investigate asymptotic stability and discuss the errors introduced by finite boundary conditions.

2 Phase conditions for orbits in ODEs

Consider a dynamical system generated by an autonomous $n-$dimensional ordinary differential equation

$$u_t = f(u), \quad u(t) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n \text{ smooth.} \tag{1}$$

Due to its autonomous character the nonlinear differential operator

$$\mathcal{L}u = u_t - f(u),$$

has a simple equivariance with respect to time shifts. I.e. for all $\gamma \in \mathbb{R}$ and for all u in some function space we have

$$[\mathcal{L}u](\cdot - \gamma) = \mathcal{L}[u(\cdot - \gamma)]. \tag{2}$$

Depending on the application appropriate function spaces may be chosen as the Sobolev space $\mathcal{H}^1(\mathbb{R}, \mathbb{R}^n)$, the space of bounded uniformly continuous C^1-functions $C^1_{\text{unif}}(\mathbb{R}, \mathbb{R}^n)$ or the space of one-periodic functions $C^1_{\text{per}}(\mathbb{R}, \mathbb{R}^n)$.

2.1 Periodic orbits

In order to determine a periodic orbit of the system (1) we should find a period $T > 0$ and a solution $u(t)$ of the boundary value problem

$$u_t = f(u),\ t \in [0,T],\ u(0) = u(T).$$

Introducing the scaled function $v(t) = u(tT),\ t \in [0,1]$ the boundary value problem for $(v,T) \in C^1([0,1],\mathbb{R}^n) \times \mathbb{R}$ now reads

$$v_t = Tf(v),\ t \in [0,1], \quad v(0) = v(1). \tag{3}$$

Due to equivariance (2) the solutions of (3) are only determined up to a phase shift and a further condition is needed to make the solution unique. In the first publication on the continuation package AUTO [13] Doedel suggested to use an *anchor equation* (as it was called in [13]) that tries to minimize the $\mathcal{L}_2$-distance to some template function $\hat{v} \in C^1_{\text{per}}(\mathbb{R},\mathbb{R}^n)$, i.e. tries to minimize

$$\rho(v,\gamma) = \int_0^1 ||v(t-\gamma) - \hat{v}(t)||_2^2\, dt = ||v(\cdot - \gamma) - \hat{v}||^2_{\mathcal{L}_2}. \tag{4}$$

By differentiating with respect to γ a necessary condition for a local minimum is

$$\int_0^1 (v(t-\gamma) - \hat{v}(t))^T \hat{v}_t(t)\, dt = 0. \tag{5}$$

A more formal statement is contained in the following lemma; see [4] for a proof.

Lemma 2.1. *Suppose that $\hat{v} \in C^1_{\text{per}}(\mathbb{R},\mathbb{R}^n)$ is a nonconstant $1-$periodic function. Then there exist neighborhoods U of $\hat{v}$ in the C^1-topology and $\Gamma \subset \mathbb{R}$ of 0 such that for any $v \in U$ the $\mathcal{L}_2$-distance from (4) has a unique minimum at $\gamma = \gamma(v) \in \Gamma$ where $\gamma : U \to V$ is a C^1-mapping satisfying $\gamma(\hat{v}) = 0$ and condition (5).*

During computations one selects v such that the phase condition (5) holds at $\gamma = 0$, i.e.

$$\int_0^1 (v(t) - \hat{v}(t))^T \hat{v}_t(t) dt = 0. \tag{6}$$

This condition has several advantages over a Poincaré type condition such as

$$(v(0) - \hat{v}(0))^T f(\hat{v}(0)) = 0. \tag{7}$$

If $\hat{v}$ is a good approximation of v obtained from continuation along a branch, then condition (6) tries to keep a steep front or a peak of the solution in the same place. Usually this facilitates mesh adaptation and simultaneously allows for larger step-sizes along branches. This phase condition is now built into standard continuation packages such as AUTO (with HOMCONT) [14], CONTENT [25] and MATCONT [12]. It has proved to be most reliable in many applications.

For an illustration we take the model example from [13], namely

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} (1-\lambda)u_1 - u_2 \\ u_1 + u_1^2 \end{pmatrix}.$$

This system is in fact Hamiltonian at $\lambda = 1$ and a continuous family of cycles bifurcating from the origin and ending in a homoclinic orbit exists. For a detailed treatment of the Hamiltonian case we refer to [32]. Figure 1 shows the result of continuing the periodic orbits with both phase conditions (6) and (7).

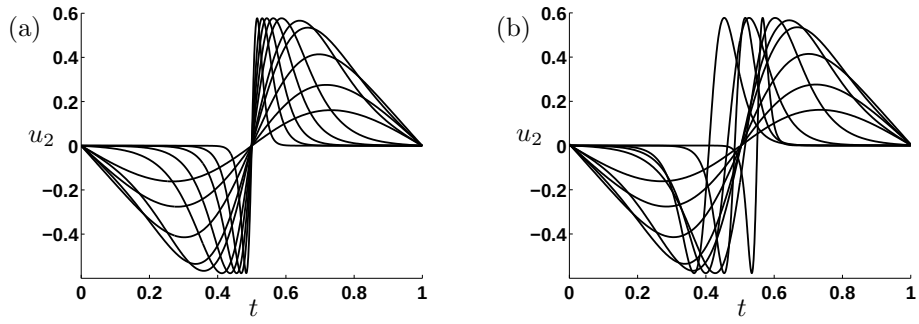

Figure 1: Effect of an integral phase condition (a) and of a point phase condition (b).

We finally mention that conditions (6) and (7) are special cases of the general form $\psi(v) = 0$, where $\psi : C^1([0,1], \mathbb{R}^n) \to \mathbb{R}$ is a C^1 mapping. Following [4, 17], one can characterize the admissible phase conditions that lead to a regular solution $(v, T) \in C^1([0,1], \mathbb{R}^n) \times \mathbb{R}$ of the operator equation

$$F(v, T) = (v_t - Tf(v), v(0) - v(1), \psi(v)) = 0 \tag{8}$$

as follows. Let $u(t)$ be a T-periodic solution of (1) such that $v(t) = u(tT)$ satisfies $\psi(v) = 0$. Then the pair (v, T) is a regular solution of (8), if and only if, 1 is a simple Floquet multiplier and $D\psi(v)v_t \neq 0$ where $D\psi$ denotes the Frechet derivative of ψ. An easy calculation shows, that the last condition requires $\langle v_t, \hat{v}_t \rangle_{\mathcal{L}_2} \neq 0$ for (6) and $v_t(0)^T f(\hat{v}(0)) \neq 0$ for (7). One may call (8) a defining equation for an isolated periodic orbit. The recent paper [16] provides a considerable extension of this general approach to defining equations for all codimension 1 bifurcations of periodic orbits, namely: fold (saddle-node), flip (period-doubling) and Neimark-Sacker bifurcations; see also [26].

2.2 Homoclinic and heteroclinic orbits

It is natural to extend the numerical methods for periodic orbits to orbits that connect stationary points in infinite time. Such orbits typically occur in

parametrized systems

$$u_t = f(u,\lambda), \quad u(t) \in \mathbb{R}^n, \quad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \text{ smooth.} \tag{9}$$

Connecting orbits have numerous applications, in particular they appear as traveling waves of PDEs; see Section 3.1.

Definition 2.2. A pair $(\bar{u}, \bar{\lambda}) \in C^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}^p$ is called a connecting orbit pair of the system (9) if $\bar{u}$ is a solution of (9) at $\lambda = \bar{\lambda}$ and if the following limits exist

$$\lim_{t\to\infty} \bar{u}(t) = \bar{u}_+, \quad \lim_{t\to-\infty} \bar{u}(t) = \bar{u}_-. \tag{10}$$

The connecting orbit is called homoclinic if $\bar{u}_+ = \bar{u}_-$ and heteroclinic otherwise.

From (10) we infer that $\bar{u}_\pm$ are stationary points, i.e. $f(\bar{u}_\pm, \bar{\lambda}) = 0$ and that $\bar{u} \in C^1_b(\mathbb{R}, \mathbb{R}^n)$, i.e. C^1 and bounded on $\mathbb{R}$. Nondegeneracy of a connecting orbit may be defined as follows; cf. [3].

Definition 2.3. A connecting orbit pair $(\bar{u}, \bar{\lambda})$ is called nondegenerate if the following conditions hold:

(i) The matrices $f_u(\bar{u}_\pm, \bar{\lambda}) \in \mathbb{R}^{n,n}$ are hyperbolic with $n_{\pm s}$ eigenvalues of negative real part and $n_{\pm u} = n - n_{\pm s}$ eigenvalues of positive real part.

(ii) $p = n_{-s} - n_{+s} + 1$

(iii) If $u \in C^1_b(\mathbb{R}, \mathbb{R}^n), \lambda \in \mathbb{R}^p$ satisfies the variational equation $u_t = f_u(\bar{u}, \bar{\lambda})u + f_\lambda(\bar{u}, \bar{\lambda})\lambda$, then $\lambda = 0$ and $u = c\bar{u}_t$ for some $c \in \mathbb{R}$.

Conditions (i) and (ii) ensure that the dimension $n_{-u} + p$ of the center-unstable manifold of $(\bar{u}_-, \bar{\lambda})$ in the extended phase space $\mathbb{R}^n \times \mathbb{R}^p$ and the dimension $n_{+s} + p$ of the center-stable manifold of $(\bar{u}_+, \bar{\lambda})$ add up to $n + p + 1$ which is one more than the dimension of the extended phase space $\mathbb{R}^n \times \mathbb{R}^p$. Condition (iii) then guarantees that these two manifolds intersect transversely in the connecting orbit $\{(\bar{u}(t), \bar{\lambda}) : t \in \mathbb{R}\}$. Similar to the periodic case one can characterize connecting orbit pairs as regular solutions of an operator equation

$$F(u,\lambda) = (u_t - f(u,\lambda), \psi(u,\lambda)) = 0, \tag{11}$$

where the phase condition is defined by a smooth map
$\psi : C^1_b(\mathbb{R}, \mathbb{R}^{n,n}) \times \mathbb{R}^p \to \mathbb{R}$; see [3] for a proof.

Proposition 2.4. *Let $(\bar{u}, \bar{\lambda})$ be a connecting orbit pair satisfying $\psi(\bar{u}, \bar{\lambda}) = 0$ and conditions (i) and (ii) of Definition 2.3. Then $(\bar{u}, \bar{\lambda})$ is a regular solution of (11) if and only if the orbit pair is nondegenerate and $\psi_u(\bar{u}, \bar{\lambda})\bar{u}_t \neq 0$.*

The analog of the functional (4) to be minimized is

$$\rho(u,\gamma) = \int_{-\infty}^{\infty} ||u(t-\gamma) - \hat{u}(t)||_2^2 \, dt = ||u(\cdot - \gamma) - \hat{u}||_{\mathcal{L}_2}^2 = ||u - \hat{u}(\cdot + \gamma)||_{\mathcal{L}_2}^2,$$

where we take $u \in \hat{u} + \mathcal{H}^1$ and assume that $\hat{u} \in C_b^2(\mathbb{R}, \mathbb{R}^n)$ is a template function that satisfies $\hat{u}_t \in \mathcal{H}^1$. Then the phase condition is again obtained from the necessary condition of a minimum $\langle u - \hat{u}, \hat{u}_t \rangle_{\mathcal{L}_2} = 0$. In applications it may be unrealistic to assume that such a template function is known because it essentially requires to know $\bar{u}_\pm$ beforehand and to choose $\hat{u}$ such that $||\hat{u}(t) - \bar{u}_\pm|| = \mathcal{O}(e^{-\alpha|t|})$. In general, $\bar{u}_\pm$ will depend on λ and be determined by $f(\bar{u}_\pm, \lambda) = 0$. Therefore, Doedel and Friedman [15, 20] suggested to minimize $||u_t - \hat{u}_t(\cdot - \gamma)||_{\mathcal{L}_2}$, which leads to the phase condition

$$\langle u_t - \hat{u}_t, \hat{u}_{tt} \rangle_{\mathcal{L}_2} = 0. \tag{12}$$

There are several ways to solve the boundary value problem (11) on the infinite line. One may discretize using globally defined Galerkin functions or transform the domain $\mathbb{R}$ to a bounded interval and then devise methods that handle the resulting singularities; see [27, 30, 31]. Perhaps the simplest method that allows to employ existing boundary value solvers is to approximate (11) by a finite boundary value problem on some large interval $J = [T_-, T_+]$. This approach was proposed and analyzed in [] and implemented in the HomCont part of Auto [14].

For $u \in C^1(J, \mathbb{R}^n), \lambda \in \mathbb{R}^p$ we consider the finite boundary value problem

$$F_J(u, \lambda) = (u_t - f(u, \lambda), B(u(T_-), u(T_+)), \lambda), \psi_J(u, \lambda)) = 0, \tag{13}$$

where the smooth maps $B : \mathbb{R}^{2n+p} \to \mathbb{R}^{n+p-1}$,$(u_-, u_+, \lambda) \mapsto B(u_-, u_+, \lambda)$ and $\psi_J : C^1(J, \mathbb{R}^n) \times \mathbb{R}^p \to \mathbb{R}$ determine the boundary condition and the approximate phase condition respectively. The error introduced by this approximation can be estimated as follows (see [3],[20],[41]).

Theorem 2.5. *Let $(\bar{u}, \bar{\lambda})$ be a nondegenerate connecting orbit pair of (9) such that*

(i) $B(\bar{u}_-, \bar{u}_+, \bar{\lambda}) = 0$ and the matrix

$$\Big(\tfrac{\partial B}{\partial u_-}(\bar{u}_-, \bar{u}_+, \bar{\lambda}) X_{-s} \quad \tfrac{\partial B}{\partial u_+}(\bar{u}_-, \bar{u}_+, \bar{\lambda}) X_{+u} \Big) \in \mathbb{R}^{n+p-1, n+p-1}$$

is nonsingular, where the columns of $X_{-s} \in \mathbb{R}^{n, n_{-s}}$ and $X_{+u} \in \mathbb{R}^{n, n_{+u}}$ form a basis of the stable subspace of $f_u(\bar{u}_-, \bar{\lambda})$ and of the unstable subspace of $f_u(\bar{u}_+, \bar{\lambda})$ respectively;

(ii) $\psi(\bar{u}, \bar{\lambda}) = 0$, $\psi_J(\bar{u}_{|J}, \bar{\lambda}) \to 0$ as $J \to \mathbb{R}$, the derivatives $D\psi_J$ are equicontinuous in a uniform neighborhood of $(\bar{u}_{|J}, \bar{\lambda})$ and $|D\psi_J(\bar{u}_{|J}, \bar{\lambda})\bar{u}'_{|J}| \geq \delta > 0$ for some $\delta > 0$.

Then there exist constants $\rho, K > 0$ and an interval $J_0 \subset \mathbb{R}$ with the following properties. For all $J_0 \subset J$ the boundary value problem (13) has a unique solution (u_J, λ_J) in a C^1-ball of radius ρ and center $(\bar{u}_{|J}, \bar{\lambda})$. Furthermore, there is a unique phase shift γ_J near zero such that $\tilde{u} = \bar{u}(\cdot - \gamma_J)$ satisfies $\psi_J(\tilde{u}_{|J}, \lambda_J) = 0$ and the following estimate holds

$$||\tilde{u}_{|J} - u_J||_{C^1} + ||\bar{\lambda} - \lambda_J|| \leq C||B(\tilde{u}(T_-), \tilde{u}(T_+), \bar{\lambda})||. \tag{14}$$

In view of (12) and (13) it is natural to take the phase conditions

$$\psi_J(u,\lambda) = \langle u - \hat{u}, \hat{u}_t \rangle_{\mathcal{L}_2(J)} \quad \text{or} \quad \psi_J(u,\lambda) = \langle u_t - \hat{u}_t, \hat{u}_{tt} \rangle_{\mathcal{L}_2(J)}.$$

The most natural choice for boundary conditions are so called projection boundary conditions that force the endpoints $u(T_-), u(T_+)$ to lie in the tangent space of the unstable manifold at $\bar{u}_-$ and the stable manifold at $\bar{u}_+$. These conditions may be written as

$$B(u_-, u_+, \lambda) = \begin{pmatrix} Y_{-s}^T(\lambda)(u_- - u_-(\lambda)) \\ Y_{+u}^T(\lambda)(u_+ - u_+(\lambda)) \end{pmatrix}, \tag{15}$$

where $f(u_\pm(\lambda), \lambda) = 0$ and the columns of $Y_{-s}(\lambda) \in \mathbb{R}^{n,n_{-s}}$ and $Y_{+u}(\lambda) \in \mathbb{R}^{n,n_{+u}}$ form a basis of the stable subspace of $f_u^T(u_-(\lambda), \lambda)$ and of the unstable subspace of $f_u^T(u_+(\lambda), \lambda)$ respectively. Note that, by Definition 2.3, equation (15) imposes $n_{-s} + n_{+u} = n + p - 1$ boundary conditions. Methods to compute these matrices such that they depend smoothly on the parameter λ were proposed in [3] and, more recently, via a smooth block Schur decomposition in [11]. For numerous computations that apply this approach to specific examples we refer to [3, 14, 15, 20].

We finally notice that projection boundary conditions imply exponential decay of the term on the right hand side of (14). More precisely, we have

$$||\tilde{u}_{|J} - u_J||_{C^1} + ||\bar{\lambda} - \lambda_J|| = \mathcal{O}(e^{2\alpha_- T_-} + e^{-2\alpha_+ T_+}),$$

where $0 < \alpha_- < \Re(\mu)$ for all eigenvalues μ of $f_u(\bar{u}_-, \lambda)$ with positive real part and $\Re(\mu) < -\alpha_+ < 0$ for all eigenvalues eigenvalues of $f_u(\bar{u}_+, \bar{\lambda})$ with negative real part. For the parameter a superconvergence behavior was observed in [3] and a corresponding estimate proved in [35], namely:

$$||\bar{\lambda} - \lambda_J|| = \mathcal{O}(\mathrm{e}^{(2\alpha_- + \alpha_+)T_-} + e^{-(2\alpha_+ + \alpha_-)T_+}).$$

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3 Phase conditions and equivariant PDEs

In this section we consider time-dependent PDEs that have continuous symmetries in the spatial operator. Therefore we will be concerned with phase conditions that act on the spatial variables of the solutions. First, we introduce the method of freezing that employs phase conditions in order to decompose a time dependent solution into a time-dependent group orbit and a spatial profile that varies as little as possible. Second, this method will be used to compute relative equilibria, i.e. spatial profiles the group orbits of which are invariant under the PDE flow. The underlying general approach was developed independently in [34] and in [7].

3.1 Traveling waves

A special class of relative equilibria in one space dimension are traveling wave solutions $u(x,t) = \bar{v}(x - \bar{\lambda}t)$ of parabolic PDEs

$$u_t = Au_{xx} + f(u), \quad u(\cdot,0) = u^0, \qquad x \in \mathbb{R},\ u(x,t) \in \mathbb{R}^m, \tag{16}$$

where $A \in \mathbb{R}^{m,m}$ is positive definite, $\bar{v}$ denotes the profile of the wave and $\bar{\lambda} \in \mathbb{R}$ its velocity.

These solutions are stationary in the moving coordinate system which is obtained via the transformation $v(\xi,t) = u(x,t)$, $\xi = x - \bar{\lambda}t$, i.e. $\bar{v}$ and $\bar{\lambda}$ solve

$$0 = A\bar{v}_{xx} + f(\bar{v}) + \bar{\lambda}\bar{v}_x. \tag{17}$$

Given a stationary solution $\bar{v}$, each shifted version $\bar{v}_\gamma = \bar{v}(\cdot - \gamma)$ is also a solution of (17). As in Section 2 we add a phase condition defined by some functional ψ in order to obtain a well posed boundary value problem for (v,λ), namely:

$$\begin{aligned} 0 &= Av_{xx} + f(v) + \lambda v_x, \\ 0 &= \psi(v, v_x, \lambda). \end{aligned}$$

The natural choice for ψ stems from the phase condition discussed in Section 2. One minimizes the $\mathcal{H}^1$-distance or the $\mathcal{L}_2$-distance to a template function $\hat{v}$. This leads to the functional $\psi(v) = \langle \hat{v}_x, v - \hat{v}\rangle_{\mathcal{H}^1}$ or

$$\psi(v) = \langle \hat{v}_x, v - \hat{v}\rangle_{\mathcal{L}_2}. \tag{18}$$

Transforming to a first order system we can apply the results from Section 2 for studying well-posedness (Proposition 2.4) and approximation (Theorem 2.5).

In our next step we are going to use phase conditions also for the non stationary case. Now we let the transformation into the moving frame depend on time in the following way

$$u(x,t) = v(x - \gamma(t), t), \tag{19}$$

where $\gamma(0) = 0$ and we define $\lambda(t) = \dot{\gamma}(t)$. With this setting equation (16) together with the phase condition transforms into a partial differential algebraic equation (PDAE) for (v,λ), namely:

$$\begin{aligned} v_t &= Av_{xx} + f(v) + \lambda v_x, \quad v(\cdot,0) = u^0, \\ 0 &= \psi(v, v_x, \lambda). \end{aligned} \tag{20}$$

Note that the initial value $\lambda(0)$ is not prescribed but, as usual with DAEs, is determined by differentiating the constraint $\psi = 0$ with respect to time and using the differential equation. In Section 3.3 we will discuss possible choices for the phase condition that lead to PDAEs of different index. System (20) can be completed by the simple ODE $\dot{\gamma} = \lambda(t)$, $\gamma(0) = 0$ (called the reconstruction equation in [34]). The traveling wave $(\bar{v}, \bar{\lambda})$ now appears as a stationary solution of the system (20) and, in case of stability, we expect the solution of (20) to converge to $(\bar{v}, \bar{\lambda})$ during time evolution; see Section 4.

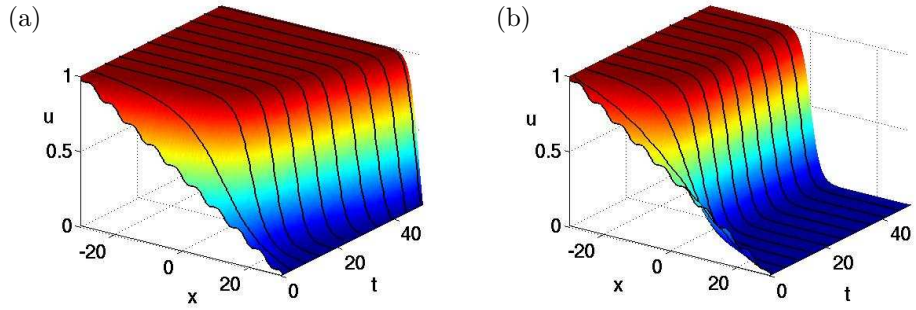

Figure 2: Calculation of a wave in the Nagumo equation (21): traveling wave (a) and frozen wave (b).

Example 3.1. The standard toy example of a traveling wave is a heteroclinic orbit between two metastable states in the Nagumo equation [23]

$$u_t = u_{xx} + u(1-u)(u-a), \quad u(x,t) \in \mathbb{R},\ x \in \mathbb{R},\ t > 0, \tag{21}$$

where $a \in (0, \frac{1}{2})$. An explicit traveling wave connecting the stationary points $u_- = 1$, $u_+ = 0$ is

$$\bar{v}(x) = 1 - \left(1 + \mathrm{e}^{\frac{-x}{\sqrt{2}}}\right)^{-1}, \quad \bar{\lambda} = \sqrt{2}\ (\tfrac{1}{2} - a). \tag{22}$$

In Figure 2 we show the results of a numerical computation for $a = 0.25$ with finite differences in space ($\Delta x = 0.1$) and the implicit Euler method in time ($\Delta t = 0.1$). Panel (a) is for the the non-frozen system (21) and panel (b) for the frozen system (20). In both cases the spatial interval is $J = [-30, 30]$ and we use Dirichlet boundary conditions. Similar to Section 2, the frozen system has the advantage that steep gradients stay in approximately the same place and the front does not leave the computational domain in finite time.

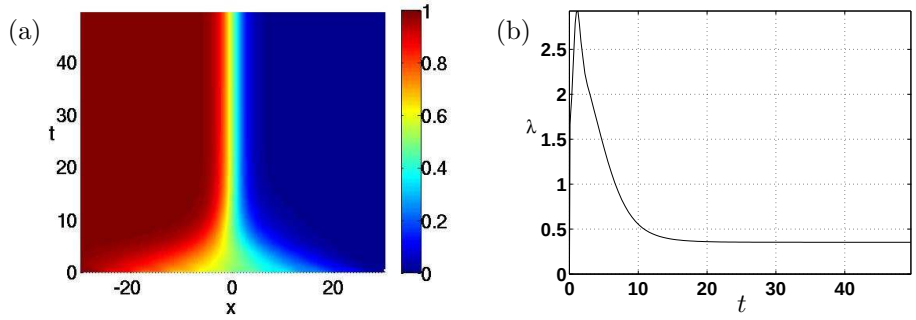

Figure 3: Calculation of a wave in the Nagumo equation (21): (t, x)-plot of the frozen wave (a), and time evolution of λ (b).

In Figure 3 we show another representation of the frozen wave as a color-coded (t, x)-plot in panel (a), while panel (b) is the time evolution of the velocity λ. This kind of plot will be used throughout the paper.

Example 3.2. We consider an autocatalytic system [1, 28] as given by

$$\begin{aligned} u_t &= a u_{xx} - u f(v), \quad a > 0,\ u, v : \mathbb{R} \to \mathbb{R} \\ v_t &= v_{xx} + u f(v), \end{aligned} \tag{23}$$

where $f(v) = v^m$ for $v \geq 0$ and zero otherwise. This system has traveling wave solutions if the parameter $m \geq 2$ is not too large. As in [1, 28] we choose limit values $(u_-, v_-) = (0, 1)$, $(u_+, v_+) = (1, 0)$ in order to eliminate a scaling invariance.

Figure 4(a) and (b) show the solution of the original and of the frozen system, respectively, in an interval of length 100 for the original system and of length 30 for the frozen system. Here $a = 0.1$, $m = 2$, and we used the Crank-Nicholson method ($\Delta x = 0.1$, $\Delta t = 0.1$) and Dirichlet boundary conditions. Figure 4(c) and (d) show the u- and v-components of the frozen system and the time evolution of μ. Again, the example shows how the method of freezing allows to observe phenomena that become visible only after a transient phase, while in a direct numerical simulation the solution may leave the finite domain before the steady profile appears.

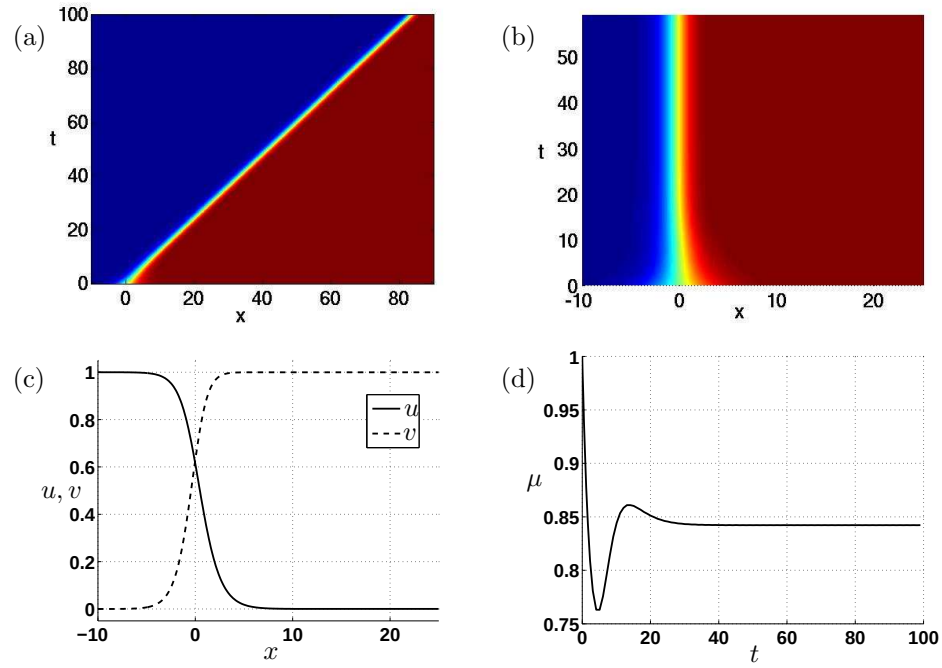

Figure 4: Calculation of a wave in autocatalytic system (23): traveling wave (a), frozen wave (b), u- and v-component at $t = 100$ (c), and time evolution of μ (d).

3.2 Freezing solutions of equivariant PDEs

Let M be a manifold modelled over some Banach space X and let N be a submanifold modelled over some dense subspace $Y \subset X$ [9]. Consider an evolution equation

$$u_t = F(u), \quad u(0) = u^0, \tag{24}$$

with a vector field $F : N \to TM$ where TM denotes the tangent bundle of M. We assume that (24) is equivariant with respect to a finite dimensional (possibly non-compact) Lie group G acting on M via

$$a : G \times M \to M, \ (\gamma, v) \mapsto a(\gamma, v),$$

with the property

$$a(\gamma_1 \circ \gamma_2, v) = a(\gamma_1, a(\gamma_2, v)), \quad a(\mathbb{1}, v) = v, \quad \mathbb{1} = \text{unit element in } G.$$

By equivariance we mean that the following relation holds

$$\begin{aligned} a(\gamma, N) &\subset N \quad \forall \gamma \in G, \\ F(a(\gamma, u)) &= Ta(\gamma, u)F(u), \quad \forall u \in N, \gamma \in G, \end{aligned}$$

where $Ta : G \times TM \to TM$ denotes the tangent action of a. We assume that the linear map

$$Ta(\gamma, v) : \ T_vM \to T_{a(\gamma,v)}M, \quad w \mapsto Ta(\gamma, v)w$$

is a homeomorphism for each $v \in M$ (Note that $a(g, \cdot)$ corresponds to $\Phi_g : N \to N$ in [29, 34] and $Ta(g, \cdot)$ corresponds to $\Psi_g : TM \to TM$) . Furthermore, we assume that for any $v \in M$ the map

$$a(\cdot, v) : G \to M, \quad \gamma \mapsto a(\gamma, v)$$

is continuous and that it is continuously differentiable for any $v \in N$ with derivative denoted by

$$da(\gamma, v) : T_\gamma G \to T_{a(\gamma)v}M, \quad \lambda \mapsto da(\gamma, v)\lambda.$$

For the construction of some spaces that satisfy this smoothness requirement we refer to [7]. Finally, we denote by $L_\gamma : G \to G, \ g \mapsto \gamma \circ g$ the multiplication by $\gamma \in G$ from the left and by $dL_\gamma(g) : T_gG \to T_{\gamma \circ g}G$ its derivative. Then we define the exponential $\exp(t\mu)$ for μ in the Lie algebra $T_{\mathbb{1}}G$ as the solution of

$$\dot{\gamma} = dL_\gamma(\mathbb{1})\mu.$$

The evolution of $\gamma(t)$ describes the motion on the group. Other equivalent definitions of exp are in common use [9, 10, 29].

Generalizing the ansatz (19) to $u(t) = a(\gamma(t), v(t))$ equation (24) can be transformed into a system for the unknowns $v(t) \in M$, $\gamma(t) \in G$, $\mu(t) \in T_{\mathbb{1}}G$ as follows (cf. [7, 34]).

$$v_t = F(v) - da(\mathbb{1}, v)\mu, \qquad v(0) = u^0 \tag{25a}$$

$$\dot{\gamma} = dL_\gamma(\mathbb{1})\mu, \qquad \gamma(0) = \mathbb{1}. \tag{25b}$$

Lemma 3.3. *For some $T > 0$ let $u \in C^1((0,T],M) \cap C([0,T],N)$ be a solution of (24) and let $\gamma \in C^1([0,T],G)$ be arbitrary with $\gamma(0) = \mathbb{1}$. Then $v(t)$ defined by $u(t) = a(\gamma(t), v(t))$ and $\mu(t)$ defined by (25b) are solutions of equation (25a). Conversely, assume that $v \in C^1((0,T],M) \cap C([0,T],N)$ and $\mu \in C^1([0,T],T_{\mathbb{1}}G)$ solve equation (25a) and define $\gamma \in C^1([0,T],G)$ as the solution of (25b). Then $u(t) = a(\gamma(t), v(t))$ solves equation (24) on $[0,T]$.*

Proof. Insert the ansatz $u(t) = a(\gamma(t), v(t))$ into (24) and use equivariance to obtain

$$da(\gamma, v)\dot{\gamma} + Ta(\gamma, v)v_t = u_t = F(u) = F(a(\gamma, v)) = Ta(\gamma, v)F(v). \tag{26}$$

Differentiating the relation $a(\gamma \circ g, v) = a(\gamma, a(g, v))$, $g, \gamma \in G$, $v \in N$ with respect to g at $g = \mathbb{1}$ leads to

$$Ta(\gamma, v)da(\mathbb{1}, v)\mu = da(\gamma, v)dL_\gamma(\mathbb{1})\mu, \quad \forall \mu \in T_{\mathbb{1}}G. \tag{27}$$

Finally, define $\mu(t)$ by $\dot{\gamma}(t) = dL_\gamma(\mathbb{1})\mu$ and combine (26), (27) to find

$$Ta(\gamma, v)\left[v_t - F(v) + da(\mathbb{1}, v)\mu\right] = 0,$$

and, hence, (25a) by the invertibility of $Ta(\gamma, v)$. The converse is proved in a similar way. □

Lemma 3.3 shows that system (25) does not have a unique solution (v, μ, γ). Rather we have $p = \dim G$ additional degrees of freedom that will be fixed by a phase condition $\psi : N \times T_{\mathbb{1}}G \to \mathbb{R}^p$. The phase condition together with (25a) yields a PDAE for v and μ, namely:

$$\begin{aligned} v_t &= F(v) - da(\mathbb{1}, v)\mu, \\ 0 &= \psi(v, \mu). \end{aligned} \tag{28}$$

Equation (25b) is called the *reconstruction equation* in [34]. It is decoupled from system (28) and can be solved by an a-posteriori process.

The traveling waves in Examples 3.1 and 3.2 easily fit into the abstract framework.

Example 3.4. For the Lie group $G = \mathbb{R}$ consider the shift action $a(\gamma, u)(x) = u(x - \gamma)$. There are different possiblities for the choice of spaces M and N. Either take $M = C_{\text{unif}}$, $N = C^2_{\text{unif}}$ or $M = w + \mathcal{L}_2 \supset N = w + \mathcal{H}^2$ where $w \in C^2_b(\mathbb{R}, \mathbb{R}^2)$ satisfies $w_x, w_{xx} \in \mathcal{L}_2$ and has the correct limit behavior, e.g. $w(x) = \bar{u}_\pm + \mathcal{O}(e^{-\alpha|x|})$ as $x \to \pm\infty$. For the last choice we actually use the manifold structure of M and N. In both cases we have $da(\mathbb{1}, v)\mu = -\mu v_x$ and using a template function $\hat{v} \in N$ the system (28) is given by (20) with ψ given in (18).

Example 3.5. Consider a system (16) of dimension $m = 2$ such that the nonlinearity is equivariant with respect to rotations, i.e.

$$f(R_\rho v) = R_\rho f(v)\ \forall v \in \mathbb{R}^2,\ \rho \in \mathbb{R}, \text{ where } R_\rho = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix}.$$

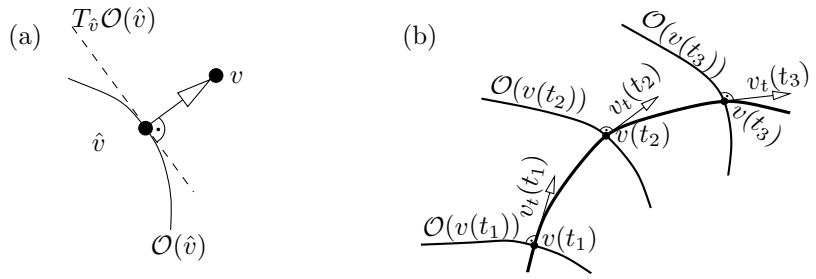

Figure 5: Conditions ψ_{fix} (a) and $\psi_{\min}$ (b).

Equations of this type arise as real versions of complex-valued systems for example the Ginzburg-Landau equation. The Lie group is $G = S^1 \times \mathbb{R}$ and the action $a : G \times \mathcal{L}_2 \to \mathcal{L}_2$ on $u : \mathbb{R} \to \mathbb{R}^2$ at $\gamma = (\rho, \tau)$ is given by

$$a(\gamma, u)(x) = R_{-\rho} u(x - \tau).$$

With $M = \mathcal{L}_2, N = \mathcal{H}^2$ we obtain $da(\mathbb{1}, v)(\mu_\tau, \mu_\rho) = -v_x \mu_\tau - R_{\frac{\pi}{2}} v \mu_\rho$ and (28) has the form

$$\begin{aligned} v_t &= A v_{xx} + \mu_\tau v_x + \mu_\rho R_{\frac{\pi}{2}} v + f(v), \\ 0 &= \langle \hat{v}', v - \hat{v} \rangle_{\mathcal{L}_2}, \quad 0 = \langle R_{\frac{\pi}{2}} v, v - \hat{v} \rangle_{\mathcal{L}_2}. \end{aligned}$$

The reconstruction equations read $\dot{\tau} = \mu_\tau,\ \tau(0) = 0$ and $\dot{\rho} = \mu_\rho,\ \rho(0) = 0$.

3.3 Fixed versus minimizing phase conditions

In the abstract setting of Section 3.2 assume that M is a Banach space in which we have a continuous inner product $\langle \cdot, \cdot \rangle_2$ with associated norm $||v||_2$. One way to set up a phase condition (in the spirit of Section 2) is to minimize the distance of the frozen solution v from the group orbit $\mathcal{O}(\hat{v}) = \{a(\gamma, \hat{v}) : \gamma \in G\}$ of a template function $\hat{v}$; see Figure 5(a). The necessary condition for a minimum of $\|a(\gamma, \hat{v}) - v\|_2$ to occur at $\gamma = \mathbb{1}$ is

$$\psi_{\text{fix}}(v)\mu = \langle da(\mathbb{1}, \hat{v})\mu,\, v - \hat{v} \rangle_2 = 0 \quad \forall \mu \in T_{\mathbb{1}} G. \tag{29}$$

In the beginning one may choose as template the initial value $\hat{v} = u^0$. Note that ψ in (29) maps into the dual $T^*_{\mathbb{1}} G$ of the Lie algebra which is isomorphic to $\mathbb{R}^p$.

Another possibility is to minimize the temporal change $\|v_t\|_2$ at each time instance which leads to the condition

$$\psi_{\min}(v)\mu = \langle da(\mathbb{1}, v)\mu,\, v_t \rangle_2 = 0 \quad \forall \mu \in T_{\mathbb{1}} G. \tag{30}$$

As illustrated in Figure 5(b) this condition requires the frozen trajectory $v(t)$ to be orthogonal to the group orbit of $v(t)$ at all times.

For the case of traveling waves we show how to transform solutions of the PDAE for both phase conditions into each other, i.e., we transform solutions

(v, λ) of

$$\begin{aligned} v_t &= v_{xx} + f(v) + \lambda v_x \\ 0 &= \langle \bar{v}_x, v - \bar{v} \rangle_{\mathcal{L}_2} \end{aligned} \tag{31}$$

into solutions (w, μ) of

$$\begin{aligned} w_t &= w_{xx} + f(w) + \lambda w_x \\ 0 &= \langle w_x, w_t \rangle_{\mathcal{L}_2}. \end{aligned} \tag{32}$$

The following Lemma will be used for the stability analysis in Section 4.3.

Lemma 3.6. *Let (v, λ) be a solution of (31). Then (w, μ) defined by*

$$w(x,t) = v(x - \eta(t), t), \quad \mu = \lambda - \dot{\eta},$$

and

$$\eta(t) = \int_0^t \frac{\langle v_x(\cdot,\tau), v_t(\cdot,\tau) \rangle_{\mathcal{L}_2}}{\|v_x(\cdot,\tau)\|^2_{\mathcal{L}_2}} \, d\tau$$

is a solution of the system (32).

Proof. We have

$$w_t = v_t - v_x \dot{\eta} = v_{xx} + f(v) + (\lambda - \dot{\eta}) v_x = w_{xx} + f(w) + \mu w_x.$$

With the shift invariance of $\langle \cdot, \cdot \rangle_{\mathcal{L}_2}$ and the definition of η we get

$$\langle w_t, w_x \rangle_{\mathcal{L}_2} = \langle v_t - \dot{\eta} v_x, v_x \rangle_{\mathcal{L}_2} = 0.$$

□

4 Relative equilibria and stability

In this section we study relative equilibria of equivariant evolution equations. In particular, we consider relative equilibria of parabolic systems in one space dimension. We use phase conditions to approximate relative equilibria on finite intervals and study their asymptotic stability in the Lyapunov sense via the freezing method.

4.1 Relative Equilibria

We seek for solutions of (24) which have the special form $u(t) = a(\gamma(t), \bar{v})$ for some time independent function $\bar{v}$.

Definition 4.1. A solution $\bar{u}$ of (24) is called a relative equilibrium if it has the form $\bar{u}(t) = a(\bar{\gamma}(t), \bar{v})$ for some $\bar{v} \in N$ and for some function $\bar{\gamma} \in C^1([0, \infty), G)$.

Without loss of generality we can assume $\bar{\gamma}(0) = \mathbb{1}$. Usually the whole group orbit $\mathcal{O}(\bar{v}) = \{a(\gamma, \bar{v}),\ \gamma \in G\}$ is called a relative equilibrium, if it is invariant under the semi-flow; see [10, 29]. We found the equivalent constructive definition above more convenient from a numerical point of view [7], because it includes explicitly the orbit $\bar{\gamma}(t)$ on the group. The following lemma shows the connection between $\bar{u}$, $\bar{\gamma}$ and $\bar{v}$.

Lemma 4.2. *Let $\bar{u}(t) = a(\bar{\gamma}(t), \bar{v})$ be a relative equilibrium with trivial stabilizer $\mathcal{S}_{\bar{v}} = \{\gamma \in G : \ a(\gamma, \bar{v}) = \bar{v}\}$. Then there exists $\bar{\mu} \in T_{\mathbb{1}}G$ such that $(\bar{v}, \bar{\mu})$ solve*

$$0 = F(\bar{v}) - da(\mathbb{1}, \bar{v})\bar{\mu} \tag{33}$$

and $\dot{\bar{\gamma}} = dL_{\bar{\gamma}}(\mathbb{1})\bar{\mu}$, $\gamma(0) = \mathbb{1}$.

Conversely, if (33) holds for $(\bar{v}, \bar{\mu})$ then $\bar{u}(t) = a(\bar{\gamma}(t), \bar{v})$ with $\bar{\gamma} = \exp(t\bar{\mu})$ is a relative equilibrium of (24).

Proof. The orbit $\mathcal{O}(\bar{v})$ has tangent space $T_{\bar{v}}\mathcal{O}(\bar{v}) = \text{range}(da(\mathbb{1}, \bar{v}))$ and it is well known [10, Lemma 4.10.4] that $\dim T_{\bar{v}}\mathcal{O}(\bar{v}) = \dim G - \dim \mathcal{S}_{\bar{v}}$. Hence the stabilizer is trivial, if and only if, $da(\mathbb{1}, \bar{v})$ is one-to-one. By Lemma 3.3 we find that $\bar{\mu}(t) = dL_{\gamma}(\mathbb{1})^{-1}\dot{\bar{\gamma}} \in T_{\mathbb{1}}G$ is continuous and satisfies (33). Since $\bar{v}$ is independent of t and $da(\mathbb{1}, \bar{v})$ is one-to-one, we obtain that $\bar{\mu}$ is independent of t as well. □

Remark 4.3. If $\bar{v}$ has nontrivial stabilizer then one can still write $\bar{\gamma}$ as an exponential in terms of the Lie algebra of the stabilizer and its normalizer; see [10, Theorem 7.2.4].

Choosing a basis $\{e^1, \dots, e^p\}$ in $T_{\mathbb{1}}G$ we can identify the Lie algebra with $\mathbb{R}^p$ via $\mu = \sum_{i=1}^p \mu_i e^i$. Further, setting $S^i(v) = -da(\mathbb{1}, v)e^i$ we find from (28) and Lemma 4.2 the equation to be solved for $(\bar{v}, \bar{\mu})$, namely:

$$\begin{aligned} 0 &= F(v) + S(v)\mu, \quad \text{where } S(v)\mu = \sum_{i=1}^{p} \mu_i S^i(v) \\ 0 &= \psi(v, \mu). \end{aligned}$$

4.2 Approximation of relative equilibria on finite intervals

We now treat the special case when the evolution equation (24) is a parabolic system of the form (16). We assume that the operators S^i are differential operators $S^i(v)(x) = S_0^i v(x) + S_1^i v_x(x)$ for suitable matrices $S_0^i, S_1^i \in \mathbb{R}^{m,m}$.

For the numerical computation of relative equilibria for (16) we solve a boundary value problem on a finite interval $J = [x_-, x_+]$, namely:

$$0 = Av_{xx} + S(v)\mu + f(v), \quad x \in [x_-, x_+], \tag{34a}$$

$$\eta = \mathcal{B}v, \tag{34b}$$

$$0 = \langle S^i(\hat{v})_{|_J}, v - \hat{v}_{|_J} \rangle_J, \quad i = 1, \dots, p. \tag{34c}$$

Here $\hat{v}$ is a template function and $\mathcal{B}$ is the two-point boundary operator

$$\mathcal{B}v = P_- v(x_-) + Q_- v_x(x_-) + P_+ v(x_+) + Q_+ v_x(x_+), \quad P_\pm, Q_\pm \in \mathbb{R}^{2m,m}.$$

The linearization of (34a) with respect to v at $(\bar{v}, \bar{\mu})$ is given by

$$\Lambda u = Au_{xx} + Bu_x + Cu, \quad B = \sum_{i=1}^{p} \bar{\mu}_i S_1^i, \ C(x) = f'(\bar{v}(x)) + \sum_{i=1}^{p} \bar{\mu}_i S_0^i. \tag{35}$$

If $\lim_{x\to\pm\infty}\bar{v}(x)=v_\pm$ and $\lim_{x\to\pm\infty}\bar{v}_x(x)=0$ then Λ turns for $x\to\pm\infty$ into the constant coefficient operator

$$\Lambda_\pm v = Av_{xx}+Bv_x+C_\pm v, \qquad C_\pm=\lim_{x\to\pm\infty}C(x).$$

The main spectral assumptions on Λ are the following:

Hypothesis 4.4 (**spectral condition).** The eigenvalue 0 lies in the connected component of $\mathbb{C}\setminus\{\Sigma_+\cup\Sigma_-\}$ that contains a right half plane, where

$$\Sigma_\pm=\{s\in\mathbb{C}:\det(-\kappa^2A+i\kappa B+C_\pm-sI)=0,\ \text{for some }\kappa\in\mathbb{R}\}.$$

Hypothesis 4.5 (**eigenvalue condition).** The functions $S^i(\bar{v})=-da(\mathbb{1},\bar{v})e^i$, $i=1,\dots,p$ lie in $\mathcal{H}^2$, are linearly independent and span the null space of $\Lambda:\mathcal{H}^2\to\mathcal{L}_2$ i.e. $\ker(\Lambda)=\operatorname{span}\{S^1(\bar{v}),\dots,S^p(\bar{v})\}$. Moreover, the algebraic and the geometric multiplicity of zero are both equal to p.

Hypothesis 4.4 guarantees that the quadratic eigenvalue problem associated with $\Lambda_\pm$ has m stable and m unstable eigenvalues; cf. [5]. In view of condition (i) of Theorem 2.5 we consider the following determinant (cf. [5])

$$\mathcal{D}=\det\left(\begin{pmatrix}P_- & Q_-\end{pmatrix}\begin{pmatrix}Y_-^s\\ Y_-^s\Sigma_-^s\end{pmatrix}\quad\begin{pmatrix}P_+ & Q_+\end{pmatrix}\begin{pmatrix}Y_+^u\\ Y_+^u\Sigma_+^u\end{pmatrix}\right), \tag{36}$$

where (Σ_-^s,Y_-^s), $(\Sigma_+^u,Y_+^u)\in\mathbb{R}^{m,m}\times\mathbb{R}^{m,m}$ solve the quadratic eigenvalue problems

$$AY\Sigma^2+BY\Sigma+C_\pm Y=0$$

with $\Re\sigma(\Sigma_-^s)<0$ and $\Re\sigma(\Sigma_+^u)>0$. Then we can formulate the determinant condition and a consistency assumption for the boundary conditions.

Hypothesis 4.6 (**boundary conditions).** The boundary condition (34b) is satisfied at the stationary points $\bar{v}_\pm$, i.e. $\eta=P_-\bar{v}_-+P_+\bar{v}_+$ and the determinant $\mathcal{D}$ defined in (36) is non-zero.

As in Section 2 the boundary conditions have to control the terms that grow in forward time on the positive axis and in backward time on the negative axis. These are given by the stable/unstable manifolds of the stationary points. Note that the determinant condition is satisfied for Dirichlet, Neumann and periodic boundary conditions; cf. [5].

For simplicity we first formulate the theorem for pulses, i.e. we use $M=\mathcal{L}_2$, $N=\mathcal{H}^2$. In order to generalize this to fronts one needs the additional condition $a(\gamma,\hat{v})-\bar{v}\in\mathcal{H}^2$.

Hypothesis 4.7 (**phase condition).** The phase condition is satisfied by $\bar{v}$, i.e. $\langle S(\hat{v}),\bar{v}-\hat{v}\rangle_{\mathcal{L}_2}=0$, $\bar{v}-\hat{v}\in\mathcal{H}^1$, $S(\hat{v})\in\mathcal{L}_2$ and the matrix

$$\langle S(\hat{v}),S(\bar{v})\rangle_{\mathcal{L}_2}=\left(\int_{\mathbb{R}}[S^i(\hat{v})](x)^T[S^j(\bar{v})](x)dx\right)_{i,j=1}^p\in\mathbb{R}^{p,p}$$

is non-singular.

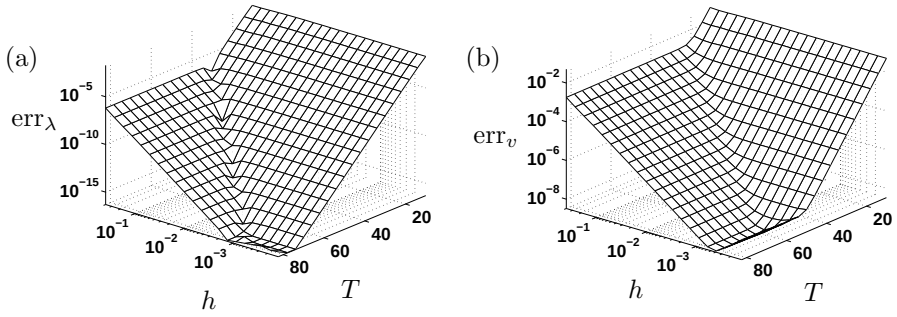

Figure 6: Approximation error $\mathrm{err}_\lambda = |\lambda_h - \bar{\lambda}|$ (a) and $\mathrm{err}_v = \|v_h - \bar{v}_{|J_h}\|_{\mathcal{H}_h^2}$ (b).

The following approximation result is an adaptation of Theorem 2.5 to the current situation; see [40] for a proof.

Theorem 4.8 (**Approximation of relative equilibria on finite intervals).** *Assume Hypotheses 4.4–4.7 hold. Then there exist $\varrho > 0$, $T > 0$, such that for $\min\{-x_-, x_+\} > T$ the boundary value problem* (34) *has a unique solution (v_J, μ_J) in a ball $B_\varrho(\bar{v}_{|J}, \bar{\mu}) = \{(v, \mu) \in \mathcal{H}^2(J, \mathbb{R}^m) \times \mathbb{R}^p : \|\bar{v}_{|J} - v\|_{\mathcal{H}^2} + \|\bar{\mu} - \mu\| < \varrho\}$. Further, there exist group elements $\gamma_J \in G$ such that $\tilde{v} = a(\gamma_J, \bar{v})$ satisfies the following estimate for some $\alpha > 0$*

$$\|v_J - \tilde{v}_{|J}\|_{\mathcal{H}^2} + \|\mu_J - \bar{\mu}\| \leq \text{const } \mathrm{e}^{-\alpha \min\{-x_-, x_+\}}.$$

A similar version for a full discretization with finite differences can be found in [43]. In that case one obtains an error estimate on the grid $J_h = \{hn,\ n_- \leq n \leq n_+\}$ for the approximate solution (v_h, μ_h), namely:

$$\|v_h - \bar{v}_{|J_h}\|_{\mathcal{H}_h^2} + \|\mu_h - \bar{\mu}\| \leq \text{const } (h^2 + \mathrm{e}^{-\alpha h \min\{-n_-, n_+\}}), \tag{37}$$

where $\|\cdot\|_{\mathcal{H}_h^2}$ is the discrete analog of the $\mathcal{H}^2$ norm. A similar result holds for the norm $\|\cdot\|_\infty$.

Example 4.9. In case of the Nagumo equation from Example 3.1 we can compare the approximation with the exact solution. Figure 6 shows the approximation error of the traveling wave for Dirichlet boundary conditions. The grid size h was varied logarithmically from 10^{-4} to 10^{-1} and the size of the symmetric interval $[-T, T]$ linearly from 20 to 80. We observe that the convergence of $\tilde{v}$ and $\tilde{\lambda}$ to the exact solution $\bar{v}$ and $\bar{\lambda}$ given in (22) is exponential in T and quadratic in h. This is in good agreement with the approximation result (37).

4.3 Stability of relative equilibria in one space dimension

Stability results for traveling waves on the real line, or more generally, relative equilibria are well known for parabolic systems [22], [45]. Here the notion of asymptotic stability with asymptotic phase is used. By the freezing method

this notion is converted into the usual asymptotic (Lyapunov-) stability. In the following we present a stability result for relative equilibria in the frozen setting. For stability the spectral assumptions 4.4 and 4.5 have to be tightened as follows.

Hypothesis 4.10. The curves $\Sigma_+ \cup \Sigma_-$ lie in the open left half plane and zero is the only eigenvalue with real part greater equal zero.

Theorem 4.11 (**Stability of relative equilibria).** *Assume Hypotheses 4.4, 4.5, 4.10 and 4.7 hold. Then there exist $\varepsilon, \nu > 0$ such that for all $u_0 \in \hat{v} + \mathcal{H}^1(\mathbb{R})$ with $||u_0 - \bar{v}||_{\mathcal{H}^1} \leq \varepsilon$ the system*

$$\begin{aligned} v_t &= Av_{xx} + f(v) + S(v)\mu, \quad v(\cdot, 0) = u_0 \\ 0 &= \langle S(\hat{v}), v - \hat{v} \rangle_{\mathcal{L}_2} \end{aligned}$$

has a unique solution $v \in C^1((0,\infty), \hat{v} + \mathcal{H}^1(\mathbb{R})) \cap C([0,\infty), \hat{v} + \mathcal{H}^1(\mathbb{R}))$ and $\mu \in C([0,\infty), \mathbb{R}^p)$. Moreover, this solution satisfies

$$||v(\cdot, t) - \bar{v}||_{\mathcal{H}^1} + \|\mu(t) - \bar{\mu}\| \leq \text{const } \mathrm{e}^{-\nu t} \|v^0 - \bar{v}\|_{\mathcal{H}^1} \quad \forall t \geq 0.$$

Remark 4.12. For the case of traveling waves a proof of this theorem can be found in [42]. The generalization to arbitrary groups is straightforward by the techniques used for Theorem 4.8. It is also shown in [42] that one can allow more general nonlinearities $f(v, v_x)$ of the form

$$f(u, v) = f_1(u)v + f_2(u), \quad f_1 \in C^1(\mathbb{R}^m, \mathbb{R}^{m,m}), f_2 \in C^1(\mathbb{R}^m, \mathbb{R}^m),$$

where f_1, f_2, f_1', f_2' are globally Lipschitz. This includes the case of the nonlinearity uu_x in Burgers equation.

An analogous result for a spatial discretization with finite differences is given in [42] for traveling waves and in [44] for general relative equilibria in one space dimension.

Remark 4.13. We note that a general stability theorem for finite dimensional equivariant systems is given in [10, Th. 7.4.2].

For the special case of stationary solutions of (31) the local stability estimate reads

$$\|v(\cdot, t) - \bar{v}\|_{\mathcal{H}^1} + |\lambda(t) - \bar{\lambda}| \leq \text{const } \mathrm{e}^{-\alpha t} \|v^0 - \bar{v}\|_{\mathcal{H}^1} \quad \forall t \geq 0. \tag{38}$$

Using the transformation between the different phase conditions ψ_{fix} and $\psi_{\min}$ in Lemma 3.6 we will show how stability transfers to the $\psi_{\min}$ case.

We define the bilinear form $b : \mathcal{H}^1 \times \mathcal{H}^1 \to \mathbb{R}$ via

$$b(u, v) = \int_{\mathbb{R}} -u_x(x)^T A v_x(x) + u(x)^T (B v_x(x) + C(x)v(x))\, dx$$

where $A, B, C(\cdot)$ are the bounded matrix functions defined in (35). Then we get via integration by parts

$$b(\hat{v}_x, v) = \langle \hat{v}_x, \Lambda v \rangle_{\mathcal{L}_2} \quad \text{for} \quad v \in \mathcal{H}^2 \quad \text{and} \quad |b(\hat{v}_x, v)| \leq \text{const } \|v\|_{\mathcal{H}^1}.$$

We define the projector P onto $\hat{v}_x^{\perp}$ along $\bar{v}_x$ and the projected differential operator Λ_P through

$$Pv = v - \hat{v}_x \langle \hat{v}_x, \bar{v}_x \rangle_{\mathcal{L}_2}^{-1} \langle \hat{v}_x, v \rangle_{\mathcal{L}_2}, \quad \Lambda_P = P\Lambda_{|range(P)}.$$

The following lemma gives the main estimate for solutions of the non-autonomous PDAE

$$\begin{aligned} v_t &= \Lambda v + \mu \bar{v}_x + g(t, v, \mu), \quad v(0) = v^0, \\ 0 &= \langle \hat{v}_x, v \rangle. \end{aligned} \tag{39}$$

Lemma 4.14. *Assume that g satisfies*

$$\|g(t, v, \mu)\| \leq \text{const } \mathrm{e}^{-\beta t}(\|v\|_{\mathcal{H}^1} + \|\mu\|), \quad \beta > 0. \tag{40}$$

Then there exist $\rho > 0$ and $\nu \in (0, \beta)$ such that any solution (v, μ) of (39) with $\|v^0\|_{\mathcal{H}^1} < \rho$ obeys the exponential estimate

$$\|v(t)\|_{\mathcal{H}^1} + \|\mu(t)\| \leq \text{const } \mathrm{e}^{-\nu t}\|v^0\|_{\mathcal{H}^1}. \tag{41}$$

Proof. The proof relies on the estimates for $r \in \text{range}(P)$ and some $\alpha > 0$

$$\|\mathrm{e}^{\Lambda_P t} r\|_{\mathcal{L}_2} \leq K\mathrm{e}^{-\alpha t}\|r\|_{\mathcal{L}_2}, \quad \|\mathrm{e}^{\Lambda_P t} r\|_{\mathcal{H}^1} \leq K\mathrm{e}^{-\alpha t} t^{-\frac{1}{2}}\|r\|_{\mathcal{L}_2} \tag{42}$$

which follow from the fact that the eigenvalue 0 has been eliminated from the spectrum of Λ_P; cf. [42, Lemma 1.24]. By the variation of constants formula the PDAE (39) can be written equivalently as

$$\begin{aligned} v(t) &= \mathrm{e}^{\Lambda_P t} v^0 + \int_0^t \mathrm{e}^{\Lambda_P (t-s)} P\, g(s, v(s), \mu(s)) ds \\ \mu(t) &= -\langle \hat{v}_x, \bar{v}_x \rangle_{\mathcal{L}_2}^{-1} [b(\hat{v}_x, v(t)) + \langle \hat{v}_x, g(t, v(t), \mu(t)) \rangle_{\mathcal{L}_2}]. \end{aligned} \tag{43}$$

Using this form, one first shows via Gronwall estimates as in [42] a global bound

$$\|v(t)\|_{\mathcal{H}^1} + \|\mu(t)\| \leq C\|v^0\|_{\mathcal{H}^1} \quad \forall t \geq 0. \tag{44}$$

From the second equation in (43) we find with (40) that

$$\|\mu(t)\| \leq C[\|v(t)\|_{\mathcal{H}^1} + \mathrm{e}^{-\beta t}(\|v(t)\|_{\mathcal{H}^1} + \|\mu(t)\|)].$$

Choose $T > 0$ such that $C\mathrm{e}^{-\beta T} \leq \frac{1}{2}$ and obtain

$$\|\mu(t)\| \leq C\|v(t)\|_{\mathcal{H}^1} \quad \forall t \geq T. \tag{45}$$

Now choose $0 < \nu < \min(\alpha, \beta)$ and use (42),(44) and (45) in the first equation

of (43) to obtain

$$\begin{aligned} n(t) = \|v(t)\|_{\mathcal{H}^1} \mathrm{e}^{\nu t} &\le C\Big(\mathrm{e}^{(\nu-\alpha)t}\|v^0\|_{\mathcal{H}^1} \\ &\quad + \int_0^t \frac{\mathrm{e}^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} \mathrm{e}^{\nu s}\mathrm{e}^{-\beta s}(\|v(s)\|_{\mathcal{H}^1} + \|\mu(s)\|)ds\Big) \\ &\le C\Big(\mathrm{e}^{(\nu-\alpha)t}\|v^0\|_{\mathcal{H}^1} + \mathrm{e}^{(\nu-\alpha)(t-T)} \int_0^T \frac{\mathrm{e}^{(\nu-\alpha)(T-s)}}{\sqrt{t-s}}\|\mu(s)\|ds \\ &\quad + \int_0^t \frac{\mathrm{e}^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} \mathrm{e}^{\nu s}\|v(s)\|_{\mathcal{H}^1}ds\Big) \\ &\le C\Big(\mathrm{e}^{(\nu-\alpha)t}\|v^0\|_{\mathcal{H}^1} + \int_0^t \frac{\mathrm{e}^{(\nu-\alpha)(t-s)}}{\sqrt{t-s}} n(s)ds\Big). \end{aligned}$$

The Gronwall inequality with weak singularities (cf. [22, Lemma 7.1.1]) yields the assertion. □

Lemma 4.15. *Let the assumptions of Theorem 4.11 be satisfied for a nonconstant solution $(\bar v, \bar\lambda) \in \mathbb{C}_b^2 \times \mathbb{R}$ of (31). Then there exists a shift $\gamma \in \mathbb{R}$ such that $(\bar v(\cdot+\gamma), \bar\lambda)$ is an asymptotically stable solution of (32).*

Proof. Note that $w = v_t$ solves

$$\begin{aligned} w_t &= w_{xx} + f'(v)w + w_x\lambda + v_x\dot\lambda \\ 0 &= \langle \hat v_x, w\rangle \end{aligned}$$

and that the first equation with $\mu := \dot\lambda$ is equivalent to

$$w_t = \Lambda w + \bar v_x\mu + (f'(v) - f'(\bar v))w + (\lambda - \bar\lambda)w_x + (v_x - \bar v_x)\mu.$$

Now we apply Lemma 4.14 for small $\|v^0\|_{\mathcal{H}^1}$ with

$$g(t, w, \mu) = (f'(v(t)) - f'(\bar v))w + (\lambda(t) - \bar\lambda)w_x + (v_x(t) - \bar v_x)\mu.$$

Note that the exponential decay (40) follows from the stability estimate (38). Since g is linear in w and μ, we obtain for all v^0 from (41) the estimate

$$\|v_t\|_{\mathcal{L}_2} + \|\dot\lambda\| \le \text{const } \mathrm{e}^{-\nu t}\|v^0\|_{\mathcal{H}^1} \quad \forall t \ge 0.$$

By this estimate the following integral exists

$$\eta_\infty = \int_0^\infty \frac{\langle v_x(\cdot,\tau), v_t(\cdot,\tau)\rangle}{\|v_x(\cdot,\tau)\|_{\mathcal{L}_2}^2}\, d\tau$$

and we have

$$\begin{aligned} |\eta(t) - \eta_\infty| &\le \int_t^\infty \frac{|\langle v_x(\cdot,\tau), v_t(\cdot,\tau)\rangle|}{\|v_x(\cdot,\tau)\|_{\mathcal{L}_2}^2}\, d\tau \le \int_t^\infty \frac{\|v(\cdot,\tau)\|_{\mathcal{H}^1}\|v_t(\cdot,\tau)\|_{\mathcal{L}_2}}{\|v_x(\cdot,\tau)\|_{\mathcal{L}_2}^2}\, d\tau \\ &\le \int_t^\infty \frac{(\|\bar v\|_{\mathcal{H}^1} + C\delta)C\mathrm{e}^{-\alpha\tau}}{(\|\bar v_x\|_{\mathcal{L}_2} - C\delta)^2}\, d\tau \le \text{const } \mathrm{e}^{-\alpha t}. \end{aligned}$$

Together with the local stability estimate (38) this leads to

$$\begin{aligned}
&\|w(\cdot,t) - \bar{v}(\cdot - \eta_\infty)\|_{\mathcal{H}^1} + |\mu(t) - \bar{\lambda}| \\
&\leq \|v(\cdot - \eta(t), t) - \bar{v}(\cdot - \eta_\infty)\|_{\mathcal{H}^1} + |\lambda(t) - \dot{\eta}(t) - \bar{\lambda}| \\
&\leq \|v(\cdot,t) - \bar{v}\|_{\mathcal{H}^1} + \|\bar{v} - \bar{v}(\cdot + \eta(t) - \eta_\infty)\|_{\mathcal{H}^1} + |\lambda(t) - \bar{\lambda}| + |\dot{\eta}(t)| \\
&\leq \text{const } \ \mathrm{e}^{-\alpha t}.
\end{aligned}$$

□

5 Spiral waves and beyond

Embedding spiral waves of parabolic systems in $\mathbb{R}^2$ into the abstract framework of Section 3.2 is a considerable task [37, 46]. Therefore we do not pursue this in detail here.

5.1 Spiral waves in two space dimensions

Consider a PDE in two space dimensions

$$\begin{aligned}
u_t &= \Delta u + f(u), \quad t \geq 0 \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}^2.
\end{aligned} \tag{46}$$

This equation is equivariant with respect to the Euclidean group $SE(2) = S^1 \ltimes \mathbb{R}^2 \ni (\phi, \tau)$ with action $a(\gamma, v)(x) = v(R_{-\phi}(x - \tau))$ and group multiplication

$$(\phi_1, \tau_1) \circ (\phi_2, \tau_2) = (\phi_1 + \phi_2, \tau_1 + R_{\phi_1}\tau_2),$$

where R_ϕ again denotes rotations; cf. Example 3.5.

Take, for example, $v \in C_{\text{unif}} = M$ and $N = C^2_{\text{unif}}$ or the subspace $C_{\text{eucl}} \subset C_{\text{unif}}$ on which $SE(2)$ acts continuously [46]. Then the infinitesimal generators $da(\mathbb{1}, v)e^i,\ i = 1,2,3$ read $da(\mathbb{1}, v)e^1 = x_2 v_{x_1} - x_1 v_{x_2}$, $da(\mathbb{1}, v)e^2 = -v_{x_1}$, $da(\mathbb{1}, v)e^3 = -v_{x_2}$. For a relative equilibrium $u(x,t) = \bar{v}(R_{-\phi}(x - \tau))$ the wave form $\bar{v}$ is a solution of

$$0 = \Delta v + f(v) + \bar{\mu}_1(x_1 v_{x_2} - x_2 v_{x_1}) + \bar{\mu}_2 v_{x_1} + v_{x_2}\bar{\mu}_3$$

for some $\bar{\mu} \in se(2)$. The motion on the group orbit is given by

$$\dot{\gamma} = dL_\gamma(\mathbb{1})\bar{\mu} = \begin{pmatrix} 1 & 0 \\ 0 & R_\phi \end{pmatrix} \bar{\mu}$$

with solution

$$\bar{\gamma}(t) = \begin{pmatrix} \bar{\mu}_1 t \\ (I - R_{\bar{\mu}_1 t})\xi \end{pmatrix}, \quad \text{where} \quad \xi = \frac{1}{\bar{\mu}_1}\begin{pmatrix} -\bar{\mu}_3 \\ \bar{\mu}_2 \end{pmatrix}.$$

Then we can represent the relative equilibrium as follows

$$\bar{u}(x,t) = \bar{v}(R_{-\bar{\mu}_1 t}(x + (R_{\bar{\mu}_1 t} - I)\xi)) = \bar{v}(R_{-\bar{\mu}_1 t}(x - \xi) + \xi).$$

A fixed reference point $\bar{x} \in \mathbb{R}^2$ with value $\bar{v}(\bar{x})$ traces the curve

$$x(t) = R_{\bar{\mu}_1 t}(\bar{x} - \xi) + \xi, \tag{47}$$

i.e. a circle of radius $\|\bar{x} - \xi\|$ and center ξ. If one uses a geometric definition for the tip x_{tip} of a spiral wave (see e.g. [8, 46]) then the tip moves on a circle of radius $r_{\text{tip}} = \|x_{\text{tip}} - \xi\|$. A special case are rigidly rotating Archimedian spirals which can be written in polar coordinates as $\bar{v}(x) = w(r, \phi)$ with $w(r, \phi) \to w_\infty(kr + \phi)$ as $r \to \infty$ for some periodic function w_∞.

Example 5.1. We use a diffusive version of Barkley's system [2] as an example, namely:

$$\begin{aligned} u_t &= \Delta u + \tfrac{1}{\epsilon} u(1-u)(u - \tfrac{1}{a}(v+b)), \\ v_t &= D_v \Delta v + u - v. \end{aligned} \tag{48}$$

We solve the corresponding PDAE

$$\begin{aligned} u_t &= \Delta u + \tfrac{1}{\epsilon} u(1-u)(u - \tfrac{1}{a}(v+b)) + \lambda_1(y u_x - x u_y) + \lambda_2 u_x + \lambda_3 u_y \\ v_t &= D_v \Delta v + u - v + \lambda_1(y v_x - x v_y) + \lambda_2 v_x + \lambda_3 v_y \\ 0 &= \langle y\hat{u}_x - x\hat{u}_y, u - \hat{u}\rangle_{\mathcal{L}_2}, \quad 0 = \langle \hat{u}_x, u - \hat{u}\rangle_{\mathcal{L}_2}, \quad 0 = \langle \hat{u}_y, u - \hat{u}\rangle_{\mathcal{L}_2} \\ 0 &= \langle y\hat{v}_x - x\hat{v}_y, v - \hat{v}\rangle_{\mathcal{L}_2}, \quad 0 = \langle \hat{v}_x, v - \hat{v}\rangle_{\mathcal{L}_2}, \quad 0 = \langle \hat{v}_y, v - \hat{v}\rangle_{\mathcal{L}_2} \end{aligned}$$

with $(\hat{u}, \hat{v}) = (u^0, v^0)$ numerically for the parameters $D_v = 0.5$, $a = 0.5$, $b = 0.05$, $\epsilon = \frac{1}{50}$ by using the Finite Element package COMSOL MULTIPHYSICS™.

In Figure 7 the time evolution of the u-component and the parameter μ is displayed.

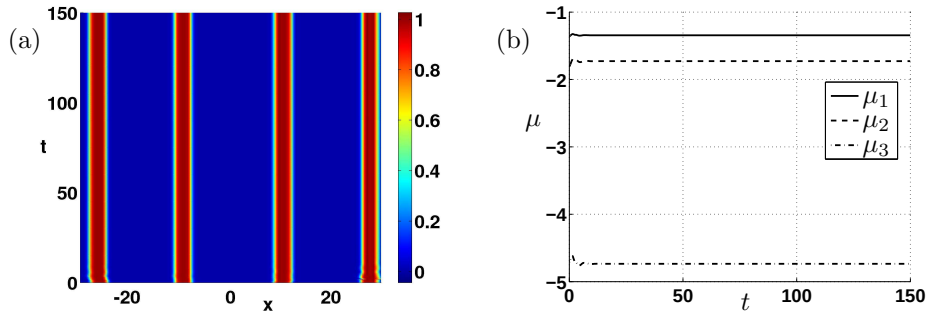

Figure 7: Frozen spiral wave of (48): time evolution of u-component (a) and time evolution of μ (b).

In Figure 8 the prediction of the motion of the tip via (47) (red circle) is compared to the tip-motion of the non-frozen spiral starting from the same initial condition (white trace). For the definition of x_{tip} we used the condition $u = \frac{1}{2}$, $v = \frac{a}{2} - b$ from [2].

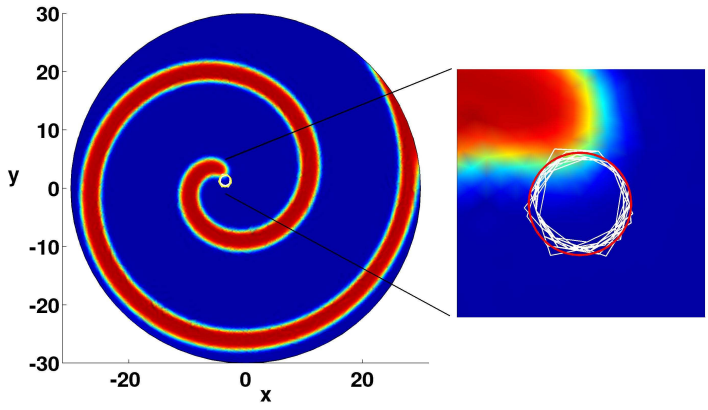

Figure 8: Prediction of the tip-motion of the spiral wave from Figure 7 via (47) (red circle) and tip-motion of the non-frozen spiral starting from the same initial condition (white trace).

5.2 A Scroll wave in 3D

In three space dimensions (46) is equivariant with respect to $G = SO(3) \ltimes \mathbb{R}^3 = SE(3)$ with action $a(\gamma, v)(x) = v(\mathcal{R}^{-1}(x - \tau))$, $\gamma = (\mathcal{R}, \tau)$, $\tau = (\tau_1, \tau_2, \tau_3)$ and group operation $\gamma \circ \tilde{\gamma} = (\mathcal{R}\tilde{\mathcal{R}}, \tau + \mathcal{R}\tilde{\tau})$. We denote the rotations about the x_1,x_2 and x_3 axis by R_{x_1}, R_{x_2} and R_{x_3} and for $(S^1)^3 \ni \rho = (\phi_1, \phi_2, \phi_3)$ define $\mathcal{R}_\rho = R_{x_1}(\phi_1)R_{x_2}(\phi_2)R_{x_3}(\phi_3)$. An easy computation shows that

$$\begin{aligned} -da(\mathbb{1})v\mu = {} & \mu_1(v_{x_2}x_3 - v_{x_3}x_2) + \mu_2(v_{x_3}x_1 - v_{x_1}x_3) + \mu_3(v_{x_1}x_2 - v_{x_2}x_1) \\ & + \mu_4 v_{x_1} + \mu_5 v_{x_2} + \mu_6 v_{x_3}. \end{aligned}$$

Example 5.2. We consider the following $\lambda - \omega$ system in complex form

$$u_t = \Delta u + (1 - |u|^2 - i|u|^2)u, \quad x \in \mathbb{R}^3, \ u(x,t) \in \mathbb{C} \tag{49}$$

for which rigidly rotating waves exist [24].

We use an adapted version of the code EZSCROLL [18] and start in a box of length 40 with $\Delta x = 0.1$ from an initial function

$$u_0(r, \varphi, z) = \mathrm{e}^{\frac{iz}{2\pi}} \tfrac{r}{40}(\cos(\varphi) + i\sin(\varphi))),$$

which ensures that in each z-slice a rotating spiral develops. We use periodic boundary conditions on the z-faces and Neumann boundary conditions on the x- and y-faces. Therefore, the initial function initiates a scroll wave twisted once in the z-direction; see [18, 19] for more on scroll waves and scroll rings.

Figure 9 shows the real part of the solution of the frozen system at the final time-instance, as well as the time evolution of μ. The solution in panel (a) is shown in the form of slices in x,y,z-direction through the origin $(0, 0, 0)$ which

have been projected to the boundaries to increase visibility. From panel (b) one can see that first the rotation around and the translations along the z-axis are active. However, after some transient time only the z-translation is used to freeze the solution.

We expect that this solution has nontrivial stabilizer since vertical motions in the z-direction and rotations about the z-axis can be exchanged. Our method seems to work nevertheless. In fact, system (49) is actually equivariant with respect to the seven-dimensional group $S^1 \times SE(3)$, where $\theta = \mathrm{e}^{i\rho} \in S^1$ acts as in Example 3.5. Including this symmetry in the computations leads to ill-conditioned systems when resolving the phase conditions for the seven parameters.

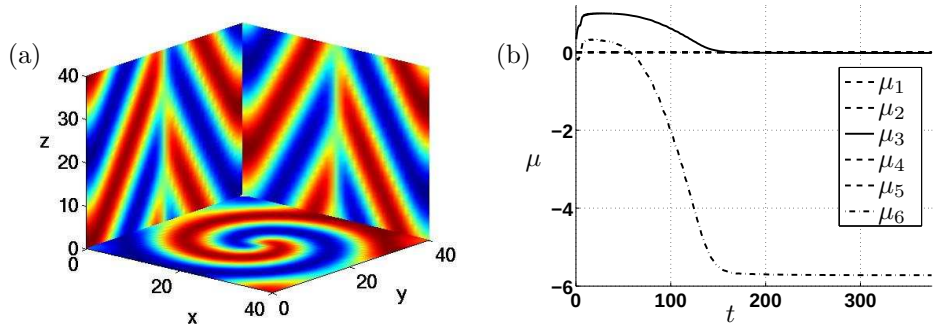

Figure 9: Three dimensional scroll wave of (49): slices of the development of u (a) and time evolution of μ (b).

5.3 Relative periodic orbits

For relative periodic orbits we have a similar definition as for relative equilibria. We seek for solutions of (24) which have the special form $\bar{u}(t) = a(\bar{\gamma}(t), \bar{v}(t))$ for some time periodic function $\bar{v}$.

Definition 5.3. A solution $\bar{u}$ of (24) is called a relative periodic orbit if it has the form $\bar{u}(t) = a(\bar{\gamma}(t), \bar{v}(t))$ where $\bar{\gamma} : \mathbb{R} \to G$ is a smooth curve satisfying $\bar{\gamma}(0) = \mathbb{1}$ and $\bar{v} : \mathbb{R} \to N$ satisfies $\bar{v}, \bar{v}_t, F(\bar{v}) \in C(\mathbb{R}, M)$, is non-constant and time periodic, i.e. $\bar{v}(\cdot + T) = \bar{v}$ for some period $T > 0$.

As for relative equilibria (see Lemma 4.1) we can relate $\bar{u}$, $\bar{\gamma}$ and $\bar{v}$.

Lemma 5.4. *Let $\bar{u}(t) = a(\bar{\gamma}(t), \bar{v}(t))$ be a relative periodic orbit with trivial stabilizer $\mathcal{S}_{\bar{v}(t)}$, $t \in [0, T]$. Then there exists a T-periodic function $\bar{\mu} \in C(\mathbb{R}, T_{\mathbb{1}}G)$ such that for all $t \in \mathbb{R}$*

$$\bar{v}_t = F(\bar{v}) - da(\mathbb{1}, \bar{v})\bar{\mu} \tag{50}$$

$$\bar{\gamma}_t = dL_\gamma(\mathbb{1})\bar{\mu}, \quad \gamma(0) = \mathbb{1}. \tag{51}$$

Conversely, if $\bar{\mu} \in C(\mathbb{R}, T_{\mathbb{1}}G)$ *and* $\bar{v} \in C^1(\mathbb{R}, M)$ *with* $\bar{v}(t) \in N \ \forall t \in \mathbb{R}$ *are* T*-periodic and solve* (50)*, then* $a(\bar{\gamma}(t), \bar{v})$ *with* $\bar{\gamma}$ *defined by* (51) *is a relative periodic orbit of* (24)*.*

Proof. The proof is similar to that of Lemma 4.2. For the first assertion define $\bar{\mu}(t)$ by (51) and obtain (50) from Lemma 3.3. Equation (50) then shows that $\bar{\mu}$ is T-periodic. The converse follows in a similar manner. □

Scaling with the period T as in Section 2 we find that $v(t) = \bar{v}(tT)$, $\mu(t)$ solve

$$v_t = T[F(v) - da(\mathbb{1}, v)\mu], \quad t \in [0, 1], \quad v(0) = v(1).$$

Example 5.5. Examples of relative periodic orbits in one space dimension are modulated traveling waves [38]. These have the form $\bar{u}(x,t) = \bar{v}(x - \bar{\lambda}t, t)$ with $\bar{v}(x,t) = \bar{v}(x, t+T)$. Such solutions occur for example in the autocatalytic system already considered in Example 3.2. We solve the system for $a = 0.1$, $m = 9$ on an interval of length 35 in the frozen case, and of length 300 for the direct simulation with $\Delta x = 0.1$, Dirichlet boundary conditions and with a θ-method with $\Delta t = 0.1$. In Figure 10(a) and (b) the numerical solutions for $\theta = \frac{1}{2}$ of the PDE (23) and of the corresponding PDAE (defining the frozen system) are shown. Panel (c) shows the solution at the last time instance, while panel (d) displays the time-evolution of μ. The periodicity of the wave and the velocity can be seen clearly.

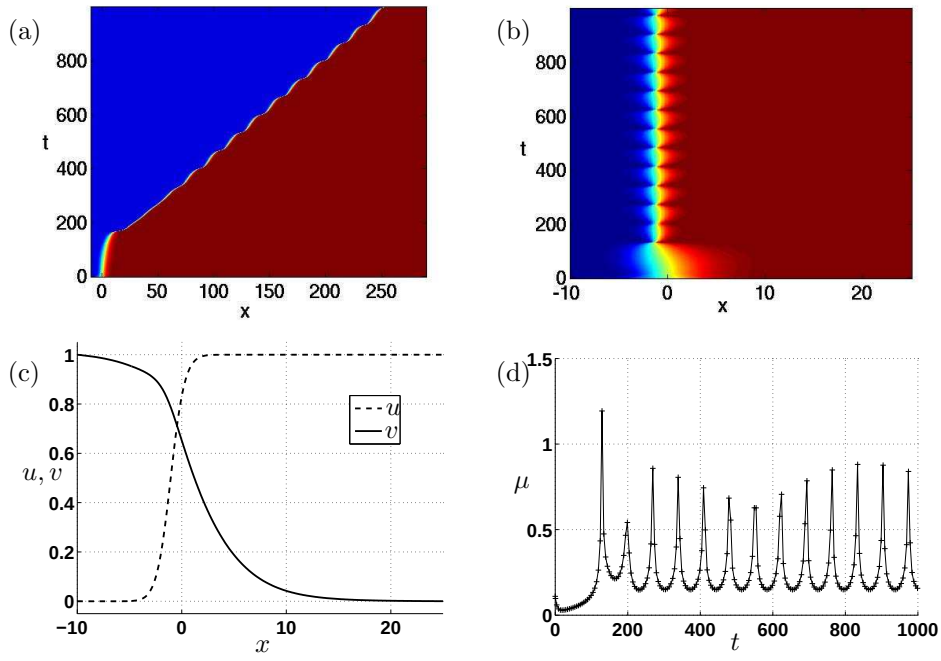

Figure 10: Calculation of a modulated traveling wave in autocatalytic system (23) for $\theta = \frac{1}{2}$: traveling wave (a), frozen wave (b), u- and v-component at $t = 1000$ (c), and time evolution of μ (d).

Figure 11 shows the modulated traveling wave and the frozen wave when the implicit Euler method is used, that is, $\theta = 1$. Note that the oscillations are strongly damped for the standard simulation in panel (a), whereas they are clearly visible in the frozen case in panel (b).

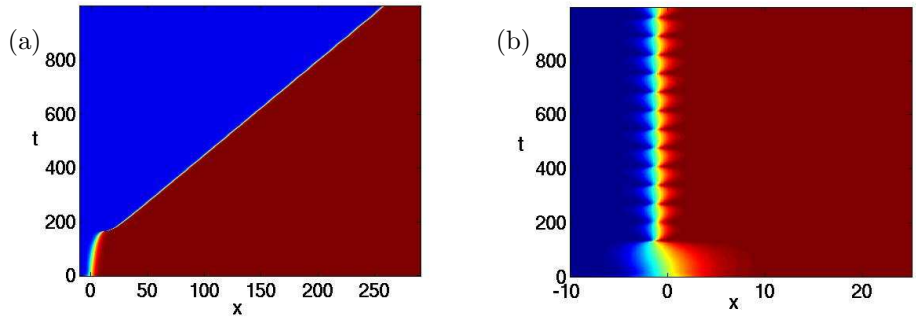

Figure 11: Calculation of a modulated traveling wave in autocatalytic system (23) for $\theta = 1$ (implicit Euler method): traveling wave (a), frozen wave (b).

Combining the principles from Sections 2 and 3 for the computation of relative periodic orbits we arrive at a boundary value problem (in space and time) for $v \in C([0,1], M)$,$\mu \in C([0,1], T_{\mathbb{1}}G)$ and $T \in \mathbb{R}$ as follows:

$$
\begin{aligned}
& v_t = T[F(v) - da(\mathbb{1}, v)\mu], \quad v(0) = v(1), \\
& 0 = \psi(v)\lambda = \int_0^1 \langle da(\mathbb{1}, \hat{v})\lambda, v - \hat{v} \rangle \, dt \quad \forall \lambda \in T_{\mathbb{1}}G, \\
& 0 = \phi(v) = \int_0^1 \langle \hat{v}_t, v - \hat{v} \rangle \, dt.
\end{aligned}
$$

Here $\langle \cdot, \cdot \rangle$ denotes an inner product on M and $\hat{v} \in C([0,1], M)$ is a template function from continuation.

5.4 Conclusions and perspectives

Phase conditions are an effective tool in selecting specific orbits in equivariant evolution equations. When based on minimization principles they facilitate mesh adaptation and speed up continuation along branches. In many applications the underlying symmetry is induced by the Euclidean group $SE(d)$ acting on functions defined in the whole space $\mathbb{R}^d$. For numerical computations one has to truncate to bounded domains and use asymptotic boundary conditions. Truncation in combination with the method of freezing spatio-temporal patterns in a co-moving frame raises several numerical as well as theoretical problems. Only a few of them have been tackled in this paper, mainly for parabolic systems in one space dimension.

Considerable challenges remain, and we expect the further development of the field to address theoretical and numerical issues, including the following:

1. Computation and continuation of relative equilibria and relative periodic orbits in equivariant systems, the detection of bifurcation points and branch switching at symmetry breaking bifurcations; see, for example, the recent progress by Wulff and Schebesch [47].

2. Adaptation of the freezing method to relative equilibria with non-trivial stabilizers.

3. Consideration of linear versus nonlinear stability for spatio-temporal patterns in space dimensions ≥ 2. There are extensive studies of the spectra associated with systems linearized about spiral waves and their truncation to bounded domains (see [36, 37, 39]), but a result on nonlinear stability still seems to be lacking.

4. Development of (implementable) asymptotic boundary conditions for spiral waves, scroll waves and the like.

5. Application of the freezing method to viscous conservation laws. As for modulated or spiral waves, this case is difficult because of the fact, that the essential spectrum has a quadratic tangency with the imaginary axis; see [6].

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