Shadowing and inverse shadowing in set-valued dynamical systems. Hyperbolic case

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Abstract

We introduce a new hyperbolicity condition for set-valued dynamical systems and show that this condition implies the shadowing and inverse shadowing properties.

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1 Introduction

The shadowing property of classical dynamical systems (diffeomorphisms or flows) is now well-studied (see, for example, the monographs [10] and [11]). This property means that, near approximate trajectories (so-called pseudotrajectories), there exist exact trajectories of the system.

Another type of shadowing properties (inverse shadowing properties) are related to the following problem: given a family of mappings that approximate the defining mapping for the dynamical system considered, can we find, for a chosen exact trajectory, a close pseudotrajectory generated by the given family? Such properties were considered by various authors (let us mention, for example, the papers [3], [4], and [5]).

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Clearly, the study of such properties is important for the theory of perturbations of dynamical systems.

In this paper, we work with shadowing properties of set-valued dynamical systems.

Recently, it was shown that so-called contractive set-valued dynamical systems possess shadowing properties (see, for example, [7, 8] and [12]). We discuss contractive systems in Sec. 4 of this paper.

The main goal of the present paper is to intruduce a new hyperbolicity property of set-valued dynamical systems and to show that this property (in its "global" and "local" variants) implies the shadowing and inverse shadowing properties.

In the literature, one can find two kinds of hyperbolicity properties for set-valued dynamical systems (to be exact, these properties were introduced for systems defined by relations, but on the level of definitions these two classes of objects do not differ significantly).

One of the definitions was given by Akin in the monoghaph [1]. In fact, this definition is "inner" (i.e., it is formulated not in terms of the mapping generating the system but in terms of behavior of trajectories): the author calls a set hyperbolic if this set is expansive and the system has on it the shadowing property.

A different approach was used by Sander in [9] and [13], where hyperbolicity was defined for smooth relations. The Sander hyperbolicity condition is designed to study features of classical dynamical systems such as stable and unstable manifolds in the framework of non-invertible maps, and it implies the shadowing property. Due to the nature of the analyzed objects, this hyperbolicity condition does not allow the graph of a relation to have nonempty interior, which is generically the case in the set-up discussed in the present paper.

It is important to note that both hyperbolicity conditions of Akin and Sander imply the uniqueness of a shadowing trajectory, which is quite unnatural for set-valued dynamical systems (it was shown in [12] that even contractive set-valued systems may fail to have the property of uniqueness of a shadowing trajectory). Since contractive systems in the sense of [12] satisfy our condition of hyperbolicity (see Sec. 4), the latter condition does not imply the uniqueness of a shadowing trajectory (and hence, this condition differs qualitatively from conditions of Akin and Sander).

Let us pass to basic notation. Consider a metric space $(\mathcal{M}, \mathrm{dist})$ and denote by $\mathcal{C}(\mathcal{M})$ the set of closed subsets of $\mathcal{M}$.

A set-valued dynamical system on $\mathcal{M}$ is determined by a set-valued mapping $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$ and its iterates. In what follows, we identify the mapping F and the corresponding dynamical system.

A sequence $\eta = \{p_k\}$ is a trajectory of the system F if

$$p_{k+1} \in F(p_k) \text{ for any } k \in \mathbb{Z}. \tag{1}$$

A sequence $\xi = \{x_k\}$ is called a d-pseudotrajectory of F if an error of size $d > 0$ is allowed in every step, i.e., if

$$\operatorname{dist}(x_{k+1}, F(x_k)) \leq d \quad \text{for any } k \in \mathbb{Z}. \tag{2}$$

We say that the system F has the shadowing property if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}$ of F there exists a trajectory $\eta = \{p_k\}$ with

$$\operatorname{dist}(x_k, p_k) \leq \epsilon \quad \text{for any } k \in \mathbb{Z}.$$

We specify the shadowing properties which we study in the statements of our theorems.

The distance between two nonempty compact subsets A and B of $\mathbb{R}^m$ is measured by the deviation

$$\operatorname{dev}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|$$

or by the Hausdorff distance

$$\operatorname{dist}_H(A, B) = \max\{\operatorname{dev}(A, B), \operatorname{dev}(B, A)\}.$$

If A and B are nonempty compact sets, there exists a (possibly not unique) vector $\operatorname{Dev}(A, B) \in \mathbb{R}^m$ such that $\operatorname{Dev}(A, B) = b - a$ for some $a \in A$ and $b \in B$ with $|b - a| = \operatorname{dev}(A, B)$. Let us note that if A is a point and B is a convex set, then the vector $\operatorname{Dev}(A, B)$ is defined uniquely. In addition, if $B(t)$ is a continuous (w.r.t. dist_H) family of convex sets, then the vector-function $\operatorname{Dev}(A, B(t))$ is continuous in t as well (cf. Theorem 1.7.1 of [2]).

The collection of nonempty, compact, and convex subsets of $\mathbb{R}^m$ will be denoted by $\mathcal{CC}(\mathbb{R}^m)$. As usual, for a sequence $\eta = \{\eta_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$,

$$||\eta||_\infty = \sup_{k \in \mathbb{Z}} |\eta_k|.$$

Let us formulate hyperbolicity conditions under which we establish the shadowing result. We consider a set-valued mapping F of the form

$$F(x) = L(x) + M(x), \tag{3}$$

where $L : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous single-valued mapping and $M : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ is a set-valued mapping with compact and convex images. We say that a mapping (3) is hyperbolic in $\mathbb{R}^m$ if there exist constants $N \geq 1$, $a, \kappa, l > 0$, and $\lambda \in (0, 1)$ such that the following conditions hold:

(P1) For any point $x \in \mathbb{R}^m$ there exist linear subspaces $U(x), S(x) \subset \mathbb{R}^m$ such that

$$S(x) \oplus U(x) = \mathbb{R}^m,$$

and if $P(x)$ and $Q(x)$ are the corresponding complementary projections from $\mathbb{R}^m$ to $U(x)$ and $S(x)$, then

$$\|P(x)\|, \|Q(x)\| \leq N. \tag{4}$$

Below, the spaces $U(x)$ and $S(x)$ are called the unstable and stable subspaces, respectively.

(P2) If $x, y, v \in \mathbb{R}^m$ satisfy the inequalities $|v| \leq a$ and

$$\text{dist}(y, F(x)) \leq a,$$

then we can represent $L(x+v)$ as

$$L(x+v) = L(x) + A(x)v + B(x,v), \tag{5}$$

where $A(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear mapping that is continuous with respect to x and such that

$$\begin{aligned} |Q(y)A(x)v| &\leq \lambda|v| \text{ for } v \in S(x), & (6)\\ |Q(y)A(x)v| &\leq \kappa|v| \text{ for } v \in U(x), & (7)\\ |P(y)A(x)v| &\leq \kappa|v| \text{ for } v \in S(x), & (8) \end{aligned}$$

and the restriction $P(y)A(x)|_{U(x)} : U(x) \to U(y)$ is a linear isomorphism satisfying

$$|P(y)A(x)v| \geq \frac{1}{\lambda}|v| \text{ for } v \in U(x). \tag{9}$$

Note that since $L(x)$ and $A(x)$ are assumed to be continuous, $B(x,v)$ is continuous for any x and v with $|v| \leq a$.

(P3) If $v \in \mathbb{R}^m$ satisfies the inequality $|v| \leq a$, then

$$|B(x,v)| \leq l|v| \tag{10}$$

and

$$\text{dist}_H(M(x), M(x+v)) \leq l|v| \text{ for } x \in \mathbb{R}^m. \tag{11}$$

Note that condition (11) implies the continuity of M w.r.t. the Hausdorff distance.

A simple example of a set-valued hyperbolic mapping is as follows. Assume that A is an $m \times m$ hyperbolic matrix (this means that the eigenvalues λ_j of A satisfy the inequalities $|\lambda_j| \neq 1$). Let S and U be the invariant subspaces of A that correspond to the parts of its spectrum inside and outside the unit disk, respectively.

Let M be a fixed compact convex subset of $\mathbb{R}^m$. Then the set-valued mapping $F(x) = Ax + M$ is hyperbolic (in this case, the spaces $S(x)$ and $U(x)$ coincide with S and U, respectively, for any x).

The structure of the paper is as follows. In Section 2, we prove a shadowing result while Section 3 is devoted to inverse shadowing. In Section 4, we study relations between hyperbolic and contractive set-valued system and prove a general shadowing theorem for contractive systems.

2 Shadowing

Theorem 1. *Let F be a set-valued hyperbolic mapping as described above. If*

$$\lambda + \kappa + 4lN < 1, \tag{12}$$

then F has the Lipschitz shadowing property: there exists a constant $d_0 > 0$ such that if $\{x_k\}$ is a d-pseudotrajectory of F with $d \leq d_0$, then there exists a trajectory $\{p_k\}$ of F such that

$$\|\{x_k\} - \{p_k\}\|_\infty \leq \mathcal{L}d,$$

where

$$\mathcal{L}^{-1} = \frac{1}{2N}\left(1 - \lambda - \kappa - 4lN\right).$$

Remark 2. *Note that condition* (12) *implies the inequality*

$$\lambda(1 + \kappa + 4lN) < 1. \tag{13}$$

In addition,

$$\mathcal{L}^{-1} \leq \frac{1}{2N}\left(\frac{1}{\lambda} - 1 - \kappa - 4lN\right). \tag{14}$$

Proof of Theorem 1. Set $d_0 = a/\mathcal{L}$ and consider a d-pseudotrajectory $\{x_k\}$ of F with $d \leq d_0$. Note that $d \leq a$ since $N \geq 1$. Our goal is to find a sequence $V = \{v_k \in \mathbb{R}^m : k \in \mathbb{Z}\}$ such that

$$x_{k+1} + v_{k+1} \in F(x_k + v_k) \tag{15}$$

and

$$\|V\|_\infty \le \mathcal{L}d; \tag{16}$$

in this case, $\{p_k = x_k + v_k\}$ is the desired trajectory of F.

By (3), relations (15) take the form

$$x_{k+1} + v_{k+1} \in L(x_k + v_k) + M(x_k + v_k). \tag{17}$$

If $|v_k| \le a$, it follows from property (P2) that we can represent

$$L(x_k + v_k) = L(x_k) + A(x_k)v_k + B(x_k, v_k). \tag{18}$$

Thus, relation (17) takes the form

$$x_{k+1} + v_{k+1} \in L(x_k) + A(x_k)v_k + B(x_k, v_k) + M(x_k + v_k),$$

or

$$v_{k+1} \in L(x_k) + A(x_k)v_k + B(x_k, v_k) + M(x_k + v_k) - x_{k+1}. \tag{19}$$

Consider the vector

$$\sigma_k = \mathrm{Dev}(x_{k+1}, L(x_k) + B(x_k, v_k) + M(x_k + v_k)).$$

The compact and convex set $L(x_k) + B(x_k, v_k) + M(x_k + v_k)$ depends continuously on v_k w.r.t. the Hausdorff distance for $|v_k| \le a$ (see the definition of F and properties (P2) and (P3)). By (10),

$$|B(x_k, v_k)| \le l|v_k|;$$

by (11),

$$\mathrm{dist}_H(M(x_k), M(x_k + v_k)) \le l|v_k|.$$

Since

$$\mathrm{dist}(x_{k+1}, L(x_k) + M(x_k)) < d,$$

we conclude that

$$|\sigma_k| = \mathrm{dist}(x_{k+1}, L(x_k) + B(x_k, v_k) + M(x_k + v_k)) \le d + 2l|v_k|. \tag{20}$$

If

$$\sigma_k = v_{k+1} - A(x_k)v_k, \tag{21}$$

then the inclusion

$$x_{k+1} + \sigma_k \in L(x_k) + B(x_k, v_k) + M(x_k + v_k)$$

implies that

$$x_{k+1} + v_{k+1} - A(x_k)v_k \in L(x_k) + B(x_k, v_k) + M(x_k + v_k),$$

which is equivalent to the desired inclusion (15). Thus, a solution $V = \{v_k\}$ of (21) gives us a shadowing trajectory.

Let us project equality (21) to $S(x_{k+1})$ and $U(x_{k+1})$, respectively:

$$Q(x_{k+1})v_{k+1} = Q(x_{k+1})A(x_k)v_k + Q(x_{k+1})\sigma_k, \tag{22}$$

$$P(x_{k+1})v_{k+1} = P(x_{k+1})A(x_k)v_k + P(x_{k+1})\sigma_k \tag{23}$$

(let us note that in the case of classical dynamical systems, a similar method of projecting an equation to stable and unstable subspaces has been used in [5]).

Denote $b := d\mathcal{L}/2$ and let

$$H_k = \{v_k \in \mathbb{R}^m : |P(x_k)v_k|, |Q(x_k)v_k| \le b\}$$

and

$$H = \prod_{k\in\mathbb{Z}} H_k. \tag{24}$$

Obviously, each H_k is compact and convex; hence, H is convex and compact w.r.t. the Tikhonov product topology.

If $V = \{v_k\} \in H$, then

$$|v_k| \le |P(x_k)v_k| + |Q(x_k)v_k| \le 2b = \mathcal{L}d \le a; \tag{25}$$

hence, all the terms in (22) and (23) are defined. We define an operator T that maps a sequence $V = \{v_k \in \mathbb{R}^m\}$ to a sequence $W = \{w_k \in \mathbb{R}^m\}$ as follows: the stable components of w_k are defined by

$$Q(x_{k+1})w_{k+1} = Q(x_{k+1})A(x_k)v_k + Q(x_{k+1})\sigma_k. \tag{26}$$

To obtain the unstable components, we transform equation (23). Consider the mapping

$$G(w) = P(x_{k+1})A(x_k)w,\ \ w \in U(x_k). \tag{27}$$

Clearly, $G(0) = 0$. It follows from (9) that

$$|G(w) - G(w')| \ge \frac{1}{\lambda}|w - w'|,\ \ w, w' \in U(x_k). \tag{28}$$

Since the restriction of $P(x_{k+1})A(x_k)$ to $U(x_k)$ is assumed to be a linear isomorphism,

$$G(D(b, x_k)) \supset D(b', x_{k+1}), \tag{29}$$

where $b' = b/\lambda$,

$$D(b, x_k) = \{z \in U(x_k) : |w| \le b\}, \text{ and } D(b', x_{k+1}) = \{z \in U(x_{k+1}) : |z| \le b'\}.$$

By (28) and (29), the inverse Γ of G is defined on $D(b', x_{k+1})$. By (28),

$$|\Gamma(z) - \Gamma(z')| \le \lambda |z - z'|, \; z, z' \in D(b', x_{k+1}). \tag{30}$$

Now we define the unstable components of w_k by

$$P(x_k)w_k = \Gamma\{P(x_{k+1})[v_{k+1} - \sigma_k - A(x_k)Q(x_k)v_k]\}. \tag{31}$$

Lemma 3. *The operator T maps H to itself.*

Proof of Lemma 3. Let us estimate

$$\begin{aligned} |Q(x_{k+1})w_{k+1}| &\le |Q(x_{k+1})A(x_k)v_k| + |Q(x_{k+1})\sigma_k| \\ &\le |Q(x_{k+1})A(x_k)P(x_k)v_k| + |Q(x_{k+1})A(x_k)Q(x_k)v_k| + |Q(x_{k+1})\sigma_k| \\ &\le \kappa |P(x_k)v_k| + \lambda |Q(x_k)v_k| + N(d + 2l|v_k|) \end{aligned}$$

(we refer to (4), (6), (7), and (20)). Since $|v_k| \le 2b$ (see (25)),

$$|Q(x_{k+1})w_{k+1}| \le (\lambda + \kappa + 4lN)b + Nd = \left(\lambda + \kappa + 4lN + \frac{2N}{\mathcal{L}}\right) b \le b \tag{32}$$

by the definition of b and inequality (14). Let us estimate the argument of Γ on the right in (31). Since $v_{k+1} \in H_{k+1}$,

$$|P(x_{k+1})v_{k+1}| \le b. \tag{33}$$

By (4) and (20),

$$|P(x_{k+1})\sigma_k| \le N(d + 2l|v_k|) \le N(d + 4lb). \tag{34}$$

By (8),

$$|P(x_{k+1})A(x_k)Q(x_k)v_k| \le \kappa |Q(x_k)v_k| \le \kappa b. \tag{35}$$

Adding (33)–(35), we see that the argument of Γ is estimated by

$$\begin{aligned} |P(x_{k+1})[v_{k+1} - \sigma_k - A(x_k)Q(x_k)v_k]| &\le (1 + \kappa + 4lN)b + Nd \\ &= \left(1 + \kappa + 4lN + \frac{2N}{\mathcal{L}}\right) b \\ &\le \frac{b}{\lambda} = b' \end{aligned} \tag{36}$$

by the definition of L and b. Thus, $\Gamma\{\ldots\}$ is defined, and it follows from (36) and (30) that

$$|P(x_k)w_k| \le b. \tag{37}$$

Inequalities (32) and (37) show that if $V \in H$ and $W = T(V)$, then $W \in H$. □

Since σ_k depends on v_k only, formulas (26) and (31) show that $(T(V))_k$ depends on v_{k-1}, v_k, v_{k+1}. Hence, the operator is continuous w.r.t. the Tikhonov topology on H.

The Tikhonov-Schauder fixed point theorem implies that T has a fixed point in H. To complete the proof of Theorem 1, it remains to show that if $T(V) = V$, then V solves equation (21). By (26),

$$Q(x_{k+1})v_{k+1} = Q(x_{k+1})A(x_k)v_k + Q(x_{k+1})\sigma_k \tag{38}$$

if $T(V) = V$. Let us apply G to the equality

$$P(x_k)v_k = \Gamma\{P(x_{k+1})[v_{k+1} - \sigma_k - A(x_k)Q(x_k)v_k]\}$$

to show that

$$P(x_{k+1})A(x_k)P(x_k)v_k = G(P(x_k)v_k) = P(x_{k+1})[v_{k+1} - \sigma_k - A(x_k)Q(x_k)v_k].$$

Hence,

$$\begin{aligned} & P(x_{k+1})v_{k+1} \\ = \ & P(x_{k+1})\sigma_k + P(x_{k+1})A(x_k)Q(x_k)v_k + P(x_{k+1})A(x_k)P(x_k)v_k \\ = \ & P(x_{k+1})[\sigma_k + A(x_k)v_k]. \end{aligned} \tag{39}$$

Adding (38) and (39), we see that

$$v_{k+1} = \sigma_k + A(x_k)v_k,$$

i.e., V solves equation (21). Since

$$\|V\|_\infty \le \mathcal{L}d$$

by (25), the proof is complete. □

3 Inverse Shadowing

Our result on inverse shadowing is, in a sense, local (in contrast to the case of shadowing) – we consider a fixed trajectory of the set-valued mapping F and look for close trajectories of sequences of mappings that approximate F.

Thus, let us fix a sequence of points $p_k \in \mathbb{R}^m$ such that $p_{k+1} \in F(p_k)$.

We assume that the mapping F is hyperbolic at the trajectory $\eta = \{p_k\}$ in the following sense: there exist constants $N \ge 1$, $a, \kappa, l > 0$, and $\lambda \in (0, 1)$ such that condition (P1) holds for points $x = p_k$, condition (P2) holds for points $x = p_k, y = p_{k+1}$, and vectors v with $|v| \le a$, and, finally, condition (P3) holds for points $x = p_k$ and vectors v with $|v| \le a$.

We also fix a number $d > 0$ and a sequence of mappings

$$\Phi = \{\Phi_k : \mathbb{R}^m \to CC(\mathbb{R}^m)\}$$

such that each Φ_k is continuous w.r.t. dist_H and

$$\mathrm{dist}_H(F(p_k + v), \Phi_k(p_k + v)) \le d \text{ for } k \in \mathbb{Z} \text{ and } |v| \le a. \tag{40}$$

We say that a sequence of points $x_k \in \mathbb{R}^m$ is a trajectory of the sequence Φ if $x_{k+1} \in \Phi_k(x_k)$.

Theorem 4. *Assume that a trajectory $\eta = \{p_k\}$ of F is hyperbolic in the above sense. If*

$$\lambda + \kappa + 4lN < 1, \tag{41}$$

then F has the inverse Lipschitz shadowing property: there exists a constant $d_0 > 0$ such that if a family of mappings Φ satisfies inequalities (40), where $d < d_0$, then there exists a trajectory $\{x_k\}$ of Φ such that

$$\|\{x_k\} - \{p_k\}\|_\infty \le \mathcal{L}d,$$

where

$$\mathcal{L}^{-1} = \frac{1}{2N}(1 - \lambda - \kappa - 4lN).$$

Proof. The line of argument is very similar to the proof of Theorem 1. Here, we construct a trajectory $\{x_k\}$ of Φ by proving the existence of a sequence $\{v_k\}$ such that

$$p_{k+1} + v_{k+1} \in \Phi_k(p_k + v_k)$$

and

$$\|V\|_\infty \le \mathcal{L}d. \tag{42}$$

We represent

$$\Phi_k(p_k + v) = L(p_k) + A(p_k)v + B(p_k, v) + \tilde{M}_k(p_k + v) \tag{43}$$

for small v, where each $\tilde{M}_k : \mathbb{R}^m \to CC(\mathbb{R}^m)$ is a continuous mapping w.r.t. dist_H such that

$$\mathrm{dist}_H(M(p_k + v), \tilde{M}_k(p_k + v)) \le d; \tag{44}$$

here we take into account that

$$F(p_k + v) = L(p_k) + A(p_k)v + B(p_k, v) + M(p_k + v),$$

inequalities (40) hold, and the Hausdorff distance between the sets $\Phi_k(p_k+v)$ and $F(p_k+v)$ is preserved when we shift these sets by the same vector $-(L(p_k)+A(p_k)v+B(p_k,v))$.

We must prove that there is a sequence $\{v_k\}$ such that

$$p_{k+1}+v_{k+1}\in L(p_k)+A(p_k)v_k+B(p_k,v_k)+\tilde{M}_k(p_k+v_k),\quad k\in\mathbb{Z}. \tag{45}$$

Similarly to the reasoning of the previous proof, we define vectors

$$\tilde{\sigma}_k=\mathrm{Dev}(p_{k+1},L(p_k)+B(p_k,v_k)+\tilde{M}(p_k+v_k)).$$

Our goal is to find a sequence of vectors $V=\{v_k\}$ such that

$$\tilde{\sigma}_k=v_{k+1}-A(p_k)v_k \tag{46}$$

and inequality (42) holds.

Indeed, it follows from (46) that

$$p_{k+1}+v_{k+1}=p_{k+1}+\tilde{\sigma}_k+A(p_k)v_k$$
$$\in L(p_k)+B(p_k,v_k)+A(p_k)v_k+\tilde{M}(p_k+v_k)=\Phi_k(p_k+v_k).$$

Now we estimate (for $|v_k|\le a$)

$$|\tilde{\sigma}_k|=\mathrm{dist}(p_{k+1},L(p_k)+B(p_k,v_k)+\tilde{M}(p_k+v_k))$$
$$\le \mathrm{dist}(p_{k+1},F(p_k))+\mathrm{dist}_H(L(p_k)+M(p_k),L(p_k)+B(p_k,v_k)+\tilde{M}(p_k+v_k))$$
$$=\mathrm{dist}_H(M(p_k),B(p_k+v_k)+\tilde{M}(p_k+v_k))$$

(we note that $\mathrm{dist}(p_{k+1},F(p_k))=0$ and shift the sets in the second term by $-L(p_k)$)

$$\le |B(p_k+v_k)|+\mathrm{dist}_H(M(p_k),M(p_k+v_k))+\mathrm{dist}_H(M(p_k+v_k),\tilde{M}_k(p_k+v_k))$$
$$\le d+2l|v_k|.$$

This estimate is similar to (20).

Now the operator $\tilde{T}:H\to H$ is defined by

$$Q(p_{k+1})w_{k+1}=Q(p_{k+1})A(p_k)v_k+Q(p_{k+1})\tilde{\sigma}_k \tag{47}$$

and

$$P(p_k)w_k=\Gamma(P(p_{k+1})[v_{k+1}-\tilde{\sigma}_k-A(p_k)Q(p_k)v_k]) \tag{48}$$

with H and Γ defined in (24) and via (27). (Of course, we replace $P(x_k)$ by $P(p_k)$ etc. in these definitions.) Since we have the same estimates of $|\sigma_k|$ and $|\tilde{\sigma}_k|$ and the operators $A(p_k)$ have the same properties as the operators $A(x_k)$ in Theorem 1, the rest of the proof is identical with that of Theorem 1, and all the constants are the same. □

4 Contractive Case

Various authors studied shadowing properties of set-valued dynamical systems with contractive properties ([7], [9], [12], [13], ...).

A set-valued dynamical system on a metric space $(\mathcal{M}, \mathrm{dist})$ determined by a set-valued mapping $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$ is called contractive if there exist constants $a > 0$ and $l \in (0, 1)$ such that if $p, q \in \mathcal{M}$ and $\mathrm{dist}(p, q) \leq a$, then

$$\mathrm{dist}_H(F(p), F(q)) \leq l \, \mathrm{dist}(p, q) \tag{49}$$

(we give here one of possible variants of the definition).

Shadowing results for contractive set-valued dynamical systems were established in [12] (for the case of mappings of $\mathbb{R}^m$ whose images are either convex or have "large continuous convex kernels") and in [7] (without the convexity assumption; unfortunately, the proof in [7] contains an error). Inverse shadowing results for contractive set-valued dynamical systems were also obtained in [12].

Let us note that in the case of $\mathcal{M} = \mathbb{R}^m$, a contractive set-valued dynamical system with convex and compact images $F(x)$ is a particular case of a system defined by a hyperbolic mapping (in the sense of the definition given in the Introduction). Indeed, in this case we may take any $\lambda \in (0, 1)$, $S(x) = \mathbb{R}^m, U(x) = \{0\}$, and $L(x) = 0$ (thus, $A = 0$) for any $x \in \mathbb{R}^m$. Then conditions (P1), (P2), and (10) hold with $N = 1$ and any $l, \kappa > 0$, while inequalities (11) are a reformulation of (49).

It was shown in [12] (see Remark 3) that there exist contractive set-valued dynamical systems such that, for pseudotrajectories with arbitrarily small errors, the shadowing trajectories are not necessarily unique. This means that our definition of hyperbolicity for set-valued dynamical systems does not imply the uniqueness of shadowing trajectories (this fact was mentioned in the Introduction).

Let us explain how to establish shadowing results for set-valued dynamical systems not assuming the convexity of images for generating set-valued mappings (the authors were informed about the fixed point theorem 3.1 of [6] by V. Glavan and V. Gutu in a private communication). We include this proof into the present paper due to the absence of such a proof in the literature.

In what follows, we assume that $(\mathcal{M}, \mathrm{dist})$ is a complete metric space.

Let $\mathcal{X}$ be the space of sequences $\xi = \{\xi_k \in \mathcal{M} : k \in \mathbb{Z}\}$ with the metric

$$\mathrm{Dist}(\xi, \eta) = \sup_k \mathrm{dist}(\xi_k, \eta_k).$$

Clearly, $(\mathcal{X}, \mathrm{Dist})$ is a complete metric space.

Let Dist_H be the Hausdorff metric on $\mathcal{X}$ generated by Dist.

Consider a set-valued mapping $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$; we relate to this mapping a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{C}(\mathcal{X})$ by the following rule:

$$\mathcal{T}(\xi) = \{\eta \in \mathcal{X} : \eta_{k+1} \in F(\xi_k) \ \forall k \in \mathbb{Z}\}.$$

A point $\xi \in \mathcal{X}$ (a sequence $\xi = \{\xi_k \in \mathcal{M}\}$) is called a fixed point of $\mathcal{T}$ if $\xi \in \mathcal{T}(\xi)$ (i.e., if $\xi_{k+1} \in F(\xi_k)$).

Now we formulate a fixed point theorem for $\mathcal{T}$ proved by Frigon and Granas in [6] (we denote elements of $\mathcal{X}$ by x, y etc).

Denote by $N(a, x)$ the closed a-neighborhood of $x \in \mathcal{X}$.

Theorem 5. *Assume that there exist numbers $r > 0$ and $\kappa \in (0, 1)$ and an element $x_0 \in \mathcal{X}$ such that*

$$\mathrm{Dist}_H(\mathcal{T}(x), \mathcal{T}(y)) \le \kappa \, \mathrm{Dist}(x, y), \quad x, y \in N(r, x_0), \tag{50}$$

and

$$\mathrm{Dist}(x_0, \mathcal{T}(x_0)) \le (1 - \kappa) r.$$

Then $\mathcal{T}$ has a fixed point in $N(r, x_0)$.

We deduce from this statement the following shadowing result.

Theorem 6. *Let $(\mathcal{M}, \mathrm{dist})$ be a complete metric space. Assume that for a set-valued mapping $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$ there exist constants $a > 0$ and $l \in (0, 1)$ such that if $p, q \in \mathcal{M}$ and $\mathrm{dist}(p, q) \le a$, then inequality (49) holds.*

Then F has the Lipschitz shadowing property: there exists a constant $d_0 > 0$ such that if $\{x_k\}$ is a d-pseudotrajectory of F with $d \le d_0$, then there exists a trajectory $\{p_k\}$ of F such that

$$\mathrm{Dist}(\{x_k\}, \{p_k\}) \le \mathcal{L} d,$$

where $\mathcal{L}^{-1} = 1 - l$.

Proof of Theorem 6. Take $d_0 = a/(2\mathcal{L})$ and let $x^0 = \{x_k^0 \in \mathcal{M}\}$ be a d-pseudotrajectory of F with $d \le d_0$.

We claim that for $r = \mathcal{L} d$, the mapping $\mathcal{T} : N(r, x^0) \to \mathcal{C}(\mathcal{X})$ has property (50) with $\kappa = l$. Indeed, if $x, y \in N(r, x^0)$, then $\mathrm{Dist}(x, y) \le 2r \le a$.

Thus, if $\xi \in \mathcal{T}(x)$, then

$$\mathrm{Dist}(\xi, \mathcal{T}(y)) \le \sup_k \mathrm{dist}(\xi_{k+1}, F(y_k)) \le \sup_k \mathrm{dist}_H(F(x_k), F(y_k))$$

$$\le \sup_k l \, \mathrm{dist}(x_k, y_k) = l \, \mathrm{Dist}(x, y),$$

and property (50) with $\kappa = l$ follows. In addition,

$$\mathrm{Dist}(x^0, \mathcal{T}(x^0)) \le \sup_k \mathrm{dist}(x^0_{k+1}, F(x^0_k)) \le d = (1-l)\mathcal{L}d = (1-l)r.$$

By the Frigon-Granas theorem, there is a fixed point x of $\mathcal{T}$ such that

$$\mathrm{Dist}(x, x_0) \le r = \mathcal{L}d.$$

It remains to note that the definition of a fixed point x of $\mathcal{T}$ implies that x is a trajectory of F. □

It is natural to try to apply a similar idea to a hyperbolic set-valued dynamical system in $\mathbb{R}^m$ that is not contractive. Analyzing the proofs of Theorems 1 and 6, one can see that in this case, one has to impose the following condition on the sets $F(x_k + v_k)$:

$$F(x_k + v_k) = P(x_{k+1})F(x_k + v_k) + Q(x_{k+1})F(x_k + v_k)$$

(i.e., to consider sets that are "products" of their projections to stable and unstable subspaces). In our opinion, such class of sets is of significantly less importance for applications (for example, to set-valued dynamical systems generated by differential inclusions) than the class of convex sets studied in this paper.

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