Spectral analysis of coupled hyperbolic-parabolic systems on finite and infinite intervals

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1 Introduction

In many applied problems in biology, physics or chemistry traveling waves arise as solutions of systems of partial differential equations of the form

$$U_t = f(U, U_x, U_{xx}) \quad \text{in } [0,\infty)\times \mathbb{R}. \tag{1}$$

A traveling wave has the special property that it is constant if one looks at it in a comoving frame. More precisely this means if U is a traveling wave solution of (1) with speed c, the function $\tilde{U}(t,x) := U(t, x+ct)$ is a steady state of the transformed PDE

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} U(t, x+ct) = f(\tilde{U}, \tilde{U}_x, \tilde{U}_{xx}) + c\tilde{U}_x \quad \text{in } [0,\infty)\times \mathbb{R}. \tag{2}$$

For the stability analysis of traveling waves it is important to know where the point spectrum of the linearized right hand side of (2) lies. We will show how this can be approximated by computing the spectrum of boundary value problems.

2 General assumptions on the problem's structure

We consider a linear coupled hyperbolic-parabolic PDE of the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = P \begin{pmatrix} u \\ v \end{pmatrix}_t := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3}$$

where $P : H^2(\mathbb{R},\mathbb{C}^n)\times H^1(\mathbb{R},\mathbb{C}^m) \to L^2(\mathbb{R},\mathbb{C}^n)\times L^2(\mathbb{R},\mathbb{C}^m)$. This structure, for example, may arise by linearizing the nonlinear PDE (2) at the wave's profile (see also Section 4).

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Assumptions. We now state the basic assumptions on the operator P.

(0) The matrix valued functions B_{ij} and C_{ij} are twice continuously differentiable and converge to constant limit matrices:
- $B_{11}(x) \to B_{11\pm} \in \mathbb{C}^{n,n}$, $C_{11}(x) \to C_{11\pm} \in \mathbb{C}^{n,n}$ as $x \to \pm\infty$.
- $B_{22}(x) \to B_{22\pm} \in \mathbb{C}^{m,m}$, $B_{22,x}(x) \to 0 \in \mathbb{C}^{m,m}$, and $C_{22}(x) \to C_{22\pm} \in \mathbb{C}^{m,m}$ as $x \to \pm\infty$, $\|B_{22,xx}\|_\infty < \infty$, $\|C_{22,x}\|_\infty < \infty$.
- $B_{12}(x) \to B_{12\pm}$, $C_{12}(x) \to C_{12\pm} \in \mathbb{C}^{n,m}$, and $C_{21}(x) \to C_{21\pm} \in \mathbb{C}^{m,n}$, as $x \to \pm\infty$, $\|B_{12,x}\|_\infty < \infty$.

(P) The matrix $A \in \mathbb{C}^{n,n}$ satisfies $A + A^* \geq \alpha I > 0$ as a quadratic form for some $\alpha \in \mathbb{R}$.

(H) The matrix function B_{22} is real diagonal valued and there exist $b_0, \gamma > 0$ such that for all $x \in \mathbb{R}$ the diagonal elements satisfy
$|b_{ii}(x) - b_{jj}(x)| \geq \gamma$, for $i \neq j$,
$b_{ii}(x) \geq b_0$, for $1 \leq i \leq r$, and $-b_{ii}(x) \geq b_0$, for $r+1 \leq i \leq m$.
Furthermore the real part of the diagonal entries of the limit matrices $C_{22\pm}$ is bounded from above by -2δ for some $\delta > 0$.

(D) There exists $\delta > 0$ such that for all $\omega \in \mathbb{R}$ and for all $s \in \mathbb{C}$ the equality

$$\det\left(-\omega^2\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix} + \mathrm{i}\omega\begin{pmatrix} B_{11\pm} & B_{12\pm} \\ 0 & B_{22\pm}\end{pmatrix} + \omega\begin{pmatrix} C_{11\pm} & C_{12\pm} \\ C_{21\pm} & C_{22\pm}\end{pmatrix}\right) = 0 \quad (4)$$

implies $\Re s \leq -\delta$.

Note that assumption (0) is generically satisfied if the profile of the traveling wave is a smooth connecting orbit of rest states of the nonlinear PDE (1).

For our analysis of the spectral properties of P we consider the resolvent equation

$$(sI - P)\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} f \\ g\end{pmatrix} \quad \text{in } L^2(\mathbb{R}, \mathbb{C}^n) \times L^2(\mathbb{R}, \mathbb{C}^m) \quad (5)$$

and transform it into a first order system using the variables $(u, Au_x, v) = z$

$$L(s)z = z_x - M(x,s)z = (0, -f + B_{12}B_{22}^{-1}g, -B_{22}^{-1}g)^T. \quad (6)$$

Here we consider the operator $L(s)$ as a mapping $H^2 \times H^1 \times H^1 \to H^1 \times L^2 \times L^2$. The matrix M is of the form

$$M(\cdot, s) = \begin{pmatrix} 0 & A^{-1} & 0 \\ B_{12}B_{22}^{-1}C_{21} + (sI - C_{11}) & -B_{11}A^{-1} & -C_{12} - B_{12}B_{22}^{-1}(sI - C_{22}) \\ -B_{22}^{-1}C_{21} & 0 & B_{22}^{-1}(sI - C_{22})\end{pmatrix}. \quad (7)$$

Recall that an operator $L : z \mapsto z_x - M(x)z$, $M(x) \in \mathbb{C}^{l,l}$ is said to have an exponential dichotomy on a closed interval J if there exist positive numbers K, β, and for every $x \in J$ there is a projection $\pi(x) \in \mathbb{C}^{l,l}$ such that

$$\begin{aligned} S(x,y)\pi(y) &= \pi(x)S(x,y),\ \forall x,y \in J,\\ |S(x,y)\pi(y)| &\le Ke^{-\beta(x-y)},\ \forall x \ge y \in J,\\ |S(x,y)(I-\pi(y))| &\le Ke^{-\beta(y-x)},\ \forall x < y \in J, \end{aligned} \tag{8}$$

where $S(\cdot,\cdot)$ is the solution operator to L. We call (K,β,π) the data of the exponential dichotomy. For general results about systems with exponential dichotomies see [4].

By assumption (0) the limit matrices $M_\pm(s) := \lim_{x\to\pm\infty} M(x,s)$ exist and the dispersion relation (D) implies that these are hyperbolic matrices for all $s \in \mathbb{C}$ with $\Re s > -\delta$. More generally $M_\pm(s)$ are non-hyperbolic if and only if the number s lies on the algebraic curves defined by (4). But for the stability analysis we are only interested in the rightmost part of the spectrum.

We denote by $V_\pm^I(s)$ and $V_\pm^{II}(s)$ bases of the stable and unstable subspace of $M_\pm(s)$, respectively. By the Roughness Theorem [4, 2] the hyperbolicity of the limit matrices implies that for each $s \in \{\Re s > -\delta\}$ the operator $L(s)$ has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$.

Using the transformation $\tilde z = \begin{pmatrix} I & 0 & 0\\ 0 & \frac{1}{\sqrt{s}}I & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} I & 0 & 0\\ 0 & I & B_{12\pm}\\ 0 & 0 & I\end{pmatrix} z$ for $\Re s$ sufficiently large, it is shown in [9] that the dimensions of $V_+^I(s)$ and $V_-^I(s)$, and of $V_+^{II}(s)$ and $V_-^{II}(s)$ coincide for $\Re s > -\delta'$. Then a result from [8] shows that $L(s)$ is a Fredholm operator of index zero for $\Re s > -\delta$. Utilizing the relation of $L(s)$ and $sI - P$ we obtain that for all $s \in \mathbb{C}$ with $\Re s > -\delta$ the operator $sI - P$ is Fredholm of index zero.

It is well known (e.g. [6]) that the operator $sI - P$ is one to one for $s \in \mathbb{R}$ sufficiently large, such that the following lemma is implied.

Lemma 1. *The operator P only has isolated eigenvalues of finite algebraic multiplicity in the right half plane $\{\Re s > -\delta\}$.*

Hence we can decompose the spectrum of P according to $\sigma(P) = \sigma_P \dot\cup \sigma_{\text{ess}}$, where σ_P is the point spectrum which consists of all isolated eigenvalues of finite multiplicity, and $\sigma_{\text{ess}} := \sigma(P) \setminus \sigma_P$ is the essential spectrum.

3 Statement of the main result

As explained in Section 2 there only is point spectrum to the right of the axis $\{\Re s = -\delta\}$. Since it is in general not possible to determine the location of the point spectrum analytically one has to approximate it numerically. One possibility is to implement the "Evans function" which is an analytic function whose roots coincide with eigenvalues of the operator, for details see [5] and the references therein. Another possibility is to restrict the operator P to a finite but large interval and to impose suitable boundary conditions, see for example [2]. Then one uses the spectrum of these boundary value problems as an approximation.

We follow the latter approach and consider the restriction of the resolvent equation (5) to a finite interval $J = [x_-, x_+]$

$$(sI - P_{|J}) \begin{pmatrix} u_J \\ v_J \end{pmatrix} = \begin{pmatrix} f_J \\ g_J \end{pmatrix} \quad \text{in } L^2(J, \mathbb{C}^n) \times L^2(J, \mathbb{C}^m), \tag{9}$$

where $P_{|J}$ is defined the same way as P, but only on the interval J. In order to obtain a well-posed problem on the finite interval J, we impose boundary conditions

$$R \begin{pmatrix} u_J \\ v_J \end{pmatrix} = \eta \in \mathbb{C}^{2n+m}, \tag{10}$$

where R is a two point boundary operator of the form

$$R \begin{pmatrix} u_J \\ v_J \end{pmatrix} := \begin{pmatrix} R_-^I & R_-^{II} & R_-^{III} \end{pmatrix} \begin{pmatrix} u_J(x_-) \\ u_{J,x}(x_-) \\ v_J(x_-) \end{pmatrix} + \begin{pmatrix} R_+^I & R_+^{II} & R_+^{III} \end{pmatrix} \begin{pmatrix} u_J(x_+) \\ u_{J,x}(x_+) \\ v_J(x_+) \end{pmatrix}. \tag{11}$$

The crucial assumption we must make on the boundary operator is the determinant condition

$$D(s) := \det \left[\begin{pmatrix} R_-^I & R_-^{II} A^{-1} & R_-^{III} \end{pmatrix} V_-^{II}(s) \,,\, \begin{pmatrix} R_+^I & R_+^{II} A^{-1} & R_+^{III} \end{pmatrix} V_+^I(s) \right]. \tag{12}$$

This condition basically states that the stable and unstable subspaces of the solutions can be controlled at the left and right endpoint of the interval, respectively. As above $V_\pm^{I/II}(s)$ are bases of the stable and unstable subspaces of $M_\pm(s)$.

Using a general convergence result, the following proposition, which gives quantitative resolvent estimates of the finite interval problem, is proven in [3]. It is also possible to prove this proposition applying techniques from the theory of exponential dichotomies as is done in [2] for the purely parabolic case.

Proposition 1. *Let $\Omega \subset \{\Re s > -\delta'\}$ be a compact subset of the resolvent set $\rho(P)$ of P and assume $D(s) \neq 0$ for all $s \in \Omega$.*

Then there is a compact interval J_0 and $K_0 > 0$ such that for all $J = [x_-, x_+] \supset J_0$ and for all $s \in \Omega$ the finite interval problem (9), (10) has for every right hand side $f_J \in L_2(J, \mathbb{C}^n)$, $g_J \in L_2(J, \mathbb{C}^m)$, $\eta \in \mathbb{C}^{2n+m}$ a unique solution $(u_J, v_J) \in H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m)$. Moreover the solution can be estimated by

$$\|u_J\|_{H^2} + \|v_J\|_{H^1} + |u_J|_\Gamma + |u_{J,x}|_\Gamma + |v_J|_\Gamma \leq K_0 \big(\|f\|_{L_2} + \|g\|_{L_2} + |\eta| \big),$$

where $|u|_\Gamma^2 := |u(x_-)|^2 + |u(x_+)|^2$.

Together with Lemma 1 this proposition also gives a qualitative result that at most the point spectrum of the operator P is approximated in the right half plane. But note that it does not say whether each eigenvalue is really approximated, and how close the approximate spectrum is to the spectrum of P. This will be the statement of our Theorem 1. Before we state this theorem we introduce some notation. As usual we denote by $\mathcal{N}(A)$ the kernel of an operator A. Furthermore we will use the Banach spaces

$$\begin{array}{ll} E := H^2(\mathbb{R}, \mathbb{C}^n) \times H^1(\mathbb{R}, \mathbb{C}^m), & F := L^2(\mathbb{R}, \mathbb{C}^n) \times L^2(\mathbb{R}, \mathbb{C}^m), \\ E_J := H^2(J, \mathbb{C}^n) \times H^1(J, \mathbb{C}^m), & F := L^2(J, \mathbb{C}^n) \times L^2(J, \mathbb{C}^m) \times \mathbb{C}^{2n+m}. \end{array}$$

Finally we denote by

$$\mathcal{A}(s) := sI - P \in L(E, F) \text{ and } \mathcal{A}_J(s) := \begin{pmatrix} sI - P_{|J} \\ R \end{pmatrix} \in L(E_J, F_J)$$

the operator polynomials, corresponding to the all line operator and to its finite interval approximation. Note that the boundary operator R is included in the definition of the finite interval approximation.

For an element s_0 in the point spectrum σ_P of P choose $\varepsilon > 0$ such that s_0 is the only element of the spectrum of P in the closed ball $\overline{B_\varepsilon(s_0)}$. Then we call $\sigma_J := \{s \in \overline{B_\varepsilon(s_0)} : s \text{ is an eigenvalue } \mathcal{A}_J(\cdot)\}$ the s_0–group of eigenvalues of $\mathcal{A}_J$[2].

Theorem 1. *Let the assumptions (0), (P), (H), and (D) hold and let Σ be an open neighborhood of the isolated eigenvalue s_0.*
Assume $D(s) \neq 0$ for all $s \in \Sigma$ and choose $\varepsilon > 0$ such that $\overline{B_\varepsilon(s_0)} \subset \Sigma$. Let $\beta_\pm$ denote the dichotomy exponents of $L(s_0)$ on $\mathbb{R}_\pm$.
Then there is a compact interval $J_0 \subset \mathbb{R}$ such that for every interval $J = [x_-, x_+] \supset J_0$ the following properties hold.

The s_0-group of eigenvalues σ_J converges to the eigenvalue s_0, in the sense that for every $0 < \beta' < \min(\beta_-, \beta_+)$ there is $c = c(\beta') > 0$ such that

$$\max_{s \in \sigma_J} |s - s_0| = \operatorname{dist}(\sigma_J, s_0) \leq c e^{-\frac{\beta'}{\kappa} \min(x_+, -x_-)}, \tag{13}$$

where κ is the maximal multiplicity of an eigenvector of $\mathcal{A}(\cdot)$ to the eigenvalue s_0.

The eigenspace of $\mathcal{A}_J(\cdot)$ for σ_J converges to the eigenspace of $\mathcal{A}(\cdot)$ to the eigenvalue s_0. Moreover the following estimate holds

$$\sup_{\left[\begin{array}{c} \left\|\binom{u_J}{v_J}\right\|_{E_J} = 1 \\ s_J \in \sigma_J, \\ \binom{u_J}{v_J} \in \mathcal{N}(\mathcal{A}_J(s_J)) \end{array}\right]} \inf_{\binom{u_0}{v_0} \in \mathcal{N}(\mathcal{A}(s_0))} \left\| \begin{pmatrix} u_J \\ v_J \end{pmatrix} - \begin{pmatrix} u_{0|J} \\ v_{0|J} \end{pmatrix} \right\|_{E_J} \leq c\, e^{-\frac{\beta'}{\kappa} \min(x_+, -x_-)}. \tag{14}$$

Furthermore the dimensions of the generalized eigenspace to $\mathcal{A}(\cdot)$ to s_0 and of $\mathcal{A}_J(\cdot)$ to the s_0–group σ_J coincide.

[2] For a definition of eigenvalue, eigenvector, and multiplicity of eigenvectors for operator polynomials see, for example, [7].

The theorem is state in a more quantitative version in [9], where also a closeness result for the generalized eigenspaces is stated. The proof is in [9] and it applies the abstract theory of discrete approximations (see [10]). The main problem in the proof is to show that for every $s \in \Sigma$ the operators $\mathcal{A}_J(s)$ regularly converge to the all line operator $\mathcal{A}(s)$. This is shown in [3].

4 Approximating the point spectrum of the FitzHugh-Nagumo system

As an example we consider the FitzHugh-Nagumo system. The system reads

$$u_t = u_{xx} + u - \frac{1}{3}u^3 - v, \quad v_t = \Phi(u + a - bv), \tag{15}$$

where we choose the parameter values $a = 0.7$, $b = 0.8$, $\Phi = 0.08$. It is well known that for this choice of parameters the system (15) has a stable and an unstable traveling pulse. We denote by $(\overline{u}, \overline{v})^T(x)$ the profile of the pulse and by c its speed. Furthermore let $(\overline{u}_\infty, \overline{v}_\infty)^T$ denote the limit for $x \to \infty$.

Linearizing the system in the comoving frame at the profile leads to the linear PDE (cf. (3))

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 - \overline{u}^2 & -1 \\ \Phi & -\Phi b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{16}$$

Obviously the assumptions (0), (P), and (H) are satisfied. It follows from the following observation that also assumption (D) is fulfilled.

If there are matrices $H_\pm = \begin{pmatrix} H_\pm^i & 0 \\ 0 & H_\pm^{ii} \end{pmatrix}$, $H_\pm = H_\pm^* > 0$, $H_\pm^{ii}$ diagonal, with

$$\begin{gathered} H_\pm^i A + A^* H_\pm^i > 0, \quad H_\pm B_\pm - B_\pm^* H_\pm = 0, \\ H_\pm C_\pm + C_\pm^* H_\pm < -2\delta H \quad \text{for some } \delta > 0 \end{gathered}$$

then assumption (D) holds.

Choosing $H = \operatorname{diag}(1, \Phi^{-1})$ shows that the FitzHugh-Nagumo system also satisfies (D).

For our computations we approximate the unstable traveling wave on a large interval $J = [0, 65]$ with projection boundary conditions, see [1]. For the computation of the spectrum we then linearize at this approximation and discretize using central differences and suitable boundary conditions. For the figures shown in this paper we always used periodic boundary conditions since they obviously satisfy the determinant condition (12) for all s with $\Re s > -\delta$ if the profile is a pulse.

In Figure 1 we plot the spectrum of (16) with periodic boundary conditions corresponding to the unstable wave. One can see the isolated eigenvalue at 0 which always appears for traveling waves and also that the unstable eigenvalue is present in the spectrum.

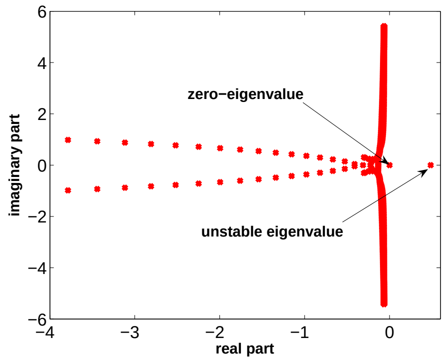

Fig. 1. The spectrum of the finite interval operator using central differences and periodic boundary conditions.

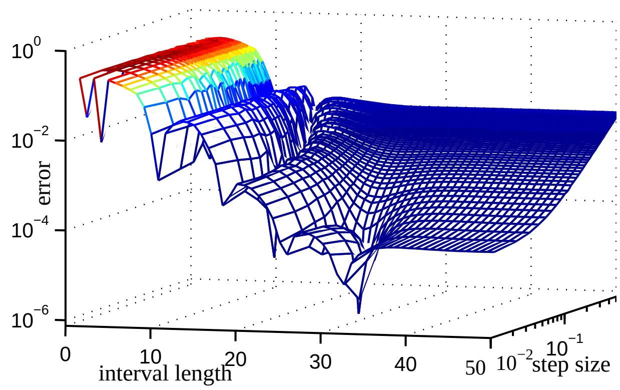

Fig. 2. Convergence of the zero eigenvalue for the unstable traveling wave for different step sizes and interval lengths.

In Figure 2 we compute the length of the eigenvalue closest to zero of the discretized operator for different step sizes and interval lengths. Similarly in Figure 3 we compute the distance of the eigenvalue with maximal real part of the discretized operator to the unstable eigenvalue of the pulse. Since the exact value of the unstable eigenvalue is not known explicitly we treated the eigenvalue with maximal real part of the operator on the interval $J = [0, 65]$ with step size 0.005 as the exact unstable eigenvalue.

One can observe the exponential rate of convergence of the eigenvalues depending on the interval length as predicted in Theorem 1. Furthermore one can see that the eigenvalues seem to converge exponentially fast, also in the step size of the discretization. Note that the convergence to the unstable eigenvalue seems to be much better. This is, because we only use an approximation of it which is not exact. So the picture is probably to optimistic.

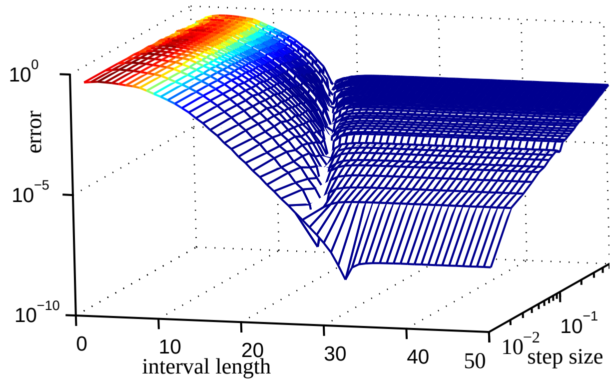

Fig. 3. Convergence of the unstable eigenvalue.

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