Shadowing and the Viability Kernel Algorithm

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Abstract

The aim of this paper is to derive estimates for the accuracy of the viability algorithm for systems which have shadowing properties. Recently developped shadowing results are applied in order to prove that the algorithm is linearly convergent for a certain class of right hand sides.

Key words. viability kernel algorithm, shadowing
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1 Introduction

Viability kernels of differential inclusions are of considerable interest, because many theoretical and practical problems can be reformulated as viability problems. A standard reference on viability theory is the book [2] by Aubin. The viability kernel $\mathrm{Viab}_F(K)$ of a set K is the largest subset of K such that for every $x_0 \in \mathrm{Viab}_F(K)$ there exists at least one solution of the differential inclusion which starts in x_0 and stays in $\mathrm{Viab}_F(K)$ for all time, or, in other words, the maximal weakly invariant subset of K. In applications, K usually denotes the set of acceptable states of a system, while the viability kernel is the subset of all states from which it is possible not to leave K.

Unfortunately, it is very difficult to calculate viability kernels analytically and thus reliable numerical methods are required. In [5], Frankowska and Quincampoix proposed a first algorithm for the computation of viability kernels, and Saint-Pierre succeeded to prove the convergence of a fully discretized and hence implementable algorithm in [11]. This viability kernel algorithm was later generalized to impulsive differential inclusions in [4]. However, it is still an open question how fast this algorithm converges.

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The shadowing property has been thoroughly investigated in the framework of dynamical systems, see [7] and [8]. A discrete time dynamical system has the shadowing property, if for every pseudotrajectory, i.e. any bi-infinite sequence which satisfies the defining relations of the system in every time step up to a small error, there exists a real trajectory which is close to it. Pseudotrajectories arise quite naturally from inaccuracies caused by imprecise models or numerical approximations. If a dynamical system has the shadowing property, it is guaranteed that numerical simulations yield a realistic picture of its behaviour without an explosion of the error of approximation in time. In set-valued systems, shadowing theory is still under development. First attempts to generalize the classical ideas can be found in [1], [6], [9], [10], and [12].

As viability theory describes the behaviour of trajectories on the unbounded time interval $[0, \infty)$, it seems natural to use shadowing results as tools in this context. The aim of this paper is to derive estimates for the accuracy of the fully discretized viability algorithm for systems which have the shadowing property. In Section 4, a shadowing theorem from [10] is applied in order to show that the viability algorithm is linearly convergent for a certain class of differential inclusions. Since viability kernels have to be computed online in critical applications such as air traffic control and control of unmanned vehicles, estimates for the accuracy of the algorithm are of practical importance.

2 Definitions

Let X be a finite-dimensional vector space with unit ball $B \subset X$, and let $K \subset X$ be a compact subset. By $\text{dist}(\cdot, \cdot)$ and $\text{dist}_H(\cdot, \cdot)$ we will denote the one-sided and the symmetric Hausdorff distance, respectively.

Consider the autonomous differential inclusion

$$\dot{x}(t) \in F(x(t)) \text{ for almost every } t \geq 0, \quad x(0) = x_0 \in X \tag{1}$$

and its ρ-flow

$$G_\rho : X \rightrightarrows X, \quad x \mapsto \mathcal{R}(0, \rho, x), \tag{2}$$

where $\mathcal{R}(0, \rho, x)$ denotes the reachable set of (1) at time ρ. For any $h > 0$ let $X_h \subset X$ be a countable subset with

$$\forall x \in X, \ \exists x_h \in X_h \text{ with } |x - x_h| \leq \alpha(h) \tag{3}$$

and $\lim_{h \to \infty} \alpha(h) = 0$. Given any subset $A \subset X$ and $\varepsilon > 0$, we define

$$A^\varepsilon := A + \varepsilon B \text{ and } A_h^\varepsilon := A^\varepsilon \cap X_h. \tag{4}$$

Consider the semi-discretized Euler scheme

$$\Gamma_\rho : X \rightrightarrows X, \quad x \mapsto x + \rho F(x) \tag{5}$$

and the fully discretized scheme

$$\Gamma_{\rho,h} : X_h \rightrightarrows X_h, \quad x_h \mapsto \big(x_h + \rho F(x_h) + k(\rho,h)B\big) \cap X_h, \tag{6}$$

with $k(\rho,h) := (2 + L\rho)\alpha(h)$, where $L > 0$ will be a Lipschitz constant of a restriction of F (see assumption (iii) below).

Definition 1. *Let $G : X \rightrightarrows X$ be a set-valued mapping. Every sequence $(p_n)_{n\in\mathbb{N}}$ with $p_{n+1} \in G(p_n)$ for all $n \in \mathbb{N}$ is called an orbit of G. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $\operatorname{dist}(x_{n+1}, G(x_n)) \leq d$ for every $n \in \mathbb{N}$ is called a d-pseudotrajectory of G.*

Definition 2. *A mapping G has the (d,ε)-shadowing property in K, if for every d-pseudotrajectory $(x_n)_{n\in\mathbb{N}} \subset K$ there is an orbit $(p_n)_{n\in\mathbb{N}}$ of G such that $|x_n - p_n| \leq \varepsilon$ for all $n \in \mathbb{N}$.*

Definition 3. *A mapping G has the inverse (d,ε)-shadowing property in K, if for every continuous mapping $\Phi : X \rightrightarrows X$ with compact and convex values satisfying*

$$\operatorname{dist}_H(\Phi(x), G(x)) \leq d$$

for all x in a neighbourhood of K and every orbit $(p_n)_{n\in\mathbb{N}} \in K$ of G there is an orbit $(x_n)_{n\in\mathbb{N}}$ of Φ such that $|p_n - x_n| \leq \varepsilon$ for all $n \in \mathbb{N}$.

The definition of the inverse shadowing property is kept slightly vague on purpose, because the required size of the neighbourhood depends on the shadowing theorem which is applied in order to ensure that the system has this property (see Section 4 for a concrete example).

Definition 4. *A subset $D \subset X$ is a discrete viability domain of a set-valued mapping G, if*

$$G(x_0) \cap D \neq \emptyset \text{ for all } x_0 \in D.$$

The discrete viability kernel $\operatorname{Viab}_G(K)$ is the largest closed discrete viability domain contained in K.

Analogously, a subset $D \subset X$ is a viability domain of the differential inclusion (1), if for any $x_0 \in D$ there exists a solution $x : [0,\infty) \to D$ such that $x(0) = x_0$. The viability kernel $\operatorname{Viab}_F(K)$ is the largest closed viability domain of (1) contained in K.

Under mild assumptions on F and G, both types of viability kernels are well-defined, compare e.g. [2].

The viability kernel algorithm presented in [11] computes the viability kernel $\mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^\varepsilon)$ for a suitable $\varepsilon > 0$ instead of the analytical object $\mathrm{Viab}_F(K)$. As K^ε is compact, K_h^ε is a finite set, and thus the algorithm terminates after at most $\#K_h^\varepsilon$ steps. Estimating the accuracy of the algorithm amounts to estimating the Hausdorff distance $\mathrm{dist}_H(\mathrm{Viab}_F(K), \mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^\varepsilon))$. Please note that the mappings Γ_ρ, $\Gamma_{\rho,h}$, etc. are defined in a slightly different way in the present paper for technical reasons, but that the basic idea has been adopted without any changes.

Throughout this paper, we will suppose the following assumptions:

(i) The viability kernel $\mathrm{Viab}_F(K)$ is stable in the sense that there exists an $\varepsilon_0 > 0$ such that $\mathrm{Viab}_F(K) = \mathrm{Viab}_F(K^{\varepsilon_0})$.

(ii) There exist a $d_0^{(s)} > 0$ and a $d_0^{(is)} > 0$ such that the ρ-flow G_ρ has

(iia) the $(d, \varphi(d))$-shadowing property in K^{ε_0} for $d \in (0, d_0^{(s)}]$ and

(iib) the inverse $(d, \psi(d))$-shadowing property in K for $d \in (0, d_0^{(is)}]$,

where $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ are increasing functions with $\lim_{d\to 0} \varphi(d) = 0$ and $\lim_{d\to 0} \psi(d) = 0$.

(iii) The mapping F is Lipschitz-continuous in K^{ε_0} with Lipschitz constant $L > 0$ and has compact and convex values.

3 Results

3.1 General estimates

Observation 1. *For any compact set $A \subset X$, the viability kernels can be characterized by*

$$\mathrm{Viab}_G(A) = \{x_0 \in A : \exists (p_n)_{n\in\mathbb{N}} \subset A \text{ with } p_0 = x_0 \text{ and } p_{n+1} \in G(p_n)\ \forall n \in \mathbb{N}\}$$

and

$$\mathrm{Viab}_F(A) = \{x_0 \in A : \exists \text{ a solution } x : [0,\infty) \to A \text{ of (1) with } x(0) = x_0\}.$$

It is obvious that the right hand sides are the largest viability domains contained in A. Under mild assumptions on F and G they are closed, compare e.g. Theorem 3.5.3 in [2].

Observation 2. *Because of assumption (iii), F is bounded on K^{ε_0} by $\|F\|_\infty = M < \infty$. Thus any solution x of (1) remaining in K^{ε_0} satisfies*

$$|x(t) - x(0)| \le \int_0^t |\dot{x}(s)| ds \le Mt. \tag{7}$$

If $x(0) \in K^\varepsilon$ with $0 < \varepsilon < \varepsilon_0$, we have $x(t) \in K^{\varepsilon_0}$ for all $t \in [0, \rho]$ with $0 < \rho < \frac{\varepsilon_0 - \varepsilon}{M}$. Otherwise $0 < t_0 := \inf\{t \in [0, \rho] : x(t) \notin K^{\varepsilon_0}\} < \rho$ and (7) holds for all $0 \le t \le t_0$. But then

$$|x(t_0) - x(0)| \le Mt_0 < M\rho \le \varepsilon_0 - \varepsilon$$

implies that $x(t_0)$ is in the interior of K^{ε_0}, which is a contradiction. Thus we can apply (7) for all $t \in [0, \rho]$. If $x(0) \in K^\varepsilon$ and $x(\rho) \in K^\varepsilon$, we can apply it forwards in time from $x(0)$ and backwards in time from $x(\rho)$ in order to obtain

$$\mathrm{dist}(x(s), K^\varepsilon) \le \frac{1}{2} M\rho \ \forall s \in [0, \rho]. \tag{8}$$

Lemma 3. *We have*

$$\mathrm{Viab}_F(K) = \mathrm{Viab}_{G_\rho}(K^\varepsilon)$$

for all $0 \le \varepsilon < \varepsilon_0$ and $0 < M\rho < \varepsilon_0 - \varepsilon$.

Proof. Obviously $\mathrm{Viab}_F(K) \subset \mathrm{Viab}_{G_\rho}(K^\varepsilon)$. But

$$\begin{aligned} x_0 \in \mathrm{Viab}_{G_\rho}(K^\varepsilon) &\Rightarrow \exists (p_n)_{n\in\mathbb{N}} \subset K^\varepsilon : \ p_0 = x_0, \ p_{n+1} \in G_\rho(p_n) \ \forall n \in \mathbb{N} \\ &\Rightarrow \exists \text{ a solution } x : [0, \infty) \to X \text{ of } (1) : \ x(n\rho) = p_n \in K^\varepsilon \\ &\Rightarrow \mathrm{dist}(x(t), K^\varepsilon) \le \frac{1}{2} M\rho \ \forall t \ge 0 \\ &\Rightarrow x_0 \in \mathrm{Viab}_F(K^{\varepsilon + \frac{1}{2} M\rho}) = \mathrm{Viab}_F(K) \end{aligned}$$

by assumption (i) and Observation 2. □

The following Lemma is contained implicitly in many works, because it estimates the local error of the semi-discretized Euler-scheme. Nevertheless it is included explicitly in this paper for the sake of self-containedness and readability.

Lemma 4. *The error of approximation between G_ρ and Γ_ρ is*

$$\mathrm{dist}_H(G_\rho(x_0), \Gamma_\rho(x_0)) \le M\rho(e^{L\rho} - 1)$$

for all $0 < \varepsilon < \varepsilon_0$, $x_0 \in K^\varepsilon$, and $\rho > 0$ such that $M\rho e^{L\rho} \le \varepsilon_0 - \varepsilon$.

Proof. Let a solution $x : [0, \rho] \to X$ of (1) with $x(0) = x_0$ be given. Because of (7), $x(s) \in K^{\varepsilon_0}$ for all $s \in [0, \rho]$. As F has convex values, we can identify the Euler-step $\Gamma_\rho(x_0)$ with the reachable set of the constant differential inclusion

$$\dot{e}(t) \in F(x_0), \quad e(0) = x_0 \tag{9}$$

at time ρ. Since

$$\mathrm{dist}(\dot{x}(t), F(x(0))) \leq \mathrm{dist}(F(x(t)), F(x(0))) \leq L|x(t) - x(0)| \leq LMt,$$

the Filippov theorem (cf. Theorem 2.4.1 in [3]) guarantees the existence of a solution $e : [0, \rho] \to X$ of (9) satisfying

$$|x(t) - e(t)| \leq \int_0^t e^{L(t-s)} LMt ds = Mt(e^{Lt} - 1)$$

for all $t \in [0, \rho]$, and in particular

$$\mathrm{dist}(x(\rho), \Gamma_\rho(x_0)) \leq |x(\rho) - e(\rho)| \leq M\rho(e^{L\rho} - 1).$$

Conversely, let $\eta \in F(x_0)$ be given and consider the corresponding linear trajectory $e(t) := x_0 + t\eta$ for $t \in [0, \rho]$ of Γ_ρ. As

$$\mathrm{dist}(\dot{e}(t), F(e(t))) \leq \mathrm{dist}(F(x_0), F(e(t))) \leq LMt,$$

the Filippov theorem yields a solution $x : [0, \rho] \to X$ of (1) with $x(0) = x_0$ and

$$|x(t) - e(t)| \leq \int_0^t e^{L(t-s)} LMt ds = Mt(e^{Lt} - 1)$$

for all $t \in [0, \rho]$, and in particular

$$\mathrm{dist}(e(\rho), G_\rho(x_0)) \leq |e(\rho) - x(\rho)| \leq M\rho(e^{L\rho} - 1).$$

□

3.2 Estimates using the shadowing and the inverse shadowing property

Lemma 5. *If* $M\rho(e^{L\rho} - 1) \leq d_0^{(is)}$, $\varepsilon_1 := \psi(M\rho(e^{L\rho} - 1)) \leq \varepsilon_0$ *and* $M\rho e^{L\rho} \leq \varepsilon_0$ *then*

$$\mathrm{dist}(\mathrm{Viab}_F(K), \mathrm{Viab}_{\Gamma_\rho}(K^{\varepsilon_1})) \leq \varepsilon_1,$$

Proof. Let $p_0 \in \mathrm{Viab}_F(K) = \mathrm{Viab}_{G_\rho}(K)$ be given. Hence there exists an orbit $(p_n)_{n\in\mathbb{N}}$ of G_ρ such that $p_n \in K$ for all $n \in \mathbb{N}$. As Γ_ρ is continuous with compact and convex values, Lemma 4 ensures that Γ_ρ is an approximation of G_ρ in the sense of assumption (iib), which in turn yields the existence of an orbit $(x_n)_{n\in\mathbb{N}}$ of Γ_ρ such that such that $|p_n - x_n| \le \varepsilon_1$. Thus $x_n \in K^{\varepsilon_1}$ for all $n \in \mathbb{N}$, and $x_0 \in \mathrm{Viab}_{\Gamma_\rho}(K^{\varepsilon_1})$ by Observation 1. □

The following lemma uses a simple calculation: For any $A \subset X$, we have $\mathrm{dist}(A, A_h^{\alpha(h)}) \le \alpha(h)$, because for every $a \in A$ there is an $x_h \in X_h$ such that $|a - x_h| \le \alpha(h)$, and then $x_h \in A^{\alpha(h)} \cap X_h$.

Lemma 6. *If $\varepsilon_1 + \alpha(h) \le \varepsilon_0$, then*

$$\mathrm{dist}(\mathrm{Viab}_{\Gamma_\rho}(K^{\varepsilon_1}), \mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^{\varepsilon_1+\alpha(h)})) \le \alpha(h).$$

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a viable orbit of Γ_ρ in K^{ε_1}. By definition, there exists a $\xi_0 \in K_h^{\varepsilon_1+\alpha(h)}$ such that $|x_0 - \xi_0| \le \alpha(h)$. Since

$$\mathrm{dist}(x_0 + \rho F(x_0), \xi_0 + \rho F(\xi_0)) \le (1 + L\rho)\alpha(h), \tag{10}$$

we have

$$\mathrm{dist}(x_0 + \rho F(x_0), \xi_0 + \rho F(\xi_0) + (1 + L\rho)\alpha(h)B) = 0, \tag{11}$$

and thus

$$\begin{aligned} & \mathrm{dist}(\Gamma_\rho(x_0), \Gamma_{\rho,h}(\xi_0)) \\ = \; & \mathrm{dist}(x_0 + \rho F(x_0), (\xi_0 + \rho F(\xi_0) + (2 + L\rho)\alpha(h)B) \cap X_h) \\ \le \; & \alpha(h). \end{aligned}$$

Thus there exists a $\xi_1 \in \Gamma_{\rho,h}(\xi_0)$ such that $|x_1 - \xi_1| \le \alpha(h)$, and by induction there exists a whole orbit $(\xi_n)_{n\in\mathbb{N}}$ of $\Gamma_{\rho,h}$ with $|x_n - \xi_n| \le \alpha(h)$ for all $n \in \mathbb{N}$. Consequently $\xi_n \in K_h^{\varepsilon_1+\alpha(h)}$ for all $n \in \mathbb{N}$, and $\xi_0 \in \mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^{\varepsilon_1+\alpha(h)})$. □

Lemma 7. *Let $\varepsilon_2 := \varphi(k(\rho, h) + M\rho(e^{L\rho} - 1))$. If $M\rho e^{L\rho} \le \varepsilon_0 - \varepsilon_1 - \alpha(h)$, $M\rho \le \varepsilon_0 - \varepsilon_1 - \alpha(h) - \varepsilon_2$, and $k(\rho, h) + M\rho(e^{L\rho} - 1) \le d_0^{(s)}$,*

$$\mathrm{dist}(\mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^{\varepsilon_1+\alpha(h)}), \mathrm{Viab}_F(K)) \le \varepsilon_2.$$

Proof. By Lemma 4,

$$\begin{aligned} & \mathrm{dist}(\Gamma_{\rho,h}(x_h), G_\rho(x_h)) \\ \le \; & \mathrm{dist}(x_h + \rho F(x_h) + k(\rho, h)B, x_h + \rho F(x_h)) + \mathrm{dist}(x_h + \rho F(x_h), G_\rho(x_h)) \\ \le \; & k(\rho, h) + M\rho(e^{L\rho} - 1) =: d \end{aligned}$$

for every $x_h \in K_h^{\varepsilon_1+\alpha(h)}$. Thus any trajectory $(\xi_n)_{n\in\mathbb{N}}$ of $\Gamma_{\rho,h}$ which is viable in $K_h^{\varepsilon_1+\alpha(h)}$ is a d-pseudotrajectory of G_ρ, and assumption (iia) implies the existence of an orbit $(p_n)_{n\in\mathbb{N}}$ of G_ρ such that $|p_n - \xi_n| \leq \varepsilon_2$ for all $n \in \mathbb{N}$. Hence $x_0 \in \mathrm{Viab}_{G_\rho}(K^{\varepsilon_1+\alpha(h)+\varepsilon_2}) = \mathrm{Viab}_F(K)$ by Lemma 3 and Observation 1. □

Altogether we obtain an estimate for the accuracy of the viability kernel algorithm:

Theorem 8. *If*

$$\begin{aligned} k(\rho,h) + M\rho(e^{L\rho}-1) &\leq d_0^{(s)}, & (12)\\ M\rho(e^{L\rho}-1) &\leq d_0^{(is)}, & (13)\\ M\rho e^{L\rho} + \varepsilon_1 + \alpha(h) &\leq \varepsilon_0, & (14)\\ M\rho + \varepsilon_1 + \varepsilon_2 + \alpha(h) &\leq \varepsilon_0, & (15)\end{aligned}$$

and if assumptions (i) to (iii) are satisfied, then

$$\mathrm{dist}_H(\mathrm{Viab}_F(K), \mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^{\varepsilon_1+\alpha(h)})) \leq \max\{\varepsilon_1 + \alpha(h), \varepsilon_2\}. \quad (16)$$

The conditions of Theorem 8 do not look very appealing. Please note that they can be verified easily in a quite practical sense: Let a desired accuracy $\delta > 0$ of the approximation of the viability kernel be prescribed. If concrete monotonous functions φ and ψ are given, the inequalities (12) to (15) together with

$$\max\{\varepsilon_1 + \alpha(h), \varepsilon_2\} \leq \delta \quad (17)$$

can be regarded as constraints which are monotonous w.r.t. ρ and $\alpha(h)$. One can fix a (sufficiently small) $\rho > 0$ and determine the maximal $\alpha(h)$ such that the pair $(\rho, \alpha(h))$ satisfies all constraints using a simple interval subdivision algorithm. The same method can be used in the context of Theorem 11.

3.3 Estimates using the shadowing property only

It is possible to dispense with the inverse shadowing property by inflating the right hand sides of the numerical schemes so much that the numerical errors are 'swallowed' by the inflation. Therefore, we define

$$\tilde{\Gamma}_\rho : X \rightrightarrows X, \quad x \mapsto x + \rho F(x) + M\rho(e^{L\rho}-1)B \quad (18)$$

and

$$\tilde{\Gamma}_{\rho,h} : X_h \rightrightarrows X_h, \quad x_h \mapsto \big(x_h + \rho F(x_h) + M\rho(e^{L\rho}-1)B + k(\rho,h)B\big) \cap X_h. \quad (19)$$

For these schemes we have

Lemma 9. *If $M\rho e^{L\rho} \le \varepsilon_0$ and $\alpha(h) \le \varepsilon_0$,*

$$\mathrm{dist}(\mathrm{Viab}_F(K), \mathrm{Viab}_{\tilde{\Gamma}_{\rho,h}}(K_h^{\alpha(h)})) \le \alpha(h).$$

Proof. According to Lemma 4,

$$\mathrm{dist}(G_\rho(x_0), \Gamma_\rho(x_0)) \le M\rho(e^{L\rho} - 1)$$

for all $x_0 \in K$, and thus

$$\mathrm{dist}(G_\rho(x_0), \tilde{\Gamma}_\rho(x_0)) = 0$$

and

$$\mathrm{dist}(\mathrm{Viab}_{G_\rho}(K), \mathrm{Viab}_{\tilde{\Gamma}_\rho}(K)) = 0.$$

Adapting the proof of Lemma 6 to $\tilde{\Gamma}_\rho$ and $\tilde{\Gamma}_{\rho,h}$ yields the desired result. □

Lemma 10. *If $\varepsilon_3 := \varphi(2M\rho(e^{L\rho} - 1) + k(\rho, h)) \le \varepsilon_0 - \alpha(h)$, $M\rho e^{L\rho} \le \varepsilon_0 - \alpha(h)$, and $2M\rho(e^{L\rho} - 1) + k(\rho, h) \le d_0^{(s)}$, then*

$$\mathrm{dist}(\mathrm{Viab}_{\tilde{\Gamma}_{\rho,h}}(K_h^{\alpha(h)}), \mathrm{Viab}_F(K)) \le \varepsilon_3.$$

Proof. By Lemma 4,

$$\begin{aligned}
&\mathrm{dist}(\tilde{\Gamma}_{\rho,h}(x_h), G_\rho(x_h)) \\
\le\ &\mathrm{dist}(x_h + \rho F(x_h) + M\rho(e^{L\rho} - 1)B + k(\rho, h)B, x_h + \rho F(x_h)) \\
&+ \mathrm{dist}(x_h + \rho F(x_h), G_\rho(x_h)) \\
\le\ &2M\rho(e^{L\rho} - 1) + k(\rho, h) =: \tilde{d}
\end{aligned}$$

for any $x_h \in K_h^{\alpha(h)}$.

Thus any trajectory $(\xi_n)_{n\in\mathbb{N}}$ of $\tilde{\Gamma}_{\rho,h}$ which is viable in $K_h^{\alpha(h)}$ is a $\tilde{d}$-pseudotrajectory of G_ρ, and assumption (iia) implies that there exists an orbit $(p_n)_{n\in\mathbb{N}}$ of G_ρ such that $|p_n - \xi_n| \le \varepsilon_3$ for all $n \in \mathbb{N}$. Hence $p_0 \in \mathrm{Viab}_{G_\rho}(K^{\alpha(h)+\varepsilon_3}) = \mathrm{Viab}_F(K)$ by Lemma 3. □

Summarizing we obtain an estimate for the accuracy of the viability kernel algorithm for systems which have the shadowing but not the inverse shadowing property:

Theorem 11. *If*

$$M\rho e^{L\rho} + \alpha(h) \le \varepsilon_0, \tag{20}$$

$$2M\rho(e^{L\rho} - 1) + k(\rho, h) \le d_0^{(s)}, \tag{21}$$

$$\varepsilon_3 + \alpha(h) \le \varepsilon_0, \tag{22}$$

and if assumptions (i), (iia), and (iii) are satisfied, then

$$\mathrm{dist}_H(\mathrm{Viab}_F(K), \mathrm{Viab}_{\tilde{\Gamma}_{\rho,h}}(K_h^{\alpha(h)})) \le \max\{\alpha(h), \varepsilon_3\}. \tag{23}$$

4 Example

It was shown in [9] that certain differential inclusions with a contractive ρ-flow have the shadowing and the inverse shadowing property. The results which are relevant for this paper are summarized in the following

Theorem 12. *Let the right hand side $F : X \rightrightarrows X$ of* (1) *be Lipschitz continuous with constant L and with compact and convex values, and let $\|F(x)\| \leq M$ for some $M > 0$ and all $x \in X$. Assume that there exist numbers $a > 0$ and $\lambda \in (0,1)$ such that the ρ-flow defined by* (2) *satisfies the stability condition*

$$\mathrm{dist}_H(G_\rho(x), G_\rho(x+v)) \leq \lambda |v| \tag{24}$$

for any $x \in X$ and $|v| \leq a$. If $\rho \leq \frac{d}{2M}$ and $d < d_0 := \frac{1-\lambda}{2}a$, then G_ρ has the $(d, \frac{2d}{1-\lambda})$-shadowing property and the inverse $(d, \frac{2d}{1-\lambda})$-shadowing property in X.

In the same paper it was shown that a differential inclusion satisfies (24) if its right hand side has the relaxed one-sided Lipschitz property with a negative Lipschitz constant.

Theorem 12 is formulated in such a way that all assumptions are required to be true on the whole space. However, it is also valid if the assumptions are satisfied on a subset K^{ε_0},

- the given d-pseudotrajectory of G_ρ is contained in K^ε (shadowing)
- the given orbit of G_ρ is contained in K^ε (inverse shadowing)

for some $\varepsilon > 0$, and $a \leq \varepsilon_0 - \varepsilon$, because the behaviour of G_ρ is only relevant on an a-neighbourhood of the given trajectory.

Repeating the line of argument of section 3.2 in this setup, we obtain

Theorem 13. *Let the right hand side $F : X \rightrightarrows X$ of* (1) *be Lipschitz continuous in K^{ε_0} with constant L and with compact and convex values, and let $\|F(x)\| \leq M$ for some $M > 0$ and all $x \in K^{\varepsilon_0}$. Let $\lambda \in (0,1)$ such that the ρ-flow G_ρ satisfies* (24) *for any $x \in K^{\varepsilon_0}$ and $v \in X$ with $x + v \in K^{\varepsilon_0}$. If*

$$\frac{5-\lambda}{1-\lambda}\alpha(h) + \frac{2L}{1-\lambda}\alpha(h)\rho + \frac{4}{1-\lambda}M\rho(e^{L\rho} - \frac{3+\lambda}{4}) \leq \varepsilon_0$$

then

$$\mathrm{dist}_H(\mathrm{Viab}_F(K), \mathrm{Viab}_{\Gamma_{\rho,h}}(K_h^{\varepsilon_1+\alpha(h)})) \leq \frac{4}{1-\lambda}\alpha(h) + \frac{2L}{1-\lambda}\alpha(h)\rho + \varepsilon_1,$$

where $\varepsilon_1 = \frac{2}{1-\lambda}M\rho(e^{L\rho} - 1)$.

Following section 3.3, we obtain a slightly but not substantially worse estimate under similar conditions.

In classical numerical analysis, the order of convergence of a scheme is regarded as one of the most important indicators for its quality. It is doubtful if this way of thinking is appropriate here, but it is possible to obtain linear convergence w.r.t. the spatial discretization $\alpha(h)$ by setting $\alpha(h) := \rho$.

Please note that every shadowing theorem for the ρ-flow of a differential inclusion can be used to derive a concrete error estimate for the viability kernel algorithm in the way sketched above. As the shadowing theory for set-valued systems is still being developped, there is hope that the reasoning of section 3 can soon be applied to more general classes of right hand sides.

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