A C^∞ density theorem for differential inclusions with Lipschitz continuous right hand sides

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Abstract

By definition, solutions of differential inclusions are absolutely continuous functions with L^1 derivatives. We prove that at least for a class of Lipschitz continuous right hand sides the C^∞-solutions are dense in the set of all solutions with respect to the supremum norm.

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1 Introduction

Differential inclusions have been intensively studied since the fifties (cf. [6]) when it was discovered that rigorous existence results for solutions of differential equations with a non-smooth right hand side could be proved within this new framework. When the basics of control theory were established, control problems could also be reformulated as differential inclusions (see Chapter 10 of [1]).

It is well-known that the C^1-solutions of the initial value problem

$$\dot{x}(t) \in F(x(t)) \text{ } a.e. \text{ } in \text{ } [0,T], \quad x(0) = x_0 \tag{1}$$

are dense in the set of all solutions w.r.t the supremum norm for certain classes of right-hand sides.

Filippov showed (cf. [5]) that this statement holds true whenever F is Lipschitz continuous and has closed and uniformly locally connected values. He constructs a sequence of absolutely continuous solutions in such a way that a suitable measure of discontinuity of the derivatives converges to zero along the sequence.

In a later article (see [13]), Wolenski provided a simpler proof for locally Lipschitz continuous right hand sides with convex and compact values. In contrast to Filippov, he uses a sequence of continuously differentiable approximations obtained by a modified Picard-Lindelöf iteration which converge uniformly to a solution.

The aim of this paper is to prove that even the C^∞-solutions are dense in the set of all solutions provided that the right hand sides are *stout* (i.e. arbitrarily small blowups of Lipschitz continuous and convex and compact valued mappings, see Definition 1). Unlike the articles mentioned above, the proof uses convolutions in order to smoothen the functions. For the sake of completeness, an alternative convolution-based proof for the classical results is provided.

This result might be of some theoretical interest, because many control problems and systems with uncertainties are equivalent to a differential inclusion with a *stout* right hand side. On the other hand, many authors (cf. e.g. [3], [4], [7], [8], [11], and the survey [9]) have introduced numerical methods for differential inclusions, but nobody has yet been able to establish higher order convergence of a set-valued numerical scheme for some sufficiently general class of right-hand sides. Any insight into the smoothness properties of the set of solutions might be valuable for further advances in this area.

While intuition suggests that the C^∞ solutions should also be dense in the set of all solutions whenever the right hand side is continuous and

$$F(x) = \overline{\mathrm{Int} F(x)} \text{ for all } x \in \mathbb{R}^d,$$

it seems very difficult to either prove this claim or give a counterexample. As our proof heavily relies on the comparatively rigid concept of Lipschitz continuity, it is impossible to adapt it to this more delicate problem.

2 The result

Let $|\cdot|$ be a vector norm in $\mathbb{R}^d$. As usual, $B(x,\delta) := \{\xi \in \mathbb{R}^d : |\xi - x| \leq \delta\}$, and for $A \subset \mathbb{R}^d$, we set

$$||A|| := \sup_{a \in A} |a|.$$

Let $\mathcal{CC}(\mathbb{R}^d)$ denote the set of all compact convex subsets of $\mathbb{R}^d$ and let dist and dist_H be the nonsymmetric and the symmetric Hausdorff distance, resprectively. The convex hull of a set A will be denoted by $co(A)$.

Definition 1. *A set-valued mapping $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ is called (L, δ_0)-*stout *with constants $\delta_0 > 0$ and $L > 0$ if there exists a Lipschitz continuous mapping $F_{\delta_0} : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ with Lipschitz constant L such that*

$$F(x) = B(F_{\delta_0}(x), \delta_0) \ \forall x \in \mathbb{R}^d. \tag{2}$$

Remark 2. *Since F_{δ_0} is Lipschitz continuous with Lipschitz constant L, the mappings defined by $x \mapsto B(F_{\delta_0}(x), \delta)$ with $\delta > 0$, and in particular F, are Lipschitz continuous with the same constant. Obviously, the images $B(F_{\delta_0}(x), \delta)$ are compact and convex. Thus an (L, δ_0)-stout mapping is (L, δ)-stout for every $\delta \in (0, \delta_0]$, where*

$$F_\delta(x) := B(F_{\delta_0}(x), \delta_0 - \delta). \tag{3}$$

The following remark formalizes a simple geometric principle.

Remark 3. *Let $F_i \subset \mathbb{R}^d$, $i \in I$ and $G \subset \mathbb{R}^d$ be closed and convex, where I is some index set. Then*

$$\text{dist}(co(\cup_{i\in I} F_i), G) \le \sup_{i\in I} \text{dist}(F_i, G), \tag{4}$$

Proof. Let $f \in co(\cup_{i\in I} F_i)$. There exist $\lambda_0, \dots, \lambda_d \in [0, 1]$ and $f_0, \dots, f_d \in \cup_{i\in I} F_i$, $f_j \in F_{i_j}$ such that $f = \sum_{j=0}^d \lambda_j f_j$ and $\sum_{j=0}^d \lambda_j = 1$. Let $g_j \in G$ be such that

$$|f_j - g_j| = \text{dist}(f_j, G) \le \text{dist}(F_{i_j}, G).$$

Then $g := \sum_{j=0}^d \lambda_j g_j \in G$, and

$$|f - g| \ \le \ \sum_{j=0}^d \lambda_j |f_j - g_j| \le \sum_{j=0}^d \lambda_j \,\text{dist}(F_{i_j}, G) \le \sup_{i\in I} \text{dist}(F_i, G).$$

□

Now we formulate our main result:

Theorem 4. *Let $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be (L, δ_0)-stout with $\delta_0 \in (0, 1]$. Then the infinitely many times differentiable solutions of the initial value problem*

$$\dot{x}(t) \in F(x(t)) \ a.e. \ in \ [0, T], \quad x(0) = x_0 \tag{5}$$

are dense in the set of all solutions with respect to the maximum norm.

Proof. Let $x(\cdot)$ be a solution of (5). We will construct smooth solutions $a_\delta(\cdot)$ arbitrarily close to $x(\cdot)$.

Step 1: A-priori bounds. The solution x is à-priori bounded: Let $z(s) \in F(x(0))$ such that $|\dot{x}(s) - z(s)| = \text{dist}(\dot{x}(s), F(x(0)))$. Then

$$\begin{aligned} |x(t) - x(0)| &\leq \int_0^t |\dot{x}(s)| ds \\ &\leq \int_0^t |\dot{x}(s) - z(s)| + |z(s)| ds \\ &\leq \int_0^t \text{dist}_H(F(x(s)), F(x(0))) + ||F(x(0))|| ds \\ &\leq t||F(x(0))|| + \int_0^t L|x(s) - x(0)| ds, \end{aligned}$$

and by the Gronwall lemma,

$$\begin{aligned} |x(t) - x(0)| &\leq t||F(x(0))|| + \int_0^t s||F(x(0))|| L e^{L(t-s)} ds \\ &= t||F(x(0))|| + \frac{1}{L}||F(x(0))||(e^{Lt} - Lt - 1) \\ &= \frac{1}{L}||F(x(0))||(e^{Lt} - 1). \end{aligned}$$

In particular,

$$\begin{aligned} |x(t+\eta) - x(t)| &\leq \frac{1}{L}||F(x(t))||(e^{L\eta} - 1) \\ &\leq \frac{1}{L}\left[||F(x(0))|| + \text{dist}_H(F(x(t)), F(x(0)))\right](e^{L\eta} - 1) \\ &\leq \frac{1}{L}\left[||F(x(0))|| + L|x(t) - x(0)|\right](e^{L\eta} - 1) \\ &\leq \frac{1}{L}\left[||F(x(0))|| + ||F(x(0))||(e^{Lt} - 1)\right](e^{L\eta} - 1) \\ &\leq \underbrace{\frac{1}{L}||F(x(0))|| e^{LT}}_{=:C_1}(e^{L\eta} - 1). \end{aligned} \tag{6}$$

Step 2: Regular approximation. Now we construct a regular approximation x_δ of x. Without loss of generality we can assume that

$$\dot{x}(t) \in F(x(t)) \; \forall t \in [0, T]$$

as a function. We formally continue it as a function $\dot{x} \in L^1_{loc}(\mathbb{R}, \mathbb{R}^d)$ by setting

$$\dot{x}(t) := \begin{cases} \dot{x}(T), & T < t \\ \dot{x}(t), & 0 < t \leq T \\ \dot{x}(0), & t \leq 0. \end{cases} \tag{7}$$

For given $\delta \in (0, \delta_0]$, there exists a function $\varphi_\delta \in C_0^\infty(\mathbb{R}, \mathbb{R}_+)$ satisfying $\text{supp}(\varphi_\delta) \subset [-\delta, \delta]$ and $\int_{\mathbb{R}} \varphi_\delta(\tau) d\tau = 1$ such that

$$y_\delta(s) := \int_{\mathbb{R}} \varphi_\delta(\tau) \dot{x}(s - \tau) d\tau$$

is a function $y_\delta \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ (see Theorem 2.16 in [10]) with

$$\int_0^T |y_\delta(s) - \dot{x}(s)| ds \leq \delta.$$

Hence $x_\delta \in C^\infty(\mathbb{R}, \mathbb{R}^d)$ given by

$$x_\delta(t) := x(0) + \int_0^t y_\delta(s) ds$$

satisfies

$$|x_\delta(t) - x(t)| \leq \delta \ \forall t \in [0, T].$$

Note that y_δ is Lipschitz continuous in $[-1, T+1]$ with Lipschitz constant $K_\delta > 0$.

Step 3: Construction of a regular selection. Consider the time dependent mappings

$$\tilde{F} : \mathbb{R} \times \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d), \tilde{F}(t, x) := F(x) - y_\delta(t) \tag{8}$$

and

$$\tilde{F}_\delta : \mathbb{R} \times \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d), \tilde{F}_\delta(t, x) := F_\delta(x) - y_\delta(t), \tag{9}$$

where the F_δ is the δ-retract of F defined in (3). Since $\tilde{F}_\delta(t, x)$ is Lipschitz continuous in t and x, its minimal selection

$$(t, x) \mapsto m(\tilde{F}_\delta(t, x))$$

is continuous by theorem 1.7.1 in [2]. Take a $\psi_\delta \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{R}_+)$ with $\text{supp}(\psi_\delta) \subset B(0, \frac{\delta}{2K_\delta}) \times B(0, \frac{\delta}{2L})$ and $\int_{\mathbb{R} \times \mathbb{R}^d} \psi_\delta(t, x) d(t, x) = 1$, so that the function

$$\tilde{m}(t, x) := \int_{\mathbb{R} \times \mathbb{R}^d} \psi_\delta(\theta, \xi) \, m(\tilde{F}_\delta(t - \theta, x - \xi)) \, d(\theta, \xi) \tag{10}$$

is an element of $C^\infty(\mathbb{R}\times\mathbb{R}^d,\mathbb{R}^d)$. According to Theorem 1.6.13 in [12], it satisfies

$$\begin{aligned}
\tilde{m}(t,x) &\in \overline{co}\{m(\tilde{F}_\delta(t-\theta,x-\xi)) : (\theta,\xi)\in \operatorname{supp}(\psi_\delta)\}\\
&\subset \overline{co}(\tilde{F}_\delta(B(t,\frac{\delta}{2K_\delta}),B(x,\frac{\delta}{2L})))\\
&\subset \overline{co}(B(\tilde{F}_\delta(B(t,\frac{\delta}{2K_\delta},x),\frac{\delta}{2})))\\
&\subset \overline{co}(B(\tilde{F}_\delta(t,x),\delta)) = \tilde{F}(t,x)
\end{aligned}$$

for $t\in[0,T]$ and $x\in\mathbb{R}^d$, which implies

$$y_\delta(t)+\tilde{m}(t,x)\in F(x). \tag{11}$$

On the other hand,

$$\begin{aligned}
|\tilde{m}(t,x)| &\le \|\overline{co}\{m(\tilde{F}_\delta(t-\theta,x-\xi)) : (\theta,\xi)\in \operatorname{supp}(\psi_\delta)\}\|\\
&= \|\{m(\tilde{F}_\delta(t-\theta,x-\xi)) : (\theta,\xi)\in \operatorname{supp}(\psi_\delta)\}\|\\
&\le \sup\{\operatorname{dist}(0,\tilde{F}_\delta(\theta,\xi)) : \theta\in B(t,\frac{\delta}{2K_\delta}),\xi\in B(x,\frac{\delta}{2L})\}\\
&\le \sup\{\operatorname{dist}(0,\tilde{F}(\theta,\xi)) : \theta\in B(t,\frac{\delta}{2K_\delta}),\xi\in B(x,\frac{\delta}{2L})\}+\delta\\
&\le \sup\{\operatorname{dist}(0,\tilde{F}(t,x))+\operatorname{dist}(\tilde{F}(t,x),\tilde{F}(\theta,x))\\
&+ \operatorname{dist}(\tilde{F}(\theta,x),\tilde{F}(\theta,\xi)) : \theta\in B(t,\frac{\delta}{2K_\delta}),\xi\in B(x,\frac{\delta}{2L})\}+\delta\\
&\le \operatorname{dist}(0,\tilde{F}(t,x))+2\delta \quad = \quad \operatorname{dist}(y_\delta(t),F(x))+2\delta.
\end{aligned}$$

Step 4: Corresponding solution. By the Cauchy-Peano theorem, the initial value problem

$$\dot{a}_\delta(t)=y_\delta(t)+\tilde{m}(t,a_\delta(t)),\quad a_\delta(0)=x(0) \tag{12}$$

admits a solution $a_\delta(\cdot)$ on a maximal subinterval $J\subset[0,T]$ with $0\in J$. It is an element of $C^\infty(J,\mathbb{R}^d)$, and because of (11) it is also a solution of the original differential inclusion (5). For $t\in J$ one obtains

$$\begin{aligned}
|x_\delta(t)-a_\delta(t)| &\le \int_0^t |y_\delta(s)-(y_\delta(s)+\tilde{m}(s,a_\delta(s)))|ds\\
&= \int_0^t |\tilde{m}(s,a_\delta(s))|ds\\
&= \int_0^t \operatorname{dist}(y_\delta(s),F(a_\delta(s)))+2\delta\,ds.
\end{aligned}$$

By Theorem 1.6.13 in [12],

$$y_\delta(s) \in \overline{co}\{\dot{x}(\tau) : \tau \in s - \operatorname{supp}(\varphi_\delta)\}. \tag{13}$$

Hence

$$\begin{aligned}
|x_\delta(t) - a_\delta(t)| &\leq \int_0^t \operatorname{dist}(\overline{co}\{\cup_{s-\operatorname{supp}(\varphi_\delta)} F(x(\tau))\}, F(a_\delta(s))) + 2\delta\, ds \\
&\overset{(4)}{\leq} \int_0^t \sup_{\tau\in[s-\delta,s+\delta]} \operatorname{dist}(F(x(\tau)), F(a_\delta(s))) + 2\delta\, ds \\
&\leq \int_0^t \sup_{\tau\in[s-\delta,s+\delta]} L|x(\tau) - a_\delta(s)| + 2\delta\, ds \\
&\overset{(6)}{\leq} \int_0^t L(|x(s) - a_\delta(s)| + C_1(e^{L\delta} - 1)) + 2\delta\, ds \\
&\leq \int_0^t L|x_\delta(s) - a_\delta(s)| + \underbrace{L(\delta + C_1(e^{L\delta} - 1)) + 2\delta}_{=:C_2(\delta)}\, ds.
\end{aligned}$$

The Gronwall lemma yields

$$\begin{aligned}
|x_\delta(t) - a_\delta(t)| &\leq C_2(\delta)t + \int_0^t C_2(\delta) s L e^{L(t-s)} ds \\
&= C_2(\delta)t + \frac{1}{L} C_2(\delta)(e^{Lt} - Lt - 1) \\
&= \frac{1}{L} C_2(\delta) t(e^{Lt} - 1),
\end{aligned}$$

and thus

$$\begin{aligned}
|x(t) - a_\delta(t)| &\leq \delta + \frac{1}{L} C_2(\delta) t(e^{Lt} - 1) \\
&\leq \delta + \frac{1}{L} C_2(\delta) T(e^{LT} - 1). \qquad (14)
\end{aligned}$$

In particular, a_δ is bounded on J. Hence $J = [0, T]$, and

$$\begin{aligned}
||x - a_\delta||_\infty &\leq \delta + \frac{1}{L} C_2(\delta) T(e^{LT} - 1) \qquad (15) \\
&\longrightarrow 0 \ as\ \delta \to 0.
\end{aligned}$$

□

Let us sketch an alternative proof for the classical result of Filippov and Wolenski using the above techniques. Please note that here it is not necessary to smoothen the minimal selection and thus we don't need to assume that the set-valued mapping F has a δ-retract.

Theorem 5. *Let* $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ *be Lipschitz continuous. Then the continuously differentiable solutions of the initial value problem*

$$\dot{x}(t) \in F(x(t)) \text{ a.e. in } [0,T], \quad x(0) = x_0 \tag{16}$$

are dense in the set of all solutions with respect to the maximum norm.

Proof. The à-priori estimate and the regular approximation $x_\delta(\cdot)$ can be obtained exactly as in the previous proof. For the construction of a regular selection, we can consider the time dependent mapping

$$\tilde{F} : \mathbb{R} \times \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d), \tilde{F}(t,x) := F(x) - y_\delta(t).$$

Since y_δ is continuous, $\tilde{F}$ is continuous w.r.t the Hausdorff metric and consequently, the minimal selection $(t,x) \mapsto m(t,x)$ of $\tilde{F}$ is also continuous. Obviously

$$|m(t,x)| = \operatorname{dist}(y_\delta(t), F(x)),$$

and

$$y_\delta(t) + m(t,x) \in F(x) \; \forall t \in [0,T], \; \forall x \in \mathbb{R}^d.$$

By the Cauchy-Peano theorem, the initial value problem

$$\dot{a}_\delta(t) = y_\delta(t) + m(t, a_\delta(t)), \quad a_\delta(0) = x(0) \tag{17}$$

admits a solution $a_\delta(\cdot)$ on a maximal subinterval $J \subset [0,T]$ with $0 \in J$. The following estimates are merely a simplified version of the previous calculations. Of course, the solution $a_\delta(\cdot)$ is defined on the whole interval $[0,T]$ and it is continuously differentiable, because the right hand side of (17) is a continuous function. □

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