A General Approach to Hyperbolicity for Set-Valued Maps

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Abstract

We introduce a general notion of hyperbolicity for set-valued dynamical systems and discuss it in the framework of polytope-valued maps.

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1 Introduction

In the context of classical dynamical systems (diffeomorphisms or flows), the shadowing property has been extensively studied (see, for example, the monographs [9] and [10]). This property means that near approximate trajectories there exist exact trajectories of the system.

There is another type of shadowing properties (inverse shadowing properties) which are related to the following question: Given a family of mappings that approximate the defining mapping of the dynamical system and an arbitrary exact trajectory, is it possible to guarantee the existence of a pseudotrajectory generated by the given family which is close to the exact trajectory? Such properties were considered by various authors (see, for example, [3], [4], and [5]).

The study of such properties is important for the theory of perturbations of dynamical systems.

It was shown that so-called contractive set-valued dynamical systems possess shadowing properties (see, for example, [6, 7] and [11]).

In the literature, one can find some concepts of hyperbolicity for set-valued dynamical systems. One of the definitions was given by Akin in [1]. The author calls a set hyperbolic if it is expansive and the system already has the shadowing property on it.

A different approach was used by Sander in [8] and [13], where hyperbolicity was defined for smooth relations. Due to the nature of the analyzed objects, this

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hyperbolicity condition does not allow the graph of a relation to have nonempty interior, which is generically the case in the set-up discussed in the present paper.

Please note that both hyperbolicity conditions of Akin and Sander imply the uniqueness of a shadowing trajectory, which is quite unnatural for set-valued dynamical systems.

In an earlier essay (cf. [12]) we proposed a condition which was a generalization of the classical single-valued hyperbolicity concept as well as of the set-valued contractive case discussed in [11]. The drawback of this approach is that it imposes rigid restrictions on the behaviour of the set-valued mapping.

We hope to improve this aspect in the present paper, where we propose a more general selection-based hyperbolicity condition for set-valued dynamical systems and prove that every such system has the Lipschitz shadowing and inverse shadowing property. Further we examine this condition in the special case of polytope-valued mappings, where it is reduced to a simple criterion for the vertices of the polytopes.

Let us pass to basic notation. Let $C(\mathbb{R}^m)$ be the collection of all closed subsets and let $CC(\mathbb{R}^m)$ denote the set of compact and convex subsets of $\mathbb{R}^m$. The distance between two compact subsets A and B of $\mathbb{R}^m$ is measured by the deviation

$$\mathrm{dev}(A,B) = \max_{a\in A}\min_{b\in B}|a-b|$$

and by the symmetric Hausdorff distance

$$\mathrm{dist}_H(A,B) = \max(\mathrm{dev}(A,B), \mathrm{dev}(B,A)),$$

respectively. The convex hull of a set $A \subset \mathbb{R}^m$ will be denoted by $co(A)$.

If $x \in \mathbb{R}^m$ and $a > 0$, we denote by $B_a(x)$ the ball of radius a centered at x.

If $x \in \mathbb{R}^m$ and $M \in C(\mathbb{R}^m)$, $\mathrm{Proj}(x, M)$ is the set of elements of M that satisfy the inequalities $|x - \mathrm{Proj}(x, M)| \le |x - y|$ for all $y \in M$. If $M \in CC(\mathbb{R}^m)$, then $\mathrm{Proj}(x, M)$ is a singleton; in this case, we also consider the vector $\mathrm{Dev}(x, M) = \mathrm{Proj}(x, M) - x$.

A set-valued dynamical system on $\mathbb{R}^m$ is determined by a set-valued mapping $F : \mathbb{R}^m \to C(\mathbb{R}^m)\backslash\{\emptyset\}$ and its iterates. In what follows, we identify the mapping F and the corresponding dynamical system.

A sequence $\eta = \{p_k\}$ is a trajectory of the system F if

$$p_{k+1} \in F(p_k) \text{ for any } k \in \mathbb{Z}. \tag{1}$$

A sequence $\xi = \{x_k\}$ is called a d-pseudotrajectory of F if an error of size $d > 0$ is allowed in every step, i.e., if

$$\mathrm{dist}(x_{k+1}, F(x_k)) < d \text{ for any } k \in \mathbb{Z}. \tag{2}$$

We say that the system F has the shadowing property if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}$ of F there exists a trajectory $\eta = \{p_k\}$ with

$$\mathrm{dist}(x_k, p_k) < \epsilon \text{ for any } k \in \mathbb{Z}.$$

As usual, for a sequence $\eta = \{\eta_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$,

$$||\eta||_\infty = \sup_{k\in\mathbb{Z}} |\eta_k|.$$

2 Shadowing

Let $F : \mathbb{R}^m \to C(\mathbb{R}^m)$ be a set-valued mapping. Any single-valued function $f : \mathbb{R}^m \to \mathbb{R}^m$ with $f(x) \in F(x)$ for all $x \in \mathbb{R}^m$ is called a selection of F.

Our definition of hyperbolicity for set-valued mappings introduced in this paper is as follows.

We say that F is hyperbolic if it is locally parametrized by a family of hyperbolic selections:

(P1) For every $x \in \mathbb{R}^m$ there exist linear subspaces $U(x), S(x) \subset \mathbb{R}^m$ such that

$$U(x) \oplus S(x) = \mathbb{R}^m. \tag{3}$$

If $P(x)$ and $Q(x)$ are the corresponding complementary projections from $\mathbb{R}^m$ to $U(x)$ and $S(x)$, then there exists an $N \geq 1$ such that

$$|P(x)|, |Q(x)| \leq N \tag{4}$$

for all $x \in \mathbb{R}^m$.

(P2) There exist constants $\lambda \in (0, 1)$, $\kappa > 0$, $l > 0$, and $a > 0$ such that for every point $(x, z) \in \text{graph}(F)$ there exists a local selection f_z of F, which is a single-valued function $f_z : B_a(x) \to \mathbb{R}^m$ with $f_z(x) = z$, $f_z(x') \in F(x')$ for all $x' \in B_a(x)$, and such that the following property holds: For any $y, v \in \mathbb{R}^m$ with $|v| \leq a$ and $|z - y| \leq a$ we have

$$f_z(x + v) = z + A_z(x)v + b_z(x, v), \tag{5}$$

where the $A_z(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear map, the restriction

$$P(y)A_z(x)_{|_{U(x)}} : U(x) \to U(y) \tag{6}$$

is an isomorphism such that

$$|P(y)A_z(x)P(x)v| \geq \lambda^{-1}|P(x)v|, \tag{7}$$

$$|P(y)A_z(x)Q(x)v| \leq \kappa|Q(x)v|, \tag{8}$$

$$|Q(y)A_z(x)P(x)v| \leq \kappa|P(x)v|, \tag{9}$$

$$|Q(y)A_z(x)Q(x)v| \leq \lambda|Q(x)v|, \tag{10}$$

and $b_z(x, \cdot)$ is a small perturbation continuous in v and bounded by

$$|b_z(x, v)| \leq l|v|. \tag{11}$$

Formula (5) and the above condition on b_z imply that f_z is continuous for $|v| \leq a$.

Remark 1.

Let us note that the definition of hyperbolicity of a set-valued mapping suggested in [12] is a particular case of the general definition given above.

Recall that in [12], set-valued mappings of the form

$$F(x) = L(x) + M(x) \tag{12}$$

were considered, where $L : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous single-valued mapping, and $M : \mathbb{R}^m \to CC(\mathbb{R}^m)$ is a set-valued mapping with compact and convex images.

It was assumed that there exist constants $N \geq 1$, $\lambda \in (0, 1)$, $\kappa > 0$, $l > 0$, and $a > 0$ such that

• condition (P1) above is satisfied;

• if $x, y, v \in \mathbb{R}^m$ satisfy the inequalities $|v| \leq a$ and $\operatorname{dist}(y, F(x)) \leq a$, then we can represent $L(x + v)$ as

$$L(x + v) = L(x) + A(x)v + b(x, v),$$

where $A(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear map that is continuous in x and such that (after the replacement of $A_z(x)$ by $A(x)$ in (P2)) the restriction (6) is an isomorphism that satisfies estimate (7), and inequalities (8)-(10) hold;

• $|b(x, v)| \leq l|v|$;

• $\operatorname{dist}_H(M(x + v), M(x)) \leq l|v|$.

Now let us take a mapping F of the form (12) that satisfies the above conditions, a point $z \in F(x)$, and define the corresponding local selection f_z by

$$f_z(x + v) = z + A(x)v + \operatorname{Dev}(z + A(x)v, F(x + v));$$

thus, we take $A(x)$ az $A_z(x)$ and set

$$b_z(x + v) = \operatorname{Dev}(z + A(x)v, F(x + v)).$$

Clearly, $f_z(x) = z$, $f_z(x + v) \in F(x + v)$, and $A_z(x) = A(x)$ satisfies the corresponding properties formulated in (P2). Since F is convex and continuous, b_z is continuous.

The inclusion $z \in F(x)$ implies that

$$\operatorname{dist}(z + A(x)v, F(x + v)) \leq \operatorname{dev}(F(x) + A(x)v, F(x + v))$$

$$= \operatorname{dev}(L(x) + A(x)v + M(x), L(x) + A(x)v + b(x, v) + M(x + v))$$

$$= \operatorname{dev}(M(x), b(x, v) + M(x + v))$$

$$\leq |b(x, v)| + \operatorname{dist}_H(M(x), M(x + v)) \leq 2l|v|,$$

and inequality (11) is verified (with l replaced by $2l$).

We show that the set-valued dynamical system

$$x \mapsto F(x) \tag{13}$$

has the Lipschitz shadowing property if F is hyperbolic in the above sense.

Theorem 1. *Let $F : \mathbb{R}^m \to C(\mathbb{R}^m)$ be a hyperbolic set-valued mapping such that*

$$\lambda + \kappa + 2lN < 1.$$

If $\xi = \{x_k\}_{k\in\mathbb{Z}}$ *is a d-pseudotrajectory of* (13) *for some* $d < a/L$, *there exists a trajectory* $\eta = \{p_k\}_{k\in\mathbb{Z}}$ *of*(13) *with*

$$\|\xi - \eta\| \le Ld,$$

where

$$L^{-1} = \frac{1}{2N}(1 - \lambda - \kappa - 2lN).$$

Proof. For every $k \in \mathbb{Z}$, we fix $y = x_{k+1}$ and a point $z \in \mathrm{Proj}(x_{k+1}, F(x_k))$.

Clearly, $L > 1$ and $d < a$. Then $|y - z| < a$, and there exists a continuous hyperbolic selection f_z of F (denoted below f_k) such that

$$f_k(x_k + v) = f_k(x_k) + A_k(x_k)v + b_k(x_k, v) \in F(x_k + v), \quad |v| \le a,$$

according to (P2).

Thus, to find a shadowing trajectory, it is enough to find a sequence v_k with $|v_k| \le Ld$ such that

$$x_{k+1} + v_{k+1} = f_k(x_k + v_k).$$

Take $b = d/(2L)$ and define

$$H_k := \{v \in \mathbb{R}^m : |P(x_k)v|, |Q(x_k)v| \le b\}$$

and $H := \prod_{k\in\mathbb{Z}} H_k$.

Note that if $v \in H_k$, then $|v| \le 2b = Ld$.

Since each H_k is compact and convex, so is H w.r.t. the Tikhonov topology. The mapping $G_k : U(x_k) \to U(x_{k+1})$ given by

$$G_k(w) := -P(x_{k+1})A_k(x_k)w \tag{14}$$

satisfies $G_k(0) = 0$,

$$|G_k(w)| \ge \lambda^{-1}|w|, \quad w \in U(x_k), \tag{15}$$

and $G_k(B^b_{x_k}) \supset B^{b/\lambda}_{x_{k+1}}$, where

$$B^c_x := \{z \in U(x) : |z| \le c\}, \tag{16}$$

because of property (P2). Thus the inverse G_k^{-1} of G_k is defined on $B^{b/\lambda}_{x_{k+1}}$, and

$$|G_k^{-1}(z) - G_k^{-1}(z')| \le \lambda|z - z'|, \quad z, z' \in B^{b/\lambda}_{x_{k+1}}. \tag{17}$$

The operator $T : H \to H$ which is given by

$$Q(x_{k+1})T_{k+1}(V) := Q(x_{k+1})\left(f_k(x_k + v_k) - x_{k+1}\right), \tag{18}$$

$$P(x_k)T_k(V) := G_k^{-1}\left(P(x_{k+1})\left\{b_k(x_k, v_k) + A_k(x_k)Q(x_k)v_k + f_k(x_k) - x_{k+1} - v_{k+1}\right\}\right) \tag{19}$$

for $V = \{v_k\}_{k\in\mathbb{Z}} \in H$, is well-defined. The argument in (19) satisfies

$$|P(x_{k+1})\left\{b_k(x_k, v_k) + A_k(x_k)Q(x_k)v_k + f_k(x_k) - x_{k+1} - v_{k+1}\right\}|$$

$$\le Nl|v_k| + \kappa|Q(x_k)v_k| + Nd + b \le 2lNb + \kappa b + Nd + b$$

$$\leq (2lN + \kappa + \frac{Nd}{b} + 1)b \leq \lambda^{-1}b$$

for $V = \{v_k\}_{k\in\mathbb{Z}} \in H$, because

$$b^{-1} = \frac{1}{Nd}(1 - \lambda - \kappa - 2lN) \leq \frac{1}{Nd}(\lambda^{-1} - 1 - \kappa - 2lN),$$

so that the argument in (19) is an element of $B^{b/\lambda}_{x_{k+1}}$. Furthermore,

$$|Q(x_{k+1})T_{k+1}(V)|$$

$$\leq |Q(x_{k+1})A_k(x_k)P(x_k)v_k| + |Q(x_{k+1})A_k(x_k)Q(x_k)v_k|$$
$$+|Q(x_{k+1})b_k(x_k, v_k)| + |Q(x_{k+1})(f_k(x_k) - x_{k+1})|$$
$$\leq \kappa|P(x_k)v_k| + \lambda|Q(x_k)v_k| + lN|v_k| + Nd \leq \kappa b + \lambda b + 2lNb + \frac{2N}{L}b = b,$$

and $T(V) \in H$. The operator T is continuous w.r.t. the Tikhonov topology, because every component T_k depends on v_{k-1}, v_k, v_{k+1} only. Hence T has a fixed point $V \in H$, which implies that

$$Q(x_{k+1})v_{k+1} = Q(x_{k+1})\left(f_k(x_k + v_k) - x_{k+1}\right) \tag{20}$$

and

$$-P(x_{k+1})A_k(x_k)P(x_k)v_k = G_k(P(x_k)v_k) \tag{21}$$
$$= \quad P(x_{k+1})\left\{b_k(x_k, v_k) + A_k(x_k)Q(x_k)v_k + f_k(x_k) - x_{k+1} - v_{k+1}\right\} \tag{22}$$

or

$$P(x_{k+1})v_{k+1} = P(x_{k+1})(f_k(x_k + v_k) - x_{k+1}). \tag{23}$$

By (20) and (23), the sequence $\eta := \{p_k\}_{k\in\mathbb{Z}}$ with $p_k = x_k + v_k$ is the desired shadowing trajectory. □

3 Inverse Shadowing

As the line of argument is very similar to the previous section, we will only highlight which elements of the setup have to be modified. Though it is possible to formulate a global result, we prefer a local version which allows the approximating mappings to be locally defined.

Let us assume that the mapping F is hyperbolic at a given trajectory $\eta = \{p_k\}$ in the following sense: There exist constants $N \geq 1$, $a, \kappa, l > 0$, and $\lambda \in (0, 1)$ such that condition (P1) holds for points $x = p_k$, and condition (P2) holds for points $x = p_k$, $y = z = p_{k+1}$, and vectors v with $|v| \leq a$.

In our theorem on inverse shadowing, we consider two classes of sequences of mappings that approximate the set-valued mapping F. Fix a number $d > 0$.

Class 1.
Consider a sequence of mappings

$$\Phi = \{\Phi_k : \mathbb{R}^m \to CC(\mathbb{R}^m)\}$$

such that each Φ_k is continuous w.r.t. dist_H and

$$\mathrm{dev}(F(p_k+v), \Phi_k(p_k+v)) \le d \text{ for } k \in \mathbb{Z} \text{ and } |v| \le a. \tag{24}$$

Class 2.
Let

$$CS(\Psi, x, a) = \{\psi \in C(B_a(x), \mathbb{R}^m) \ : \ \psi(y) \in \Psi(y), \quad y \in B_a(x)\}$$

be the set of all local continuous selections of a set-valued mapping Ψ; we equip $C(B_a(x), \mathbb{R}^m)$ with the supremum norm.

Consider a sequence of mappings

$$\Phi = \{\Phi_k : \mathbb{R}^m \to C(\mathbb{R}^m)\}$$

such that

$$\mathrm{dev}(CS(F, p_k, a), CS(\Phi_k, p_k, a)) \le d, \quad k \in \mathbb{Z}. \tag{25}$$

For both classes, we say that a sequence of points $x_k \in \mathbb{R}^m$ is a trajectory of the sequence Φ if $x_{k+1} \in \Phi_k(x_k)$.

Remark 2.
Though we have seen that it is not necessary to assume convexity of the values of F for proving that it has the shadowing property, we are not able to dispense with convexity of the approximating mappings Φ_k of Class 1 for the inverse shadowing property.

Theorem 2. *Assume that a trajectory $\eta = \{p_k\}$ of F is hyperbolic in the above sense. If*

$$\lambda + \kappa + 2lN < 1, \tag{26}$$

then F has the inverse Lipschitz shadowing property: Whenever a family Φ of mappings is defined as above with $d < a/L$, there exists a trajectory $\xi = \{x_k\}$ of Φ such that

$$\|\xi - \eta\|_\infty \le Ld,$$

where

$$L^{-1} = \frac{1}{2N}(1 - \lambda - \kappa - 2lN).$$

Proof. By assumption, there exist hyperbolic selections f_k of F such that $f_k(p_k) = p_{k+1}$,

$$f_k(p_k+v) = f_k(p_k) + A_k(p_k)v + b_k(p_k, v),$$

$$|P(p_{k+1})A_k(p_k)P(p_k)v| \ge \lambda^{-1}|P(p_k)v|,$$

and so on.

Case 1. Because of (24), $\phi_k(p_k+v) := \mathrm{Proj}(f_k(p_k+v), \Phi_k(p_k+v))$ is a selection of Φ_k such that $|f_k(p_k+v) - \phi_k(p_k+v)| \le d$ for all $|v| \le a$. Since Φ_k is continuous w.r.t. the Hausdorff distance and has convex values, the ϕ_k are also continuous according to Theorem 1.7.1 of [2].

Case 2. Assumption (25) implies the existence of continuous selections ϕ_k of Φ_k such that

$$|f_k(p_k+v) - \phi_k(p_k+v)| \le d, \quad |v| \le a.$$

In both cases, we search for a sequence v_k with $|v_k| \le Ld$ such that

$$p_{k+1} + v_{k+1} = \phi_k(p_k + v_k) \in \Phi_k(p_k + v_k).$$

As before, $b = d/(2L)$, $H_k := \{v \in \mathbb{R}^m : |P(p_k)v|, |Q(p_k)v| \le b\}$, and $H := \prod_{k\in\mathbb{Z}} H_k$. Here,

$$G_k(w) := -P(p_{k+1})A_k(p_k)w, \tag{27}$$

and the operator $T : H \to H$ is defined by

$$\begin{aligned} Q(p_{k+1})T_{k+1}(V) &:= Q(p_{k+1})\left(\phi_k(p_k+v_k) - p_{k+1}\right), && (28)\\ P(p_k)T_k(V) &:= G_k^{-1}\left(P(p_{k+1})\{b_k(p_k,v_k) + A_k(p_k)Q(p_k)v_k\right. && (29)\\ &\quad \left. -(f_k-\phi_k)(p_k+v_k) + f_k(p_k) - p_{k+1} - v_{k+1}\}\right). \end{aligned}$$

The estimates are essentially unchanged, merely the error Nd is now caused by the term $P(\cdot)(\phi_k - f_k)(\cdot)$ instead of $P(\cdot)(f_k(x_k) - x_{k+1})$ as before.

□

4 Polytope-valued Mappings

Let $F : \mathbb{R}^m \to CC(\mathbb{R}^m)$ be a polytope-valued mapping, i.e. a set-valued mapping which is characterized by its vertices $s_1, \ldots, s_n : \mathbb{R}^m \to \mathbb{R}^m$ via

$$F(x) = co\{s_1(x), \ldots, s_n(x)\} \text{ for all } x \in \mathbb{R}^m. \tag{30}$$

Assume that there exist $N \ge 1$, $a, \kappa, l > 0$, and $\lambda \in [0,1]$ such that

(P1′) condition (P1) of Sec. 2 holds, and the dimensions of the spaces $U(x)$ are the same for $x \in \mathbb{R}^m$.

(P2′) For any $x, y, v \in \mathbb{R}^m$ with $|v| \le a$ and $|s_i(x) - y| \le a$, we can represent

$$s_i(x+v) = s_i(x) + A_i(x)v + b_i(x,v) \tag{31}$$

for $1 \le i \le n$, where any $A_i(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear map such that for each v there exists a direction of expansion $p(x,v) \in \mathbb{R}^m$ with $|p(x,v)| = 1$ and

$$\langle p(x,v), P(y)A_i(x)P(x)v\rangle \ge \lambda^{-1}|P(x)v|, \tag{32}$$

analogs of conditions (8)-(10) hold (with $A_z(x)$ replaced by $A_i(x)$), and $b_i(x,\cdot)$ are small continuous perturbations for which analog of condition (11) is valid.

Remark 2. From the geometric point of view, inequality (32) ensures that the unstable perturbations $P(y)A_i(x)P(x)v$ drive all the vertices in the same direction, so that their movements cannot cancel each other when combined.

Now we show that in the case of polytope-valued mappings, the general condition of hyperbolicity introduced in Sec. 2 is implied by conditions on behavior of a finite set of points.

Theorem 3. *Let $F : \mathbb{R}^m \to CC(\mathbb{R}^m)$ be a polytope-valued mapping such that its vertices satisfy conditions (P1′) and (P2′). Assume that the projections P and Q are Lipschitz continuous with Lipschitz constant $K > 0$ such that*

$$K \operatorname{diam}(F(x)) \max_{1\le i\le n} \|A_i(x)\| < \lambda^{-1}, \quad x \in \mathbb{R}^m. \tag{33}$$

If

$$\lambda_0 := \sup_{x\in\mathbb{R}^m} \max(\lambda_1(x), \lambda_2(x)) < 1,$$

where

$$\lambda_1(x) := (\lambda^{-1} - K \operatorname{diam}(F(x)) \max_{1\le i\le n} \|A_i(x)\|)^{-1}$$

and

$$\lambda_2(x) := \lambda + K \operatorname{diam}(F(x)) \max_{1\le i\le n} \|A_i(x)\|,$$

then F is a hyperbolic set-valued mapping with constants N, λ_0,

$$\kappa_0 := \kappa + \sup_{x\in\mathbb{R}^m} K \operatorname{diam}(F(x)) \max_{1\le i\le n} \|A_i(x)\|,$$

l, and a.

Proof. Let any point $(x, z) \in \operatorname{graph}(F)$ be given. Because of (30), there exist $\theta_1, \dots, \theta_n \in [0, 1]$ with $\sum_{i=1}^n \theta_i = 1$ and

$$z = \sum_{i=1}^n \theta_i s_i(x).$$

Define the selection $f_z : \mathbb{R}^m \to \mathbb{R}^m$ as the convex combination

$$f_z(x') := \sum_{i=1}^n \theta_i s_i(x'). \tag{34}$$

of the vertices of F with the above coefficients. Then

$$\begin{aligned} f_z(x+v) &= \sum_{i=1}^n \theta_i s_i(x+v) = \sum_{i=1}^n \theta_i \left(s_i(x) + A_i(x)v + b_i(x,v)\right) \\ &= s(x) + \sum_{i=1}^n \theta_i A_i(x) v + \sum_{i=1}^n \theta_i b_i(x,v) =: z + A(x)v + b(x,v). \end{aligned}$$

Let us check condition (P2).

Take y with $|y-z| \le a$ and define $y_i = y - z + s_i(x)$, so that $|y_i - s_i(x)| \le a$.

Since the projections P are Lipschitz continuous with Lipschitz constant K, we deduce from estimates (32) that

$$\begin{aligned}
& |P(y)A(x)P(x)v| \\
\geq\ & \langle p(x,v), P(y)A(x)P(x)v\rangle \\
=\ & \langle p(x,v), \sum_{i=1}^{n}\theta_i P(y)A_i(x)P(x)v\rangle \\
=\ & \langle p(x,v), \sum_{i=1}^{n}\theta_i P(y_i)A_i(x)P(x)v\rangle \\
& +\langle p(x,v), \sum_{i=1}^{n}\theta_i \left(P(y)-P(y_i)\right)A_i(x)P(x)v\rangle \\
\geq\ & \lambda^{-1}|P(x)v| - K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\||P(x)v| \\
=\ & \left(\lambda^{-1} - K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\|\right)|P(x)v| \\
=\ & \lambda_1^{-1}(x)|P(x)v|,
\end{aligned}$$

which implies that the restriction

$$P(y)A(x)_{|_{U(x)}} : U(x) \to U(y)$$

is an isomorphism (let us recall that the dimensions of $U(x)$ and $U(y)$ coincide). The same estimate proves inequality (7).

To prove inequalities (8)-(10), we note that

$$\begin{aligned}
& |P(y)A(x)Q(x)v| \\
=\ & |P(y)\sum_{i=1}^{n}\theta_i A_i(x)Q(x)v| \\
\leq\ & |\sum_{i=1}^{n}\theta_i P(y_i)A_i(x)Q(x)v| + |\sum_{i=1}^{n}\theta_i\left(P(y)-P(y_i)\right)A_i(x)Q(x)v| \\
\leq\ & \kappa|Q(x)v| + K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\||Q(x)v| \\
=\ & \left(\kappa + K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\|\right)|Q(x)v| \\
\leq\ & \kappa_0|Q(x)v|,
\end{aligned}$$

$$\begin{aligned}
& |Q(y)A(x)P(x)v| \\
=\ & |Q(y)\sum_{i=1}^{n}\theta_i A_i(x)P(x)v| \\
\leq\ & |\sum_{i=1}^{n}\theta_i Q(y_i)A_i(x)P(x)v| + |\sum_{i=1}^{n}\theta_i\left(Q(y)-Q(y_i)\right)A_i(x)P(x)v| \\
\leq\ & \kappa|P(x)v| + K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\||P(x)v| \\
=\ & \left(\kappa + K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\|\right)|P(x)v| \\
\leq\ & \kappa_0|P(x)v|,
\end{aligned}$$

and

$$
\begin{aligned}
& |Q(y)A(x)Q(x)v| \\
= \; & |Q(y)\sum_{i=1}^{n}\theta_i A_i(x)Q(x)v| \\
\leq \; & |\sum_{i=1}^{n}\theta_i Q(y_i)A_i(x)Q(x)v| + |\sum_{i=1}^{n}\theta_i \left(Q(y)-Q(y_i)\right)A_i(x)Q(x)v| \\
\leq \; & \lambda|Q(x)v| + K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\||Q(x)v| \\
= \; & \left(\lambda + K\,\mathrm{diam}(F(x)) \max_{1\leq i\leq n}\|A_i(x)\|\right)|Q(x)v| \\
= \; & \lambda_2(x)|Q(x)v|.
\end{aligned}
$$

Finally,

$$|b(x,v)| \leq \sum_{i=1}^{n}\theta_i|b_i(x,v)| \leq l|v|,$$

which proves estimate (11).

□

Corollary. If

$$\lambda_0 + \kappa_0 + 2lN < 1,$$

then F has the Lipschitz shadowing property due to Theorem 1 and the inverse shadowing property in the sense of Theorem 2.

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