

# The implicit Euler scheme for one-sided Lipschitz differential inclusions

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## Abstract

We propose a set-valued version of the implicit Euler scheme for relaxed one-sided Lipschitz differential inclusions and prove that the defining implicit inclusions have a well-defined solution. Furthermore, we give a convergence analysis based on stability theorems, which shows that the set-valued implicit Euler method inherits all favourable stability properties from the single-valued scheme. The impact of spatial discretization is discussed, a fully discretized version of the scheme is analyzed, and a numerical example is given.

## Introduction

We consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \text{ a.e. in } [0, T], \quad x(0) = x_0 \quad (1)$$

in the Euclidean space  $\mathbb{R}^m$ , where  $F$  is continuous, has convex and compact values, and satisfies the relaxed one-sided Lipschitz condition (ROSL, see Definition 1 in Section 1 and assumption (A2) in Section 3). According to

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Theorem 2 in [4], the set of solutions of (1) is a nonempty compact subset of the space of continuous functions  $C([0, T], \mathbb{R}^m)$  equipped with the supremum norm.

In most articles which deal with the set-valued explicit Euler scheme, the right-hand side of the differential inclusion is required to be Lipschitz continuous. One of the first attempts to give error estimates for the set-valued Euler scheme

$$\Phi_h^{\text{exp}}(t, x) := \{y \in \mathbb{R}^m : y \in x + hF(t, x)\} \quad (2)$$

was presented in [8]. The Euler method proposed in [9] only uses the extremal points of the right-hand side and adjusts the time discretization to this predefined discretization in the velocity space.

In [13], the spatial discretization is considered explicitly, and a more detailed analysis of spatial discretization effects has been given in [3]. The performance of the explicit Euler scheme in the presence of state constraints has been thoroughly investigated in the remarkable paper [2]. Recently, the error analysis of the set-valued Euler scheme has been extended to nonconvex-valued differential inclusions in [17]. The existence of attractors of differential inclusions and the corresponding Euler discretizations has been established in [12].

The explicit Euler scheme has been examined in the context of relaxed one-sided Lipschitz right-hand sides as well, see [6] and [7]. In [6], a ROSL version of the Filippov Theorem is proved which serves as the main tool for the error estimates, while [7] gives a detailed analysis of the dynamics of (1) and the corresponding Euler solutions.

The aim of the present paper is to establish an implicit set-valued Euler scheme for continuous and ROSL right-hand sides. In Section 1, we present a solvability theorem for ROSL maps which extends results from [5] and is needed in Section 2 to show that implicit Euler steps are well-defined and Lipschitz continuous with nonempty and compact values. For the system (1) a single implicit Euler step with stepsize  $h$  is given by

$$\Phi_h^{\text{imp}}(t, x) := \{y \in \mathbb{R}^m : y \in x + hF(t + h, y)\}. \quad (3)$$

In Section 3, we present error estimates for the implicit Euler scheme on bounded and unbounded intervals. We prove a version of the Filippov Theorem which is very similar to the one presented in [6] and a discrete stability

theorem for the implicit Euler scheme from which all further estimates follow in a natural way.

Finally, we introduce a fully discretized implicit Euler scheme in Section 4 and show how a solution of the implicit inclusion (3) can be approximated. We apply this method to the well-known Michaelis-Menten model in order to demonstrate that for stiff differential inclusions the implicit Euler scheme (3) respects the stability properties of (1), while the explicit Euler fails to recognize them. In a way this shows that the favourable properties of A-stable and B-stable methods for stiff ODEs (see [10]) transfer to the implicit Euler method for differential inclusions.

The notation used in this paper is standard: The Euclidean norm is denoted by  $|\cdot|$ , while  $\|A\| := \sup_{a \in A} |a|$  denotes the maximal norm of the elements of a set  $A \subset \mathbb{R}^m$ . The spaces of the compact and the convex and compact subsets of  $\mathbb{R}^m$  are denoted by  $\mathcal{C}(\mathbb{R}^m)$  and  $\mathcal{CC}(\mathbb{R}^m)$ , respectively. For  $A, B \in \mathcal{C}(\mathbb{R}^m)$ , the one-sided and the symmetric Hausdorff distance are given by

$$\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b|$$

and

$$\text{dist}_H(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\}.$$

For a set-valued mapping  $F : \mathbb{R}^m \rightarrow \mathcal{C}(\mathbb{R}^m)$ , we define  $F(A) := \cup_{a \in A} F(a)$ . Thus, the composition  $F \circ G$  of two set-valued mappings  $F$  and  $G$  is given by  $(F \circ G)(x) := \cup_{y \in G(x)} F(y)$ , and  $F^k(x) := (F \circ \dots \circ F)(x)$  is defined by induction.

## 1 A solvability theorem for ROSL maps

The notion of relaxed one-sided Lipschitz (ROSL) set-valued mappings was coined by Tzanko Donchev, see e.g. [4]. It generalizes the concepts of Lipschitz continuity and the (strong) one-sided Lipschitz property. A detailed analysis of this property can be found in [5] and several other works of this author.

**Definition 1.** *A mapping  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  is called relaxed one-sided Lipschitz with constant  $l \in \mathbb{R}$  if for every  $x, x' \in \mathbb{R}^m$  and  $y \in F(x)$  there*

exists some  $y' \in F(x')$  such that

$$\langle y - y', x - x' \rangle \leq l|x - x'|^2. \quad (4)$$

It was shown in [5], Theorem 3, that ROSL mappings with negative Lipschitz constant are onto for a quite general class of Banach spaces. The following results are more specific, because they provide information about the location of a point where a desired value is attained.

**Theorem 2.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be usc and ROSL with constant  $l < 0$ . Then the inclusion  $0 \in F(x)$  has a solution  $\bar{x}$  with*

$$|\bar{x}| \leq -\frac{1}{l} \text{dist}(0, F(0)). \quad (5)$$

*Proof.* Consider the inclusion  $0 \in F(x)$  and the usc mapping  $G : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  defined by

$$G(x) := x + \alpha F(x)$$

with some  $\alpha > 0$ . Let  $y_0 \in F(0)$  be the element with minimal norm. Because of the ROSL property, for any  $x \in \mathbb{R}^m$  there exists a  $y \in F(x)$  such that

$$\langle y - y_0, x \rangle \leq l|x|^2.$$

Consequently,  $z := x + \alpha y$  satisfies

$$\begin{aligned} |z|^2 &= |x|^2 + 2\alpha \langle y, x \rangle + \alpha^2 |y|^2 \\ &= |x|^2 + 2\alpha \langle y - y_0, x \rangle + 2\alpha \langle y_0, x \rangle + \alpha^2 |y|^2 \\ &\leq |x|^2 + 2\alpha l|x|^2 + 2\alpha |x| \text{dist}(0, F(0)) + \alpha^2 |y|^2. \end{aligned}$$

Thus, if  $R > -\frac{1}{l} \text{dist}(0, F(0))$ ,  $|x| \leq R$ , and  $\alpha$  is so small that  $1 + 2\alpha l \geq 0$ , then

$$\begin{aligned} |z|^2 &\leq R^2 + 2\alpha(lR + \text{dist}(0, F(0)))R + \alpha^2 |y|^2 \\ &< R^2 + \alpha^2 |y|^2. \end{aligned} \quad (6)$$

As  $F$  is usc,

$$M_R := \sup_{x \in B_R(0)} \|F(x)\| < \infty,$$

and there exists an  $\alpha > 0$  such that  $|z|^2 \leq R^2$  follows from (6). This means that for this fixed  $\alpha$ ,

$$H(x) := G(x) \cap B_R(0) \neq \emptyset \text{ for all } x \in B_R(0),$$

and  $H(\cdot)$  is also usc. By the Kakutani Theorem,  $H$  and thus also  $G$  have a fixed point  $x_R$  in  $B_R(0)$ , which implies that  $0 \in F(x_R)$ .

In particular, we find elements  $x_n \in B(0, -\frac{1}{l} \text{dist}(0, F(0)) + 1/n)$  for all  $n \in \mathbb{N}$  such that  $0 \in F(x_n)$ . As  $B(0, -\frac{1}{l} \text{dist}(0, F(0)) + 1)$  is compact, there exists a convergent subsequence of  $\{x_n\}_{n \in \mathbb{N}}$  with limit

$$\bar{x} \in B(0, -\frac{1}{l} \text{dist}(0, F(0))).$$

Since  $F$  is usc,

$$0 \in F(\bar{x}).$$

□

**Corollary 3.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be usc and ROSL with constant  $l < 0$ , and let  $x \in \mathbb{R}^m$  and  $y \in \mathbb{R}^m$  be given. Then there exists an  $\bar{x} \in \mathbb{R}^m$  with  $y \in F(\bar{x})$  and*

$$|x - \bar{x}| \leq -\frac{1}{l} \text{dist}(y, F(x)). \quad (7)$$

*Proof.* Consider the set-valued mapping

$$G(z) := F(z + x) - y,$$

which is ROSL with constant  $l$ . By the above theorem, there exists some

$$z \in B(0, -\frac{1}{l} \text{dist}(0, G(0))) = B(0, -\frac{1}{l} \text{dist}(y, F(x)))$$

such that  $0 \in G(z)$  or  $y \in F(\bar{x})$ , where  $\bar{x} = x + z$ . □

## 2 Properties of the implicit Euler map

In this section we set up the implicit Euler map associated with an ROSL map. Then we discuss some properties of the implicit Euler sets such as compactness, diameter, dependence on initial conditions, convexity, and connectedness.

## 2.1 Existence and estimates of solutions

The following theorem is the central existence and stability statement for the implicit Euler method. It follows from Corollary 3, but the roles of  $x$  and  $y$  are interchanged because of the implicit character of the scheme.

**Theorem 4.** *If  $F$  is usc and ROSL with constant  $l$  such that  $lh < 1$ , and if  $x, y \in \mathbb{R}^m$ , then there exists a solution  $\bar{y} \in \Phi_h(x)$  in the implicit Euler set*

$$\Phi_h(x) := \{z \in \mathbb{R}^m : z \in x + hF(z)\}$$

with

$$|\bar{y} - y| \leq \frac{1}{1 - lh} \text{dist}(y, x + hF(y)). \quad (8)$$

*Proof.* Consider the set-valued mapping

$$G_h(z) := hF(z) - z.$$

Since  $F$  is ROSL, for any  $z, z' \in \mathbb{R}^m$  and  $\xi \in F(z)$  there exists  $\xi' \in F(z')$  such that

$$\langle \xi - \xi', z - z' \rangle \leq l|z - z'|^2.$$

In particular,

$$\langle (h\xi - z) - (h\xi' - z'), z - z' \rangle \leq -(1 - lh)|z - z'|^2,$$

which means that  $G_h$  satisfies the ROSL condition with constant  $-(1 - lh) < 0$ . According to Corollary 3, there exists a  $\bar{y} \in \mathbb{R}^m$  such that  $-x \in G_h(\bar{y})$ , i.e.  $\bar{y} \in \Phi_h(x)$ , and

$$|\bar{y} - y| \leq \frac{1}{1 - lh} \text{dist}(-x, G_h(y)) = \frac{1}{1 - lh} \text{dist}(y, x + hF(y)).$$

□

**Corollary 5.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be an usc and relaxed one-sided Lipschitz set-valued mapping with constant  $l$ . If  $hl < 1$ , the implicit Euler scheme is Lipschitz continuous with Lipschitz constant  $\frac{1}{1-hl}$ , i.e.*

$$\text{dist}_H(\Phi_h(x), \Phi_h(x')) \leq \frac{1}{1 - hl} |x - x'|. \quad (9)$$

*Proof.* Let  $y \in \Phi_h(x)$ , i.e.  $y \in x + hF(y)$ . For any  $x' \in \mathbb{R}^m$ , Theorem 4 guarantees the existence of a  $y'$  with  $y' \in x' + hF(y')$  such that

$$\begin{aligned} |y - y'| &\leq \frac{1}{1 - hl} \text{dist}(y, x' + hF(y)) \\ &\leq \frac{1}{1 - hl} (\text{dist}(y, x + hF(y)) + \text{dist}(x + hF(y), x' + hF(y))) \\ &\leq \frac{1}{1 - hl} |x - x'|. \end{aligned}$$

□

The implicit Euler scheme is robust under perturbations of the map  $F$ .

**Corollary 6.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be an usc and relaxed one-sided Lipschitz set-valued mapping with constant  $l$  such that  $hl < 1$ . If  $G : \mathbb{R}^m \rightarrow \mathcal{C}(\mathbb{R}^m)$  is any mapping and the corresponding perturbed implicit Euler scheme satisfies  $\Phi_h^G(x) \neq \emptyset$ , then*

$$\text{dist}(\Phi_h^G(x), \Phi_h^F(x)) \leq \frac{h}{1 - hl} \sup_{\xi \in \mathbb{R}^m} \text{dist}(G(\xi), F(\xi)). \quad (10)$$

Moreover, if  $G$  is usc and one-sided Lipschitz with the same constant and if it has convex and compact values, then

$$\text{dist}_H(\Phi_h^F(x), \Phi_h^G(x)) \leq \frac{h}{1 - hl} \sup_{\xi \in \mathbb{R}^m} \text{dist}_H(F(\xi), G(\xi)) \quad (11)$$

holds for all  $x \in \mathbb{R}^m$ .

*Proof.* For  $z \in \Phi_h^G(x)$ , we have

$$\begin{aligned} \text{dist}(z, x + hF(z)) &\leq \text{dist}(z, x + hG(z)) + \text{dist}(x + hG(z), x + hF(z)) \\ &\leq h \text{dist}(G(z), F(z)), \end{aligned}$$

so that (10) follows from Theorem 4, and (11) follows by symmetry. □

**Lemma 7.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be an usc and relaxed one-sided Lipschitz set-valued mapping with constant  $l$ . If  $hl < 1$ , the diameter of the values of the implicit Euler scheme is bounded by*

$$\text{diam}(\Phi_h(x)) \leq \frac{h}{1 - lh} \sup_{y \in \Phi_h(x)} \text{diam}(F(y)). \quad (12)$$

*Proof.* Let  $y, y' \in \Phi_h(x)$ , i.e.  $y \in x + hF(y)$  and  $y' \in x + hF(y')$ . Because of the ROSL property, there exists an  $\eta \in x + hF(y')$  such that

$$\begin{aligned} hl|y - y'|^2 &\geq \langle y - \eta, y - y' \rangle \\ &= \langle y - y', y - y' \rangle + \langle y' - \eta, y - y' \rangle \\ &= |y - y'|^2 + \langle y' - \eta, y - y' \rangle. \end{aligned}$$

Thus

$$(1 - hl)|y - y'|^2 \leq \langle \eta - y', y - y' \rangle \leq h \operatorname{diam}(F(y'))|y - y'|,$$

and the statement of the lemma follows.  $\square$

## 2.2 Structure of the implicit Euler set

Estimate (12) for the diameters and estimates (43) and (45) for the location of the images of the method inherit the implicit character of the scheme, which is inconvenient for practical implementation, because only crude upper bounds for the diameters or the norm of the images of  $F$  can be used.

**Theorem 8.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be an usc and relaxed one-sided Lipschitz set-valued mapping with constant  $l$ . If  $hl < 1$ , the values of the implicit Euler scheme*

$$\Phi_h(x) := \{y \in \mathbb{R}^m : y \in x + hF(y)\}$$

*are nonempty and compact.*

*Proof.* Theorem 4 guarantees that the images of the implicit Euler scheme are non-empty.

Let  $\{y_k\}_{k \in \mathbb{N}}$  be a convergent sequence in  $\Phi_h(x)$  with  $\lim_{k \rightarrow \infty} y_k = y$ . Then  $x \in y_k - hF(y_k)$  implies

$$\begin{aligned} \operatorname{dist}(x, y - hF(y)) &\leq \operatorname{dist}(x, y_k - hF(y_k)) + \operatorname{dist}(y_k - hF(y_k), y - hF(y)) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

because the mapping  $y \mapsto y - hF(y)$  is usc. Consequently,  $x \in y - hF(y)$ , and the images of  $\Phi_h$  are closed.

Assume that  $\Phi_h(x)$  is unbounded for some  $x \in \mathbb{R}^m$ . Without loss of generality,  $x = 0$ . Thus there exists a sequence  $\{y_k\}_{k \in \mathbb{N}}$  in  $\Phi_h(0)$  with  $|y_k| \rightarrow$



$\infty$  as  $k \rightarrow \infty$ . By definition,  $y_k \in hF(y_k)$ . As  $hF(\cdot)$  is ROSL with constant  $hl$ , there exists a sequence  $\{y'_k\}_{k \in \mathbb{N}}$  in  $F(0)$  such that for all  $k \in \mathbb{N}$ ,

$$|y_k|^2 - \langle y'_k, y_k \rangle = \langle y_k - y'_k, y_k - 0 \rangle \leq hl|y_k|^2,$$

and consequently

$$(1 - hl)|y_k|^2 \leq \langle y'_k, y_k \rangle \leq |y'_k||y_k|$$

implies

$$|y'_k| \geq (1 - hl)|y_k| \rightarrow \infty,$$

which contradicts the assumption that  $F(0)$  is bounded.  $\square$

Below we will show that implicit Euler sets of a mapping  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  that satisfies the ROSL condition need not be convex in dimension  $m > 1$ . However, in dimension one the answer is affirmative.

**Lemma 9.** *Let  $F : \mathbb{R} \rightarrow \mathcal{CC}(\mathbb{R})$  be a relaxed one-sided Lipschitz set-valued mapping with constant  $l$ . If  $hl < 1$ , the values of the implicit Euler scheme are convex.*

*Proof.* The set-valued mapping  $F$  can be represented as

$$F(x) = [\alpha(x), \beta(x)], \tag{13}$$

where  $\alpha(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  and  $\beta(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  are single-valued functions.

The ROSL condition states that for all  $x, x' \in \mathbb{R}$  and  $y \in F(x)$  there exists a  $y' \in F(x')$  such that

$$(y - y')(x - x') \leq l(x - x')^2. \tag{14}$$

In this particular setting, this is equivalent with

$$\begin{aligned} \beta(x) - \beta(x') &\leq l(x - x'), \\ \alpha(x) - \alpha(x') &\leq l(x - x') \end{aligned}$$

whenever  $x \geq x'$ .

Because of (13), the implicit Euler set is

$$\Phi_h(x) := \{y \in \mathbb{R} : x \in [y - h\beta(y), y - h\alpha(y)].\}$$

Assume that  $\Phi_h(x)$  is not convex for some fixed  $x \in \mathbb{R}$ , i.e. there exist  $y_0 < y_1 < y_2$  such that

$$x \in [y_0 - h\beta(y_0), y_0 - h\alpha(y_0)] \text{ and } x \in [y_2 - h\beta(y_2), y_2 - h\alpha(y_2)],$$

but

$$x \notin [y_1 - h\beta(y_1), y_1 - h\alpha(y_1)].$$

Then either  $x > y_1 - h\alpha(y_1)$  or  $x < y_1 - h\beta(y_1)$ . In the first case,

$$y_0 - h\alpha(y_0) \geq x > y_1 - h\alpha(y_1)$$

implies

$$y_1 - y_0 < h(\alpha(y_1) - \alpha(y_0)) \leq hl(y_1 - y_0),$$

which yields  $hl > 1$ . In the second case,

$$y_2 - h\beta(y_2) \leq x < y_1 - h\beta(y_1)$$

and

$$y_2 - y_1 < h(\beta(y_2) - \beta(y_1)) \leq hl(y_2 - y_1)$$

also imply  $hl > 1$ .

Hence the image of the implicit Euler scheme in dimension one is convex whenever the scheme is well-defined according to Theorem 8.  $\square$

In  $\mathbb{R}^m$  with  $m > 1$ , the ROSL condition does not seem to be as powerful as in dimension one, because it controls the expansion of the images in only one direction. The following example provides a continuous map  $G : \mathbb{R}^2 \rightarrow \mathcal{CC}(\mathbb{R}^2)$  which satisfies the ROSL property and which has nonconvex implicit Euler sets.

**Example 10.** Let a set-valued mapping  $F : \mathbb{R} \times \{0\} \rightarrow \mathcal{CC}(\mathbb{R}^2)$  be given by

$$F(x, 0) = \{z^-(x) + \eta(z^+(x) - z^-(x)) : 0 \leq \eta \leq 1\},$$

where

$$z^-(x) = \begin{cases} (-x, 1), & x < -1 \\ (-x, -x), & |x| \leq 1 \\ (-x, -1), & x > 1 \end{cases}$$

and

$$z^+(x) = \begin{cases} (-\frac{1}{2} - 2x, -\frac{1}{2}), & x < -1 \\ (1 - \frac{x}{2}, \frac{x}{2}), & |x| \leq 1 \\ (1 - \frac{x}{2}, \frac{1}{2}), & x > 1. \end{cases}$$

Consider its extension  $F : \mathbb{R}^2 \rightarrow \mathcal{CC}(\mathbb{R}^2)$  defined by

$$F(x_1, x_2) = F(x_1, 0) - \frac{5}{4}(0, x_2) \quad \text{for } (x_1, x_2) \in \mathbb{R}^2.$$

A short discussion shows that

$$F^{-1}(0) = \left\{ (x_1, \frac{8}{5} \min(x_1, 1)(x_1 - 1)(2 + x_1)^{-1}) : 0 \leq x_1 \leq 2 \right\}$$

is a nonconvex set. Next we prove that  $F$  satisfies (ROSL) with  $l = -\frac{1}{4}$ . Introduce the function  $g(x_1, \eta) = z^-(x_1) + \eta(z^+(x_1) - z^-(x_1))$ . From the negative slopes in the first component of  $z^\pm$  one finds that for all  $x_1, x'_1 \in \mathbb{R}$  and  $0 \leq \eta \leq 1$

$$(g_1(x_1, \eta) - g_1(x'_1, \eta))(x_1 - x'_1) \leq -\frac{1}{2}(x_1 - x'_1)^2.$$

Thus for any two points  $x = (x_1, 0), x' = (x'_1, 0)$  and  $y = g(x_1, \eta) \in F(x_1, 0)$  we can take  $y' = g(x'_1, \eta) \in F(x'_1, 0)$  and obtain that (ROSL) holds with  $l = -\frac{1}{2}$ . Moreover, we have  $|y_2 - y'_2| = |g_2(x_1, \eta) - g_2(x'_1, \eta)| \leq |x_1 - x'_1|$ . In the general case, given  $x, x' \in \mathbb{R}^2$  and  $y \in F(x)$  we find  $y = z - \frac{5}{4}(0, x_2)$  for some  $z \in F(x_1, 0)$ . Therefore, there exists a  $z' \in F(x'_1, 0)$  such that

$$(z_1 - z'_1)(x_1 - x'_1) \leq -\frac{1}{2}(x_1 - x'_1)^2, \quad |z_2 - z'_2| \leq |x_1 - x'_1|.$$

Setting  $y' = z' - \frac{5}{4}(0, x'_2)$  we obtain  $y' \in F(x')$  and the estimate

$$\begin{aligned} \langle y - y', x - x' \rangle &= (z_1 - z'_1)(x_1 - x'_1) + (z_2 - z'_2)(x_2 - x'_2) - \frac{5}{4}(x_2 - x'_2)^2 \\ &\leq -\frac{1}{2}(x_1 - x'_1)^2 + |x_1 - x'_1||x_2 - x'_2| - \frac{5}{4}(x_2 - x'_2)^2 \\ &\leq -\frac{1}{4}(x_1 - x'_1)^2 - \frac{1}{4}(x_2 - x'_2)^2. \end{aligned}$$

Finally, the mapping

$$G(y) = y + F(y), \quad y \in \mathbb{R}^2$$

satisfies the ROSL condition with constant  $l = \frac{3}{4}$ . (Note that  $G$  satisfies the conditions of Lemma 12.) The implicit Euler sets associated with  $G$  are

$$\Phi_h(x) = \{y \in \mathbb{R}^2 : y \in x + hG(y)\}.$$

For  $x = 0$  and  $h = 1$  we have  $hl < 1$  as in Theorem 4, but

$$\Phi_1(0) = \{y \in \mathbb{R}^2 : y \in 0 + y + F(y)\} = F^{-1}(0)$$

is nonconvex. A more detailed computation reveals that  $\Phi_h(0)$  is nonconvex for all  $0 < h \leq 1$ .

The following geometric characterization of convexity still holds.

**Lemma 11.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be any set-valued mapping. The images of the implicit Euler scheme are convex if and only if for any  $x \in \mathbb{R}^m$ , the intersection*

$$\{(z, z) : z \in \mathbb{R}^m\} \cap \text{graph}\{x + hF(\cdot)\} \quad (15)$$

*is (either empty or) convex. In particular, this is satisfied if  $\text{graph } F(\cdot)$  is convex.*

*Proof.* If  $y, y' \in \Phi_h(x)$ , then

$$y \in x + hF(y) \text{ and } y' \in x + hF(y')$$

and

$$(y, y) \in \text{graph}\{x + hF(\cdot)\} \text{ and } (y', y') \in \text{graph}\{x + hF(\cdot)\}$$

hold.

At the same time, convexity of  $\Phi_h(x)$  is equivalent with

$$\lambda y + (1 - \lambda)y' \in \Phi_h(x),$$

$$\lambda y + (1 - \lambda)y' \in x + hF(\lambda y + (1 - \lambda)y'),$$

and

$$(\lambda y + (1 - \lambda)y', \lambda y + (1 - \lambda)y') \in \text{graph}\{x + hF(\cdot)\} \quad (16)$$

for any  $\lambda \in [0, 1]$  and  $y, y' \in \Phi_h(x)$ .

It remains to note that convexity of the set (15) and inclusion (16) are equivalent.

If  $\text{graph } F(\cdot)$  is convex then  $\text{graph}\{x + hF(\cdot)\}$  and its intersection with the diagonal are also convex.  $\square$

The question of connectedness of the implicit Euler sets could not be fully resolved. The following lemma gives an affirmative answer if the right-hand side  $F$  is parametrized by a continuous family of ROSL selections.

**Lemma 12.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be a set-valued mapping, let  $U \subset \mathbb{R}^d$  be any path-connected set, and let  $f : \mathbb{R}^m \times U \rightarrow \mathbb{R}^m$  be a single-valued function such that*

- $F(x) = \cup_{u \in U} f(x, u)$ ;

- $f(\cdot, u)$  is continuous for any  $u \in U$ ;
- there exists some  $l \in \mathbb{R}$  such that  $lh < 1$  and  $f(\cdot, u)$  is ROSL with constant  $l$  for any  $u \in U$ ;
- $f(x, \cdot)$  is continuous for any  $x \in \mathbb{R}^m$ .

Then the images of the implicit Euler scheme are path-connected.

*Proof.* Theorem 4 implies that the implicit Euler scheme applied to the function  $f(\cdot, u)$  has a solution. It is well-known that this solution is unique, because  $y = x + hf(y, u)$  and  $y' = x + hf(y', u)$  imply

$$y - y' = h(f(y, u) - f(y', u))$$

and thus

$$|y - y'|^2 = h \langle f(y, u) - f(y', u), y - y' \rangle \leq lh |y - y'|^2.$$

As  $lh < 1$ , it follows that  $y = y'$ .

If  $u, u' \in U$  and  $y$  satisfies  $y = x + hf(y, u)$ , then Theorem 4 and

$$|y - (x + hf(y, u'))| \leq \underbrace{|y - x - hf(y, u)|}_{=0} + h|f(y, u) - f(y, u')|$$

imply that the unique solution  $y'$  of  $y' = x + hf(y', u')$  satisfies

$$|y - y'| \leq \frac{h}{1 - lh} |f(y, u) - f(y, u')|.$$

Now let  $y, y' \in \Phi_h(x)$  be arbitrary solutions of the implicit Euler scheme, and let  $u, u' \in U$  be the corresponding parameters such that  $y = x + hf(y, u)$  and  $y' = x + hf(y', u')$ . As  $U$  is path-connected, there exists a continuous path  $\sigma : [0, 1] \rightarrow U$  such that  $\sigma(0) = u$  and  $\sigma(1) = u'$ . Define  $\varphi : [0, 1] \rightarrow \Phi_h(x)$  by  $\varphi(s) := \{z \in \mathbb{R}^m : z = x + hf(z, \sigma(s))\}$ . By the argument above,  $\varphi(\cdot)$  is single-valued, well-defined and continuous, and thus  $\Phi_h(x)$  is path-connected.  $\square$

### 3 Error estimates

Throughout this section, we assume the following hypotheses to be satisfied:

- (A1) The mapping  $F : [0, T] \times \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  is (simultaneously) continuous in both arguments.
- (A2) There is a continuous function  $l : [0, T] \rightarrow \mathbb{R}$  such that for  $t \in [0, T]$ ,  $l(t)h < 1$  and the mapping  $F(t, \cdot)$  is ROSL with constant  $l(t)$ , i.e. for every  $x, x' \in \mathbb{R}^m$  and  $y \in F(t, x)$  there exists some  $y' \in F(t, x')$  such that

$$\langle y - y', x - x' \rangle \leq l(t)|x - x'|^2. \quad (17)$$

Consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \text{ a.e. in } [0, T], \quad x(0) = x_0, \quad (18)$$

where  $T \in \mathbb{R}_+$ . The set of solutions to (18) will be denoted  $\mathcal{S}(T, 0, x_0)$ , while

$$\mathcal{R}(t, 0, x_0) := \{x(t) : x(\cdot) \in \mathcal{S}(t, 0, x_0)\}$$

denotes the reachable set at time  $t$  with  $0 \leq t \leq T$ .

Consider also the discrete-time (but spatially continuous) system

$$y_{k+1} \in \Phi_h(t_k, y_k) := \{y \in \mathbb{R}^m : y \in y_k + hF(t_{k+1}, y)\} \text{ for } k \in \mathbb{N}, \quad y_0 = x_0 \quad (19)$$

with step-size  $h := T/N$ , where  $N \in \mathbb{N}$ ,  $t_k := kh$ . The sets  $\mathcal{S}_h(T, 0, x_0)$  and

$$\mathcal{R}_h(kh, 0, x_0) := \{y_k : \{y_j\}_{j=0, \dots, k} \in \mathcal{S}_h(kh, 0, x_0)\} \quad (20)$$

for  $k \in \mathbb{N}$  with  $kh \leq T$  are the discrete-time solution set and reachable set at time  $kh$ , respectively.

#### 3.1 Stability theorems

The following theorem is a continuous version of the Filippov Theorem for ROSL right-hand sides. The proof follows the argument in [6], but existence of a solution is guaranteed by continuity instead of a growth condition.

**Theorem 13.** For any given  $y \in C^1([0, T], \mathbb{R}^m)$ , there exists a solution  $x(\cdot)$  of (18) such that

$$|x(t) - y(t)| \leq e^{\int_0^t l(s) ds} |x(0) - y(0)| + \int_0^t e^{\int_s^t l(\tau) d\tau} g(s) ds \quad (21)$$

for all  $t \in [0, T]$ , where  $g \in C([0, T], \mathbb{R}_+)$  is defined by

$$g(t) := \text{dist}(\dot{y}(t), F(t, y(t))).$$

If  $F$  is defined on  $[0, \infty) \times \mathbb{R}^m$  and satisfies (A1) and (A2) on the infinite time interval, and if  $y \in C^1([0, \infty), \mathbb{R}^m)$  is given, then there exists a solution  $x(\cdot)$  of (18) on  $[0, \infty)$  such that (21) holds for any  $t \geq 0$ .

*Proof.* Define the set-valued mapping

$$G(t, x) := F(t, x) \cap H(t, x),$$

where

$$H(t, x) := \{v \in \mathbb{R}^m : \langle y(t) - x, \dot{y}(t) - v \rangle \leq l(t)|y(t) - x|^2 + g(t)|y(t) - x|\}. \quad (22)$$

The values of  $H(\cdot, \cdot)$  are nonempty: For given  $t$  and  $x$ , there exists some  $w \in F(t, y(t))$  such that  $|\dot{y}(t) - w| = g(t)$ . The ROSL condition implies the existence of some  $v \in F(t, x)$  with

$$\langle y(t) - x, w - v \rangle \leq l(t)|y(t) - x|^2.$$

Thus,

$$\begin{aligned} \langle y(t) - x, \dot{y}(t) - v \rangle &= \langle y(t) - x, \dot{y}(t) - w \rangle + \langle y(t) - x, w - v \rangle \\ &\leq |y(t) - x|g(t) + l(t)|y(t) - x|^2, \end{aligned}$$

which means that  $v \in G(t, x)$ .

As  $y(\cdot)$ ,  $\dot{y}(\cdot)$ , the scalar product,  $l(\cdot)$ , and  $g(\cdot)$  are continuous, the graph of  $H(\cdot, \cdot)$  is closed. By Theorem 1.1.1 in [1],  $G(\cdot, \cdot)$  is usc. As  $G$  has convex and compact values, Theorem 2.1.3 in [1] guarantees the existence of a solution  $x(\cdot)$  of the differential inclusion

$$\dot{x}(t) \in G(t, x(t)), \quad x(0) = x_0 \quad (23)$$

in a neighbourhood of  $t = 0$ . Let  $[0, t_+)$  be the maximal subinterval of  $[0, T]$  on which a solution  $x(\cdot)$  of (23) exists, and define

$$s(t) := |x(t) - y(t)| \text{ for } t \in [0, t_+).$$

By construction,

$$s(t)\dot{s}(t) = \langle x(t) - y(t), \dot{x}(t) - \dot{y}(t) \rangle \leq l(t)s(t)^2 + g(t)s(t). \quad (24)$$

Consider the set  $Z := \{t \in [0, T] : s(t) = 0\}$ , and let  $Z'$  be the set of all density points of  $Z$ . Because of the Lebesgue Density Theorem (see Theorem 2.2.1 in [11]),  $\text{meas}(Z) = \text{meas}(Z')$ . If  $t \notin Z$ , then

$$\dot{s}(t) \leq l(t)s(t) + g(t) \quad (25)$$

holds, because we can divide (24) by  $s(t)$ . If  $t \in Z'$  and  $\dot{s}(t)$  exists, then  $\dot{s}(t) = 0$ , and (25) holds, because  $s(t) = 0$  and  $g(t) \geq 0$ . Thus,  $\dot{s}$  satisfies (25) almost everywhere, and the Gronwall Lemma yields that

$$|x(t) - y(t)| \leq e^{\int_0^t l(s)ds} |x(0) - y(0)| + \int_0^t e^{\int_s^t l(\tau)d\tau} g(s)ds \quad (26)$$

for all  $t \in [0, t_+)$ . In particular,  $x(\cdot)$  is bounded on  $[0, t_+)$ . As  $[0, t_+)$  was assumed to be maximal,  $t_+ = T$ .

Precisely the same argument yields the desired result on the infinite time interval.  $\square$

The following theorem is a discrete version of the Filippov Theorem for the implicit Euler scheme.

**Theorem 14.** *For any sequence  $\{x_k\}_{k=0}^N$  in  $\mathbb{R}^m$  there exists a solution  $\{y_k\}_{k=0}^N$  of the implicit Euler scheme such that*

$$|x_k - y_k| \leq e^{h \sum_{j=1}^k \mu(t_j)} |x_0 - y_0| + h \sum_{j=1}^k e^{h \sum_{i=j}^k \mu(t_i)} g_j, \quad (27)$$

where  $\mu(t) := \frac{l(t)}{1-l(t)h}$  and  $g_j := \text{dist}(\frac{1}{h}(x_j - x_{j-1}), F(t_j, x_j))$ .



*Proof.* Let  $\{x_j\}_{j=0}^N$  be given, and assume that a solution  $\{y_j\}$  of (19) has already been constructed for  $j \in \{0, \dots, k\}$ ,  $k < N$ . Because of Theorem 4, there exists a solution  $y_{k+1} \in \Phi_h(y_k)$  such that

$$\begin{aligned} |x_{k+1} - y_{k+1}| &\leq \frac{1}{1 - l(t_{k+1})h} \text{dist}(x_{k+1}, y_k + hF(t_{k+1}, x_{k+1})) \\ &\leq \frac{1}{1 - l(t_{k+1})h} |x_k - y_k| + \frac{h}{1 - l(t_{k+1})h} g_{k+1}. \end{aligned}$$

By induction,

$$\begin{aligned} |x_k - y_k| &\leq \left( \prod_{j=1}^k \frac{1}{1 - l(t_j)h} \right) |x_0 - y_0| + h \sum_{j=1}^k \left( \prod_{i=j}^k \frac{1}{1 - l(t_i)h} \right) g_j \\ &= \left( \prod_{j=1}^k \left( 1 + \frac{l(t_j)h}{1 - l(t_j)h} \right) \right) |x_0 - y_0| + h \sum_{j=1}^k \left( \prod_{i=j}^k \left( 1 + \frac{l(t_i)h}{1 - l(t_i)h} \right) \right) g_j \\ &\leq e^{\sum_{j=1}^k \frac{l(t_j)h}{1 - l(t_j)h}} |x_0 - y_0| + h \sum_{j=1}^k e^{\sum_{i=j}^k \frac{l(t_i)h}{1 - l(t_i)h}} g_j. \end{aligned}$$

□

### 3.2 The finite time interval

The boundedness of the continuous as well as of the numerical solutions follows immediately from the continuous and the discrete Filippov Theorems.

**Corollary 15.** *Any solution  $x(\cdot)$  of (18) is bounded by*

$$|x(t)| \leq e^{\int_0^t l(\tau) d\tau} |x_0| + \int_0^t e^{\int_s^t l(\tau) d\tau} \text{dist}(0, F(s, 0)) ds \quad (28)$$

for all  $t \in [0, T]$ .

*Proof.* The statement follows from Theorem 13 with  $y(t) = 0$  for all  $t \in [0, T]$ . □

**Corollary 16.** *Every solution  $\{y_k\}_{k=0}^N$  of the implicit Euler scheme is bounded by*

$$|y_k| \leq e^{\sum_{j=1}^k h\mu(t_j)} |y_0| + h \sum_{j=1}^k e^{\sum_{i=j}^k h\mu(t_i)} \text{dist}(0, F(t_j, 0)), \quad (29)$$

where  $\mu(t) := \frac{l(t)}{1 - l(t)h}$ .

*Proof.* The statement follows from Theorem 14 with  $x_k = 0$  for  $k = 0, \dots, N$ .  $\square$

Because of Corollaries 15 and 16, all solutions of (18) and (19) are contained in a ball  $B_M(0)$  with sufficiently large radius  $M > 0$ . In particular, we may assume because of continuity that  $F$  is globally bounded, i.e. that there exists some  $P > 0$  such that  $F(t, x) \subset B_P(0)$  for all  $t \in [0, T]$  and  $x \in \mathbb{R}^m$ .

It is well-known that a continuous set-valued mapping  $F : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  is uniformly continuous on every compact set  $K \subset \mathbb{R} \times \mathbb{R}^m$ , i.e. for any  $\varepsilon > 0$  there exists a  $\delta > 0$  such that

$$\text{dist}_H(F(x), F(x')) \leq \varepsilon$$

whenever  $x, x' \in K$  and  $|x - x'| \leq \delta$ . Hence the moduli of continuity

$$\chi(\delta) := \sup\{\text{dist}_H(F(t, x), F(t, x')) : t \in [0, T], x, x' \in B_M(0) \text{ with } |x - x'| \leq \delta\} \quad (30)$$

and

$$\tau(\delta) := \sup\{\text{dist}_H(F(t, x), F(t', x)) : x \in B_M(0), t, t' \in [0, T] \text{ with } |t - t'| \leq \delta\} \quad (31)$$

are well-defined.

**Theorem 17.** *For any solution  $\{y_k\}_{k=0}^N$  of (19), there exists a solution  $x(\cdot)$  of (18) such that*

$$|x(kh) - y_k| \leq (\tau(h) + \chi(Ph)) \int_0^{t_k} e^{l(t_k-s)} ds \quad (32)$$

for all  $k \in \{0, \dots, N\}$ .

*Proof.* By definition,  $y_{k+1} \in y_k + hF(t_{k+1}, y_{k+1})$ . Define the absolutely continuous function  $y(\cdot) : [0, T] \rightarrow \mathbb{R}^m$  by linear interpolation on the subintervals  $[kh, (k+1)h]$  for  $k \in \{0, \dots, N-1\}$ . On each subinterval,

$$\begin{aligned} \text{dist}(\dot{y}(t), F(t, y(t))) &\leq \text{dist}(F(t_{k+1}, y_{k+1}), F(t, y(t))) \\ &\leq \tau(t_{k+1} - t) + \chi(|y_{k+1} - y(t)|) \\ &\leq \tau(h) + \chi(Ph). \end{aligned}$$

Theorem 13, the continuous version of the Filippov Theorem, guarantees (piecewise, and thus globally) the existence of a solution  $x(\cdot)$  of (18) such that

$$|x(t) - y(t)| \leq \int_0^t e^{l(t-s)} (\tau(h) + \chi(Ph)) ds$$

for all  $t \in [0, T]$  and in particular for all  $t \in \{kh : 0 \leq k < N\}$ .  $\square$

**Theorem 18.** *For any solution  $x(\cdot)$  of (18) there exists a solution  $\{y_k\}_{k=0}^N$  of (19) with*

$$|x(t_k) - y_k| \leq (\tau(h) + \chi(Ph)) h \sum_{j=1}^k e^{h \sum_{i=j}^k \mu(t_i)}, \quad (33)$$

where  $\mu(t) := \frac{l(t)}{1-l(t)h}$

*Proof.* For  $k \geq 1$ , let

$$g_k := \text{dist}\left(\frac{1}{h}(x(t_k) - x(t_{k-1})), F(t_k, x_k)\right).$$

Then

$$\begin{aligned} g_k &= \text{dist}\left(\frac{1}{h} \int_{t_{k-1}}^{t_k} \dot{x}(t) dt, F(t_k, x(t_k))\right) \\ &\leq \text{dist}(\overline{\text{co}}(\{\dot{x}(t) : t \in [t_{k-1}, t_k]\}), F(t_k, x(t_k))) \\ &\leq \sup_{t \in [t_{k-1}, t_k]} \text{dist}(F(t, x(t)), F(t_k, x(t_k))) \\ &\leq \tau(h) + \chi(Ph), \end{aligned}$$

and the desired result follows by applying the discrete Filippov Theorem 14 with  $x_k := x(t_k)$ .  $\square$

Please note that the estimates (28), (29), (32), and (33) allow to exploit negative one-sided Lipschitz constants.

### 3.3 The infinite time-interval

Consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t)) \text{ a.e. in } [0, \infty), \quad x(0) = x_0, \quad (34)$$

where the mapping  $F : [0, \infty) \times \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  is required to be (simultaneously) continuous in both arguments and bounded at zero, i.e.

$$\sup_{t \geq 0} \text{dist}(0, F(t, 0)) < \infty. \quad (35)$$

We have to assume that the relaxed one-sided Lipschitz constant is given by a continuous function  $l : [0, \infty) \rightarrow \mathbb{R}$  such that the integrals in (28) and the Riemann sum in (29) are bounded for all  $t \geq 0$ .

In this case, Corollaries 15 and 16 guarantee that all solutions and their numerical approximations remain in a ball  $B_M(0)$  for all time.

If we assume in addition, that  $F$  is uniformly continuous on  $[0, \infty) \times B_M(0)$ , the moduli of continuity  $\tau$  and  $\chi$  are well-defined, and it follows from (35) that  $F$  is bounded on  $[0, \infty) \times B_M(0)$ .

Thus Theorems 17 and 18 hold provided that the integral in (32) is bounded on  $[0, \infty)$ .

Note that if  $l_{\max} := \sup_{t \in [0, \infty)} l(t) < 0$ , the integrals and Riemann sums mentioned above are bounded indeed. This behaviour matches the shadowing results presented in [15], where it is shown that the errors of reasonable numerical approximations of differential inclusions with strictly contractive flow are bounded on  $[0, \infty)$ . However, the above results are stronger than the shadowing theorem, because the boundedness condition on  $l(\cdot)$  is by no means necessary.

### 3.4 The influence of spatial discretization

Corollary 5 implies the following estimate for errors induced by spatial discretization. Recall that compositions of set-valued maps are inductively defined as explained in the Introduction.

**Theorem 19.** *Let  $F$  be a set-valued mapping which satisfies (A1) and (A2), and let  $\Phi_h : [0, T] \times \mathbb{R}^m \rightarrow \mathcal{C}(\mathbb{R}^m)$  be the corresponding implicit Euler scheme. Let  $\Phi_{h,\rho} : [0, T] \times \mathbb{R}^m \rightarrow \mathcal{C}(\mathbb{R}^m)$  be any set-valued mapping with*

$$\text{dist}_H(\Phi_h(t, x), \Phi_{h,\rho}(t, x)) \leq \rho \text{ for all } t \in [0, T] \text{ and } x \in \mathbb{R}^m$$

for some  $\rho \geq 0$ . Then the error caused by spatial discretization is

$$\begin{aligned} & \text{dist}_H((\Phi_h(t_{k-1}, \cdot) \circ \dots \circ \Phi_h(t_0, \cdot))(X), (\Phi_{h,\rho}(t_{k-1}, \cdot) \circ \dots \circ \Phi_{h,\rho}(t_0, \cdot))(X_\rho)) \\ & \leq \rho \sum_{j=1}^{k+1} \prod_{i=j}^k \frac{1}{1-l(t_i)h} \end{aligned} \quad (36)$$

whenever  $\text{dist}_H(X, X_\rho) \leq \rho$  and  $kh \leq T$ .

*Proof.* Assume that (36) already holds true for some  $k \in \mathbb{N}$ . Then, for any point  $x^k \in (\Phi_h(t_{k-1}, \cdot) \circ \dots \circ \Phi_h(t_0, \cdot))(X)$  there exists some  $x_\rho^k \in (\Phi_{h,\rho}(t_{k-1}, \cdot) \circ \dots \circ \Phi_{h,\rho}(t_0, \cdot))(X_\rho)$  such that  $|x^k - x_\rho^k| \leq \rho \sum_{j=1}^{k+1} \prod_{i=j}^k \frac{1}{1-l(t_j)h}$ . By Corollary 5,  $\Phi_h(t_k, \cdot)$  is Lipschitz continuous with Lipschitz constant  $\frac{1}{1-l(t_{k+1})h}$ . Thus

$$\begin{aligned} & \text{dist}_H(\Phi_h(t_k, x^k), \Phi_{h,\rho}(t_k, x_\rho^k)) \\ & \leq \text{dist}_H(\Phi_h(t_k, x^k), \Phi_h(t_k, x_\rho^k)) + \text{dist}_H(\Phi_h(t_k, x_\rho^k), \Phi_{h,\rho}(t_k, x_\rho^k)) \\ & \leq \frac{1}{1-l(t_{k+1})h} |x^k - x_\rho^k| + \rho \\ & = \rho \sum_{j=1}^{k+2} \prod_{i=j}^{k+1} \frac{1}{1-l(t_j)h}. \end{aligned}$$

□

**Remark 20.** *The detailed analysis of the explicit Euler method given in [3] shows that it is necessary to choose  $\rho := h^2$  in order to obtain convergence of order one for the fully discretized scheme. As the right-hand side of (36) behaves like the integral*

$$\frac{\rho}{h} \int_0^t e^{\int_s^t l(\tau) d\tau} ds,$$

*it is reasonable to adapt the same setting for  $\rho$  in the present context.*

## 4 Implementation

Let  $\Delta_\rho := \rho\mathbb{Z}^m$  be an equidistant grid in  $\mathbb{R}^m$ . For  $x_\rho \in \Delta_\rho$ , we define the fully discretized implicit Euler scheme  $\Phi_{h,\rho} : \Delta_\rho \rightrightarrows \Delta_\rho$  by

$$\Phi_{h,\rho}(x_\rho) := \{y_\rho \in \Delta_\rho : \text{dist}(y_\rho, x_\rho + hF(y_\rho)) \leq \rho + h\chi(\rho)\}. \quad (37)$$

As in (30), let  $\chi(\cdot)$  denote the modulus of continuity of  $F$  w.r.t. the space coordinate. The reachable set of the fully discretized implicit Euler scheme is the set

$$\mathcal{R}_{h,\rho}(kh, 0, x_0) := \Phi_{h,\rho}^k(x_0) = \Phi_{h,\rho} \circ \dots \circ \Phi_{h,\rho}(x_0).$$

The following theorem shows that the tolerance  $\rho + h\chi(\rho)$  in the right-hand side of (37) leads to a reasonable error estimate.

**Theorem 21.** *If  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  is continuous and ROSL with constant  $l$  such that  $lh < 1$ , then*

$$\text{dist}_H(\Phi_h(x_\rho), \Phi_{h,\rho}(x_\rho)) \leq \max(\rho, \frac{\rho + h\chi(\rho)}{1 - lh}) \quad (38)$$

for any  $x_\rho \in \Delta_\rho$ .

*Proof.* By definition (37),

$$\text{dist}(y_\rho, x_\rho + hF(y_\rho)) \leq \rho + h\chi(\rho)$$

for any  $y_\rho \in \Phi_{h,\rho}(x_\rho)$ , and Theorem 4 guarantees the existence of some  $y \in \Phi_h(x_\rho)$  with

$$\text{dist}(y_\rho, y) \leq \frac{\rho + h\chi(\rho)}{1 - lh}.$$

On the other hand, if  $y \in \Phi_h(x_\rho)$ , then  $y \in x_\rho + hF(y)$ . Let  $y_\rho \in \Delta_\rho$  be such that  $|y_\rho - y| \leq \rho$ . Then

$$\begin{aligned} & \text{dist}(y_\rho, x_\rho + hF(y_\rho)) \\ & \leq |y_\rho - y| + \text{dist}(y, x_\rho + hF(y)) + \text{dist}(x_\rho + hF(y), x_\rho + hF(y_\rho)) \\ & \leq |y_\rho - y| + h\chi(|y_\rho - y|) \leq \rho + h\chi(\rho), \end{aligned}$$

so that  $y_\rho \in \Phi_{h,\rho}(x_\rho)$  and

$$\text{dist}(\Phi_h(x_\rho), \Phi_{h,\rho}(x_\rho)) \leq \rho.$$

□

**Remark 22.** *There is an alternative approach to the spatial discretization of the implicit Euler scheme, which is mainly of theoretical interest. We discuss it briefly, because it is similar to techniques which have been used successfully in [16] for the fully discretized Viability Kernel Algorithm.*

*Consider the set-valued grid function  $\tilde{\Phi}_{h,\rho} : \Delta_\rho \rightrightarrows \Delta_\rho$  given by*

$$\tilde{\Phi}_{h,\rho}(x_\rho) := \{y_\rho \in \Delta_\rho : \text{dist}(y_\rho, x_\rho + hF(B_\rho(y_\rho))) \leq \rho\}, \quad (39)$$

*Let  $y \in \Phi_h(x_\rho)$ , i.e.  $y \in x_\rho + hF(y)$ . If  $|y_\rho - y| \leq \rho$ , then*

$$\text{dist}(y_\rho, x_\rho + hF(B_\rho(y_\rho))) \leq |y_\rho - y| + \text{dist}(y, x_\rho + hF(y)) \leq \rho,$$

i.e.  $y_\rho \in \tilde{\Phi}_{h,\rho}(x_\rho)$ , and thus

$$\text{dist}(\Phi_h(x_\rho), \tilde{\Phi}_{h,\rho}(x_\rho)) \leq \rho. \quad (40)$$

On the other hand, if  $y_\rho \in \tilde{\Phi}_{h,\rho}(x_\rho)$ , then there exists a  $y' \in B_\rho(y_\rho)$  with  $\text{dist}(y_\rho, x_\rho + hF(y')) \leq \rho$ , so that

$$\begin{aligned} & \text{dist}(y_\rho, x_\rho + hF(y_\rho)) \\ & \leq \text{dist}(y_\rho, x_\rho + hF(y')) + \text{dist}(x_\rho + hF(y'), x_\rho + hF(y_\rho)) \\ & \leq \rho + h\chi(\rho). \end{aligned}$$

By Theorem 4,

$$\text{dist}(\tilde{\Phi}_{h,\rho}(x_\rho), \Phi_h(x_\rho)) \leq \frac{\rho + h\chi(\rho)}{1 - lh}. \quad (41)$$

Thus both discretizations,  $\Phi_{h,\rho}$  and  $\tilde{\Phi}_{h,\rho}$ , satisfy the same error estimates, but we do not see an effective way to compute  $\tilde{\Phi}_{h,\rho}$  in applications.

The following lemmas provide information about the location of the image of the implicit Euler scheme.

**Lemma 23.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be continuous and ROSL with constant  $l$  such that  $lh < 1$ , and let  $\Phi_h^{\text{exp}} : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be the explicit set-valued Euler scheme defined by  $\Phi_h^{\text{exp}}(x) := x + hF(x)$ . Then the estimates*

$$\text{dist}(\Phi_h^{\text{exp}}(x), \Phi_h(x)) \leq \frac{h\chi(h\|F(x)\|)}{1 - lh} \quad (42)$$

and

$$\text{dist}(\Phi_h(x), \Phi_h^{\text{exp}}(x)) \leq h \sup_{y \in \Phi_h(x)} \chi(h\|F(y)\|) \quad (43)$$

hold.

*Proof.* For given  $z \in x + hF(x)$ , inequality

$$\begin{aligned} \text{dist}(z, x + hF(z)) & \leq \text{dist}(x + hF(x), x + hF(z)) \leq h \text{dist}(F(x), F(z)) \\ & \leq h\chi(|x - z|) \leq h\chi(h\|F(x)\|) \end{aligned}$$

and Theorem 4 prove (42).

Let  $y \in x + hF(y)$  be given. Then

$$\begin{aligned} \text{dist}(y, x + hF(x)) & \leq \text{dist}(x + hF(y), x + hF(x)) \leq h \text{dist}(F(y), F(x)) \\ & \leq h\chi(|y - x|) \leq h\chi(h\|F(y)\|) \end{aligned}$$

implies (43). □

**Lemma 24.** *Let  $F : \mathbb{R}^m \rightarrow \mathcal{CC}(\mathbb{R}^m)$  be continuous and ROSL with constant  $l$  such that  $lh < 1$ . Then*

$$\text{dist}(x, \Phi_h(x)) \leq \frac{h}{1-lh} \text{dist}(0, F(x)) \quad (44)$$

and

$$\text{dist}(\Phi_h(x), x) \leq h \sup_{y \in \Phi_h(x)} \|F(y)\|. \quad (45)$$

*Proof.* Because of Theorem 4,

$$\text{dist}(x, \Phi_h(x)) \leq \frac{1}{1-lh} \text{dist}(x, x + hF(x)) = \frac{h}{1-lh} \text{dist}(0, F(x)).$$

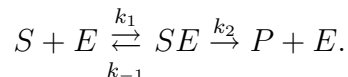
If  $y \in x + hF(y)$ , then  $|y - x| \leq h\|F(y)\|$ , which proves (45).  $\square$

We have implemented a very simple, but very robust version of the implicit Euler scheme, which does not require any assumptions apart from (A1) and (A2). In every step, we estimate the region in which the reachable set  $\mathcal{R}_{h,\rho}((k+1)h, 0, x_0)$  must be located according to Lemma 23 or 24. Unfortunately, estimates (43) and (45) are of implicit nature and allow only a rough guess for the location based on crude upper bounds for the norm of the images of  $F$ . For every grid point  $y$  in this region, we compute an approximation of  $-y + hF(y)$  with a suitable precision and check for any grid point  $x \in \mathcal{R}_{h,\rho}(kh, 0, x_0)$ , whether

$$\text{dist}(y, x + hF(y)) = \text{dist}(-x, -y + hF(y)) \leq \rho + h\chi(\rho).$$

If the distance is below the tolerance, the point  $y$  is accepted as an element of  $\mathcal{R}_{h,\rho}((k+1)h, 0, x_0)$ .

As a model example, we choose the Michaelis-Menten model which is of considerable importance in theoretical Biology. It describes a simple biochemical process, where an organic substrate molecule is changed into a product by means of an enzyme. At first, the substrate molecule forms a complex with an enzyme. In a second step, the substrate molecule reacts and becomes the product, which leaves the complex in a third step. The chemical equation





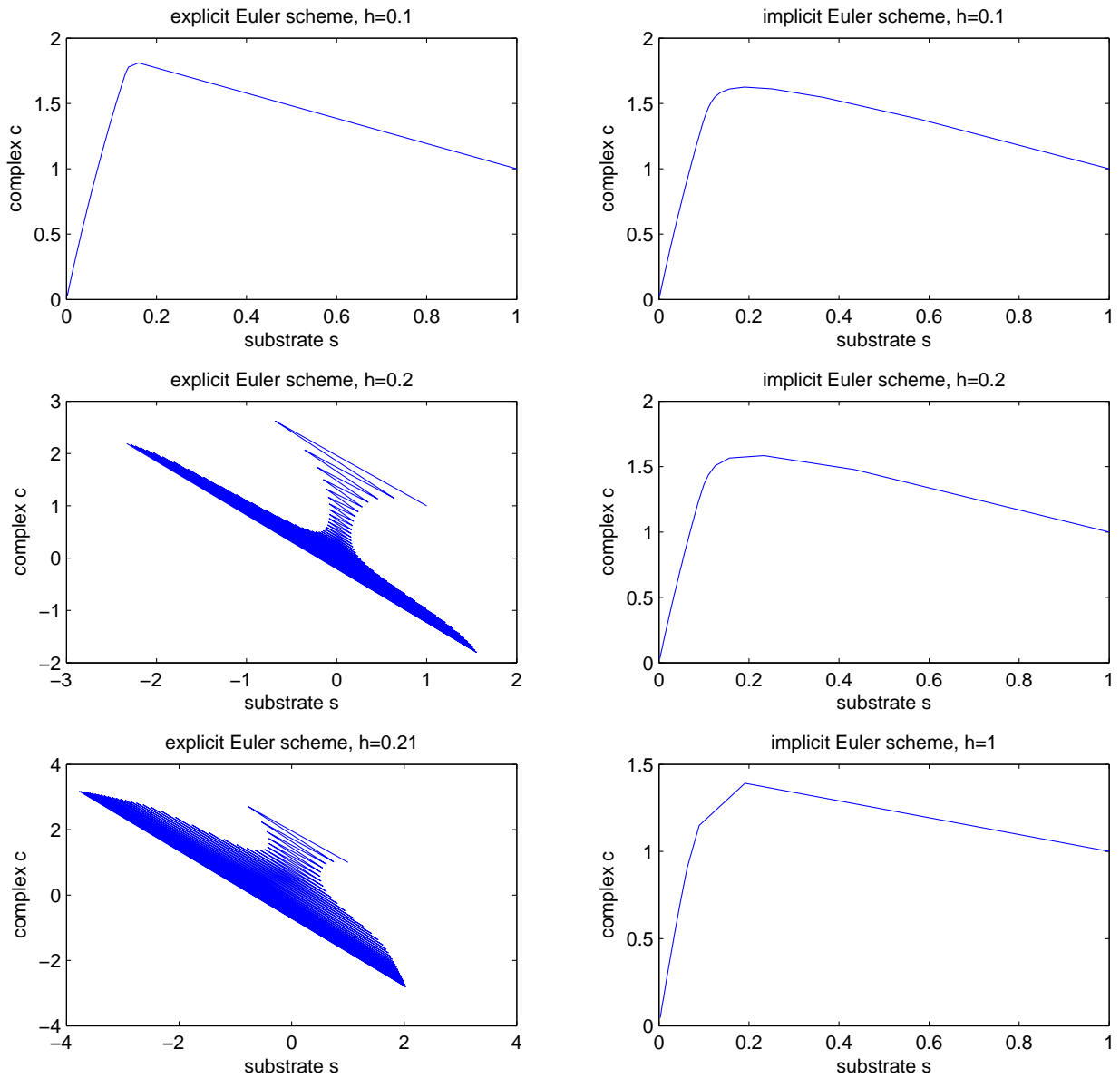


Figure 1: Performance of the single-valued explicit and implicit Euler schemes applied to the Michaelis-Menten system

can easily be expressed as a four dimensional system of differential equations which can be reduced to the system

$$\begin{pmatrix} \dot{s} \\ \dot{c} \end{pmatrix} = f\left(\begin{pmatrix} s \\ c \end{pmatrix}\right) = \begin{pmatrix} -k_1 e_0 s + (k_1 s + k_{-1})c \\ k_1 e_0 s - (k_1 s + k_{-1} + k_2)c \end{pmatrix}, \quad (46)$$

where the  $k_i$  are rate constants,  $e_0$  is the concentration of the enzyme, and  $s$  and  $c$  denote the concentrations of the substrate and substrate-enzyme complex, respectively. For a more detailed discussion of this model see e.g. [14].

If we choose the parameters  $e_0 = 10$ ,  $k_1 = 1$ ,  $k_{-1} = 0.6$ , and  $k_2 = 0.3$ , the eigenvalues of the Jacobian  $Df(0,0)$  are  $\lambda_1 = -0.28$  and  $\lambda_2 = -10.62$ , which means that the ODE is moderately stiff in a neighbourhood of zero. In such a setting, explicit numerical methods are known to yield uncontrollable errors if  $h\lambda_j$  is outside the region of absolute stability (see [10]).

The solutions of the single-valued explicit and implicit Euler schemes applied to (46) with the above parameters are displayed in Figure 1. Whereas the solution of the explicit Euler scheme with step size  $h = 0.1$  still converges to the stable equilibrium zero, the method produces an utterly erratic trajectory when performed with larger step sizes. The implicit Euler scheme, however, retains the stability properties of the underlying ODE as expected.

When we replace the parameters  $k_1$  and  $k_2$  with intervals  $[1, 1.1]$  and  $[0.3, 0.4]$ , the ODE (46) becomes

$$\begin{pmatrix} \dot{s} \\ \dot{c} \end{pmatrix} \in F\left(\begin{pmatrix} s \\ c \end{pmatrix}\right) = \begin{pmatrix} -[1, 1.1]e_0 s + ([1, 1.1]s + k_{-1})c \\ [1, 1.1]e_0 s - (k_1 s + k_{-1} + [0.3, 0.4])c \end{pmatrix}. \quad (47)$$

This differential inclusion is Lipschitz continuous and thus ROSL with constant  $l = L \approx 10$  in a neighbourhood of the origin. Figure 2 shows its reachable sets obtained using the explicit Euler scheme which we presented in [3] and those computed with the implicit Euler scheme described above. In all four computations, we choose  $\rho = 0.005$  in order to provide images of equal optical quality.

As in the single-valued case, the explicit Euler with step size  $h = 0.1$  yields a reasonable result, but for larger step-sizes its images grow rapidly and diverge, whereas the implicit Euler scheme converges to the stable equilibrium. Its performance is better than predicted e.g. by Theorem 4, because it works well even for some values  $lh > 1$ .

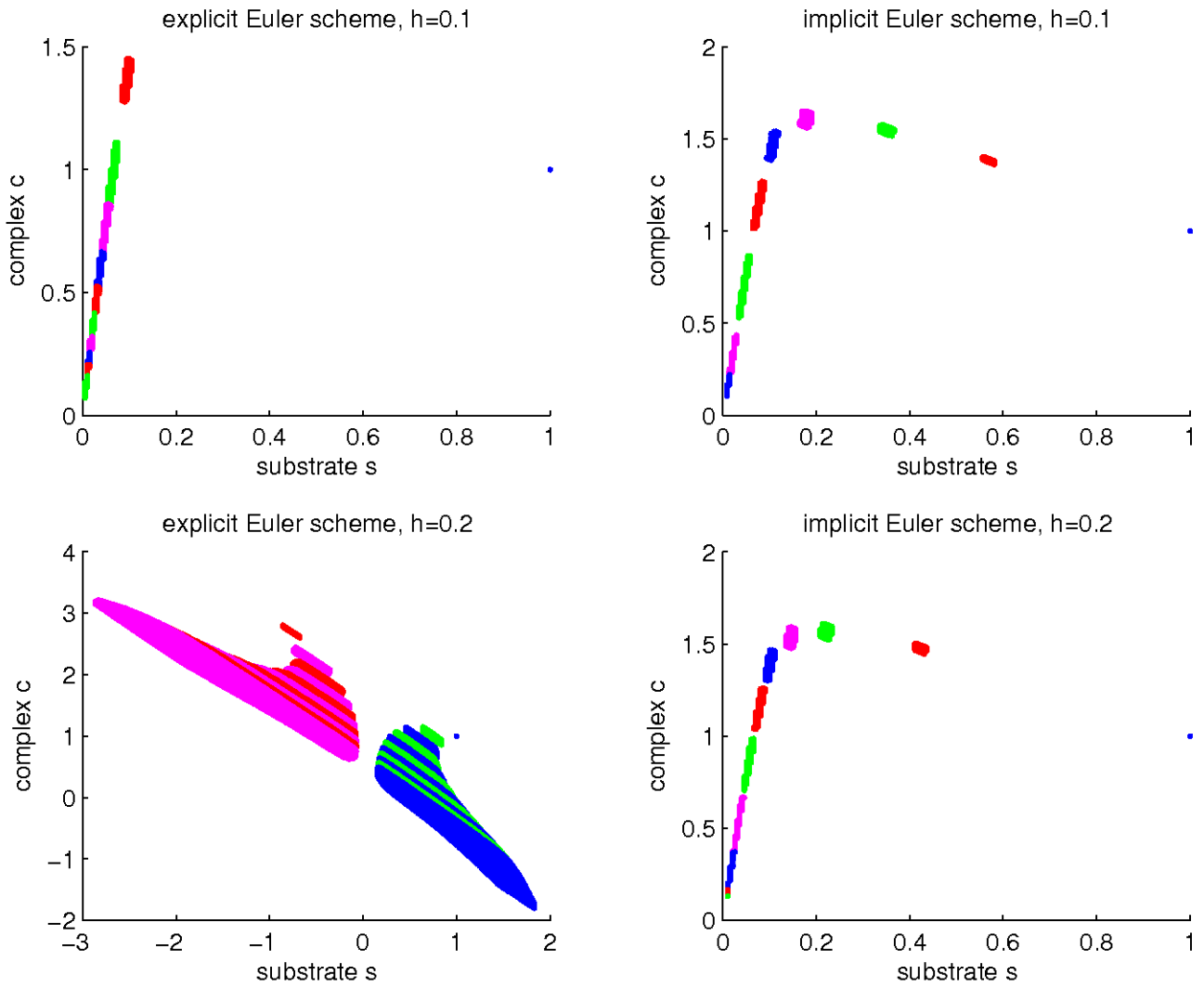


Figure 2: Performance of the set-valued explicit and implicit Euler schemes applied to the Michaelis-Menten system

Please note that neither the implicit nor the explicit Euler scheme (for  $h = 0.1$ ) converge exactly to zero but attain a fixed set near the origin in finite time. This effect is due to spatial discretization: If the norm of the right-hand side becomes too small, the numerical scheme cannot reach the neighbouring grid points any more and eventually becomes constant.

In our computations, searching the spatial grid for elements of  $\Phi_{h,\rho}$  according to Lemma 24 leads to a better performance than according to Lemma 23, which is probably due to the fact that it is necessary to compute the explicit Euler image as a predictor in the latter case. However, the computational costs are still extremely high.

The main obstacle for the construction of more efficient algorithms is our limited knowledge about the properties of the images of the implicit Euler scheme. In dimension one, Lemma 9 guarantees that the images are convex, but for any higher dimensional space it is even unclear whether they are connected. Because of Lemma 12 there is hope that connectedness of the images might be true, which would be a crucial precondition for improvements.

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