Discretizing Bifurcation Diagrams near Codimension two Singularities

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Abstract

We consider parameter-dependent, continuous-time dynamical systems under discretizations. It is shown that fold-Hopf singularities are $O(h^p)$-shifted and turned into fold-Neimark-Sacker points by one-step methods of order p. Then we analyze the effect of discretizations methods on the local bifurcation diagram near Bogdanov-Takens and fold-Hopf singularities. In particular we prove that the discretized codimension one curves intersect at the singularities in a generic manner. The results are illustrated by a numerical example.

1 Introduction

Consider a continuous-time dynamical system depending on parameters

$$\dot{x}(t) = f(x(t), \alpha), \tag{1.1}$$

where $f \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$ with open sets $0 \in \Omega \subset \mathbb{R}^N$, $0 \in \Lambda \subset \mathbb{R}^2$, $k \geq 1$ sufficiently large, $N \geq 2$. The first and commonly used tool for exploring the dynamics generated by the vector field (1.1) is numerical time-integration. To accomplish this, we can appeal to one-step methods, which consists in approximating the evolution operator by a discrete-time system

$$x \mapsto g(x, \alpha), \tag{1.2}$$

with $g \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$, where the step-size were assumed to be previously fixed. It then becomes evident the importance of establishing theoretical results that allow us to make

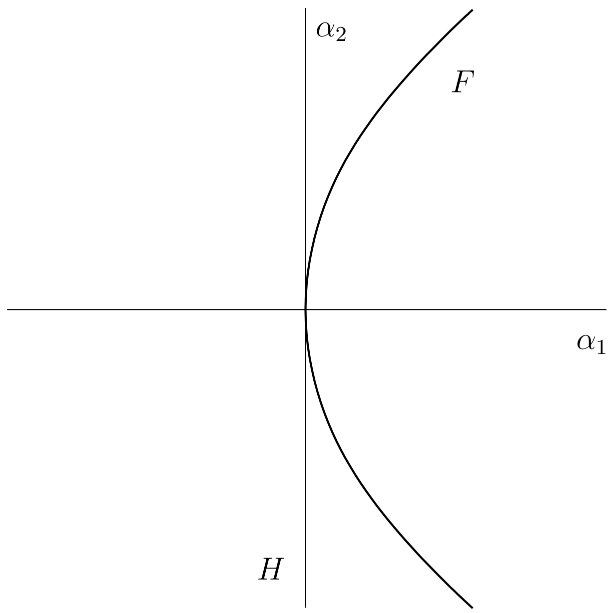

Fig. 1.1: Local bifurcation diagram near a BT_2 point.

conclusions about the real behavior of system (1.1) starting from the numerical observations obtained via (1.2). The situation turns out to be more involved if we additionally consider singularities under variation of parameters. A rigorous analysis concerning topological conjugacies of continuous-time systems and their discretizations can be found in [18]. There, elementary codimension one bifurcations are considered.

In this article we suppose that system (1.1) undergoes one of the following codimension two singularities: Bogdanov-Takens or fold-Hopf (see [15]). Further we assume that (1.1) is discretized via general one-step methods, see Section 2. Conjugacy results are not expected in this case, since e.g. homoclinic connections are turned into exponentially small sectors of transversal homoclinic orbits by one-step methods, see [4]. We do not consider such global phenomena occurring near the mentioned codimension two singularities.

In the setting described above, two main questions are tackled, i.e., does a one-step method applied to the continuous-time system reproduce by a "discrete version" the codimension two singularity? If this is so, does the discrete point remain at the same position in both, state space, as well as parameter space? The second major question is a natural consequence of a positive answer to the first one, namely, is the bifurcation picture also reproduced (and maybe shifted) by the discretization method? For cusp and Bogdanov-Takens bifurcations, it is already known that they persist at the same position under general Runge-Kutta methods, see [19]. Results in this direction for the remaining codimension two singularities seem not to be available.

As for the second major question outlined above, we have some useful results at hand. Consider for instance the local bifurcation diagram around a Bogdanov-Takens point, see Figure 1.1. In this picture, the curves labeled by F, H correspond to paths of fold and Hopf points, respectively. By [19], it is known that a fold point persists at the same position under general Runge-Kutta methods, thus the emanating curve of fold points is not affected by those one-step methods. Note that this result, together with the fact that cusp points persist under Runge-Kutta methods, lead us to the conclusion that the local bifurcation diagram near cusp points remains unaffected under those discretization methods.

On the other hand, the analysis of the path of discretized Hopf points requires more attention. Discretization of systems with Hopf singularities has been addressed to a large extent (cf. [2, 5, 6, 12, 13, 20, 24]). It has been proven that Hopf points are $O(h^p)$-shifted and turned into Neimark-Sacker points by general one-step methods of order $p \geq 1$. Approximation of regular periodic orbits originated at the Hopf bifurcation has been considered too. Nevertheless, these results strictly apply when dealing with one-dimensional sections in Figure 1.1, thus the analysis of the discretized Hopf curve has to be carried out in a codimension two context.

The present article summarizes the analysis of bifurcating dynamical systems under discretizations that appears in [21].

2 Basic setup

Let us first formally define the singularities we will deal with.

Definition 2.1. *A point* $(x_0, \alpha_0) \in \Omega \times \Lambda$ *is referred to as a Bogdanov-Takens singularity of codimension two (in short* BT_2 *point) of* (1.1) *if:*

- $f(x_0, \alpha_0) = 0$,
- *The only Jordan block of* $f_x(x_0, \alpha_0)$ *corresponding to the eigenvalue* 0 *is* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, *and there are no other eigenvalues on the imaginary axis.*

Definition 2.2. *A point* $(x_0, \alpha_0) \in \Omega \times \Lambda$ *is referred to as a fold-Hopf point (in short FH point*[1]*) of* (1.1) *if:*

- $f(x_0, \alpha_0) = 0$,
- $f_x(x_0, \alpha_0)$ *has the only critical, simple eigenvalues* $\{0, \pm i\omega_0\}$, $0 < \omega_0 \in \mathbb{R}$.

For our purposes it is useful to introduce minimally augmented systems for the continuation of fold and Hopf points of system (1.1). The jacobian of the systems:

$$\begin{cases} f(x, \alpha) = 0, \\ \det(f_x(x, \alpha)) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} f(x, \alpha) = 0, \\ \det(2f_x(x, \alpha) \odot I_N) = 0, \end{cases} \tag{2.2}$$

[1] Also called zero-Hopf, zero-pair, Hopf-saddle-node, Hopf-steady-state, Gavrilov-Guckenheimer, among others.

have full rank at any generic fold, Hopf point of (1.1), respectively (cf. [3, Theorem 5.1]). The symbol I_N stands for the identity matrix in $\mathbb{R}^{N,N}$ and $\odot$ for the bialternate product of matrices (cf. [9, 11, 15]). These systems allow us to formalize the notion of genericity of BT_2 and FH points. A BT_2 or FH point (x_0, α_0) of (1.1) is said to be generic if the system

$$\begin{cases} f(x,\alpha) = 0, \\ \det(2f_x(x,\alpha) \odot I_N) = 0, \\ \det(f_x(x,\alpha)) = 0, \end{cases} \tag{2.3}$$

is regular at (x_0, α_0). It follows immediately that the genericity of a BT_2 or FH point implies that the jacobian of systems (2.1) and (2.2) have full rank at (x_0, α_0), and hence the existence of emanating paths of fold and Hopf singularities is guaranteed. Genericity conditions are often also referred to as transversality conditions, and they guarantee that the parameters unfold the singularities in a generic manner (cf. [15, Section 2.4]).

Now let us introduce the singularities and minimally augmented systems in the discrete-time sense.

Definition 2.3. *A point $(x_0, \alpha_0) \in \Omega \times \Lambda$ is referred to as a fold-Neimark-Sacker point (in short FN point) of* (1.2) *if:*

- $g(x_0, \alpha_0) - x_0 = 0$,
- $g_x(x_0, \alpha_0)$ *has the only critical, simple eigenvalues* $\{1, e^{\pm i\theta_0}\}$, $0 < \theta_0 \in \mathbb{R}$, $e^{ik\theta_0} \neq 1$, $k = 1, 2, 3, 4$.

Minimally augmented systems for the continuation of fold and Neimark-Sacker points of (1.2) are given by (cf. [15]):

$$\begin{cases} g(x,\alpha) - x = 0, \\ \det(g_x(x,\alpha) - I_N) = 0, \end{cases} \tag{2.4}$$

$$\begin{cases} g(x,\alpha) - x = 0, \\ \det(g_x(x,\alpha) \odot g_x(x,\alpha) - I_m) = 0, \end{cases} \tag{2.5}$$

respectively, with $m := \frac{1}{2}N(N-1)$.

As described in the Introduction, our main concern is to describe the effect of discretization methods on the bifurcation diagram of dynamical systems with singularities. In this sense we consider general one-step methods of order $p \geq 1$ applied to (1.1), given by

$$x \mapsto \psi^h(x,\alpha) := x + h\Phi(h, x, \alpha), \tag{2.6}$$

with $\psi, \Phi : [-h^*, h^*] \times \tilde{\Omega} \times \tilde{\Lambda} \to \mathbb{R}^N$ sufficiently smooth, $h^* > 0$, where $0 \in \tilde{\Omega} \subset \Omega$, $0 \in \tilde{\Lambda} \subset \Lambda$ are compact sets. That the method is of order p means that there exists a positive constant C_0 (depending only on f), such that it holds

$$||\varphi^h(x,\alpha) - \psi^h(x,\alpha)|| \leq C_0 |h|^{p+1},$$

for all $(h,x,\alpha) \in [-h^*,h^*] \times \tilde{\Omega} \times \tilde{\Lambda}$, where $\varphi^t(\cdot,\alpha)$ stands for the t-flow of (1.1) and $||\cdot||$ denotes any norm[2] in $\mathbb{R}^N$. In this setting, there exist smooth functions $\Upsilon, \Xi : [-h_0,h_0] \times \tilde{\Omega} \times \tilde{\Lambda} \to \mathbb{R}^N$ such that:

$$\begin{aligned} \psi^h(x,\alpha) &= \varphi^h(x,\alpha) + \Upsilon(h,x,\alpha)h^{p+1}, \\ \psi^h_w(x,\alpha) &= \varphi^h_w(x,\alpha) + \Upsilon_w(h,x,\alpha)h^{p+1}, \\ \Phi(h,x,\alpha) &= f(x,\alpha) + \Xi(h,x,\alpha)h, \\ \Phi_w(h,x,\alpha) &= f_w(x,\alpha) + \Xi_w(h,x,\alpha)h, \end{aligned} \tag{2.7}$$

hold for all $(h,x,\alpha) \in [-h_0,h_0] \times \tilde{\Omega} \times \tilde{\Lambda}$, where $0 < h_0 < h^*$, and w stands for any of the variables of $f(\cdot,\cdot)$, see [4, 8, 22].

Moreover, the following two lemmata will be used in our analysis:

Lemma 2.4 (Banach)**.** *Let $M \in \mathbb{R}^{N,N}$ and $||\cdot||$ denote any matrix norm in $\mathbb{R}^{N,N}$ for which $||I_N|| = 1$. If $||M|| < 1$, then $(I_N + M)^{-1}$ exists, and it holds*

$$||(I_N + M)^{-1}|| \leq \frac{1}{1 - ||M||}.$$

Proof. See [17]. □

Theorem 2.5 (Local Inverse Lipschitz Mapping Theorem)**.** *Let V, W be Banach spaces and $H \in C^1(V,W)$. Let $y_0 \in V$, and assume $H'(y_0)$ to be a homeomorphism. Suppose that there exists positive constants δ, κ, σ, such that*

$$||H'(y) - H'(y_0)|| \leq \kappa < \sigma \leq \frac{1}{||(H'(y_0))^{-1}||}, \quad \forall y \in B_\delta(y_0),$$

$$||H(y_0)|| \leq (\sigma - \kappa)\delta,$$

where $B_\delta(y_0)$ denotes a closed ball of radius δ and centered at y_0. Then, H has a unique zero in $B_\delta(y_0)$, and it holds

$$||y_1 - y_2|| \leq \frac{1}{\sigma - \kappa}||H(y_1) - H(y_2)||, \quad \forall y_1, y_2 \in B_\delta(y_0).$$

Proof. See [23]. □

With the technical framework above introduced, we have all the necessary machinery at hand for presenting the main results of the present work.

[2]Throughout this article, the symbol $||\cdot||$ will be used to denote norms in different spaces. From the context, no confusion should arise.

3 Fold-Hopf singularities under discretization

In this section we will study the effect of one-step methods applied to systems having an FH point. More precisely, we will suppose we are given a continuous-time dynamical system (1.1), which undergoes an FH singularity. We assume that this system is discretized via general, p-th order one-step methods. Under these conditions, we will show that the FH point is $O(h^p)$-shifted and turned into an FN point by the one-step map, for all sufficiently small step-size. Formally speaking, we have the following:

Theorem 3.1. *Let a general one-step method of order $p \geq 1$ applied to (1.1) be given by (2.6). Assume that system (1.1) has a generic FH point at $(x_{FH}, \alpha_{FH}) \in \tilde{\Omega} \times \tilde{\Lambda}$. Then, there exists a positive constant $\rho \leq h_0$ and a neighborhood $\Omega' \times \Lambda' \subset \tilde{\Omega} \times \tilde{\Lambda}$ of (x_{FH}, α_{FH}), in which (2.6) has a unique FN point $(x_{FN}(h), \alpha_{FN}(h))$ that depends smoothly on h, for all $h \in (-\rho, \rho)$. Furthermore, the following estimate holds*

$$||(x_{FN}(h), \alpha_{FN}(h)) - (x_{FH}, \alpha_{FH})|| \leq C|h|^p, \tag{3.1}$$

for some $C > 0$ and all $h \in (-\rho, \rho)$.

Before proving this theorem, some comments are in order. A generic FH point can be seen as a regular zero of the defining system (2.3). Likewise, such a system can be constructed, so that an FN point is a regular zero of it (e.g. by combining (2.4) and (2.5)). The basic idea here is then to suitably modify the defining systems of FH and FN points, in such a way that we can establish closeness relations between them. Once this is done, the estimate of the distance between the FH and FN points (see (3.1)), and the smooth dependence of the latter on h, will follow from application of Theorem 2.5 and the Implicit Function Theorem. This technique has been applied in several contexts, e.g., for discretizations of hyperbolic equilibria of continuous-time systems (cf. [2, Section 5.5.2]), and in a much more elaborated context in [4], where the authors study the effect of one-step methods applied to systems having connecting orbits. With these remarks, we are ready to present:

Proof of Theorem 3.1. As explained before, a generic FH point of (1.1) is a regular zero of

$$\tilde{F}(x, \alpha) := \begin{pmatrix} f(x, \alpha) \\ \det(2f_x(x, \alpha) \odot I_N) \\ \det(f_x(x, \alpha)) \end{pmatrix} = 0. \tag{3.2}$$

We will try to rewrite the above equation in terms of the h-flow $\varphi^h(\cdot, \alpha)$ of (1.1). By a straightforward analysis of the variational equation of (1.1) at an equilibrium (x_0, α_0), the following relation holds (cf. [10, Section 1.3])

$$\mathrm{Sp}(\varphi^h_x(x_0, \alpha_0)) = e^{h\,\mathrm{Sp}(f_x(x_0, \alpha_0))}, \tag{3.3}$$

where $\mathrm{Sp}(A)$ denotes the spectrum of a matrix $A \in \mathbb{R}^{N,N}$. Thus, we can conclude that $\varphi^h_x(x_0, \alpha_0)$ has a pair of eigenvalues on the unit circle (resp. an eigenvalue equal to 1), if and only if $f_x(x_0, \alpha_0)$ has a pair of purely imaginary eigenvalues (resp. an eigenvalue equal to 0), for $h \neq 0$. Therefore, (3.2) can be written in terms of the h-flow $\varphi^h(\cdot, \alpha)$ as follows:

$$\begin{cases} \varphi^h(x,\alpha) - x = 0, \\ \det(\varphi^h_x(x,\alpha) \odot \varphi^h_x(x,\alpha) - I_m) = 0, \\ \det(\varphi^h_x(x,\alpha) - I_N) = 0. \end{cases}$$

However, note that this system becomes trivial at $h = 0$, which is inconvenient for our approach, as we want to perform our analysis for h small. Therefore, we will rather consider the following system:

$$F(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\varphi^h(x,\alpha) - x) \\ \det\left(\frac{1}{h}(\varphi^h_x(x,\alpha) \odot \varphi^h_x(x,\alpha) - I_m)\right) \\ \det\left(\frac{1}{h}(\varphi^h_x(x,\alpha) - I_N)\right) \end{pmatrix} = 0,$$

which will be later shown not to be trivial at $h = 0$. Similarly, an FN point of (2.6) is a solution of (cf. (2.4), (2.5)):

$$G(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\psi^h(x,\alpha) - x) \\ \det\left(\frac{1}{h}(\psi^h_x(x,\alpha) \odot \psi^h_x(x,\alpha) - I_m)\right) \\ \det\left(\frac{1}{h}(\psi^h_x(x,\alpha) - I_N)\right) \end{pmatrix} = 0. \tag{3.4}$$

The next step is to establish relations between $\tilde{F}$, F, and G, which will be crucial for our analysis. Let us begin with G and $\tilde{F}$. With the expansions in (2.7) we obtain:

$$\begin{aligned} G(h, x, \alpha) &= \begin{pmatrix} f(x,\alpha) + \Xi(h,x,\alpha)h \\ \det(2f_x(x,\alpha) \odot I_N + \Theta_1(h,x,\alpha)h) \\ \det(f_x(x,\alpha) + \Xi_x(h,x,\alpha)h) \end{pmatrix}, \\ &= \tilde{F}(x,\alpha) + \Theta(h,x,\alpha)h, \end{aligned} \tag{3.5}$$

where $\Theta_1(h,x,\alpha) := \Phi_x(h,x,\alpha) \odot \Phi_x(h,x,\alpha) + 2\Xi_x(h,x,\alpha) \odot I_N$, and Θ is some smooth function[3]. As for F and G, we obtain again by (2.7):

$$\begin{aligned} G(h, x, \alpha) &= \begin{pmatrix} \frac{1}{h}(\varphi^h(x,\alpha) - x) + \Upsilon(h,x,\alpha)h^p \\ \det\left(\frac{1}{h}(\varphi^h_x(x,\alpha) \odot \varphi^h_x(x,\alpha) - I_m) + \Psi_1(h,x,\alpha)h^p\right) \\ \det\left(\frac{1}{h}(\varphi^h_x(x,\alpha) - I_N) + \Upsilon_x(h,x,\alpha)h^p\right) \end{pmatrix}, \\ &= F(h,x,\alpha) + \Psi(h,x,\alpha)h^p, \end{aligned} \tag{3.6}$$

where $\Psi_1(h,x,\alpha) := 2\varphi^h_x(x,\alpha) \odot \Upsilon_x(h,x,\alpha) + \Upsilon_x(h,x,\alpha) \odot \Upsilon_x(h,x,\alpha)h^{p+1}$, and Ψ is some smooth function. The next step is to apply Theorem 2.5 to $H := G(h,\cdot,\cdot)$, for all h in some interval. Consequently, we need first to show that the assumptions of that theorem are fulfilled for h small. Let us then define $y := (x, \alpha)$, and take $y_0 := (x_{FH}, \alpha_{FH})$. We

[3] In what follows, by the term "(some smooth function) $\cdot\, w^k$", $k \geq 1$, w some real variable, we mean the integral remainder of a Taylor series.

will show that $G_y(h, y_0)$ is nonsingular for all h near zero. Indeed, assume $h \in (-h_0, h_0)$, then by (3.5), and recalling the genericity of the FH point, it holds

$$G_y(h, y_0) = \tilde{F}_y(y_0) + \Theta_y(h, y_0)h = \tilde{F}_y(y_0)(I_N + (\tilde{F}_y(y_0))^{-1}\Theta_y(h, y_0)h).$$

Choose $0 < \rho_1 < h_0$, so that

$$||(\tilde{F}_y(y_0))^{-1}\Theta_y(h, y_0)h|| < ||(\tilde{F}_y(y_0))^{-1}|| \left(\sup_{h\in(-h_0,h_0)} ||\Theta_y(h, y_0)|| \right) |h| < 1,$$

for all $h \in (-\rho_1, \rho_1)$. Then, Lemma 2.4 ensures the invertibility of $G_y(h, y_0)$ in $(-\rho_1, \rho_1)$, and furthermore the following estimate holds

$$||(G_y(h, y_0))^{-1}|| < \frac{||(\tilde{F}_y(y_0))^{-1}||}{1 - ||(\tilde{F}_y(y_0))^{-1}|| \sup_{h\in(-h_0,h_0)} ||\Theta_y(h, y_0)||\rho_1} =: \frac{1}{\sigma}.$$

Take $\kappa := \frac{1}{2}\sigma$. Then, by the continuity of G_y, we can find a closed ball $B_\delta(y_0)$, $\delta > 0$, and a positive constant ρ_1', such that

$$||G_y(h, y) - G_y(h, y_0)|| \le \kappa,$$

for all $y \in B_\delta(y_0)$, $h \in [-\rho_1', \rho_1']$. Likewise, by the continuity of G, and noticing that $G(0, y_0) = 0$ (see (3.5)), we can find a positive constant ρ_1'', such that

$$||G(h, y_0)|| \le (\sigma - \kappa)\delta,$$

for all $h \in [-\rho_1'', \rho_1'']$. Finally, take $\rho_2 := \min(\rho_1, \rho_1', \rho_1'')$. Then, the assumptions of Theorem 2.5 hold for $H := G(h, \cdot)$, and for all $h \in (-\rho_2, \rho_2)$. Moreover, since $G(0, y_0) = 0$ and $G_y(0, y_0) = \tilde{F}_y(y_0)$ is nonsingular, the Implicit Function Theorem guarantees the existence of a function $y_{FN} := (x_{FN}, \alpha_{FN}) : (-\rho_2', \rho_2') \to \mathbb{R}^{N+2}$, such that

$$G(h, y_{FN}(h)) = 0, \quad y_{FN}(0) = (x_{FN}(0), \alpha_{FN}(0)) = (x_{FH}, \alpha_{FH}),$$

$h \in (-\rho_2', \rho_2')$. This shows the existence, uniqueness, and smooth dependence on h of an FN point of (2.6). It is left to show the Estimate (3.1). To achieve this, choose $0 < \rho_2'' \le \rho_2'$, so that $y_{FN}(h) \in B_\delta(y_0)$ for all $h \in (-\rho_2'', \rho_2'')$. Define $\rho_3 := \min(\rho_2, \rho_2'')$, then Theorem 2.5 applied to $G(h, \cdot)$, $h \in (-\rho_3, \rho_3)$, combined with (3.6) yields:

$$\begin{aligned} ||(x_{FN}(h), \alpha_{FN}(h)) - (x_{FH}, \alpha_{FH})|| &\le \frac{1}{\sigma - \kappa}||G(h, y_{FN}(h)) - G(h, y_0)||, \\ &= \frac{2}{\sigma}||\Psi(h, y_0)|||h^p|, \\ &< \frac{2}{\sigma}\left(\sup_{h\in(-\rho_3,\rho_3)} ||\Psi(h, y_0)|| \right) |h|^p. \end{aligned}$$

To conclude, choose $0 < \rho < \rho_3$, and $C := \frac{2}{\sigma}\left(\sup_{h\in(-\rho_3,\rho_3)} ||\Psi(h, y_0)||\right)$. $\square$

As already mentioned in the Introduction, it has been shown that BT_2 bifurcations persist at the same position under Runge-Kutta methods (see [19]). However, in a more general framework, the following theorem holds:

Theorem 3.2. *Let a general one-step method of order $p \geq 1$ applied to (1.1) be given by (2.6). Assume that system (1.1) has a generic BT_2 point at $(x_{BT_2}, \alpha_{BT_2}) \in \tilde{\Omega} \times \tilde{\Lambda}$. Then, there exists a positive constant $\rho \leq h_0$ and a neighborhood $\Omega' \times \Lambda' \subset \tilde{\Omega} \times \tilde{\Lambda}$ of $(x_{BT_2}, \alpha_{BT_2})$, in which (2.6) has a unique $1:1$ resonance $(x_{R1}(h), \alpha_{R1}(h))$ that depends smoothly on h, for all $h \in (-\rho, \rho)$. Furthermore, the following estimate holds*

$$||(x_{R1}(h), \alpha_{R1}(h)) - (x_{BT_2}, \alpha_{BT_2})|| \leq C|h|^p,$$

for some $C > 0$ and all $h \in (-\rho, \rho)$.

Sketch of the proof. The approach employed to analyze FH points under discretization can be applied. A generic BT_2 point of (1.1) is a regular zero of (cf. Section 2):

$$\tilde{F}(x, \alpha) := \begin{pmatrix} f(x, \alpha) \\ \det(2f_x(x, \alpha) \odot I_N) \\ \det(f_x(x, \alpha)) \end{pmatrix} = 0.$$

As before, this system can be replaced by

$$F(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\varphi^h(x, \alpha) - x) \\ \det\left(\frac{1}{h}(\varphi^h_x(x, \alpha) \odot \varphi^h_x(x, \alpha) - I_m)\right) \\ \det\left(\frac{1}{h}(\varphi^h_x(x, \alpha) - I_N)\right) \end{pmatrix} = 0.$$

Recall that a $1:1$ resonance (in short $R1_2$ point) is a fixed point of (1.2) with a double eigenvalue equal to 1, and with geometric multiplicity equal to one. Thus, an $R1_2$ point of the one-step map (2.6) is a solution of:

$$G(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\psi^h(x, \alpha) - x) \\ \det\left(\frac{1}{h}(\psi^h_x(x, \alpha) \odot \psi^h_x(x, \alpha) - I_m)\right) \\ \det\left(\frac{1}{h}(\psi^h_x(x, \alpha) - I_N)\right) \end{pmatrix} = 0.$$

Consequently, the statements of the theorem follow from the previous discussion (see Theorem 3.1). □

4 Emanating Hopf curve under discretization

In the previous section we saw that FH and BT_2 points persist under one-step methods. As we mentioned in the Introduction, we are interested to know whether the local bifurcation diagram near these codimension two singularities is "well" reproduced by one-step methods. In this sense, the first part of this task has been achieved, namely, the organizing centers were shown to be preserved by one-step methods.

Now we tackle the problem of analyzing the discretization of the Hopf curve of system (1.1) that emanates from BT_2 and FH points. Throughout this section we denote by

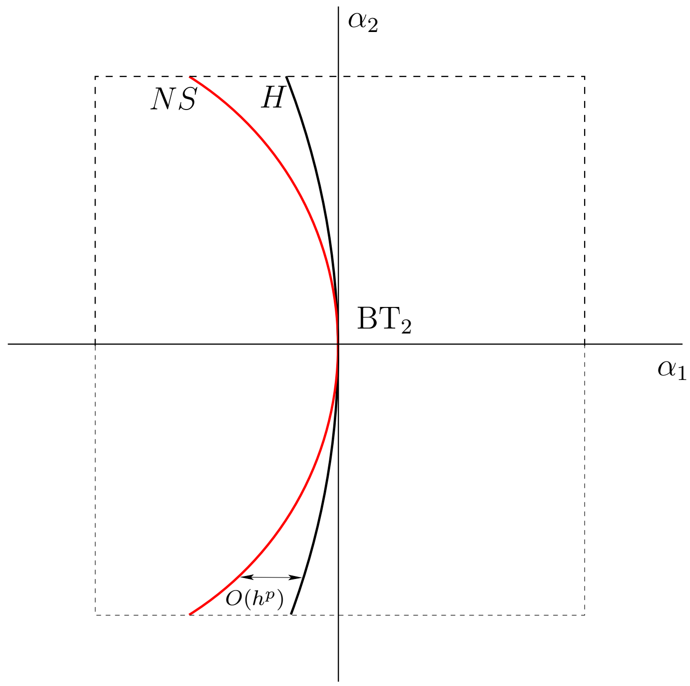

Fig. 4.1: Discretized path of Hopf points near a BT_2 singularity.

$\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ the parameters of the system. The result we are after is illustrated in Figure 4.1. The curves labeled by H, NS represent paths of Hopf, Neimark-Sacker points, respectively.

The analysis is formulated as follows. Suppose we are given a continuous-time dynamical system (1.1) which undergoes a BT_2 or an FH bifurcation at the origin. We assume that this system is discretized via general one-step methods, as in Section 3. Under these conditions, we will show that there exists a step-size-independent neighborhood (the dashed square in Figure 4.1) of a BT_2 (resp. FH) point, such that the discretized path of Hopf points (NS in Figure 4.1) approximates the original curve (H in Figure 4.1) with the order of the method. For this purpose, we do not reduce the systems, e.g. via center manifold theory, but we rather work with them in full dimension. To do so, the approach employed for the study of FH points under discretizations will be applied. In fact, the arguments are very close to those used in the proof of Theorem 3.1. For this reason we will only present a sketch of the proof of the main result. With these remarks we are ready to formulate:

Theorem 4.1. *Let a general one-step discretization method of order $p \geq 1$ applied to* (1.1) *be given by* (2.6). *Assume that system* (1.1) *has a generic BT_2 or FH point at the origin. Further consider the system:*

$$\tilde{F}(x, \alpha) := \begin{pmatrix} f(x, \alpha) \\ \det(2f_x(x, \alpha) \odot I_N) \end{pmatrix} = 0, \tag{4.1}$$

and assume that $\tilde{F}_y(0,0)$, $y := (x, \alpha_1)$ is nonsingular[4]*. Then, there exist positive constants $\rho \le h_0$, δ and curves of Hopf and Neimark-Sacker points of systems* (1.1) *and* (2.6)*, respectively, defined by:*

$$C_H(\alpha_2) := (x_H(\alpha_2), \alpha_{1H}(\alpha_2), \alpha_2),$$
$$C_{NS}(h, \alpha_2) := (x_{NS}(h, \alpha_2), \alpha_{1NS}(h, \alpha_2), \alpha_2),$$

with $x_H : (-\delta, \delta) \to \mathbb{R}^N$, $x_{NS} : (-\rho, \rho) \times (-\delta, \delta) \to \mathbb{R}^N$, $\alpha_{1H} : (-\delta, \delta) \to \mathbb{R}$, $\alpha_{1NS} : (-\rho, \rho) \times (-\delta, \delta) \to \mathbb{R}$ smooth[5]*. Furthermore, the following estimate holds for all $(h, \alpha_2) \in (-\rho, \rho) \times (-\delta, \delta)$ and uniformly in α_2*

$$||d_{NS}(h, \alpha_2) - d_H(\alpha_2)|| \le C|h|^p, \tag{4.2}$$

where $d_H(\cdot) := (x_H(\cdot), \alpha_{1H}(\cdot))$ and $d_{NS}(\cdot,\cdot) := (x_{NS}(\cdot,\cdot), \alpha_{1NS}(\cdot,\cdot))$, $C > 0$.

Sketch of the proof. Since $\tilde{F}_y(0,0)$ is nonsingular, the Implicit Function Theorem guarantees the existence of the function $d_H := (x_H, \alpha_{1H}) : (-\delta_1, \delta_1) \to \mathbb{R}^N \times \mathbb{R}$, such that

$$\tilde{F}(d_H(\alpha_2), \alpha_2) = 0, \quad d_H(0) = (x_H(0), \alpha_{1H}(0)) = (0, 0),$$

$\alpha_2 \in (-\delta_1, \delta_1)$. As in the proof of Theorem 3.1, we will rewrite (4.1) in terms of the h-flow $\varphi^h(\cdot, \alpha)$ of (1.1). Thus, we obtain the system:

$$F(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\varphi^h(x, \alpha) - x) \\ \det\left(\frac{1}{h}(\varphi^h_x(x, \alpha) \odot \varphi^h_x(x, \alpha) - I_m)\right) \end{pmatrix} = 0,$$

and it holds (see (3.3))

$$F(h, d_H(\alpha_2), \alpha_2) = 0,$$

for all $(h, \alpha_2) \in (-h_0, h_0) \times (-\delta_1, \delta_1)$. Likewise, a system whose zeroes describe a curve of Neimark-Sacker points of (2.6) is given by (see (2.5), (3.4)):

$$G(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\psi^h(x, \alpha) - x) \\ \det\left(\frac{1}{h}(\psi^h_x(x, \alpha) \odot \psi^h_x(x, \alpha) - I_m)\right) \end{pmatrix} = 0.$$

Hence, we will show the existence of a curve of Neimark-Sacker points of (2.6). By truncation of (3.5), the following relation holds locally

$$G(h, x, \alpha) = \tilde{F}(x, \alpha) + \Theta(h, x, \alpha)h, \tag{4.3}$$

where Θ is some smooth function. Thus, since $\tilde{F}_y(0,0)$ is nonsingular, we have that $G_y(0,0,0) = \tilde{F}_y(0,0)$ (see above) is nonsingular, thereby the Implicit Function Theorem

[4]This assumption is imposed only for assuring that α_2 can be used as parametrization variable. This is allowed due to the genericity of the codimension two points (see comments after system (2.3)).

[5]By the term "Neimark-Sacker curve", we mean the graph of the function $C_{NS}(h_*, \cdot)$, for $h_* \in (-\rho, \rho)$ fixed.

guarantees the existence of a smooth function $d_{NS} := (x_{NS}, \alpha_{1NS}) : (-\rho_1', \rho_1') \times (-\delta_1', \delta_1') \to \mathbb{R}^N \times \mathbb{R}$, such that

$$G(h, d_{NS}(h, \alpha_2), \alpha_2) = 0, \quad d_{NS}(0,0) = (x_{NS}(0,0), \alpha_{1NS}(0,0)) = (0,0),$$

$(h, \alpha_2) \in (-\rho_1', \rho_1') \times (-\delta_1', \delta_1')$. We have thus shown the existence and smoothness of the curves C_H, C_{NS}. Now we will show the $O(h^p)$-closeness part. Similarly as before, by truncation of (3.6), the following relation holds locally

$$G(h, x, \alpha) = F(h, x, \alpha) + \Psi(h, x, \alpha) h^p, \tag{4.4}$$

where Ψ is some smooth function. The next step is to apply Theorem 2.5. By carefully doing estimates similar to those of Theorem 3.1, we can guarantee that the assumptions of the Local Inverse Lipschitz Theorem applied to $H := G(h, \cdot, \alpha_2)$ hold for all $(h, \alpha_2) \in (-\rho_2, \rho_2) \times (-\delta_2, \delta_2)$, where ρ_2, δ_2 are some positive constants. Thus, Theorem 2.5 combined with (4.4) yields:

$$\begin{aligned} ||d_{NS}(h, \alpha_2) - d_H(\alpha_2)|| &\leq \frac{1}{\sigma - \kappa} ||G(h, d_{NS}(h, \alpha_2), \alpha_2) - G(h, d_H(\alpha_2), \alpha_2)||, \\ &= \frac{1}{\sigma - \kappa} ||\Psi(h, d_H(\alpha_2), \alpha_2)|| |h^p|, \qquad (4.5) \\ &< \frac{1}{\sigma - \kappa} \left(\sup_{(h, \alpha_2) \in (-\rho_2, \rho_2) \times (-\delta_2, \delta_2)} ||\Psi(h, d_H(\alpha_2), \alpha_2)|| \right) |h|^p. \end{aligned}$$

To conclude, take $\rho := \rho_2$, $\delta := \delta_2$, and

$$C := \frac{1}{\sigma - \kappa} \left(\sup_{(h, \alpha_2) \in (-\rho, \rho) \times (-\delta, \delta)} ||\Psi(h, d_H(\alpha_2), \alpha_2)|| \right). \qquad \square$$

Before finishing this section, it is worth deriving a particular result from the above discussion. Under assumptions and notation of the previous theorem, suppose additionally that the one step method (2.6) preserves the BT_2 point at the origin. This assumption is quite reasonable, as this is the case when dealing with general Runge-Kutta methods (cf. [19]). This means that for all sufficiently small step-size, the map (2.6) undergoes a $1:1$ resonance at the origin, and therefore it holds

$$G(h, 0, 0) = 0,$$

for all h in some interval, say $(-\rho, \rho)$. Consider the following expansion (see (4.4))

$$G(h, d_H(\alpha_2), \alpha_2) = F(h, d_H(\alpha_2), \alpha_2) + \Psi(h, d_H(\alpha_2), \alpha_2) h^p = \Psi(h, d_H(\alpha_2), \alpha_2) h^p,$$

therefore we must have

$$G(h, d_H(0), 0) = G(h, 0, 0) = \Psi(h, d_H(0), 0) h^p = 0,$$

for all $h \in (-\rho, \rho)$, and hence $\Psi(h, d_H(0), 0) = 0$ in this interval. This means, that if we expand $\Psi(h, d_H(\cdot), \cdot)$ with respect to α_2, we obtain

$$\Psi(h, d_H(\alpha_2), \alpha_2) = \Gamma(h, \alpha_2)\alpha_2,$$

where Γ is some smooth function. By taking this into account in (4.5), we obtain the improved estimate

$$||d_{NS}(h, \alpha_2) - d_H(\alpha_2)|| \leq C|\alpha_2||h|^p, \tag{4.6}$$

for all $(h, \alpha_2) \in (-\rho, \rho) \times (-\delta, \delta)$, where

$$C := \frac{1}{\sigma - \kappa}\left(\sup_{(h,\alpha_2)\in(-\rho,\rho)\times(-\delta,\delta)} ||\Gamma(h, \alpha_2)||\right).$$

Note that such an improved estimate can only be obtained when dealing with BT_2 singularities. The reason is that FH points are always shifted by one-step methods due to the Hopf eigenvalues, therefore the estimate given by Theorem 4.1 is, for FH singularities, already optimal.

5 Intersection of the discretized fold and Hopf curves

We conclude the theoretical part of this article with the analysis of the intersection of the discretized paths of fold and Hopf points. It is assumed that the continuous-time system (1.1) undergoes a generic BT_2 or FH point at the origin and that the system is discretized via general one-step methods.

Curves of fold and Hopf points are known to emanate from the mentioned codimension two singularities (see Section 2). Denote these curves by $C_F, C_H : (-\epsilon, \epsilon) \to \mathbb{R}^{N+2}$, respectively. Generically, it is expected that the projections of C_F and C_H onto the parameter space intersect tangentially at the codimension two singularity, see Figure 1.1. Likewise, C_F and C_H are known to intersect transversally (in full space $\mathbb{R}^{N+2}$) at the bifurcation. Thus, the question we are to take up is whether this generic behavior persists under general one-step methods, i.e., whether the discretized fold and Hopf curves intersect tangentially (resp. transversally) in parameter space (resp. in full space). In this sense, the genericity of the BT_2 and FH point (as explicitly defined in Section 2) will be shown to induce a generic intersection of the discretized curves. For the sake of completeness of our discussion, we will first prove that the defined genericity conditions indeed imply the expected generic intersection of C_F and C_H at the codimension two point. We accomplish this task in the following:

Theorem 5.1. *Let system* (1.1) *undergo a generic BT_2 or FH singularity at the origin. Then, there exist paths of fold and Hopf points of* (1.1) *which intersect transversally (resp. tangentially) at the codimension two point in full space (resp. parameter space).*

Proof. Recall that the genericity of the codimension two points, i.e., the invertibility of the matrix

$$A^0 := \begin{pmatrix} f_x^0 & f_\alpha^0 \\ g_x^0 & g_\alpha^0 \\ h_x^0 & h_\alpha^0 \end{pmatrix},$$

where $g := \det(2f_x \odot I_N)$, $h := \det(f_x)$, implies that the jacobian of system (2.2) has full rank at the origin (see comments after system (2.3)). Hence, we proved the existence of an emanating path of Hopf points (cf. Theorem 4.1). Similarly, since the jacobian of system (2.1) has also full rank at the origin, the existence of a fold curve is guaranteed.

Thus, consider regular, smooth parametrizations of the fold and Hopf curves denoted by $C_F := (x_F, \alpha_F) : (-\epsilon, \epsilon) \to \mathbb{R}^N \times \mathbb{R}^2$, $C_H := (x_H, \alpha_H) : (-\epsilon, \epsilon) \to \mathbb{R}^N \times \mathbb{R}^2$, $\epsilon > 0$, respectively. The intersection of these curves can be found as the solution of the system

$$\begin{cases} f(x, \alpha) = 0, \\ \det(2f_x(x, \alpha) \odot I_N) = 0, \\ \det(f_x(x, \alpha)) = 0, \end{cases}$$

which is, by assumption, regular at the origin, i.e., it possesses the isolated solution $(x, \alpha) = (0, 0)$. Thus, C_F and C_H intersect at the origin, and without loss of generality we assume $C_F(0) = C_H(0) = 0$. The next step is to show that these curves intersect tangentially in parameter space. First, note that A^0 has full rank, i.e. $\operatorname{rank}(A^0) = N + 2$, thereby we must have $\operatorname{rank}\left(\begin{pmatrix} f_x^0 & f_\alpha^0 \end{pmatrix}\right) = N$. Recall that due to the codimension two point, f_x^0 has rank defect equal to 1, thus it holds

$$(a, b) := p_0^T f_\alpha^0 \neq 0,$$

where p_0 is a left eigenvector of f_x^0 corresponding to the eigenvalue equal to 0. Since C_F and C_H represent equilibria of (1.1), we conclude that:

$$\begin{aligned} f(x_F(s), \alpha_F(s)) &= 0, \\ \Rightarrow f_x(x_F(s), \alpha_F(s))x'_F(s) + f_\alpha(x_F(s), \alpha_F(s))\alpha'_F(s) &= 0, \end{aligned}$$

for $s \in (-\epsilon, \epsilon)$, and by evaluating the above expression at $s = 0$, we arrive at

$$f_x^0 x'_F(0) + f_\alpha^0 \alpha'_F(0) = 0.$$

Likewise, we can show

$$f_x^0 x'_H(0) + f_\alpha^0 \alpha'_H(0) = 0.$$

By multiplying both sides of the above equations from the left by p_0^T, we obtain

$$(a, b)\alpha'_F(0) = (a, b)\alpha'_H(0) = 0,$$

which implies that α_F, α_H are tangential at the origin, and furthermore a common tangent vector is given by $(-b, a)^T = (-p_0^T f_{\alpha_2}^0, p_0^T f_{\alpha_1}^0)^T$. It is left to show that C_F and C_H intersect transversally in full space. Suppose the contrary, namely, that there exists a nonzero constant K, such that

$$C'_F(0) = K C'_H(0).$$

Since C_F, C_H are solutions of (2.1), (2.2), respectively, it follows

$$\begin{pmatrix} f_x^0 & f_\alpha^0 \\ g_x^0 & g_\alpha^0 \end{pmatrix} C'_H(0) = 0,$$

and

$$\begin{pmatrix} f_x^0 & f_\alpha^0 \\ h_x^0 & h_\alpha^0 \end{pmatrix} C_F'(0) = 0,$$

but since $C_F'(0) = KC_H'(0)$ holds, we conclude that

$$\begin{pmatrix} f_x^0 & f_\alpha^0 \\ g_x^0 & g_\alpha^0 \\ h_x^0 & h_\alpha^0 \end{pmatrix} C_F'(0) = 0,$$

which contradicts the invertibility of A^0. Thus, C_F and C_H intersect transversally at the origin. □

It is worth having presented the above discussion, since a similar approach will be applied for proving the main result of this section, namely, the generic intersection of the discretized fold and Hopf curves. Roughly speaking, we will see that genericity "persists" under general one-step methods, although we have not formally defined genericity of codimension two points in discrete-time systems, and that topic will not be discussed in detail in the present article. Thus, the main result of this section is presented in the following:

Theorem 5.2. *Let a general one-step discretization method of order $p \geq 1$ applied to* (1.1) *be given by* (2.6)*. Assume that system* (1.1) *has a generic BT_2 or FH point at the origin. Further let*

$$\begin{aligned} C_F &:= (x_F, \alpha_F) : (-\rho, \rho) \times (-\epsilon, \epsilon) \to \mathbb{R}^N \times \mathbb{R}^2, \\ C_{NS} &:= (x_{NS}, \alpha_{NS}) : (-\rho, \rho) \times (-\epsilon, \epsilon) \to \mathbb{R}^N \times \mathbb{R}^2, \end{aligned}$$

where $\epsilon > 0$, $0 < \rho \leq h_0$ (so that the conclusions of Theorems 3.1, 3.2 and 4.1 hold) be smooth, regular parametrizations of the curves of fold and Neimark-Sacker points of (2.6)*, respectively, with $C_F(0,0) = C_{NS}(0,0) = 0$. Then, C_F and C_{NS} intersect transversally (resp. tangentially) at the discretized codimension two point in full space (resp. parameter space) for all $h \in (-\rho, \rho)$.*

Some remarks before presenting the proof of this theorem are in order. The existence of a curve of discretized fold points can be deduced as in the Hopf case (cf. Theorem 4.1). Thus, we will not assume that this curve remains at the same position, as it happens under general Runge-Kutta methods (see the Introduction). The reason for doing so is that in this way we can establish our results in a general framework. Particular results, e.g. when dealing with Runge-Kutta methods, will be of course consistent with and covered by the approach we are to employ. With these few remarks, we are ready to present:

Proof of Theorem 5.2. We will first show that the curves C_F and C_{NS} actually intersect at the discretized codimension two point. This point, at which the curves intersect, can be found as a solution of the system

$$\begin{cases} E(h, x, \alpha) = 0, \\ H(h, x, \alpha) = 0, \\ F(h, x, \alpha) = 0, \end{cases} \tag{5.1}$$

where $E : (-\rho, \rho) \times \tilde{\Omega} \times \tilde{\Lambda} \to \mathbb{R}^N$, $H, F : (-\rho, \rho) \times \tilde{\Omega} \times \tilde{\Lambda} \to \mathbb{R}$ are defined by

$$\begin{aligned} E(h, x, \alpha) &:= \frac{1}{h}(\psi^h(x, \alpha) - x), \\ H(h, x, \alpha) &:= \det\left(\frac{1}{h}(\psi_x^h(x, \alpha) \odot \psi_x^h(x, \alpha) - I_m)\right), \\ F(h, x, \alpha) &:= \det\left(\frac{1}{h}(\psi_x^h(x, \alpha) - I_N)\right), \end{aligned}$$

(cf. (3.4)). In the proof of Theorem 3.1, we saw that system (5.1) has for every $h \in (-\rho, \rho)$ a unique solution $(x_0(h), \alpha_0(h))$ (which in this case represent an FN point or a 1 : 1 resonance of (2.6)). Thus, it follows that C_F and C_{NS} intersect at this point, and without loss of generality, we suppose that for every $h \in (-\rho, \rho)$ there exists $\tilde{s}_h, \bar{s}_h \in (-\epsilon, \epsilon)$, such that $C_F(h, \tilde{s}_h) = C_{NS}(h, \bar{s}_h) = (x_0(h), \alpha_0(h))$. Next, we will see that C_F and C_{NS} intersect tangentially in parameter space.

In what follows, we carry out the analysis for $h \neq 0$, since at $h = 0$ the conclusions of the theorem clearly hold. This is readily seen by noticing that the curves C_F, C_{NS} converge uniformly to their continuous counterpart (see estimate (4.2)), thereby for $h = 0$ the conclusions of the present theorem already follow from Theorem 5.1.

Recall that the curves C_F, C_{NS} represent equilibria of (2.6), thus it holds

$$E(h, x_F(h, s), \alpha_F(h, s)) = 0,$$

and hence[6]

$$E_x(h, x_F(h, s), \alpha_F(h, s))x'_F(h, s) + E_\alpha(h, x_F(h, s), \alpha_F(h, s))\alpha'_F(h, s) = 0,$$

for all $(h, s) \in (-\rho, \rho) \times (-\epsilon, \epsilon)$, and by evaluating the above expression at $s = \tilde{s}_h$, we arrive at

$$E_x(h, x_0(h), \alpha_0(h))x'_F(h, \tilde{s}_h) + E_\alpha(h, x_0(h), \alpha_0(h))\alpha'_F(h, \tilde{s}_h) = 0. \tag{5.2}$$

Likewise, we can show

$$E_x(h, x_0(h), \alpha_0(h))x'_{NS}(h, \bar{s}_h) + E_\alpha(h, x_0(h), \alpha_0(h))\alpha'_{NS}(h, \bar{s}_h) = 0. \tag{5.3}$$

Note that for every $0 < |h| < \rho$ there exists a nonzero vector $p_{0h} \in \mathbb{R}^N$, such that

$$p_{0h}^T E_x(h, x_0(h), \alpha_0(h)) = 0,$$

since $E_x(h, x_0(h), \alpha_0(h))$ has rank defect equal to 1 (see the third equation in (5.1)). Furthermore, due to the invertibility of (for $(x_0(h), \alpha_0(h))$ is a regular zero of (5.1))

$$A^0(h) := \begin{pmatrix} E_x(h, x_0(h), \alpha_0(h)) & E_\alpha(h, x_0(h), \alpha_0(h)) \\ H_x(h, x_0(h), \alpha_0(h)) & H_\alpha(h, x_0(h), \alpha_0(h)) \\ F_x(h, x_0(h), \alpha_0(h)) & F_\alpha(h, x_0(h), \alpha_0(h)) \end{pmatrix},$$

[6] Throughout this discussion, the symbol $'$ means derivative with respect to s.

it follows that rank $\left(\begin{pmatrix} E_x(h, x_0(h), \alpha_0(h)) & E_\alpha(h, x_0(h), \alpha_0(h)) \end{pmatrix}\right) = N$, for all $h \in (-\rho, \rho)$. Consequently, it holds

$$(a_h, b_h) := p_{0h}^T E_\alpha(h, x_0(h), \alpha_0(h)) \neq 0,$$

for all $0 < |h| < \rho$. By multiplying both sides of (5.2) and (5.3) from the left by p_{0h}^T, we obtain

$$(a_h, b_h)\alpha_F'(h, \tilde{s}_h) = (a_h, b_h)\alpha_{NS}'(h, \bar{s}_h) = 0,$$

which implies that α_F and α_{NS} meet tangentially at $\alpha = \alpha_0(h)$, for all $0 < |h| < \rho$. It remains to show the transversal intersection of the curves in full space. Suppose the contrary, namely, that there exists a $0 \neq h_c \in (-\rho, \rho)$, and a nonzero constant K, such that

$$C_F'(h_c, \tilde{s}_{h_c}) = K C_{NS}'(h_c, \bar{s}_{h_c}).$$

Since C_F, C_H are solutions of

$$\begin{cases} E(h, x, \alpha) = 0, \\ F(h, x, \alpha) = 0, \end{cases}$$

and

$$\begin{cases} E(h, x, \alpha) = 0, \\ H(h, x, \alpha) = 0, \end{cases}$$

respectively, it follows

$$\begin{pmatrix} E_x(h_c, x_0(h_c), \alpha_0(h_c)) & E_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \\ F_x(h_c, x_0(h_c), \alpha_0(h_c)) & F_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \end{pmatrix} C_F'(h_c, \tilde{s}_{h_c}) = 0,$$

and

$$\begin{pmatrix} E_x(h_c, x_0(h_c), \alpha_0(h_c)) & E_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \\ H_x(h_c, x_0(h_c), \alpha_0(h_c)) & H_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \end{pmatrix} C_{NS}'(h_c, \bar{s}_{h_c}) = 0,$$

but since $C_F'(h_c, \tilde{s}_{h_c}) = K C_{NS}'(h_c, \bar{s}_{h_c})$ holds, we conclude that

$$\begin{pmatrix} E_x(h_c, x_0(h_c), \alpha_0(h_c)) & E_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \\ H_x(h_c, x_0(h_c), \alpha_0(h_c)) & H_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \\ F_x(h_c, x_0(h_c), \alpha_0(h_c)) & F_\alpha(h_c, x_0(h_c), \alpha_0(h_c)) \end{pmatrix} C_F'(h_c, \tilde{s}_{h_c}) = 0,$$

which contradicts the invertibility of $A^0(h_c)$. Thus, C_F and C_{NS} intersect transversally at $(x_0(h), \alpha_0(h))$ for all $0 < |h| < \rho$. □

6 A Numerical Example

Consider the following continuous-time, dimensionless system:

$$\begin{aligned} \dot{x} &= -\left(\frac{\beta + \alpha}{R}\right) x + \frac{\alpha}{R} y - \frac{C}{R} x^3 + \frac{D}{R}(y - x)^3 - \frac{E}{R} x^5 + \frac{F}{R}(y - x)^5, \\ \dot{y} &= \alpha x - (\alpha + G) y - z - D(y - x)^3 - H y^3 - F(y - x)^5 - I y^5, \\ \dot{z} &= y, \end{aligned} \tag{6.1}$$

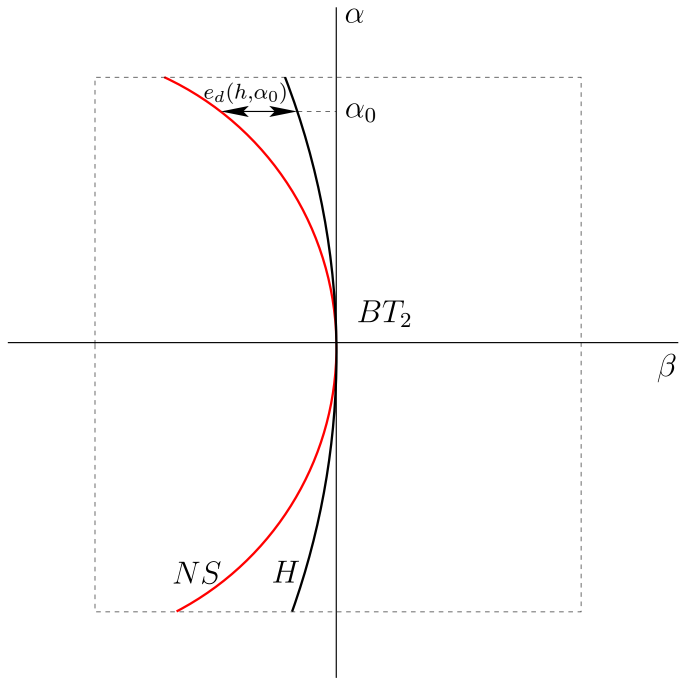

Fig. 6.1: Interpretation of the distance function on parameter space.

with state variables $(x, y, z) \in \mathbb{R}^3$ and with parameters $\beta, \alpha, C, D, E, F, G, H, I, R \in \mathbb{R}$, $R > 0$. This system describes the dynamics of a modified van der Pol-Duffing oscillator. A thorough analysis of this oscillator concerning both local, as well as global phenomena can be found in [1], [7], and a more general discussion concerning the dynamics of this type of circuits can be found in [14, Chapter 7].

For numerical purposes, we assume β, α to be our bifurcation parameters, and we let $C = 1$, $D = -5$, $E = 1$, $F = 1$, $G = -1.5$, $H = 1$, $I = 1$, $R = 3$ fixed. Moreover, the numerical computations will be done with the continuation software CONTENT, cf. [16]. Further numerical manipulations will be performed with MATLAB.

The purpose of this experiment is to observe whether the emanating path of Hopf points is $O(h^p)$-shifted by one-step methods, cf. Theorem 4.1. In particular, we will deal with the Hopf curve that emanates from a BT_2 point of (6.1) located at $(x_{BT}, y_{BT}, z_{BT}) = (-1, 0, -4.26794919243109)$, $(\alpha_{BT}, \beta_{BT}) = (8.26794919243109, -6.26794919243109)$ (cf. [19]). As for a discretization method, we will use the 3-th order method of Runge.

Under notation of Theorem 4.1, define the following distance function

$$Dist_H(h, \alpha) := ||d_{NS}(h, \alpha) - d_H(\alpha)||,$$

for $h > 0$, $|\alpha - \alpha_{BT}|$ small, where $|| \cdot ||$ represents the Euclidean norm. Thus, our aim is to investigate the behavior of $Dist_H$, as (h, α) vary. In Figure 6.1, we illustrate the meaning of the above distance function. In this picture, e_d represents the β-component of $Dist_H$. By repeatedly fixing α from $\alpha = 7.41663851586445$ to $\alpha = 9.09659824492632$, and letting h vary from $h \approx 0.05$ to $h = 0.3$, for each α fixed, we obtained a surface

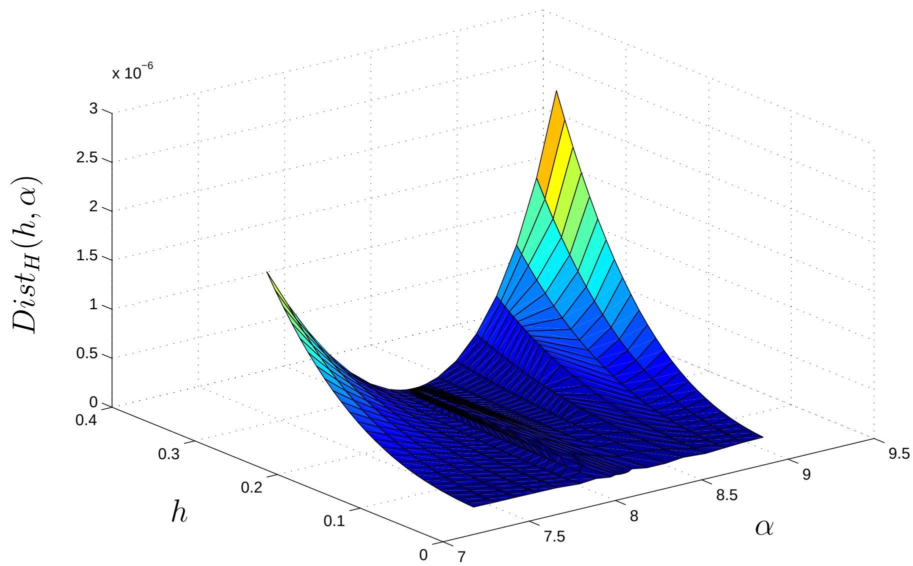

Fig. 6.2: Behavior of $Dist_H$ with respect to (h, α).

plot of $Dist_H$ which is shown in Figure 6.2. In this picture, two facts draw special attention. First recall that the singularity, i.e. the BT_2 point, is located along the line $\alpha = \alpha_{BT} \approx 8.26$. Thus, it is observed that $Dist_H(h, \alpha)$ tends to zero, as α tends to the singularity, regardless h. This fact is analytically seen in (4.6), and schematically seen in Figure 6.1. Secondly, it is also noted that $Dist_H(h, \alpha)$ tends to zero, as h tends to zero, regardless α, which means that $Dist_H$ is uniformly bounded, however, we do not know to which order can this function be bounded. For determining the order, we will analyze the behavior of $Dist_H$ with respect to h, for several, fixed α's. In Figure 6.3, this behavior is shown. In this picture, we plotted the logarithm of the variables, so that we can determine the order as the slope of the quasi-straight lines obtained. The labels on the lines represent, approximately, the corresponding fixed value of α. Thus, it is seen that the slope of the lines are approximately the same, that is, $m \approx 3.0029 \approx 3$, which is consistent with Theorem 4.1.

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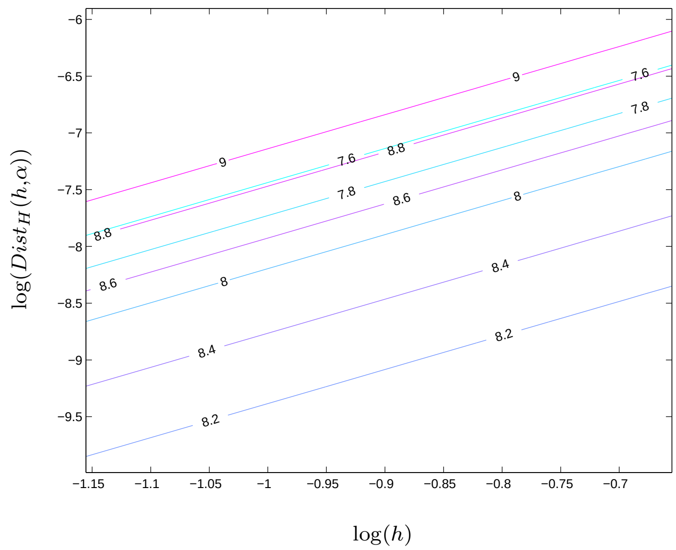

Fig. 6.3: Behavior of $Dist_H$ with respect to h, for several, fixed α's.

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