A set-valued implicit function theorem under relaxed one-sided Lipschitz conditions.

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Abstract Implicit function theorems are derived for nonlinear set valued equations that satisfy a relaxed one-sided Lipschitz condition. We discuss a local and a global version and study in detail the continuity properties of the implicit set-valued function. Applications are provided to the Crank-Nicolson scheme for differential inclusions and to the analysis of differential algebraic inclusions.

Keywords set valued implicit function theorem · one-sided Lipschitz condition · differential (algebraic) inclusions

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1 Introduction

The implicit function theorem is one of the most important tools in analysis. It guarantees that a solution (p_0, x_0) of the equation

$$F(p, x) = 0$$

is stable under small perturbations in p provided that $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and satisfies a nondegeneracy condition with respect to x near the known solution (p_0, x_0). Thus, it is a standard ingredient for an abundance of existence and robustness type results.

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The aim of the present paper is to prove an implicit function theorem for set-valued mappings in the spirit of the classical result: If $F : \mathbb{R}^k \times \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a set-valued mapping and

$$0 \in F(p_0, x_0)$$

for some $p_0 \in \mathbb{R}^k$, $x_0 \in \mathbb{R}^m$, then there exist neighbourhoods V_1 and V_2 of p_0 and x_0 such that the set $\{x \in V_2 : 0 \in F(p, x)\}$ can be described as the image $N(p)$ of a set-valued mapping $N : V_1 \rightrightarrows V_2$ with favourable properties. As in the single-valued case, it is necessary to specify matching smoothness and nondegeneracy conditions.

Several generalizations of the implicit function theorem to the set-valued setting have been proposed. These theorems can be divided into two classes.

a) The systematic investigation of optimal control problems with constraints caused a demand for analytical tools designed for nonsmooth optimization. A first generalized implicit function theorem aiming in that direction is presented in [14], which only guarantees the existence of a single-valued resolving function. In [3], a first fully set-valued implicit function theorem is proposed, which requires lower semicontinuity of the mapping F and a technical transversality condition and provides a lower semicontinuous set-valued resolving mapping $N(\cdot)$. The main result of [10] is an implicit function theorem based on traditional set-valued calculus, which allows to express the set derivative of the resolving mapping in terms of the set derivative of the original multifunction. The theorem given in [12] partly generalizes the latter one without providing much information about the nature of $N(\cdot)$, while the main result of [11] emphasizes its continuity properties. Recently, an analysis from a modern point of view and extensions of the classical result from [14] have been given in [8].

b) An implicit function theorem for maximal monotone and uniformly coercive mappings is presented in [1]. It is shown that monotonicity forces the resolving mapping $N(\cdot)$ to be single-valued. Note that the theory of nonlinear maximal monotone operators (see the monograph [16] for a survey) always requires one-sided Lipschitz estimates for all admissible arguments in contrast to the relaxed one-sided Lipschitz conditions in this paper (see below).

Several variants of set-valued implicit function theorems in Banach spaces are proved in [4]. Here, the nondegeneracy assumptions imply that the resolving mapping $N(\cdot)$ is single-valued as well, and in view of Proposition 3.2 in that article, the mapping F is essentially single-valued whenever the resolving mapping $N(\cdot)$ is continuous.

The type of implicit function theorem we develop here is designed for the study of set-valued dynamics and set-valued numerical analysis. We require the mapping F to be convex and compact-valued, continuous in the parameter, and upper semicontinuous and relaxed one-sided Lipschitz (ROSL, see Definition 1) with negative constant in space. Under these assumptions, we can guarantee that the resolving set-valued mapping $N(\cdot)$ is well-defined, compact-valued, and continuous. In addition, we can show that if F is Hölder continuous, then so is $N(\cdot)$.

It was shown in [7] (Theorem 3.2 is of particular importance) that the ROSL condition is one of the most natural and useful stability concepts for set-valued dynamics. In contrast to the stronger one-sided Lipschitz condition (OSL), it allows the mapping

to be multivalued everywhere and not just on a set of measure zero. Moreover, if the right hand side of a differential inclusion

$$\dot{x}(t) \in F(t, x(t))$$

is ROSL, there are typically many solutions, while in the OSL case, it is easy to see that there exists at most one solution.

In Section 2, we prove a local version of the solvability theorem presented in [2] by extending a locally defined ROSL mapping to $\mathbb{R}^m$ in a suitable manner. This solvability theorem can be regarded as a multivalued version of the uniform monotonicity theorem (see [13], [15]). Thus, the single-valued implicit function theorem given in Section 3 is closely related to the latter result. In Section 4 we propose a global and a local version of our set-valued implicit function theorem, which we apply in Section 5 in order to discuss perturbations of one-sided Lipschitz functions, a set-valued Crank-Nicolson scheme, and differential algebraic inclusions.

The notation used in this paper is standard: The Euclidean norm is denoted by $| \cdot |$, closed balls are defined by $B_R(x) = \{y \in \mathbb{R}^m : |y - x| \leq R\}$ and $\|A\| := \sup_{a \in A} |a|$ denotes the maximal norm of the elements of a set $A \subset \mathbb{R}^m$. The spaces of the nonempty compact and the nonempty convex and compact subsets of $\mathbb{R}^m$ are denoted by $\mathcal{C}(\mathbb{R}^m)$ and $\mathcal{CC}(\mathbb{R}^m)$, respectively. For $A, B \in \mathcal{C}(\mathbb{R}^m)$, the one-sided and the symmetric Hausdorff distance are given by

$$\mathrm{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b|$$

and

$$\mathrm{dist}_H(A, B) := \max\{\mathrm{dist}(A, B), \mathrm{dist}(B, A)\}.$$

If $A \subset \mathbb{R}^k$ and $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^m$ is a set-valued mapping, then $F(A) := \cup_{a \in A} F(a)$. The map $F : U \subset \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ is called upper semicontinuous (usc) at $x \in U$ if $\mathrm{dist}(F(x'), F(x)) \to 0$ as $x' \to x$.

2 A local solvability theorem

The notion of relaxed one-sided Lipschitz (ROSL) set-valued mappings is an important stability criterion for multivalued generators. It generalizes the concepts of Lipschitz continuity and the (strong) one-sided Lipschitz property. A detailed analysis of this property can be found in [6] and several other works of the same author.

Definition 1 A mapping $F : U \subset \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ is called relaxed one-sided Lipschitz with constant $l \in \mathbb{R}$ if for every $x, x' \in U$ and $y \in F(x)$ there exists some $y' \in F(x')$ such that

$$\langle y - y', x - x' \rangle \leq l |x - x'|^2. \tag{1}$$

The following Theorem is a local version of Theorem 2 in [2]. In order to obtain the optimal result, it is necessary to extend the locally defined mapping to the whole space in a suitable way.

Theorem 1 *Let $F : B_R(0) \to \mathcal{CC}(\mathbb{R}^m)$ be usc and ROSL with a constant $l < 0$ such that $-\frac{1}{l}\operatorname{dist}(0, F(0)) \leq R$. Then the inclusion $0 \in F(x)$ has a solution $\bar{x} \in \mathbb{R}^m$ with*

$$|\bar{x}| \leq -\frac{1}{l}\operatorname{dist}(0, F(0)).$$

Proof Let $y_0 \in F(0)$ be the element with minimal norm. We extend F from $B_R(0)$ to $\mathbb{R}^m$ by setting

$$F(x) := \frac{R - |x|}{R}y_0 + \frac{|x|}{R}F(\frac{R}{|x|}x)$$

whenever $x \notin B_R(0)$. This extension is usc: In the interior of $B_R(0)$, this is true by assumption. For $|x|, |x'| \geq R$,

$$
\begin{aligned}
&\operatorname{dist}(F(x'), F(x)) \\
&= \operatorname{dist}\big(\frac{R - |x'|}{R}y_0 + \frac{|x'|}{R}F(\frac{R}{|x'|}x'), \frac{R - |x|}{R}y_0 + \frac{|x|}{R}F(\frac{R}{|x|}x)\big) \\
&\leq \left|\frac{R - |x'|}{R}y_0 - \frac{R - |x|}{R}y_0\right| + \operatorname{dist}\big(\frac{|x'|}{R}F(\frac{R}{|x'|}x'), \frac{|x|}{R}F(\frac{R}{|x'|}x')\big) \\
&\quad + \operatorname{dist}\big(\frac{|x|}{R}F(\frac{R}{|x'|}x'), \frac{|x|}{R}F(\frac{R}{|x|}x)\big) \to 0 \text{ as } x' \to x,
\end{aligned}
$$

because F is usc and thus bounded on $B_R(0)$. In particular, the extension is usc on $\partial B_R(0)$, because by assumption, $\operatorname{dist}(F(x'), F(x)) \to 0$ as $x' \to x$ whenever $x \in \partial B_R(0)$ and $x' \in \operatorname{int}B_R(0)$.

The extended F is not necessarily ROSL, but for every $x \in \mathbb{R}^m$, there exists some $y \in F(x)$ such that

$$\langle y - y_0, x \rangle \leq l|x|^2. \tag{2}$$

Indeed, for $x \in B_R(0)$, this is just the ROSL property, and for $x \notin B_R(0)$, we define

$$y := \frac{R - |x|}{R}y_0 + \frac{|x|}{R}y',$$

where we may choose $y' \in F(\frac{R}{|x|}x)$ with $\langle y' - y_0, \frac{R}{|x|}x \rangle \leq lR^2$, because $\frac{R}{|x|}x \in B_R(0)$ where F is ROSL. But then,

$$
\begin{aligned}
\langle y - y_0, x \rangle &= \langle \frac{R - |x|}{R}y_0 + \frac{|x|}{R}y' - y_0, x \rangle = \langle \frac{|x|}{R}(y' - y_0), x \rangle \\
&= \frac{|x|^2}{R^2}\langle y' - y_0, \frac{R}{|x|}x \rangle \leq l|x|^2.
\end{aligned}
$$

Now we consider the inclusion $0 \in F(x)$, $x \in \mathbb{R}^m$, and the usc mapping $G : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ defined by

$$G(x) := x + \alpha F(x)$$

with some $\alpha > 0$. Given $x \in \mathbb{R}^m$, there exists some $y \in F(x)$ such that (2) holds. Then $z := x + \alpha y$ satisfies

$$
\begin{aligned}
|z|^2 &= |x|^2 + 2\alpha\langle y, x \rangle + \alpha^2|y|^2 \\
&= |x|^2 + 2\alpha\langle y - y_0, x \rangle + 2\alpha\langle y_0, x \rangle + \alpha^2|y|^2 \\
&\leq |x|^2 + 2\alpha l|x|^2 + 2\alpha|x|\operatorname{dist}(0, F(0)) + \alpha^2|y|^2.
\end{aligned}
$$

Thus, for any $\varrho > -\frac{1}{l}\operatorname{dist}(0, F(0))$, $|x| \leq \varrho$, and α small enough so that $1 + 2\alpha l \geq 0$, inequality

$$
\begin{aligned}
|z|^2 &\leq \varrho^2 + 2\alpha(l\varrho + \operatorname{dist}(0, F(0)))\varrho + \alpha^2 |y|^2 \\
&\leq \alpha[2\varrho \underbrace{(l\varrho + \operatorname{dist}(0, F(0)))}_{<0} + \alpha|y|^2] + \varrho^2
\end{aligned}
\tag{3}
$$

holds. As F is usc,

$$
M_\varrho := \sup_{x \in B_\varrho(0)} \|F(x)\| < \infty,
$$

and there exists an $\alpha > 0$ such that $|z|^2 \leq \varrho^2$ follows from (3). This means that for this fixed α,

$$
H(x) := G(x) \cap B_\varrho(0) \neq \emptyset \text{ for all } x \in B_\varrho(0),
$$

and $H(\cdot)$ is also usc. By the Kakutani Theorem, H and thus also G have a fixed point x_ϱ in $B_\varrho(0)$, which implies that $0 \in F(x_\varrho)$.

In particular, we find elements $x_n \in B(0, -\frac{1}{l}\operatorname{dist}(0, F(0)) + 1/n)$ for all $n \in \mathbb{N}$ such that $0 \in F(x_n)$. As $B(0, -\frac{1}{l}\operatorname{dist}(0, F(0)) + 1)$ is compact, there exists a convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ with limit

$$
\bar{x} \in B(0, -\frac{1}{l}\operatorname{dist}(0, F(0))).
$$

Since F is usc,

$$
0 \in F(\bar{x}).
$$

As $\bar{x} \in B_R(0)$, the element $\bar{x}$ is a zero of the original mapping F.

As an immediate consequence, we obtain the following corollary.

Corollary 1 *Let $x, y \in \mathbb{R}^m$ and let $F : B_R(x) \to \mathcal{CC}(\mathbb{R}^m)$ be usc and ROSL with a constant $l < 0$ such that $-\frac{1}{l}\operatorname{dist}(y, F(x)) \leq R$. Then there exists an $\bar{x} \in \mathbb{R}^m$ such that $y \in F(\bar{x})$ and*

$$
|x - \bar{x}| \leq -\frac{1}{l}\operatorname{dist}(y, F(x)).
$$

Proof Consider the usc mapping $G : B_R(0) \to \mathcal{CC}(\mathbb{R}^m)$ given by

$$
G(z) := F(z + x) - y.
$$

It is ROSL in $B_R(0)$ with the same constant $l < 0$, and

$$
-\frac{1}{l}\operatorname{dist}(0, G(0)) = -\frac{1}{l}\operatorname{dist}(y, F(x)) \leq R.
$$

By Theorem 1, there exists a solution $\tilde{x}$ of $0 \in G(\tilde{x})$ such that

$$
|\tilde{x}| \leq -\frac{1}{l}\operatorname{dist}(0, G(0)),
$$

so that, setting $\bar{x} := \tilde{x} + x$, we obtain $y \in F(\bar{x})$ and

$$
|x - \bar{x}| \leq -\frac{1}{l}\operatorname{dist}(y, F(x)).
$$

3 An implicit function theorem for single-valued OSL functions

Applying Corollary 1 to a single-valued mapping yields a variant of the classical implicit function Theorem. Recall that any single-valued function which is usc and ROSL is continuous and OSL.

Theorem 2 *Let $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{R}^m$ be open sets, and let $f : U_1 \times U_2 \to \mathbb{R}^m$ be continuous in the first and continuous and OSL in the second argument with constant $l < 0$. Assume furthermore that there exist $p_0 \in U_1$ and $x_0 \in U_2$ such that $f(p_0, x_0) = 0$. Then there exist neighbourhoods $V_1 \subset U_1$ and $V_2 \subset U_2$ with $p_0 \in V_1$ and $x_0 \in V_2$ and a continuous function $g : V_1 \to V_2$ such that*

$$f(p, x) = 0 \quad \Leftrightarrow \quad g(p) = x$$

whenever $p \in V_1$ and $x \in V_2$. Moreover, if $f(\cdot, x)$ is (L, β)-Hölder continuous for any $x \in U_2$, then g is $(-\frac{L}{l}, \beta)$-Hölder continuous.

Proof Let $R := \sup\{r > 0 : B_r(x_0) \subset U_2\}$, and set $V_2 := \text{int} B_R(x_0)$. As $f(\cdot, x_0)$ is continuous, there exists a neighbourhood $V_1 \subset U_1$ of p_0 such that $|f(p, x_0)| < -lR$ for all $p \in V_1$. Thus we can apply Corollary 1 to the mapping $f(p, \cdot)$ for any $p \in V_1$, which yields a point $x_p \in \mathbb{R}^m$ such that $f(p, x_p) = 0$ and

$$|x_p - x_0| \le -\frac{1}{l}|f(p, x_0)| < R.$$

The zero x_p is unique in $\text{int} B_R(x_0)$: Assume that there were two zeroes x_p and x_p' in $\text{int} B_R(x_0)$. Then

$$0 = \langle f(x_p) - f(x_p'), x_p - x_p' \rangle \le l |x_p - x_p'|^2 \tag{4}$$

implies that $x_p = x_p'$.

Let us define $g : V_1 \to V_2$ by $g(p) := x_p$. It is continuous, because for any $p, p' \in V_1$,

$$|g(p) - g(p')| \le -\frac{1}{l}|f(p', g(p))| \to 0 \text{ as } p' \to p$$

by Corollary 1 and continuity of $f(\cdot, g(p))$. Moreover, if $f(\cdot, g(p))$ is (L, β)-Hölder continuous, the same reasoning yields

$$|g(p) - g(p')| \le -\frac{1}{l}|f(p', g(p)) - f(p, g(p))| \le -\frac{L}{l}|p - p'|^\beta,$$

so that g is $(-\frac{L}{l}, \beta)$-Hölder continuous.

Note that if $U_2 = \mathbb{R}^m$, then $V_2 = \mathbb{R}^m$ according to the definition of R in the proof.

Remark 1 This theorem is closely related to the uniform monotonicity theorem, see e.g. [13], [15], which plays a major role in the verification of the solvability of implicit Runge-Kutta methods for ODEs. In Section 5, we will use the set-valued implicit function theorem presented below in order to prove the solvability of a Crank-Nicolson scheme for differential inclusions and to discuss differential algebraic inclusions.

Remark 2 If a mapping $f : \mathbb{R}^m \to \mathbb{R}^m$ is OSL with constant $l < 0$ and continuously differentiable at x, then its differential is invertible and the classical implicit function theorem applies.

4 Implicit function theorems for set-valued ROSL maps

The following implicit function theorem is based on Corollary 1. The existence state-ment guarantees solvability of the implicit inclusion, while the defect estimate yields the smoothness properties of the resolving mapping $N(\cdot)$.

Theorem 3 *Let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ be uniformly continuous in the first argu-ment in the sense that*

$$\mathrm{dist}(F(p,x), F(p',x)) \leq \omega(|p-p'|) \ \forall x \in \mathbb{R}^m$$

with $\omega(\tau) \to 0$ as $\tau \to 0$ and usc and ROSL with a constant $l < 0$ in the second argument. Then there exists a continuous mapping $N(\cdot) : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^m)$ such that

$$0 \in F(p,x) \quad \Leftrightarrow \quad x \in N(p),$$

and

$$\|N(p)\| \leq -\frac{1}{l}\|F(p,0)\|.$$

For the diameter of the images, the implicit estimate

$$\mathrm{diam}\, N(p) \leq -\frac{1}{l} \sup_{x \in N(p)} \mathrm{diam}\, F(x)$$

holds. Moreover, if F is uniformly (L, β)-Hölder continuous in the sense that

$$\mathrm{dist}(F(p,x), F(p',x)) \leq L|p-p'|^\beta \ \forall p, p' \in \mathbb{R}^k, \ x \in \mathbb{R}^m,$$

then $N(\cdot)$ is $(-\frac{L}{l}, \beta)$-Hölder continuous.

Proof Define $N(p) := \{x \in \mathbb{R}^m : 0 \in F(p,x)\}$. It follows immediately from Corollary 1 that $N(p) \neq \emptyset$ for all $p \in \mathbb{R}^k$.

Let $\{x_n\}_{n \in \mathbb{N}}$ be a convergent sequence in $N(p)$ with $\lim_{n \to \infty} x_n = x$. Then

$$\mathrm{dist}(0, F(p,x)) \leq \mathrm{dist}(F(p,x_n), F(p,x)) \to 0 \text{ as } n \to \infty$$

by (upper semi-)continuity of $F(p, \cdot)$. Since $F(p,x)$ is compact, $0 \in F(p,x)$ and $N(p)$ is closed.

Let $x \in N(p)$. Then $0 \in F(p,x)$, and the ROSL condition implies that there exists some $y \in F(p,0)$ such that

$$-|y||x| \leq -\langle y, x\rangle = \langle y - 0, 0 - x\rangle \leq l|x|^2,$$

and hence

$$\|N(p)\| \leq -\frac{1}{l}\|F(p,0)\|.$$

In order to estimate $\mathrm{diam}\, N(p)$, consider $x, x' \in N(p)$. As $0 \in F(p,x')$, there exists some $y \in F(p,x)$ satisfying

$$-|y| \cdot |x' - x| \leq \langle 0 - y, x' - x\rangle \leq l|x' - x|^2,$$

so that

$$|x - x'| \leq -\frac{|y|}{l} \leq -\frac{1}{l} \mathrm{diam}\, F(x)$$

and hence

$$\operatorname{diam} N(p) \leq -\frac{1}{l} \operatorname{diam} F(x).$$

If $x \in N(p)$, Corollary 1 guarantees the existence of some $x' \in N(p')$ such that

$$|x - x'| \leq -\frac{1}{l} \operatorname{dist}(0, F(p', x)) \leq -\frac{1}{l} \operatorname{dist}(F(p, x), F(p', x)).$$

Thus,

$$\operatorname{dist}(N(p), N(p')) \leq -\frac{1}{l} \sup_{x \in N(p)} \operatorname{dist}(F(p, x), F(p', x)) \leq -\frac{1}{l}\omega(|p - p'|)$$

ensures continuity of N. If, in addition, F is (L, β)-Hölder continuous in the first variable, then

$$\operatorname{dist}(N(p), N(p')) \leq -\frac{1}{l} \sup_{x \in N(p)} \operatorname{dist}(F(p, x), F(p', x)) \leq -\frac{L}{l}|p - p'|^{\beta},$$

so that N is $(-\frac{L}{l}, \beta)$-Hölder continuous.

The following theorem is a local version of the above. The domains have to be chosen carefully in order to ensure that for all parameters considered, the full sets $N(p)$ are contained in the specified region. Otherwise we encounter various problems related to the continuity properties of the solution mapping $N(\cdot)$ due to the fact that its images need not be convex, see Example 10 in [2].

Theorem 4 *Let $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{R}^m$ be open subsets, and let $F : U_1 \times U_2 \to \mathcal{CC}(\mathbb{R}^m)$ be uniformly continuous in the first argument in the sense that*

$$\operatorname{dist}(F(p, x), F(p', x)) \leq \omega(|p - p'|) \ \forall p, p' \in U_1, \ x \in U_2$$

with $\omega(\tau) \to 0$ as $\tau \to 0$ and usc and ROSL with a constant $l < 0$ in the second argument. Assume that there exist elements $p_0 \in U_1$ and $x_0 \in U_2$ such that $0 \in F(p_0, x_0)$ and $\|F(p_0, x_0)\| < -l \operatorname{dist}(x_0, \partial U_2)$.

Then there exist neighbourhoods $V_1 \subset U_1$ of p_0 and $V_2 \subset U_2$ of x_0 and a continuous mapping $N(\cdot) : V_1 \to \mathcal{C}(\mathbb{R}^m)$ such that

$$0 \in F(p, x), \ p \in V_1, \ x \in V_2 \quad \Leftrightarrow \quad x \in N(p), \ p \in V_1.$$

For the diameter of the images, the implicit estimate

$$\operatorname{diam} N(p) \leq -\frac{1}{l} \sup_{x \in N(p)} \operatorname{diam} F(x)$$

holds, and we have

$$\operatorname{dist}(N(p), x_0) \leq -\frac{1}{l}\|F(p, x_0)\| \tag{5}$$

for all $p \in V_1$. Moreover, if F is uniformly (L, β)-Hölder continuous in the sense that

$$\operatorname{dist}(F(p, x), F(p', x)) \leq L|p - p'|^{\beta} \ \forall p, p' \in U_1, \ x \in U_2,$$

then N is locally $(-\frac{L}{l}, \beta)$-Hölder continuous, i.e.

$$\operatorname{dist}(N(p), N(p')) \leq -\frac{L}{l}|p - p'|^{\beta}$$

for small $|p - p'|$.

Proof Set $V_2 := \mathrm{int}B_R(x_0)$, where $R := \mathrm{dist}(x_0, \partial U_2)$, so that $V_2 \subset U_2$.

Properties of the images:

As $F(\cdot, x_0)$ is usc, there exists a neighbourhood $V_1 \subset U_1$ of p_0 such that $\|F(p, x_0)\| < -lR$ for all $p \in V_1$. Consequently, the images of the mapping $N : V_1 \rightarrow V_2$ defined by $N(p) := \{x \in V_2 : 0 \in F(p, x)\}$ are non-empty by Corollary 1.

Besides, if $p \in V_1$ and $x \in N(p)$, then the ROSL condition implies the existence of some $y \in F(p, x_0)$ such that

$$-|y||x - x_0| \leq \langle 0 - y, x - x_0 \rangle \leq l|x - x_0|^2,$$

and hence

$$\mathrm{dist}(N(p), x_0) \leq -\frac{1}{l}\|F(p, x_0)\| < R, \tag{6}$$

so that

$$\inf_{x \in N(p)} \mathrm{dist}(x, \partial V_2) > 0. \tag{7}$$

Let $\{x_n\}_{n \in \mathbb{N}}$ be a convergent sequence in $N(p)$, $p \in V_1$, with $\lim_{n \rightarrow \infty} x_n = x$. Note that (7) ensures that $x \in V_2$. Thus,

$$\mathrm{dist}(0, F(p, x)) \leq \mathrm{dist}(F(p, x_n), F(p, x)) \rightarrow 0 \text{ as } n \rightarrow \infty$$

by upper semicontinuity of $F(p, \cdot)$. Since $F(p, x)$ is compact, $0 \in F(p, x)$. Therefore, $N(p)$ is closed, and because of (6), $N(p)$ is compact.

The implicit estimate for the diameter of $N(p)$ can be obtained precisely as in the proof of Theorem 3, and (5) is proved by the same argument.

Continuity properties of the mapping $N(\cdot)$:

Let us fix some $p \in V_1$. Then

$$\sup_{x \in N(p)} -\frac{1}{l}\mathrm{dist}(0, F(p', x)) \leq \sup_{x \in N(p)} -\frac{1}{l}\mathrm{dist}(F(p, x), F(p', x))$$

$$\leq -\frac{1}{l}\omega(|p - p'|) < \inf_{x \in N(p)} \mathrm{dist}(x, \partial V_2)$$

for sufficiently small $|p - p'|$ because of (7). Hence Corollary 1 guarantees that for any $x \in N(p)$, there exists some x' such that $0 \in F(p', x')$ and

$$|x - x'| \leq -\frac{1}{l}\omega(|p - p'|),$$

which is contained in V_2 by the above reasoning and thus an element of $N(p')$. It follows that

$$\mathrm{dist}(N(p), N(p')) \leq -\frac{1}{l}\omega(|p - p'|)$$

in a neighbourhood of p, and hence $N(\cdot)$ is lower semicontinuous.

On the other hand, inequality (6) and upper semicontinuity of $F(\cdot, x_0)$ imply that there exist positive constants ε and δ such that

$$\mathrm{dist}(N(p'), x_0) \leq -\frac{1}{l}\|F(p', x_0)\| < R - \varepsilon$$

whenever $p' \in B_\delta(p)$. In particular,

$$\inf_{x' \in N(p')} \mathrm{dist}(x', \partial V_2) > \varepsilon$$

for all $p' \in B_\delta(p)$. But then there exists some $\delta' \in (0, \delta]$ such that

$$-\frac{1}{l}\mathrm{dist}(0, F(p, x')) \leq -\frac{1}{l}\mathrm{dist}(F(p', x'), F(p, x')) \leq -\frac{1}{l}\omega(|p - p'|) < \varepsilon$$

for all $p' \in B_{\delta'}(p)$ and $x' \in N(p')$. Then Corollary 1 applied to $B_\varepsilon(x')$ yields an element $x \in V_2$ with $0 \in F(p, x)$ and

$$|x - x'| < -\frac{1}{l}\omega(|p - p'|).$$

Thus we have

$$\mathrm{dist}(N(p'), N(p)) < -\frac{1}{l}\omega(|p - p'|)$$

for all p' sufficiently close to p, and $N(\cdot)$ is upper semicontinuous.

Furthermore, if F is uniformly (L, β)-Hölder continuous, repeating the above arguments with $L|p - p'|^\beta$ instead of the modulus of continuity $\omega(|p - p'|)$ yields that $N(\cdot)$ is locally (L, β)-Hölder continuous.

As in the single-valued case, if $U_2 = \mathbb{R}^m$, then $V_2 = \mathbb{R}^m$ according to the definition of R in the proof. In particular, the assumption $\|F(p_0, x_0)\| < -l\,\mathrm{dist}(x_0, \partial U_2)$ is obsolete, and the mapping $N(\cdot)$ is globally Hölder continuous whenever F is Hölder continuous.

5 Applications

5.1 Perturbations of single-valued OSL functions

Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be continuous and OSL with constant $l < 0$, and assume that $f(0) = 0$. By the Uniform Monotonicity Theorem (see [13],[15]), this zero is unique. We will study set-valued perturbations of f and apply Theorem 4 in order to gain information about the behaviour of their zero sets.

Let $G : \mathbb{R} \times \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ be a set-valued mapping such that

a) $G(0, x) = 0$ for all $x \in \mathbb{R}^m$,
b) $G(p, \cdot)$ is usc for all $p \in \mathbb{R}$,
c) for all $p \in \mathbb{R}$, the mapping $G(p, \cdot)$ is ROSL with a constant $l_p \in \mathbb{R}$ satisfying $l_p \to 0$ as $p \to 0$, and
d) $\mathrm{dist}(G(p, x), G(p', x)) \leq \omega(|p - p'|) \; \forall x \in \mathbb{R}^m$, where $\omega(\tau) \to 0$ as $\tau \to 0$.

It is easy to see that each perturbed mapping $x \mapsto f(x) + G(p, x)$ is ROSL with constant $l + l_p$. Thus, the mapping $(p, x) \mapsto f(x) + G(p, x)$ satisfies the assumptions of Theorem 4 with $U_1 := (-p_0, p_0)$ and $U_2 := \mathbb{R}^m$, where $p_0 > 0$ is chosen so small that $l + l_p \leq l_0 < 0$ for all $p \in (-p_0, p_0)$. Thus, there exists a continuous set-valued mapping $N : (-p_0, p_0) \to \mathcal{C}(\mathbb{R}^m)$ such that $0 \in F(p, x)$ is equivalent with $x \in N(p)$. This shows that the set of zeroes of the perturbed mapping $F(p, \cdot)$ deforms continuously into the unique zero of the original mapping f as $p \to 0$.

5.2 A set-valued Crank-Nicolson method

Consider the ordinary differential equation

$$\dot{x}(t) = f(x(t)) \text{ for all } t \in [0, T], \quad x(0) = x_0, \tag{8}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$. The semi-implicit numerical scheme given by

$$x_{n+1} = x_n + \frac{h}{2} f(x_n) + \frac{h}{2} f(x_{n+1}), \ n = 0, \ldots, \lfloor T/h \rfloor, \tag{9}$$

is the classical Crank-Nicolson method, which has particularly favourable stability properties, so that it plays a major role in the treatment of Galerkin approximations of reaction-diffusion equations.

The multivalued analogs of (8) and (9) are the ordinary differential inclusion

$$\dot{x}(t) \in F(x(t)) \text{ almost everywhere in } [0, T], \quad x(0) = x_0, \tag{10}$$

where $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a set-valued mapping with convex, compact, and nonempty values, and a numerical scheme given by

$$x_{n+1} \in x_n + \frac{h}{2} F(x_n) + \frac{h}{2} F(x_{n+1}), \ n = 0, \ldots, \lfloor T/h \rfloor. \tag{11}$$

It is not obvious that relation (11) is well-defined.

In [2], we have analyzed the set-valued implicit Euler scheme

$$x_{n+1} \in x_n + hF(x_{n+1}), \ n = 0, \ldots, \lfloor T/h \rfloor.$$

in detail. Here, we just want to demonstrate that the elementary properties of the set-valued Crank-Nicolson scheme (11) follow immediately from our implicit function theorems.

Theorem 5 *If $F : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ is usc and ROSL with constant l, then the set-valued Crank-Nicolson scheme defined by (11) has a well-defined solution which depends continuously on $h \in (-\frac{2}{|l|}, \frac{2}{|l|})$. In addition, if F is uniformly continuous or Lipschitz continuous, then the values of the Crank-Nicolson scheme depend continuously or in a Lipschitz continuous way on x.*

Proof Solvability, dependence on h:
Consider some fixed $x \in \mathbb{R}^m$ and set $U_1 := (-\frac{2}{|l|}, \frac{2}{|l|}) \subset \mathbb{R}$ and $U_2 := B_R(x) \subset \mathbb{R}^m$ for some $R > 0$. Then $G : U_1 \times U_2 \to \mathcal{CC}(\mathbb{R}^m)$ defined by

$$G(h, z) := x + \frac{h}{2} F(x) + \frac{h}{2} F(z) - z$$

satisfies $G(0, x) = 0$ and is uniformly Lipschitz continuous in h and usc and ROSL in z with constant $l_G := \frac{1}{2} lh - 1 < 0$. Thus Theorem 4 guarantees that relation (11) is solvable in the sense that there exist neighbourhoods $V_1 \subset U_1$ and $V_2 \subset U_2$ of 0 and x and a Lipschitz continuous mapping $N_x : V_1 \to \mathcal{C}(\mathbb{R}^m)$ (with Lipschitz constant depending on R) such that $N_x(h) \subset V_2$ for all $h \in V_1$ and $0 \in G(h, z)$ is equivalent with $z \in N_x(h)$. In particular, $N_x(h) \to \{x\}$ as $h \to 0$.

In the proof of Theorem 4, the neighbourhood V_1 is defined by

$$h\|F(x)\| = \|G(h, x)\| < -l_G R.$$

Thus, if we choose $R > 0$ large enough, we obtain $V_1 = U_1$, so that our numerical scheme is well-defined for all $h \in (-\frac{2}{|l|}, \frac{2}{|l|})$.

In fact, our solution $N_x(h)$ is globally unique, because estimate (5) implies that the resolving mapping will not change if we further enlarge R.

Dependence on x:
Now fix $h \in (-\frac{2}{|l|}, \frac{2}{|l|})$, assume that F is uniformly continuous, and consider the mapping $H : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ given by

$$H(x, z) := x + \frac{h}{2}F(x) + \frac{h}{2}F(z) - z,$$

which is uniformly continuous in x with modulus of continuity independent of z and continuous and ROSL with constant $l_H := \frac{1}{2}lh - 1 < 0$ in z. Hence Theorem 3 applies and yields the existence of a continuous resolving mapping $N_h : \mathbb{R}^m \to \mathcal{C}(\mathbb{R}^m)$ which is Lipschitz whenever F is Lipschitz.

Note that $N_x(h) = N_h(x)$ follows from the maximality of the mappings $N_x(\cdot)$ and $N_h(\cdot)$.

Of course, the estimates for the images of $N(\cdot)$ from Theorem 4 apply to $N_x(\cdot)$ for any $x \in \mathbb{R}^m$, so that

$$\text{dist}(N_x(h), x) \leq -\frac{2}{lh - 2}\|G(h, x)\| = -\frac{2h}{lh - 2}\|F(x)\|$$

and

$$\text{diam } N_x(h) \leq -\frac{2}{lh - 2} \sup_{z \in N_{G,x}(h)} \text{diam } G(h, z)$$
$$\leq -\frac{1}{lh - 2}\left(\text{diam } F(x) + \sup_{z \in N_{G,x}(h)} \text{diam } F(z)\right).$$

5.3 Differential algebraic inclusions

A thorough treatment of the DAE

$$\dot{x} = f(x, y), \quad 0 = g(x, y) \tag{12}$$

under OSL conditions is given in [9]. The authors require that the map g satisfies a OSL for some $l < 0$ with respect to y and that f is OSL with respect to x, and both functions are Lipschitz with respect to the remaining variables. Then the algebraic equation $0 = g(x, y)$ has a unique solution $y = n(x)$ for all x, and the authors prove that the resulting differential equation

$$\dot{x} = f(x, n(x))$$

has a OSL right hand side. This leads to estimates of the longterm behavior which can be transferred to discretized versions of (12).

Now let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^k)$ be continuous and ROSL with constant $l_1 \in \mathbb{R}$ in the first and Lipschitz continuous with constant $L_1 > 0$ in the second argument. Furthermore, let $G : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ be Lipschitz continuous with constant

$L_2 > 0$ in the first and usc and ROSL with constant $l_2 < 0$ in the second argument. This setting together with the inclusion

$$\dot{x} \in F(x,y), \quad 0 \in G(x,y) \tag{13}$$

is a straight-forward generalization of system (12).

In this situation, we can apply Theorem 3, which guarantees the existence of a resolving mapping $N : \mathbb{R}^k \to \mathcal{CC}(\mathbb{R}^m)$ such that $0 \in G(x,y)$ is equivalent with $y \in N(x)$. Since $x \mapsto G(x,y)$ is Lipschitz with constant L_2, the mapping $N(\cdot)$ is Lipschitz with constant $-\frac{L_2}{l_2}$. Hence, we can rewrite inclusion (13) as

$$\dot{x} \in F(x,N(x)) =: H(x). \tag{14}$$

We claim that the set-valued mapping $H : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^k)$ is ROSL with constant $l_1 - \frac{L_1 L_2}{l_2}$. Indeed, the Lipschitz continuity of $F(x,\cdot)$ and compactness of $N(x)$ ensure that $H(x)$ is bounded. Let $(v_i) \subset H(x)$ be a convergent sequence with limit v. As $v_i \in F(x,N(x))$, there exist elements $w_i \in N(x)$ such that $v_i \in F(x,w_i)$ for all i. By compactness of $N(x)$, there exists a convergent subsequence (again denoted w_i) converging to some $w \in N(x)$. Then

$$\text{dist}(v_i, F(x,w)) \leq \text{dist}(v_i, F(x,w_i)) + \text{dist}(F(x,w_i), F(x,w))$$
$$\leq L_1 |w_i - w| \to 0$$

shows that $v \in F(x,w) \subset F(x,N(x))$, and consequently, $H(x)$ is closed.

If $x, x' \in \mathbb{R}^k$ and $y \in H(x)$ are given, then there exists some $z \in N(x)$ such that $y \in F(x,z)$. As $F(\cdot,z)$ is ROSL, there exists some $y'' \in F(x',z)$ satisfying

$$\langle y - y'', x - x' \rangle \leq l_1 |x - x'|^2.$$

Since $N(\cdot)$ is Lipschitz, there exists some $z' \in N(x')$ with

$$|z - z'| \leq -\frac{L_2}{l_2}|x - x'|,$$

and by Lipschitz continuity of $F(x',\cdot)$, there is some $y' \in F(x',z')$ such that

$$|y'' - y'| \leq -\frac{L_1 L_2}{l_2}|x - x'|.$$

In particular, we have

$$\langle y - y', x - x' \rangle = \langle y - y'', x - x' \rangle + \langle y'' - y', x - x' \rangle$$
$$\leq l_1 |x - x'|^2 + |y'' - y'| \cdot |x - x'|$$
$$\leq l_1 |x - x'|^2 - \frac{L_1 L_2}{l_2}|x - x'|^2$$
$$= (l_1 - \frac{L_1 L_2}{l_2})|x - x'|^2,$$

so that H is ROSL.

Moreover, the set-valued mapping H is continuous: We have to estimate $\text{dist}(F(x, N(x)), F(x', N(x')))$ for fixed x and x' in a neighbourhood of x. Clearly,

$$\text{dist}(F(x, N(x)), F(x', N(x')))$$
$$\leq \text{dist}(F(x, N(x)), F(x', N(x))) + \text{dist}(F(x', N(x)), F(x', N(x'))).$$

As $(x, y) \mapsto F(x, y)$ is Lipschitz continuous in y with a constant independent of x, we have

$$\text{dist}(F(x', N(x)), F(x', N(x'))) \to 0 \text{ as } x' \to x.$$

On the other hand,

$$\text{dist}(F(x, N(x)), F(x', N(x))) \leq \sup_{z \in N(x)} \text{dist}(F(x, z), F(x', z)).$$

Assume that $\sup_{z \in N(x)} \text{dist}(F(x, z), F(x', z))$ does not tend to zero as $x' \to x$. In that case, there exists some $\varepsilon > 0$ and sequences $x_n \in \mathbb{R}^m$ and $z_n \in N(x)$ such that $|x - x_n| \to 0$ as $n \to \infty$ and

$$\text{dist}(F(x, z_n), F(x_n, z_n)) > \varepsilon \text{ for all } n \in \mathbb{N}.$$

Since $N(x)$ is compact, we can extract a subsequence (without changing notation) such that $z_n \to z \in N(x)$. But then,

$$\text{dist}(F(x, z_n), F(x_n, z_n))$$
$$\leq \text{dist}(F(x, z_n), F(x, z)) + \text{dist}(F(x, z), F(x_n, z)) + \text{dist}(F(x_n, z), F(x_n, z_n))$$
$$\leq \varepsilon$$

for sufficiently large n, because in the first two terms, one of the arguments is fixed, and the third term is small, because $(x, z) \mapsto F(x, z)$ is Lipschitz in z with a constant independent of x. Thus, we obtain a contradiction, and H is continuous.

We summarize the above discussion in the following theorem.

Theorem 6 *Let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ be continuous and ROSL with constant $l_1 \in \mathbb{R}$ in the first and Lipschitz continuous with constant $L_1 > 0$ in the second argument. Moreover, let $G : \mathbb{R}^k \times \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ be a set-valued mapping which is Lipschitz continuous with constant $L_2 > 0$ in the first and usc and ROSL with constant $l_2 < 0$ in the second argument.*

Then the differential algebraic inclusion (13) can be reformulated as a differential inclusion

$$\dot{x} \in H(x)$$

with a continuous right hand side $H : \mathbb{R}^k \to \mathcal{C}(\mathbb{R}^k)$ which is ROSL with constant $l_1 - \frac{L_1 L_2}{l_2}$.

It is well-known (see e.g. [5]) that such a differential inclusion has a solution on the unbounded time interval $[0, \infty)$. On every bounded interval $[0, T]$, the set of its solutions is dense in the set of solutions of the convexified problem

$$\dot{x} \in \overline{\text{co}} F(x)$$

with respect to the supremum norm, which is compact with respect to the same topology.

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