Discretizing Dynamical Systems with Hopf-Hopf Bifurcations

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Abstract

We consider parameter-dependent, continuous-time dynamical systems under discretizations. It is shown that Hopf-Hopf bifurcations are $O(h^p)$-shifted and turned into double Neimark-Sacker points by general one-step methods of order p. Then we discuss the effect of discretization methods on the emanating Hopf curves. The numerical approximation of the critical eigenvalues is analyzed too. The results are illustrated by a numerical example.

1 Introduction

Consider a continuous-time dynamical system depending on parameters, given by

$$\dot{x}(t) = f(x(t), \alpha), \tag{1.1}$$

where $f \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$ with open sets $0 \in \Omega \subset \mathbb{R}^N$, $0 \in \Lambda \subset \mathbb{R}^2$, $k \geq 1$ sufficiently large, $N \geq 4$. The first and commonly used tool for exploring the dynamical behavior of system (1.1) is numerical time-integration. For this purpose we can employ one-step methods, which consists in approximating the evolution operator by a discrete-time system

$$x \mapsto g(x, \alpha), \tag{1.2}$$

with $g \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$, where the step-size were assumed to be previously chosen. It then becomes clear the importance of establishing theoretical results that allow us to make conclusions about the real behavior of system (1.1) starting from the numerical

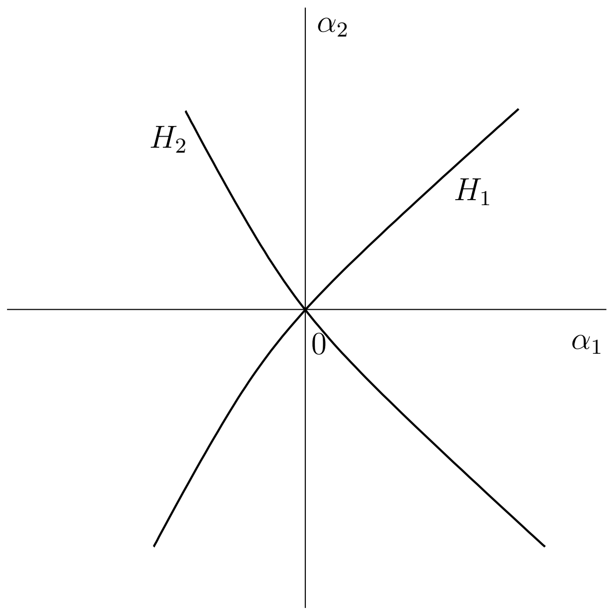

Fig. 1.1: Local bifurcation diagram near a double Hopf point.

observations obtained via (1.2). The situation turns out to be more involved if the system undergoes bifurcations under variation of parameters. In this sense, we can find in [18] rigorous results at the level of topological conjugacies of continuous-time systems and their discretizations. There, elementary codimension one bifurcations are analyzed.

In the present article we suppose that system (1.1) undergoes Hopf-Hopf bifurcations, which occur when the system presents an equilibrium with two pairs of Hopf eigenvalues, see [16]. Further we assume that (1.1) is approximated via general one-step methods, see Section 2.

Under these conditions, we will first investigate whether one-step methods applied to the continuous-time system reproduce by a "discrete version" the underlying codimension two point. When dealing with cusp and Bogdanov-Takens bifurcations, it is already known that they persist at the same position under general Runge-Kutta methods, see [19]. The fold-Hopf singularity and its discretized bifurcation picture are analyzed in [21]. Discretizations of generalized Hopf points are dealt in [22]. There, the numerical approximation of the first Lyapunov coefficient along the Hopf branches is discussed too. Results in this direction for the Hopf-Hopf bifurcation seem not to be available.

Then we analyze the effect of discretization methods on the local bifurcation picture of the Hopf-Hopf bifurcation, see Figure 1.1. In this picture, the curves labeled by $H_{1,2}$ represent paths of Hopf points which intersect at the codimension two point. The local bifurcation diagram near a Hopf-Hopf point is known to present other phenomena such as Neimark-Sacker bifurcation of cycles, and homoclinic and heteroclinic bifurcations. These objects are not considered in our discussion. Our efforts will be rather dedicated to understanding the effect of discretization methods on the emanating Hopf curves, whose intersection is, strictly speaking, what gives rise to the Hopf-Hopf bifurcation.

Discretization of systems with Hopf points has been addressed to a large extent (cf. [2, 5, 6, 14, 15, 20, 26]). It has been shown that Hopf points are $O(h^p)$-shifted and turned into Neimark-Sacker points by general one-step methods of order $p \geq 1$. The approximation of regular periodic orbits originated at the Hopf bifurcation has been addressed too. However, these results strictly apply when dealing with one-dimensional sections in Figure 1.1. The problem of discretizing the emanating Hopf curves must be then tackled in a codimension two setting. To achieve this we will appeal to the techniques employed in [21, 22]. There, suitably modified defining systems of the underlying bifurcations are manipulated in order to obtain existence and closeness results. Such defining systems for the computation of Hopf points are studied in [2, 3, 10, 12, 24, 27], and for Hopf-Hopf bifurcations in [1, 9, 29]. A broad overview of defining systems for various types of bifurcations can be found in [8].

In particular, the work of [1] constitutes the cornerstone of the present article. There, the genericity of a double Hopf point is associated to the regularity of a defining system at the point. Thereby, the approach employed in [21] for analyzing fold-Hopf bifurcations can be applied to our case too.

2 Basic Setup

Let us first formally define the bifurcations we will work with.

Definition 2.1. *A point* $(x_0, \alpha_0) \in \Omega \times \Lambda$ *is referred to as a Hopf-Hopf bifurcation (in short HH point*[1]*) of* (1.1) *if:*

- $f(x_0, \alpha_0) = 0$,
- $f_x(x_0, \alpha_0)$ *has the only critical eigenvalues*[2] $\{\pm i\omega_1, \pm i\omega_2\}$, $0 < \omega_{1,2} \in \mathbb{R}$, $\omega_1 \neq \omega_2$.

Definition 2.2. *A point* $(x_0, \alpha_0) \in \Omega \times \Lambda$ *is referred to as a double Neimark-Sacker bifurcation (in short DN point) of* (1.2) *if:*

- $g(x_0, \alpha_0) - x_0 = 0$,
- $g_x(x_0, \alpha_0)$ *has the only critical eigenvalues* $\left\{e^{\pm i\theta_1}, e^{\pm i\theta_2}\right\}$, $0 < \theta_{1,2} < \pi$, $\theta_1 \neq \theta_2$.

Note that since the eigenvalues $\pm i\omega_1, \pm i\omega_2$ of the matrix $f_x(x_0, \alpha_0)$ are simple, there exist smooth eigenvalue-continuations $\mu_{1,2}(\alpha) \pm i\sigma_{1,2}(\alpha)$, such that $\mu_{1,2}(\alpha_0) = 0$ and $\sigma_{1,2}(\alpha_0) = \omega_{1,2}$, for $||\alpha - \alpha_0||$ small, where $|| \cdot ||$ represents any norm[3] in $\mathbb{R}^2$. This allows us to introduce the notion of genericity (transversality). An HH point (x_0, α_0) of

[1]Also called double Hopf, Hopf/Hopf mode interaction, and multiple Hopf in case that more than two pairs of Hopf eigenvalues are present.

[2]In our study we do not consider $1:1$ resonances, however, other $n_1 : n_2$ resonances are covered by our approach.

[3]Throughout this article, the symbol $|| \cdot ||$ will be used to denote norms in different spaces. From the context, no confusion should arise.

(1.1) is said to be generic, if the map $\alpha \mapsto (\mu_1(\alpha), \mu_2(\alpha))$ is regular at $\alpha = \alpha_0$, that is

$$\det \begin{pmatrix} \frac{\partial \mu_1}{\partial \alpha_1} & \frac{\partial \mu_1}{\partial \alpha_2} \\ \frac{\partial \mu_2}{\partial \alpha_1} & \frac{\partial \mu_2}{\partial \alpha_2} \end{pmatrix} \Bigg|_{\alpha=\alpha_0} \neq 0. \tag{2.1}$$

The above condition guarantees (cf. [16]) that two curves of Hopf points intersect at the HH bifurcation (see Figure 1.1), and furthermore, that the HH point can be computed as a regular zero of a defining system which will be presented later.

For our analysis it is useful to introduce standard augmented systems for the continuation of Hopf points. Assume that $(x_0, \alpha_0) \in \Omega \times \Lambda$ is a generic Hopf point of (1.1) with critical eigenvalues $\pm i\omega_0$ and eigenvectors $\phi_0 := v_0 \pm iw_0 \in \mathbb{C}^N$. Consider the following system of $3n+2$ scalar equations for the variables $(x, \alpha, v, w, \omega) \in \Omega \times \Lambda \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ (cf. [10]):

$$\begin{cases} f(x,\alpha) = 0, \\ f_x(x,\alpha)v + \omega w = 0, \\ f_x(x,\alpha)w - \omega v = 0, \\ l^T w - 1 = 0, \\ l^T v = 0, \end{cases} \tag{2.2}$$

which is the real form of the complex system defining the Hopf bifurcation:

$$\begin{cases} f(x,\alpha) = 0, \\ f_x(x,\alpha)\phi - i\omega\phi = 0, \\ \langle l, \phi \rangle - i = 0, \end{cases} \tag{2.3}$$

where $\langle v_1, v_2 \rangle = \overline{v_1}^T v_2$, $v_{1,2} \in \mathbb{C}^N$, $\phi := v + iw$, and $l \in \mathbb{R}^N$ is a reference vector that is not orthogonal to the critical eigenspace spanned by $\{v_0, w_0\}$. In this setting we have that system (2.2) has full rank at the Hopf point, cf. [10, Theorem 3.1].

Similarly, a standard augmented system for the continuation of Neimark-Sacker points of (1.2) is given by (cf. [16, Section 10.3.1]):

$$\begin{cases} g(x,\alpha) - x = 0, \\ g_x(x,\alpha)v - \cos(\omega)v + \sin(\omega)w = 0, \\ g_x(x,\alpha)w - \cos(\omega)w - \sin(\omega)v = 0, \\ l^T w - 1 = 0, \\ l^T v = 0, \end{cases}$$

which is the real form of the complex system defining the Neimark-Sacker bifurcation bifurcation:

$$\begin{cases} g(x,\alpha) - x = 0, \\ g_x(x,\alpha)\phi - e^{i\omega}\phi = 0, \\ \langle l, \phi \rangle - i = 0, \end{cases} \tag{2.4}$$

where $l \in \mathbb{R}^N$ is again a suitably chosen reference vector.

Following the ideas of [21, 22], we now present a defining system for HH bifurcations, which will enable us to translate the genericity condition (2.1) into the regularity of the underlying system. This fact will be very helpful in our approach, as we will see in the forthcoming sections. Thus, an HH point of (1.1) is generic, if and only if the real form of the system:

$$\begin{cases} f(x,\alpha) = 0, \\ f_x(x,\alpha)\phi_1 - i\omega_1\phi_1 = 0, \\ \langle l_1, \phi_1\rangle - i = 0, \\ f_x(x,\alpha)\phi_2 - i\omega_2\phi_2 = 0, \\ \langle l_2, \phi_2\rangle - i = 0, \end{cases} \tag{2.5}$$

is regular at the HH point, see [1, Theorem 3.1]. Few remarks about the above presented system are in order. Note that it combines (2.3) twice in order to define the double Hopf condition. As already mentioned in the Introduction, the analysis presented in [1] constitutes the cornerstone of the present article, since there the genericity of an HH point is associated to the regularity of the real form of system (2.5). However, it is important to point out that in [1] the author considers dynamical systems with Z_2-symmetry. Nevertheless, by carefully following the proof of [1, Theorem 3.1], we can conclude that the result is valid for general systems too. Thereby, system (2.5) will allow us to adapt the approach employed in [21, 22] to the Hopf-Hopf case.

As mentioned before, our main goal is to describe the effect of discretization methods on the local bifurcation diagram of dynamical systems near an HH point. In this sense we consider general one-step methods of order $p \geq 1$ applied to (1.1), given by

$$x \mapsto \psi^h(x,\alpha) := x + h\Phi(h,x,\alpha), \tag{2.6}$$

with $\psi, \Phi : [-h^*, h^*] \times \widetilde{\Omega} \times \widetilde{\Lambda} \to \mathbb{R}^N$ sufficiently smooth, $h^* > 0$, where $0 \in \widetilde{\Omega} \subset \Omega$, $0 \in \widetilde{\Lambda} \subset \Lambda$ are compact sets. That the method is of order p means that there exists a positive constant C_0 (depending only on f), such that it holds

$$||\varphi^h(x,\alpha) - \psi^h(x,\alpha)|| \leq C_0|h|^{p+1},$$

for all $(h,x,\alpha) \in [-h^*, h^*] \times \widetilde{\Omega} \times \widetilde{\Lambda}$, where $\varphi^t(\cdot,\alpha)$ stands for the t-flow of (1.1). In this setting, there exist smooth functions $\Upsilon, \Xi : [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda} \to \mathbb{R}^N$ such that:

$$\begin{aligned} \psi^h(x,\alpha) &= \varphi^h(x,\alpha) + \Upsilon(h,x,\alpha)h^{p+1}, \\ \Phi(h,x,\alpha) &= f(x,\alpha) + \Xi(h,x,\alpha)h, \end{aligned} \tag{2.7}$$

hold for all $(h,x,\alpha) \in [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda}$, where $0 < h_0 < h^*$, see [4, 7, 25].

With the technical framework above introduced, we have all the necessary machinery at hand for showing the main results of the present work.

3 Hopf-Hopf Bifurcations under Discretization

In this section we will suppose we are given a continuous-time dynamical system (1.1) which undergoes an HH bifurcation. We assume that the system is discretized via general one-step methods of order p. Under these conditions, we will show that the HH point is

$O(h^p)$-shifted and turned into a DN point by the one-step map, for all sufficiently small step-size. For this purpose, the arguments employed in [21, 22] will be adapted to our case. For this reason we only present a sketch of the proof of the main result. Formally speaking, we have the following:

Theorem 3.1. *Let a general one-step method of order $p \geq 1$ applied to* (1.1) *be given by* (2.6)*. Assume that system* (1.1) *has a generic HH point at $(x_{HH}, \alpha_{HH}) \in \widetilde{\Omega} \times \widetilde{\Lambda}$, with critical eigenvalues $\pm i\omega_{1,2}^{HH}$ and eigenvectors $\phi_{1,2}^{HH} := v_{1,2}^{HH} \pm iw_{1,2}^{HH}$. Then, there exists a positive constant $\rho \leq h_0$ and a neighborhood $\Omega' \times \Lambda' \subset \widetilde{\Omega} \times \widetilde{\Lambda}$ of (x_{HH}, α_{HH}), in which* (2.6) *has a unique DN point $(x_{DN}(h), \alpha_{DN}(h))$, with critical eigenvalues $e^{\pm ih\omega_{1,2}^{DN}(h)}$ and eigenvectors $\phi_{1,2}^{DN}(h) := v_{1,2}^{DN}(h) \pm iw_{1,2}^{DN}(h)$, which all depend smoothly on h, for all $h \in (-\rho, \rho)$. Furthermore, the following estimate holds*

$$||(x_{DN}(h), \alpha_{DN}(h), z_{DN}(h)) - (x_{HH}, \alpha_{HH}, z_{HH})|| \leq C|h|^p, \tag{3.1}$$

for some $C > 0$ and all $h \in (-\rho, \rho)$, where $z_{HH} := (v_{1,2}^{HH}, w_{1,2}^{HH}, \omega_{1,2}^{HH})$ and $z_{DN}(h) := (v_{1,2}^{DN}(h), w_{1,2}^{DN}(h), \omega_{1,2}^{DN}(h))$.

Sketch of the proof. Denote the real form of (2.5) by

$$\widetilde{F}(x, \alpha, z) = 0, \tag{3.2}$$

with $\widetilde{F} : \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{4N+2} \to \mathbb{R}^{5N+4}$, $z := (v_1, v_2, w_1, w_2, \omega_1, \omega_2)$. Then, system (3.2) is regular at $(x_{HH}, \alpha_{HH}, z_{HH})$ (cf. Section 2). We will now rewrite system (3.2) in terms of the h-flow $\varphi^h(\cdot, \alpha)$ of (1.1). By following the analysis of [21, Theorem 3.1] and taking into account system (2.4), it can be seen that $(x_{HH}, \alpha_{HH}, z_{HH})$ is also a solution of:

$$\begin{cases} \frac{1}{h}\left(\varphi^h(x, \alpha) - x\right) = 0, \\ \frac{1}{h}\left(\varphi^h_x(x, \alpha)\phi_1 - e^{ih\omega_1}\phi_1\right) = 0, \\ \langle l_1, \phi_1 \rangle - i = 0, \\ \frac{1}{h}\left(\varphi^h_x(x, \alpha)\phi_2 - e^{ih\omega_2}\phi_2\right) = 0, \\ \langle l_2, \phi_2 \rangle - i = 0, \end{cases} \tag{3.3}$$

for all $h \in [-h_0, h_0]$. Denote then the real form of the above system by

$$F(h, x, \alpha, z) = 0,$$

with $F : [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{4N+2} \to \mathbb{R}^{5N+4}$. Now let us construct a similar system that describes a DN point of the one-step map (2.6). By a similar analysis, we obtain the following complex system:

$$\begin{cases} \frac{1}{h}\left(\psi^h(x, \alpha) - x\right) = 0, \\ \frac{1}{h}\left(\psi^h_x(x, \alpha)\phi_1 - e^{ih\omega_1}\phi_1\right) = 0, \\ \langle l_1, \phi_1 \rangle - i = 0, \\ \frac{1}{h}\left(\psi^h_x(x, \alpha)\phi_2 - e^{ih\omega_2}\phi_2\right) = 0, \\ \langle l_2, \phi_2 \rangle - i = 0, \end{cases} \tag{3.4}$$

whose solutions correspond to DN points of (2.6). Write the real form of the above system as

$$G(h,x,\alpha,z)=0,$$

with $G:[-h_0,h_0]\times\widetilde{\Omega}\times\widetilde{\Lambda}\times\mathbb{R}^{4N+2}\to\mathbb{R}^{5N+4}$. The next step is to establish relations between $\widetilde{F}$, F, and G, which will be crucial for our discussion. Let us begin with G and $\widetilde{F}$. For this purpose, it suffices to work with their complex forms. To begin with, consider the first equation of (2.5) and (3.4). By (2.7), we have that

$$\frac{1}{h}\left(\psi^h(x,\alpha)-x\right)=\Phi(h,x,\alpha)=f(x,\alpha)+\Xi(h,x,\alpha)h. \tag{3.5}$$

Similarly, for the second equation it holds that:

$$\begin{aligned}\frac{1}{h}\left(\psi^h_x(x,\alpha)\phi_1-e^{ih\omega_1}\phi_1\right)&=\frac{1}{h}\left((I_N+h\Phi_x(h,x,\alpha))\phi_1-(1+ih\omega_1+O(h^2))\phi_1\right),\\&=(f_x(x,\alpha)+\Xi_x(h,x,\alpha)h)\phi_1-(i\omega_1+O(h))\phi_1,\\&=f_x(x,\alpha)\phi_1-i\omega_1\phi_1+\Theta_1(h,x,\alpha,\phi_1,\omega_1)h,\end{aligned} \tag{3.6}$$

where Θ_1 is some smooth function[4]. The analysis for the fourth equation follows analogously. By combining the above relations, we conclude that

$$G(h,x,\alpha,z)=\widetilde{F}(x,\alpha,z)+\Theta(h,x,\alpha,z)h, \tag{3.7}$$

where Θ is some smooth function.

Now let us find closeness relations between F and G. Similarly as before, for the first equation of (3.3) and (3.4) it follows that:

$$\frac{1}{h}\left(\psi^h(x,\alpha)-x\right)=\frac{1}{h}\left(\varphi^h(x,\alpha)-x+\Upsilon(h,x,\alpha)h^{p+1}\right)=\frac{1}{h}\left(\varphi^h(x,\alpha)-x\right)+\Upsilon(h,x,\alpha)h^p, \tag{3.8}$$

and for the second (and also fourth) equation:

$$\begin{aligned}\frac{1}{h}\left(\psi^h_x(x,\alpha)\phi_1-e^{ih\omega_1}\phi_1\right)&=\frac{1}{h}\left((\varphi^h_x(x,\alpha)+\Upsilon_x(h,x,\alpha)h^{p+1})\phi_1-e^{ih\omega_1}\phi_1\right),\\&=\frac{1}{h}\left(\varphi^h_x(x,\alpha)\phi_1-e^{ih\omega_1}\phi_1\right)+\Upsilon_x(h,x,\alpha)\phi_1h^p.\end{aligned} \tag{3.9}$$

Thereby, it holds that

$$G(h,x,\alpha,z)=F(h,x,\alpha,z)+\Psi(h,x,\alpha,v_1,w_1,v_2,w_2)h^p, \tag{3.10}$$

where Ψ is some smooth function. With the relations (3.7) and (3.10) combined with the techniques employed in [21, Theorem 3.1], the assertion of the Theorem follows. □

[4]In what follows, by the term "(some smooth function) $\cdot\, s^k$", $k\geq 1$, s some real variable, we mean the integral remainder of a Taylor series.

Before switching to the next topic of this article, it is worth analyzing in detail some consequences of the above discussion. First, note that by means of the variable z which appears in the Estimate (3.1), we obtain not only closeness relations between HH points and their discretizations but also between the critical eigenvalues and eigenvectors and their discretized counterpart. In particular, we see that the critical eigenvectors are discretized with the order of the method[5]. Nevertheless, we will next see that the critical eigenvalues are approximated with an improved order, in the following sense.

Under assumptions and notation of the previous theorem, denote by

$$\tau_{1,2}^{DN}(h) := e^{ih\omega_{1,2}^{DN}(h)},$$

the critical eigenvalues of the one-step map (2.6) at the DN point $(x_{DN}(h), \alpha_{DN}(h))$, for $h \in (-\rho, \rho)$ (cf. Theorem 3.1). Then, by (3.1), it follows that:

$$\begin{aligned} \left|\tau_{1,2}^{DN}(h) - e^{ih\omega_{1,2}^{HH}}\right| &= \left|e^{ih\omega_{1,2}^{DN}(h)} - e^{ih\omega_{1,2}^{HH}}\right|, \\ &\leq L_{1,2}\left|\omega_{1,2}^{DN}(h) - \omega_{1,2}^{HH}\right| |h|, \\ &\leq C \cdot L|h|^{p+1}, \end{aligned} \tag{3.11}$$

where $L_{1,2}$ are local Lipschitz constants of the exponential function and $L := \max(L_1, L_2)$. Therefore, we conclude that

$$\tau_{1,2}^{DN}(h) = e^{ih\omega_{1,2}^{HH}} + O(h^{p+1}), \tag{3.12}$$

for h sufficiently small. This relation improves, for this particular case, the general approximation given in [26, Lemma 3.2].

4 Emanating Hopf Curves under Discretization

In the past section we showed that HH points persist under one-step methods. As we mentioned in the Introduction, we are interested to know whether the local bifurcation diagram near this bifurcation is reproduced "correctly" by general one-step methods. In this sense, the first part of this task has been accomplished, namely, the organizing center was shown to be preserved by discretization methods.

Now we tackle the problem of analyzing the discretization of the Hopf curves of system (1.1) that emanate from the HH bifurcation. The result we are after is illustrated in Figure 4.1. The curves labeled by $H_{1,2}$ and $NS_{1,2}$ represent paths of Hopf points of (1.1) and Neimark-Sacker points of (2.6), respectively.

The analysis is formulated as follows. Suppose we are given a continuous-time dynamical system (1.1) which undergoes an HH bifurcation. We assume that this system is discretized via general one-step methods, as in the previous Section. Under these conditions we will show that there exists a step-size-independent neighborhood (the dashed square in Figure 4.1) of the HH point, such that the discretized paths of Hopf points

[5] Of course, this assertion makes sense only if we consider a suitable normalization condition.

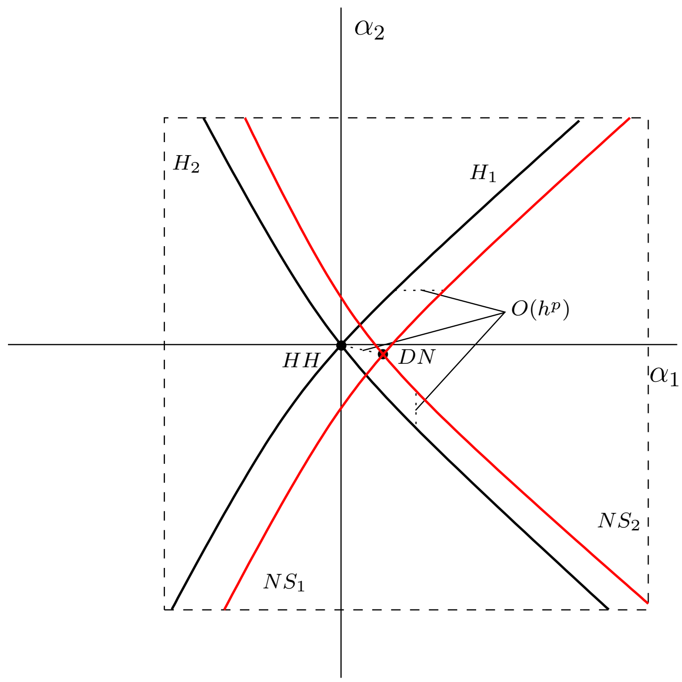

Fig. 4.1: Discretized curves of Hopf points near an HH bifurcation.

($NS_{1,2}$ in Figure 4.1) approximate the original curves ($H_{1,2}$ in Figure 4.1) with the order of the method. For this purpose, we do not reduce the systems, e.g. via center manifold theory, but we rather work with them in full dimension. To do so, the arguments employed in [21, 22] will be applied. For this reason we will only present a sketch of the proof of the main result. Throughout this section we denote by $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ the parameters of the system. With these remarks we are ready to formulate:

Theorem 4.1. *Let a general one-step discretization method of order $p \geq 1$ applied to* (1.1) *be given by* (2.6)*. Assume that system* (1.1) *has a generic HH point at the origin $(x_0, \alpha_0) := (0, 0)$, with eigenvectors $\phi^0_{1,2} := v^0_{1,2} \pm i w^0_{1,2}$ corresponding to the critical eigenvalues $\pm i\omega^0_{1,2}$. Further suppose that*[6]

$$\left.\frac{\partial \mu_1}{\partial \alpha_1} \cdot \frac{\partial \mu_2}{\partial \alpha_2}\right|_{(\alpha_1,\alpha_2)=(0,0)} \neq 0. \tag{4.1}$$

Then, there exist positive constants $\rho \leq h_0$, δ and curves of Hopf and Neimark-Sacker

[6]This assumption is imposed only for assuring that $\alpha_{2,1}$ can be used as parametrization variables for the Hopf curves $H_{1,2}$, respectively. This is allowed due to the genericity condition (2.1).

points of systems (1.1) *and* (2.6)*, respectively, defined by:*

$$\begin{aligned} C_{H_1}(\alpha_2) &:= (x_{H_1}(\alpha_2), \alpha_{1H_1}(\alpha_2), z_{H_1}(\alpha_2), \alpha_2), \\ C_{H_2}(\alpha_1) &:= (x_{H_2}(\alpha_1), \alpha_{2H_2}(\alpha_1), z_{H_2}(\alpha_1), \alpha_1), \\ C_{NS_1}(h,\alpha_2) &:= (x_{NS_1}(h,\alpha_2), \alpha_{1NS_1}(h,\alpha_2), z_{NS_1}(h,\alpha_2), \alpha_2), \\ C_{NS_2}(h,\alpha_1) &:= (x_{NS_2}(h,\alpha_1), \alpha_{2NS_2}(h,\alpha_1), z_{NS_2}(h,\alpha_1), \alpha_1), \end{aligned}$$

with $x_{H_{1,2}} : (-\delta,\delta) \to \mathbb{R}^N$*,* $x_{NS_{1,2}} : (-\rho,\rho) \times (-\delta,\delta) \to \mathbb{R}^N$*,* $\alpha_{1H_1,2H_2} : (-\delta,\delta) \to \mathbb{R}$*,* $\alpha_{1NS_1,2NS_2} : (-\rho,\rho) \times (-\delta,\delta) \to \mathbb{R}$*,* $z_{H_{1,2}} : (-\delta,\delta) \to \mathbb{R}^{2N+1}$*,* $z_{NS_{1,2}} : (-\rho,\rho) \times (-\delta,\delta) \to \mathbb{R}^{2N+1}$ *smooth*[7]*. Furthermore, the following estimates hold for all* $(h,\alpha_{2,1}) \in (-\rho,\rho) \times (-\delta,\delta)$ *and uniformly in* α_2, α_1*, respectively:*

$$\begin{aligned} ||d_{NS_1}(h,\alpha_2) - d_{H_1}(\alpha_2)|| &\le C|h|^p, \\ ||d_{NS_2}(h,\alpha_1) - d_{H_2}(\alpha_1)|| &\le C|h|^p, \end{aligned} \tag{4.2}$$

where $d_{H_{1,2}} := (x_{H_{1,2}}, \alpha_{1H_1,2H_2}, z_{H_{1,2}})$ *and* $d_{NS_{1,2}} := (x_{NS_{1,2}}, \alpha_{1NS_1,2NS_2}, z_{NS_{1,2}})$*,* $C > 0$*.*

Sketch of the proof. We will show the result for the curves C_{H_1}, C_{NS_1}. The analysis for C_{H_2}, C_{NS_2} follows analogously. A curve of Hopf points of system (1.1) can be computed as a solution of (cf. (2.3), (2.5)):

$$\left\{ \begin{aligned} f(x,\alpha) &= 0, \\ f_x(x,\alpha)\phi_1 - i\omega_1\phi_1 &= 0, \\ \langle l_1, \phi_1 \rangle - i &= 0. \end{aligned} \right. \tag{4.3}$$

Write the real form of the above system as

$$\widetilde{F}(x,\alpha,z_1) = 0,$$

with $\widetilde{F} : \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{2N+1} \to \mathbb{R}^{3N+2}$, $z_1 := (v_1, w_1, \omega_1)$. By condition (4.1), it holds that $\frac{\partial \mu_1}{\partial \alpha_1} \neq 0$, which implies that $\widetilde{F}_{(x,\alpha_1,z_1)}(0,0,z_1^0)$, $z_1^0 := (v_1^0, w_1^0, \omega_1^0)$, is nonsingular, see [10, Theorem 3.1]. Thereby, the Implicit Function Theorem guarantees the existence (and uniqueness) of a function $d_{H_1} = (x_{H_1}, \alpha_{1H_1}, z_{H_1}) : (-\delta_1, \delta_1) \to \mathbb{R}^{3N+2}$, such that

$$\widetilde{F}(d_{H_1}(\alpha_2), \alpha_2) = 0, \quad d_{H_1}(0) = (0, 0, z_1^0),$$

$\alpha_2 \in (-\delta_1, \delta_1)$. Following the ideas of [21], we will express (4.3) in terms of the h-flow $\varphi^h(\cdot,\alpha)$ of (1.1). Thus, we arrive at the following system (cf. (2.4) and (3.3)):

$$\left\{ \begin{aligned} \tfrac{1}{h}\left(\varphi^h(x,\alpha) - x\right) &= 0, \\ \tfrac{1}{h}\left(\varphi^h_x(x,\alpha)\phi_1 - e^{ih\omega_1}\phi_1\right) &= 0, \\ \langle l_1, \phi_1 \rangle - i &= 0. \end{aligned} \right. \tag{4.4}$$

Denote the real form of the above system by

$$F(h,x,\alpha,z_1) = 0,$$

[7] By the term "Neimark-Sacker curve", we mean the graph of the function $C_{NS_i}(h_*, \cdot)$, for $h_* \in (-\rho,\rho)$ fixed, $i = 1$ or 2.

with $F:[-h_0,h_0]\times\widetilde{\Omega}\times\widetilde{\Lambda}\times\mathbb{R}^{2N+1}\to\mathbb{R}^{3N+2}$. It then holds (see [21, Theorem 4.1])

$$F(h,d_{H_1}(\alpha_2),\alpha_2)=0,$$

for all $(h,\alpha_2)\in(-h_0,h_0)\times(-\delta_1,\delta_1)$. Analogously, a system whose zeroes describe a curve of Neimark-Sacker points of (2.6) is given by (cf. (3.4)):

$$\begin{cases} \frac{1}{h}\left(\psi^h(x,\alpha)-x\right)=0,\\ \frac{1}{h}\left(\psi^h_x(x,\alpha)\phi_1-e^{ih\omega_1}\phi_1\right)=0,\\ \langle l_1,\phi_1\rangle-i=0,\end{cases}\tag{4.5}$$

whose real form will be denoted by $G(h,x,\alpha,z_1)=0$. Hence we will show the existence of a curve of Neimark-Sacker points of (2.6). By (3.5) and (3.6), we can conclude that

$$G(0,x,\alpha,z_1)=\widetilde{F}(x,\alpha,z_1),$$

for all $(x,\alpha,z_1)\in\widetilde{\Omega}\times\widetilde{\Lambda}\times\mathbb{R}^{2N+1}$. Therefore, we have that $G(0,0,0,z_1^0)=0$ and $G_{(x,\alpha_1,z_1)}(0,0,0,z_1^0)$ is nonsingular. Thus, the Implicit Function Theorem guarantees the existence (and uniqueness) of a function $d_{NS_1}=(x_{NS_1},\alpha_{1NS_1},z_{NS_1}):(-\rho_1,\rho_1)\times(-\delta_2,\delta_2)\to\mathbb{R}^{3N+2}$, $0<\rho_1\le h_0$, such that

$$G(h,d_{NS_1}(h,\alpha_2),\alpha_2)=0,\quad d_{NS_1}(0,0)=(0,0,z_1^0),$$

$(h,\alpha_2)\in(-\rho_1,\rho_1)\times(-\delta_2,\delta_2)$.

It remains to establish closeness relations between the real form of the defining systems (4.4) and (4.5). For this purpose, it suffices to analyze their complex form. By (3.8) and (3.9), it is straightforward to see that:

$$G(h,x,\alpha,z_1)=F(h,x,\alpha,z_1)+\Gamma(h,x,\alpha,v_1,w_1)h^p,$$

holds locally, where Γ is some smooth function. By application of the Local Inverse Lipschitz Mapping Theorem, as in [21, Theorem 4.1], the assertion of the present Theorem follows. □

Note that the theorem above presented provides stronger results than those of [21, Theorem 4.1]. This was possible due to the defining system for Hopf points used here. In the underlying system we considered the additional variable $z_1=(v_1,w_1,\omega_1)$, which allowed us to obtain, additionally, $O(h^p)$-estimates between the critical eigenvectors. As in the analysis of the discretization of HH points (see Section 3), it is also possible to establish improved closeness relations between the critical eigenvalues of the original system and their discretizations, along the Hopf curves. Thus, under assumptions and notation of the previous theorem, denote by

$$\tau_1^{NS}(h,\alpha_2):=e^{ih\omega_1^{NS}(h,\alpha_2)},\quad \tau_2^{NS}(h,\alpha_1):=e^{ih\omega_2^{NS}(h,\alpha_1)},$$

the critical eigenvalues of the one-step map (2.6) along the Neimark-Sacker curves $C_{NS_1}(h,\alpha_2)$, $C_{NS_2}(h,\alpha_1)$, respectively, for $(h,\alpha_{2,1})\in(-\rho,\rho)\times(-\delta,\delta)$ (cf. Theorem 4.1). Further denote by $i\omega_1^H(\alpha_2)$, $i\omega_2^H(\alpha_1)$ the critical eigenvalues of (1.1) along the Hopf curves $C_{H_1}(\alpha_2)$, $C_{H_2}(\alpha_1)$, respectively, $\alpha_{2,1} \in (-\delta, \delta)$. Then, By doing estimates similar to those of (3.11), it can be shown that

$$\tau_1^{NS}(h, \alpha_2) = e^{ih\omega_1^H(\alpha_2)} + O(h^{p+1}), \quad \tau_2^{NS}(h, \alpha_1) = e^{ih\omega_2^H(\alpha_1)} + O(h^{p+1}),$$

for h sufficiently small and uniformly in α_2, α_1, respectively.

Before finishing this section, few remarks concerning the sofar obtained results are in order. According to [3, Lemma 7.3], a generic HH bifurcation of (1.1) is a simple branching point (in short SB point) of the system:

$$Q(x, \alpha) := \begin{pmatrix} f(x, \alpha) \\ \det(2f_x(x, \alpha) \odot I_N) \end{pmatrix} = 0, \tag{4.6}$$

where the symbol $\odot$ stands for the bialternate product of matrices (cf. [8, 12, 16]). In general, any C^1-perturbation of a system with an SB point destroys the connection of the intersecting curves (see e.g. Figure 1.1). For this reason we say that an SB point is a structurally unstable object. However, in the present work we proved that an HH point persists as an $O(h^p)$-shifted DN point, under one-step methods of order p. In fact, a DN bifurcation of (2.6) is an SB point of the system:

$$P(h, x, \alpha) := \begin{pmatrix} \frac{1}{h}(\psi^h(x, \alpha) - x) \\ \det\left(\frac{1}{h}(\psi_x^h(x, \alpha) \odot \psi_x^h(x, \alpha) - I_m)\right) \end{pmatrix} = 0,$$

where $m := \frac{1}{2}N(N-1)$, and P, Q are locally related as follows (cf. [21, Theorem 4.1])

$$P(h, x, \alpha) = Q(x, \alpha) + O(h).$$

This means that an HH point, regarded as an SB point of (4.6), is stable under smooth $O(h)$-perturbations coming from a general discretization method.

5 A Numerical Example

The Hopf-Hopf bifurcation appears in many parameter-dependent systems which describe the dynamics of various phenomena, e.g. neuron modeling (cf. [11]), nonlinear electric circuits (cf. [23]), and mechanical systems (cf. [28]), among others. In the present article we consider the following five-dimensional, continuous-time system:

$$\begin{aligned} \dot{x} &= A - Bx + x^2y - x, \\ \dot{y} &= Bx - x^2y, \\ \dot{z} &= x - zv, \\ \dot{v} &= Bx - zv + v^2w - v, \\ \dot{w} &= zv - v^2w, \end{aligned} \tag{5.1}$$

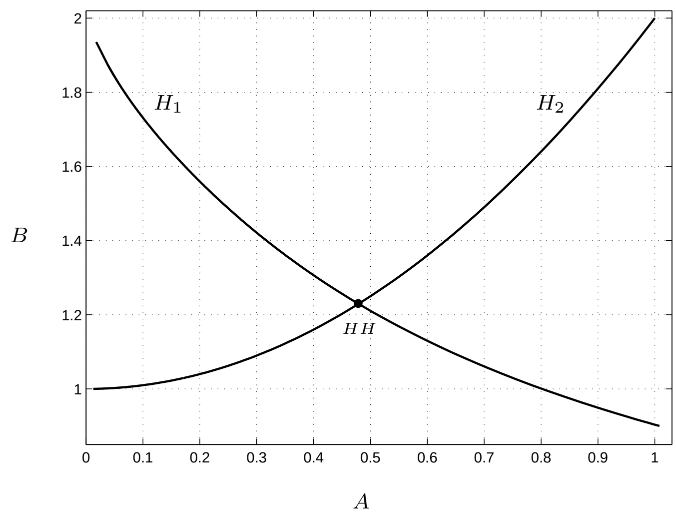

Fig. 5.1: Hopf curves near an HH point of system (5.1).

with state variables $(x, y, z, v, w) \in \mathbb{R}^5$ and with parameters $A, B \in \mathbb{R}$. This system describes the behavior of a chemical oscillator arising from the series coupling of two Brusselators, see [30].

Throughout this section, the numerical results will be computed by means of the continuation software CONTENT, cf. [17]. Further numerical manipulations will be performed with MATLAB.

The first aim of this example is to illustrate the theoretical result obtained in Theorem 3.1, namely, we want to observe whether HH points are $O(h^p)$-shifted and turned into DN points by one-step methods of order p. To achieve this, we need first to find an HH point of system (5.1). Thus, choose $x_{ini} = y_{ini} = z_{ini} = v_{ini} = w_{ini} = 0.1$, $(A_{ini}, B_{ini}) = (0.47, 1.22)$ as initial data for the continuation of equilibria, and let A freely vary. With this procedure we found two Hopf points which lie close to each other. The location of one of them is $(x, y, z, v, w) \approx (0.46, 2.6, 0.81, 0.57, 1.43)$, $(A, B) \approx (0.46, 1.22)$. Next, we switch to this point, and by continuing the Hopf curve with respect to A and B, we find an HH point at $(x_{HH}, y_{HH}, z_{HH}, v_{HH}, w_{HH}) = (0.479012, 2.566642, 0.81337, 0.588922, 1.381115)$, $(A_{HH}, B_{HH}) = (0.479012, 1.229452)$. The local bifurcation diagram near this HH point is depicted in Figure 5.1. In this picture, the curves labeled by $H_{1,2}$ correspond to paths of Hopf points which emanate from the HH bifurcation.

The next step is to discretize system (5.1) by a one-step method. For this purpose, we choose the 3-th order method of Runge (cf. [13]) with an initial step-size $h_0 = 0.13$. Define the distance function

$$Dist_{HH}(h) := ||(x_{DN}(h), y_{DN}(h), z_{DN}(h), v_{DN}(h), w_{DN}(h), A_{DN}(h), B_{DN}(h)) - (x_{HH}, y_{HH}, z_{HH}, v_{HH}, w_{HH}, A_{HH}, B_{HH})||,$$

for $h > 0$ small, where $|| \cdot ||$ represents the Euclidean norm and $(x_{DN}(h), y_{DN}(h), z_{DN}(h), v_{DN}(h), w_{DN}(h), A_{DN}(h), B_{DN}(h))$ stands for a DN point of the Runge map. We will then

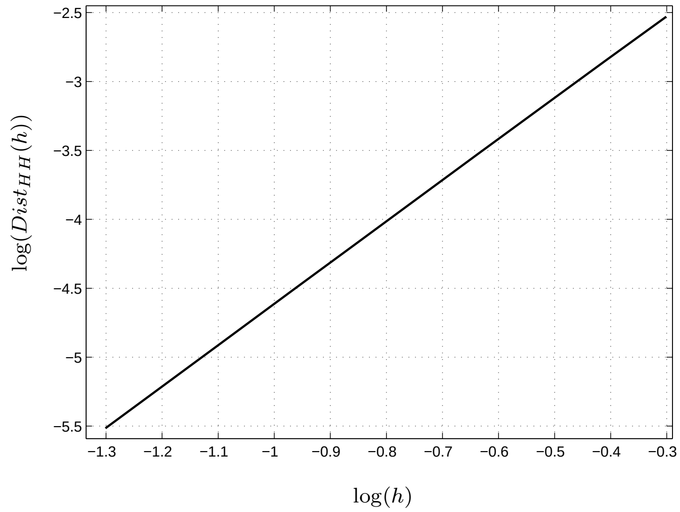

Fig. 5.2: Distance between HH and DN points for different values of step-size.

investigate how the distance between HH and DN points behaves with the step-size. The result is shown in Figure 5.2. In this picture, we let h vary from 0.05 to 0.5. For several, fixed values of h in this interval, we computed a DN point of the Runge map, and thus we obtained the curve shown in the figure. We plotted the logarithm of the variables, so that we can determine the order of approximation as the slope of the quasi-straight line obtained. This slope is approximately equal to $2.99 \approx 3$, which is of course consistent with Theorem 3.1.

For $h = 0.13$, the bifurcation diagram of the Runge map is shown in Figure 5.3. In this picture, the curves labeled by $NS_{1,2}$ represent paths of Neimark-Sacker points which emanate from the discretized HH bifurcation. As we can see, the branching point HH persists under the perturbation originated by Runge's method, cf. final remarks of Section 4. Furthermore, it can be seen that the Neimark-Sacker curves $NS_{1,2}$ lie very close to the Hopf curves $H_{1,2}$, respectively. This fact is explained by Theorem 4.1.

To finish this experiment, we will investigate how the critical eigenvalues of system (1.1) at the HH bifurcation are approximated by Runge's method, as the step-size varies. Under notation of Section 3, define the distance function

$$Dist_{eig}(h) := \left|\tau_1^{DN}(h) - e^{ih\omega_1^{HH}}\right| + \left|\tau_2^{DN}(h) - e^{ih\omega_2^{HH}}\right|,$$

for $h > 0$ small, where $|\cdot|$ represents the absolute value of complex numbers. At the HH point found before, we have that $\omega_1^{HH} = 0.479012075168911$, $\omega_2^{HH} = 0.426596163417436$. The behavior of $Dist_{eig}$ with respect to h is plotted in Figure 5.4. As in Figure 5.2, we let h vary from 0.05 to 0.5, and for some fixed values of h in this interval, we computed the critical eigenvalues of the Runge map at DN points close to the HH bifurcation of system (1.1). In this way we obtained the curve depicted in Figure 5.4. Hence we find that the order of approximation of the critical eigenvalues for this example is $3.94 \approx 4$, which is consistent with the relation (3.12).

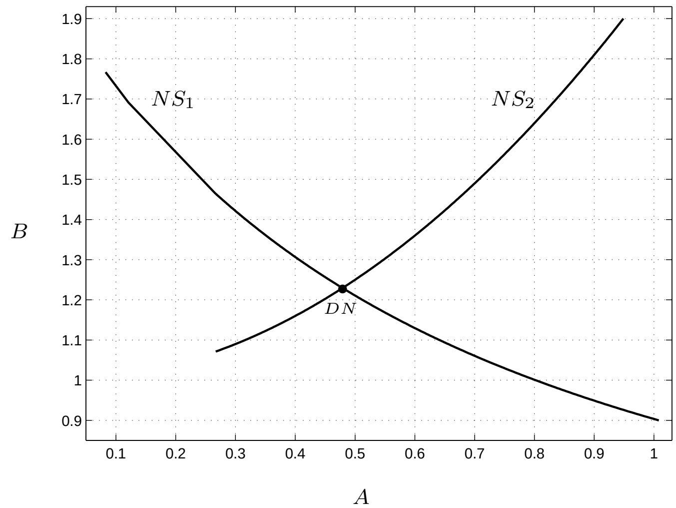

Fig. 5.3: Neimark-Sacker branches near a DN point of the one-step method.

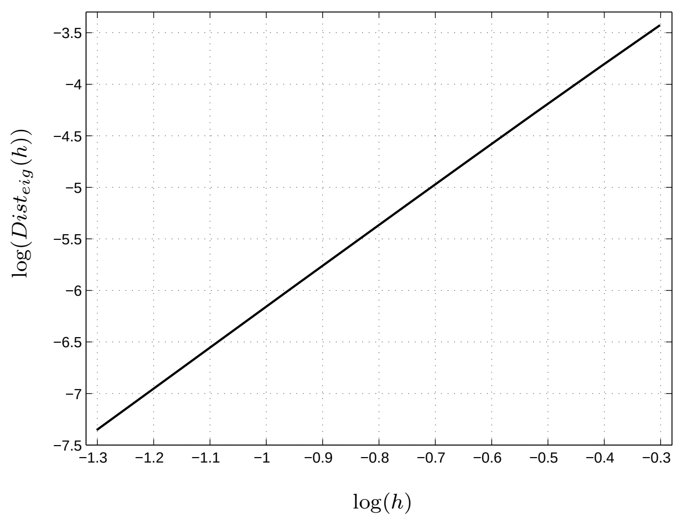

Fig. 5.4: Distance between HH eigenvalues and their discretization for different values of step-size.

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