Discretizing Dynamical Systems with Generalized Hopf Bifurcations

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Abstract

We consider parameter-dependent, continuous-time dynamical systems under discretizations. It is shown that generalized Hopf bifurcations are shifted and turned into generalized Neimark-Sacker points by general one-step methods. We analyze the effect of discretizations methods on the emanating Hopf curve. In particular, we obtain estimates of the discretized eigenvalues along this curve. A detailed analysis of the discretized first Lyapunov coefficient is also given. The results are illustrated by a numerical example.

1 Introduction

Consider a continuous-time dynamical system depending on parameters

$$\dot{x}(t) = f(x(t), \alpha), \tag{1.1}$$

where $f \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$ with open sets $0 \in \Omega \subset \mathbb{R}^N$, $0 \in \Lambda \subset \mathbb{R}^2$, $k \geq 1$ sufficiently large, $N \geq 2$. The first and commonly used tool for understanding the dynamics generated by the vector field (1.1) is numerical time-integration. For this purpose we can utilize one-step methods, which consists in approximating the evolution operator by a discrete-time system

$$x \mapsto g(x, \alpha), \tag{1.2}$$

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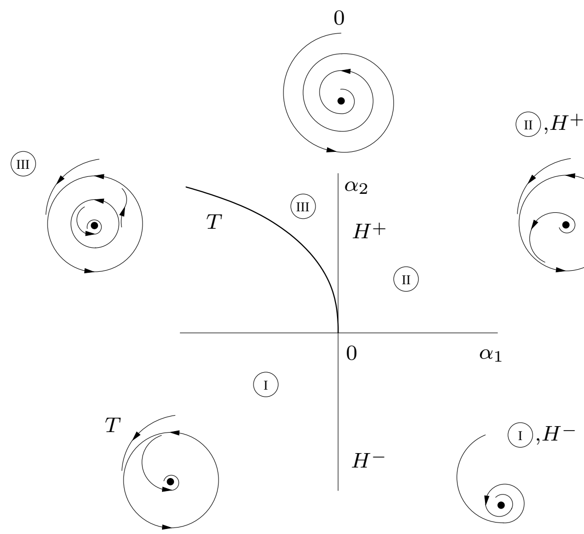

Fig. 1.1: Bifurcation diagram and generic phase portraits near a generalized Hopf point.

with $g \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$, where the step-size were assumed to be previously fixed. It then becomes evident the importance of establishing theoretical results that allow us to make conclusions about the real behavior of system (1.1) starting from the numerical observations obtained via (1.2). The situation turns out to be more involved if the system undergoes bifurcations under variation of parameters. Rigorous results concerning topological conjugacies of continuous-time systems and their discretizations can be found in [21]. There, elementary codimension one bifurcations are considered.

In this article we suppose that system (1.1) presents generalized Hopf bifurcations, which consist of Hopf points with vanishing first Lyapunov coefficient (see [18]). Further we assume that (1.1) is discretized via general one-step methods, see Section 2.

In this setting, we will first discuss whether a one-step method applied to the continuous-time system reproduce by a "discrete version" the codimension two point. For cusp and Bogdanov-Takens bifurcations, it is already known that they persist at the same position under general Runge-Kutta methods, see [22]. The fold-Hopf case, and its discretized bifurcation picture, is dealt in [24]. Results in this direction for the remaining codimension two bifurcations seem not to be available.

Then we analyze the effect of discretization methods on the local bifurcation picture of the generalized Hopf bifurcation, see Figure 1.1. In this picture, the curves labeled by H^+ (resp. H^-) and T correspond to paths of Hopf points with positive (resp. negative) first Lyapunov coefficient and nondegenerate fold bifurcation of cycles, respectively. Typical phase portraits are also depicted. In this article our efforts will be focused to studying the discretization of the emanating Hopf curve. Discretization of systems with Hopf bifurcations has been addressed to a large extent (cf. [3, 6, 7, 14, 15, 23, 27]). It has been proven that Hopf points are $O(h^p)$-shifted and turned into Neimark-Sacker points by general one-step methods of order $p \geq 1$. Approximation of regular periodic orbits originated at the Hopf bifurcation has been considered too. Nevertheless, these results strictly apply when dealing with one-dimensional sections in Figure 1.1. The problem of discretizing the emanating Hopf curve must be tackled in a codimension two context. To accomplish this we will appeal to the techniques employed in [24]. There, suitably modified defining systems of the underlying bifurcations are manipulated in order to obtain existence and closeness results. Defining systems for the computation of Hopf points are presented in [3, 4, 12, 13, 25, 28], and for generalized Hopf bifurcations in [11, 29]. A broad overview of defining systems for various types of bifurcations can be found in [10].

In the study of generalized Hopf bifurcations and their discretizations, the first Lyapunov coefficient plays an important role. For this reason, the present article also take up the analysis of the discretization of this coefficient, providing further details concerning the effect of one-step methods on the emanating Hopf branches, whose dynamical nature is determined by the above mentioned coefficient. Explicit formulae for the computation of the first Lyapunov coefficient of both continuous- as well as discrete-time dynamical systems are derived in [17, 20], respectively.

2 Basic setup

Let us first formally define the bifurcations we will deal with.

Definition 2.1. *A point* $(x_0, \alpha_0) \in \Omega \times \Lambda$ *is referred to as a generalized Hopf bifurcation (in short GH point*[1]*) of* (1.1) *if:*

- $f(x_0, \alpha_0) = 0$,
- $f_x(x_0, \alpha_0)$ *has the only critical, simple eigenvalues* $\pm i\omega_0$, $0 < \omega_0 \in \mathbb{R}$.
- *The first Lyapunov coefficient vanishes at* (x_0, α_0).

Definition 2.2. *A point* $(x_0, \alpha_0) \in \Omega \times \Lambda$ *is referred to as a generalized Neimark-Sacker bifurcation (in short GN point*[2]*) of* (1.2) *if:*

- $g(x_0, \alpha_0) - x_0 = 0$,
- $g_x(x_0, \alpha_0)$ *has the only critical, simple eigenvalues* $e^{\pm i\theta_0}$, $0 < \theta_0 \in \mathbb{R}$, $e^{ik\theta_0} \neq 1$, $k = 1, 2, 3, 4, 5, 6$.
- *The first Lyapunov coefficient vanishes at* (x_0, α_0).

For our purposes it is useful to introduce standard augmented systems for the continuation of Hopf points. Assume that $(x_0, \alpha_0) \in \Omega \times \Lambda$ is a generic (see below) GH point

[1] Also called Bautin and degenerate Hopf, though the latter term is sometimes given a wider meaning.
[2] Also called Chenciner and degenerate Neimark-Sacker.

of (1.1). Consider the following system of $3n+2$ scalar equations for $(x, \alpha, q_1, q_2, \omega) \in \Omega \times \Lambda \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ (cf. [4, Section 5.2.2]):

$$\begin{cases} f(x,\alpha) = 0, \\ f_x(x,\alpha)q_1 + \omega q_2 = 0, \\ f_x(x,\alpha)q_2 - \omega q_1 = 0, \\ q_{01}^T q_1 + q_{02}^T q_2 - 1 = 0, \\ q_{01}^T q_2 - q_{02}^T q_1 = 0, \end{cases} \tag{2.1}$$

which is the real form of the complex system defining the Hopf bifurcation:

$$\begin{cases} f(x,\alpha) = 0, \\ f_x(x,\alpha)q - i\omega q = 0, \\ \langle q, q_0 \rangle - 1 = 0, \end{cases} \tag{2.2}$$

where $\langle q, q_0 \rangle = \overline{q}^T q_0$, $q = q_1 + iq_2$, and $q_0 = q_{01} + iq_{02}$ is a reference vector that is not orthogonal to the critical complex eigenspace corresponding to $\pm i\omega_0$. In this setting we have that system (2.1) has full rank at (x_0, α_0), cf. [4, Lemma 7.4].

Similarly, a standard augmented system for the continuation of Neimark-Sacker points of (1.2) is given by (cf. [18, Section 10.3.1]):

$$\begin{cases} g(x,\alpha) - x = 0, \\ g_x(x,\alpha)q_1 - \cos(\omega)q_1 + \sin(\omega)q_2 = 0, \\ g_x(x,\alpha)q_2 - \cos(\omega)q_2 - \sin(\omega)q_1 = 0, \\ q_{01}^T q_1 + q_{02}^T q_2 - 1 = 0, \\ q_{01}^T q_2 - q_{02}^T q_1 = 0, \end{cases}$$

which is the real form of the complex system defining the Neimark-Sacker bifurcation bifurcation:

$$\begin{cases} g(x,\alpha) - x = 0, \\ g_x(x,\alpha)q - e^{i\omega} q = 0, \\ \langle q, q_0 \rangle - 1 = 0, \end{cases} \tag{2.3}$$

where $q_0 \in \mathbb{C}^N$ is again a suitably chosen reference vector.

We will now present explicit formulae for the computation of the first Lyapunov coefficients, which are necessary for Definition 2.1, 2.2 to be precise. For this purpose, denote by $A_\cdot$, $B_\cdot(\cdot,\cdot)$, and $C_\cdot(\cdot,\cdot,\cdot)$ the operators given by:

$$\begin{aligned} A_r &:= r_x(x,\alpha), \\ B_r(v,w) &:= r_{xx}(x,\alpha)[v,w] := \sum_{i=1}^N \sum_{j=1}^N \frac{\partial^2 r(x,\alpha)}{\partial x_j \partial x_i} v_i w_j, \\ C_r(v,w,z) &:= r_{xxx}(x,\alpha)[v,w,z] := \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N \frac{\partial^3 r(x,\alpha)}{\partial x_l \partial x_j \partial x_i} v_i w_j z_l, \end{aligned}$$

where $v, w, z \in \mathbb{R}^N$, and $r \in C^k(\Omega \times \Lambda, \mathbb{R}^N)$ with $k \geq 1$ sufficiently large. With the notation above introduced, the first Lyapunov coefficient at a Hopf point of (1.1) (cf. (2.2)) is given by

$$L_H := \frac{1}{2} \operatorname{Re}\langle p, C_f(q, q, \overline{q}) - 2B_f(q, A_f^{-1} B_f(q, \overline{q})) + B_f(\overline{q}, (2i\omega I_N - A_f)^{-1} B_f(q, q))\rangle,$$

where $p \in \mathbb{C}^N$ satisfies $A_f^T p = -i\omega p$, $\langle p, q\rangle = 1$. Analogously, the first Lyapunov coefficient at a Neimark-Sacker point of (1.2) (cf. (2.3)) is defined as

$$L_{NS} := \frac{1}{2} \operatorname{Re} e^{-i\omega} d, \tag{2.4}$$

where

$$d := \langle p, C_g(q, q, \overline{q}) - 2B_g(q, (A_g - I_N)^{-1} B_g(q, \overline{q})) + B_g(\overline{q}, (e^{2i\omega} I_N - A_g)^{-1} B_g(q, q))\rangle.$$

With the coefficient L_H above introduced, we are now able to define genericity of a GH point. Thus, a GH point of (1.1) is said to be generic if the real form of the system:

$$\begin{cases} f(x, \alpha) = 0, \\ f_x(x, \alpha)q - i\omega q = 0, \\ f_x^T(x, \alpha)p + \lambda p = 0, \\ \langle q, q_0\rangle - 1 = 0, \\ \langle p, q\rangle - 1 = 0, \\ A_f v - B_f(q, \overline{q}) = 0, \\ (2i\omega I_N - A_f)w - B_f(q, q) = 0, \\ \operatorname{Re}\langle p, C_f(q, q, \overline{q}) - 2B_f(q, v) + B_f(\overline{q}, w)\rangle = 0, \end{cases} \tag{2.5}$$

has full rank at the GH point, see [11, Proposition 3]. Few remarks about the above presented system are in order. Note that the first, second, and fourth equations of (2.5) define the Hopf condition (cf. (2.2)). On the other hand, the third, fifth, sixth, and seventh equations provide auxiliary data in order to force the first Lyapunov coefficient to vanish. This condition is expressed in the last equation of system (2.5). Furthermore, the complex variable $\lambda = \lambda_1 + i\lambda_2$ is artificially introduced to regularize the system. Formally, it holds $\lambda = i\omega$ at a GH point.

The analysis presented in [11] is the cornerstone of the present article, since there the genericity of a GH point is associated to the regularity of the real form of system (2.5). Thereby, the approach employed in [24] for analyzing fold-Hopf bifurcations can be applied to our case too.

As mentioned in the Introduction, our main goal is to describe the effect of discretization methods on the local bifurcation diagram of dynamical systems near a GH point. In this sense we consider general one-step methods of order $p \geq 1$ applied to (1.1), given by

$$x \mapsto \psi^h(x, \alpha) := x + h\Phi(h, x, \alpha), \tag{2.6}$$

with $\psi, \Phi : [-h^*, h^*] \times \widetilde{\Omega} \times \widetilde{\Lambda} \to \mathbb{R}^N$ sufficiently smooth, $h^* > 0$, where $0 \in \widetilde{\Omega} \subset \Omega$, $0 \in \widetilde{\Lambda} \subset \Lambda$ are compact sets. That the method is of order p means that there exists a positive constant C_0 (depending only on f), such that it holds

$$||\varphi^h(x, \alpha) - \psi^h(x, \alpha)|| \leq C_0 |h|^{p+1},$$

for all $(h, x, \alpha) \in [-h^*, h^*] \times \widetilde{\Omega} \times \widetilde{\Lambda}$, where $\varphi^t(\cdot, \alpha)$ stands for the t-flow of (1.1) and $|| \cdot ||$ denotes any norm[3] in $\mathbb{R}^N$. In this setting, there exist smooth functions $\Upsilon, \Xi : [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda} \to \mathbb{R}^N$ such that:

$$\begin{aligned} \psi^h(x, \alpha) &= \varphi^h(x, \alpha) + \Upsilon(h, x, \alpha)h^{p+1}, \\ \Phi(h, x, \alpha) &= f(x, \alpha) + \Xi(h, x, \alpha)h, \end{aligned} \tag{2.7}$$

hold for all $(h, x, \alpha) \in [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda}$, where $0 < h_0 < h^*$, see [5, 9, 26].

With the technical framework above introduced, we have all the necessary machinery at hand for presenting the main results of the present work.

3 Generalized Hopf bifurcations under discretization

In this section we will suppose we are given a continuous-time dynamical system (1.1) which undergoes a GH bifurcation. We assume that the system is discretized via general, p-th order one-step methods. Under these conditions, we will show that the GH point is shifted and turned into a GN point by the one-step map, for all sufficiently small step-size. For this purpose, the ideas applied in [24] will be adapted to our case. Formally speaking, we have the following:

Theorem 3.1. *Let a general one-step method of order $p \geq 1$ applied to* (1.1) *be given by* (2.6)*. Assume that system* (1.1) *has a generic GH point at $(x_{GH}, \alpha_{GH}) \in \widetilde{\Omega} \times \widetilde{\Lambda}$. Then, there exists a positive constant $\rho \leq h_0$ and a neighborhood $\Omega' \times \Lambda' \subset \widetilde{\Omega} \times \widetilde{\Lambda}$ of (x_{GH}, α_{GH}), in which* (2.6) *has a unique GN point $(x_{GN}(h), \alpha_{GN}(h))$ that depends smoothly on h, for all $h \in (-\rho, \rho)$. Furthermore, the following estimate holds*

$$||(x_{GN}(h), \alpha_{GN}(h)) - (x_{GH}, \alpha_{GH})|| \leq C|h|, \tag{3.1}$$

for some $C > 0$ and all $h \in (-\rho, \rho)$.

Proof. Denote the real form of (2.5) by

$$F(x, \alpha, z) = 0, \tag{3.2}$$

with $F : \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{7N+3} \to \mathbb{R}^{8N+5}$, $z := (q_1, q_2, p_1, p_2, v, w_1, w_2, \omega, \lambda_1, \lambda_2)$. Consider $z_{GH} \in \mathbb{R}^{7N+3}$, such that $F(x_{GH}, \alpha_{GH}, z_{GH}) = 0$. Then, system (3.2) is regular at $(x_{GH}, \alpha_{GH}, z_{GH})$ (cf. Section 2). Now let us construct a similar system that describes a GN point of the one-step map (2.6). By following the structure of (2.5), combined with (2.3) and (2.4),

[3] Throughout this article, the symbol $|| \cdot ||$ will be used to denote norms in different spaces. From the context, no confusion should arise.

we obtain the following complex system:

$$\begin{cases} \psi^h(x,\alpha) - x = 0, \\ \psi^h_x(x,\alpha)q - e^{i\omega}q = 0, \\ (\psi^h_x(x,\alpha))^T p - e^{-\lambda}p = 0, \\ \langle q, q_0\rangle - 1 = 0, \\ \langle p, q\rangle - 1 = 0, \\ (A_\psi - I_N)v - B_\psi(q,\overline{q}) = 0, \\ (e^{2i\omega}I_N - A_\psi)w - B_\psi(q,q) = 0, \\ \operatorname{Re} e^{-i\omega}\langle p, C_\psi(q,q,\overline{q}) - 2B_\psi(q,v) + B_\psi(\overline{q},w)\rangle = 0, \end{cases} \tag{3.3}$$

whose solutions correspond to GN points of (2.6). Note that for $h = 0$ the system becomes trivial (observe e.g. the first equation of (3.3) at $h = 0$). This is inconvenient for our approach, as we want to perform our analysis for h small, although, in general, system (3.3) may be suitable for the continuation of GN points of three-parameter discrete systems. Let us then consider the following modified system:

$$\begin{cases} \frac{1}{h}\left(\psi^h(x,\alpha) - x\right) = 0, \\ \frac{1}{h}\left(\psi^h_x(x,\alpha)q - e^{ih\omega}q\right) = 0, \\ \frac{1}{h}\left((\psi^h_x(x,\alpha))^T p - e^{-h\lambda}p\right) = 0, \\ \langle q, q_0\rangle - 1 = 0, \\ \langle p, q\rangle - 1 = 0, \\ \frac{1}{h}\left((A_\psi - I_N)v - B_\psi(q,\overline{q})\right) = 0, \\ \frac{1}{h}\left((e^{2ih\omega}I_N - A_\psi)w - B_\psi(q,q)\right) = 0, \\ \frac{1}{h}\operatorname{Re} e^{-ih\omega}\langle p, C_\psi(q,q,\overline{q}) - 2B_\psi(q,v) + B_\psi(\overline{q},w)\rangle = 0. \end{cases} \tag{3.4}$$

Note that all but fourth and fifth equations have been divided by h, and ω, λ have been replaced with $h\omega$, $h\lambda$, respectively. This change will regularize the system at $h = 0$, as we will see.

Denote the real form of (3.4) by

$$G(h, x, \alpha, z) = 0,$$

with $G : [-h_0, h_0] \times \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{7N+3} \to \mathbb{R}^{8N+5}$. We will next show that

$$G(0, x, \alpha, z) = F(x, \alpha, z) \tag{3.5}$$

for all $(x, \alpha, z) \in \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{7N+3}$. To accomplish this, it suffices to work with the complex forms. Let us begin with the first equation of (2.5) and (3.4). By (2.7), we have that

$$\frac{1}{h}\left(\psi^h(x,\alpha) - x\right)\Big|_{h=0} = \Phi(h,x,\alpha)|_{h=0} = f(x,\alpha). \tag{3.6}$$

Similarly, it follows:

$$\begin{aligned} \frac{1}{h}\left(\psi^h_x(x,\alpha)q - e^{ih\omega}q\right)\Big|_{h=0} &= \frac{1}{h}\left((I_N + h\Phi_x(h,x,\alpha))q - (1 + ih\omega + O(h^2))q\right)\Big|_{h=0}, \\ &= \left(\Phi_x(h,x,\alpha)q - (i\omega + O(h))q\right)|_{h=0}, \\ &= f_x(x,\alpha)q - i\omega q. \end{aligned} \tag{3.7}$$

The analysis for the third equation follows analogously. As for the sixth equation, it holds that:

$$\begin{aligned}\frac{1}{h}\left((A_\psi - I_N)v - B_\psi(q,\overline{q})\right)\Big|_{h=0} &= (\Phi_x(h,x,\alpha)v - B_\Phi(q,\overline{q}))|_{h=0}\,,\\ &= A_f v - B_f(q,\overline{q}).\end{aligned}$$

Now let us work with the seventh equation:

$$\begin{aligned}\frac{1}{h}\left((e^{2ih\omega}I_N - A_\psi)w - B_\psi(q,q)\right)\Big|_{h=0} &= (((2i\omega + O(h))I_N - \Phi_x(h,x,\alpha))w - B_\Phi(q,q))|_{h=0}\,,\\ &= (2i\omega I_N - A_f)w - B_f(q,q).\end{aligned}$$

Finally, it follows that:

$$\begin{aligned}\frac{1}{h}\operatorname{Re} e^{-ih\omega}\langle p, C_\psi(q,q,\overline{q}) - 2B_\psi(q,v) + B_\psi(\overline{q},w)\rangle\Big|_{h=0} &= \operatorname{Re} e^{-ih\omega}\langle p, C_\Phi(q,q,\overline{q}) - 2B_\Phi(q,v)\\ &\quad + B_\Phi(\overline{q},w)\rangle|_{h=0}\,,\\ &= \operatorname{Re}\langle p, C_f(q,q,\overline{q}) - 2B_f(q,v)\\ &\quad + B_f(\overline{q},w)\rangle.\end{aligned}$$

Thus, (3.5) holds, and thereby we have that $G(0, x_{GH}, \alpha_{GH}, z_{GH}) = 0$ and $G_{(x,\alpha,z)}(0, x_{GH}, \alpha_{GH}, z_{GH}) = F_{(x,\alpha,z)}(x_{GH}, \alpha_{GH}, z_{GH})$ is nonsingular, as (3.2) is regular at the GH point. Therefore, the Implicit Function Theorem guarantees the existence of functions $(x_{GN}, \alpha_{GN}, z_{GN}) : (-\rho, \rho) \to \mathbb{R}^{8N+5}$, $0 < \rho < h_0$, such that

$$G(h, x_{GN}(h), \alpha_{GN}(h), z_{GN}(h)) = 0, \quad (x_{GN}(0), \alpha_{GN}(0), z_{GN}(0)) = (x_{GH}, \alpha_{GH}, z_{GH}),$$

$h \in (-\rho, \rho)$. This shows the existence, uniqueness, and smooth dependence on h of a GN point of (2.6). Now expand (x_{GN}, α_{GN}) near $h = 0$. We obtain

$$(x_{GN}(h), \alpha_{GN}(h)) = (x_{GH}, \alpha_{GH}) + \Theta(h)h,$$

where Θ is some smooth function[4]. Hence, it follows:

$$\begin{aligned}||(x_{GN}(h), \alpha_{GN}(h)) - (x_{GH}, \alpha_{GH})|| &= ||\Theta(h)h||,\\ &\leq C|h|,\end{aligned}$$

where $C := \sup_{h\in(-\rho,\rho)} ||\Theta(h)||$. □

Note that the main statement of the above theorem is that a GH point persists as a shifted GN point, under general one-step methods. The shift estimate obtained is of first order, regardless the order of the method. This result may seem to be quite pessimistic, however, the numerical experiments will show that a much better result cannot be expected, see Section 5.

[4]In what follows, by the term "(some smooth function) $\cdot\, w^k$", $k \geq 1$, w some real variable, we mean the integral remainder of a Taylor series.

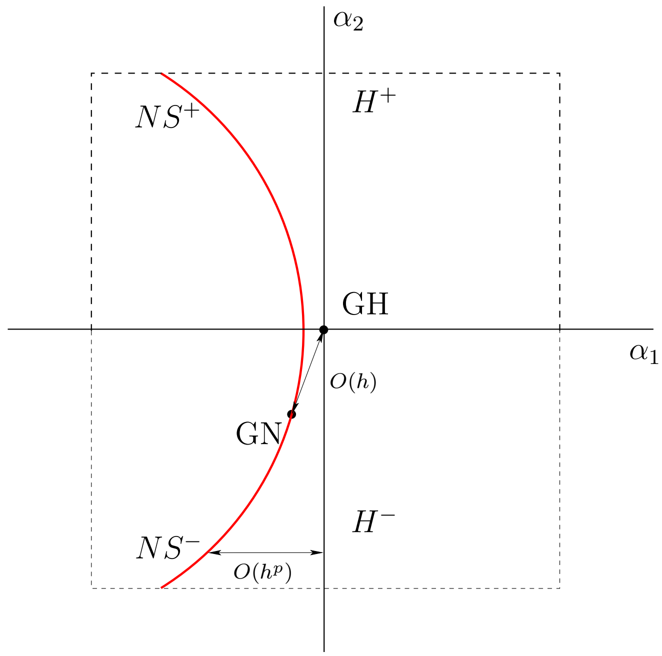

Fig. 4.1: Discretized path of Hopf points near a GH bifurcation.

4 Emanating Hopf curve under discretization

In the previous section we proved that GH points persist under one-step methods. As we mentioned in the Introduction, we are interested to know whether the local bifurcation diagram near this bifurcation is "well" reproduced by a one-step method. In this sense, the first part of this task has been achieved, namely, the organizing center was shown to be preserved by discretization methods.

Now we take up the problem of analyzing the discretization of the Hopf curve of system (1.1) that emanates from the GH bifurcation. The result we are after is illustrated in Figure 4.1. The curves labeled by H^+, NS^+ (resp. H^-, NS^-) represent paths of Hopf, Neimark-Sacker points respectively, with positive (resp. negative) Lyapunov coefficients.

The analysis is formulated as follows. Suppose we are given a continuous-time dynamical system (1.1) which undergoes a GH bifurcation. We assume that this system is discretized via general one-step methods, as in Section 3. Under these conditions, we will show that there exists a step-size-independent neighborhood (the dashed square in Figure 4.1) of the GH point, such that the discretized path of Hopf points (NS in Figure 4.1) approximates the original curve (H in Figure 4.1) with the order of the method. For this purpose, we do not reduce the systems, e.g. via center manifold theory, but we rather work with them in full dimension. To do so, the arguments employed in [24, Theorem 4.1] will be applied. For this reason we will only present a sketch of the proof of the main result. Throughout this section we denote by $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ the parameters of the system. With these remarks we are ready to formulate:

Theorem 4.1. *Let a general one-step discretization method of order $p \geq 1$ applied to*

(1.1) *be given by* (2.6). *Assume that system* (1.1) *has a generic GH point at the origin* $(x_{GH}, \alpha_{GH}) = (0,0)$, *with eigenvectors* $q_{1GH} \pm iq_{2GH}$ *corresponding to the critical eigenvalues* $\pm i\omega_{GH}$. *Further denote system* (2.1) *by*

$$\widetilde{F}(x, \alpha, z) = 0, \tag{4.1}$$

with $\widetilde{F} : \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{2N+1} \to \mathbb{R}^{3N+2}$, $z := (q_1, q_2, \omega)$, *and assume that* $\widetilde{F}_{(x,\alpha_1,z)}(x_{GH}, \alpha_{GH}, z_{GH})$ *is nonsingular*[5]. *Then, there exist positive constants* $\rho \leq h_0$, δ *and curves of Hopf and Neimark-Sacker points of systems* (1.1) *and* (2.6), *respectively, defined by:*

$$\begin{aligned} C_H(\alpha_2) &:= (x_H(\alpha_2), \alpha_{1H}(\alpha_2), z_H(\alpha_2), \alpha_2), \\ C_{NS}(h, \alpha_2) &:= (x_{NS}(h, \alpha_2), \alpha_{1NS}(h, \alpha_2), z_{NS}(h, \alpha_2), \alpha_2), \end{aligned}$$

with $x_H : (-\delta, \delta) \to \mathbb{R}^N$, $x_{NS} : (-\rho, \rho) \times (-\delta, \delta) \to \mathbb{R}^N$, $\alpha_{1H} : (-\delta, \delta) \to \mathbb{R}$, $\alpha_{1NS} : (-\rho, \rho) \times (-\delta, \delta) \to \mathbb{R}$, $z_H : (-\delta, \delta) \to \mathbb{R}^{2N+1}$, $z_{NS} : (-\rho, \rho) \times (-\delta, \delta) \to \mathbb{R}^{2N+1}$ *smooth*[6]. *Furthermore, the following estimate holds for all* $(h, \alpha_2) \in (-\rho, \rho) \times (-\delta, \delta)$ *and uniformly in* α_2

$$||d_{NS}(h, \alpha_2) - d_H(\alpha_2)|| \leq C|h|^p, \tag{4.2}$$

where $d_H := (x_H, \alpha_{1H}, z_H)$ *and* $d_{NS} := (x_{NS}, \alpha_{1NS}, z_{NS})$, $C > 0$.

Sketch of the proof. Since $\widetilde{F}_{(x,\alpha_1,z)}(0, 0, z_{GH})$ is nonsingular, the Implicit Function Theorem guarantees the existence of the function $d_H = (x_H, \alpha_{1H}, z_H) : (-\delta_1, \delta_1) \to \mathbb{R}^{3N+2}$, such that

$$\widetilde{F}(d_H(\alpha_2), \alpha_2) = 0, \quad d_H(0) = (0, 0, z_{GH}),$$

$\alpha_2 \in (-\delta_1, \delta_1)$. In the spirit of [24, Theorem 4.1], we will rewrite (4.1) in terms of the h-flow $\varphi^h(\cdot, \alpha)$ of (1.1). Thus, we obtain the system (cf. (2.3) and (3.4)):

$$\begin{cases} \frac{1}{h}\left(\varphi^h(x, \alpha) - x\right) = 0, \\ \frac{1}{h}\left(\varphi^h_x(x, \alpha)q - e^{ih\omega}q\right) = 0, \\ \langle q, q_0 \rangle - 1 = 0. \end{cases} \tag{4.3}$$

Denote the real form of the above system by $F(h, x, \alpha, z) = 0$, with $F : (-h_0, h_0) \times \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{2N+1} \to \mathbb{R}^{3N+2}$. Then, it holds (see [24, Theorem 4.1])

$$F(h, d_H(\alpha_2), \alpha_2) = 0,$$

for all $(h, \alpha_2) \in (-h_0, h_0) \times (-\delta_1, \delta_1)$. Similarly, a system whose zeroes describe a curve of Neimark-Sacker points of (2.6) is given by:

$$\begin{cases} \frac{1}{h}\left(\psi^h(x, \alpha) - x\right) = 0, \\ \frac{1}{h}\left(\psi^h_x(x, \alpha)q - e^{ih\omega}q\right) = 0, \\ \langle q, q_0 \rangle - 1 = 0, \end{cases} \tag{4.4}$$

[5]This assumption is imposed only for assuring that α_2 can be used as parametrization variable. This is allowed due to the genericity of the GH point (see system (2.5)).

[6]By the term "Neimark-Sacker curve", we mean the graph of the function $C_{NS}(h_*, \cdot)$, for $h_* \in (-\rho, \rho)$ fixed.

whose real form will be denoted by $G(h, x, \alpha, z) = 0$. Hence, we will show the existence of a curve of Neimark-Sacker points of (2.6). By (3.6) and (3.7), it holds

$$G(0, x, \alpha, z) = \widetilde{F}(x, \alpha, z),$$

for all $(x, \alpha, z) \in \widetilde{\Omega} \times \widetilde{\Lambda} \times \mathbb{R}^{2N+1}$. Hence, we have that $G(0, 0, 0, z_{GH}) = 0$ and $G_{(x,\alpha_1,z)}(0, 0, 0, z_{GH})$ is nonsingular. Thereby, the Implicit Function Theorem guarantees the existence of the function $d_{NS} = (x_{NS}, \alpha_{1NS}, z_{NS}) : (-\rho_1, \rho_1) \times (-\delta_2, \delta_2) \to \mathbb{R}^{3N+2}$, $0 < \rho_1 \leq h_0$, such that

$$G(h, d_{NS}(h, \alpha_2), \alpha_2) = 0, \quad d_{NS}(0, 0) = (0, 0, z_{GH}),$$

$(h, \alpha_2) \in (-\rho_1, \rho_1) \times (-\delta_2, \delta_2)$. Let us now establish closeness relations between the real form of the defining systems (4.3) and (4.4). For this purpose, it suffices to analyze their complex form. By (2.7), it follows that:

$$\frac{1}{h}\left(\psi^h(x, \alpha) - x\right) = \frac{1}{h}\left(\varphi^h(x, \alpha) - x + \Upsilon(h, x, \alpha)h^{p+1}\right) = \frac{1}{h}\left(\varphi^h(x, \alpha) - x\right) + \Upsilon(h, x, \alpha)h^p,$$

and

$$\begin{aligned}\frac{1}{h}\left(\psi_x^h(x, \alpha)q - e^{ih\omega}q\right) &= \frac{1}{h}\left((\varphi_x^h(x, \alpha) + \Upsilon_x(h, x, \alpha)h^{p+1})q - e^{ih\omega}q\right), \\ &= \frac{1}{h}\left(\varphi_x^h(x, \alpha)q - e^{ih\omega}q\right) + \Upsilon_x(h, x, \alpha)qh^p.\end{aligned}$$

Therefore, we conclude that

$$G(h, x, \alpha, z) = F(h, x, \alpha, z) + \Gamma(h, x, \alpha, z)h^p,$$

where Γ is some smooth function. With the above relation, combined with the techniques employed in [24, Theorem 4.1], the assertion of the Theorem follows. □

Before switching to the next topic of this section, few remarks seem to be in order. The theorem above presented provides stronger results than those of [24, Theorem 4.1]. The main difference is the defining system used. In the analysis presented here, we consider the additional variable $z = (q_1, q_2, \omega)$, and thereby we can obtain $O(h^p)$-estimates between the critical eigenvectors[7]. Moreover, it can also be derived interesting relations between the critical eigenvalues of the original system and their discretization. Under assumptions and notation of the previous theorem, denote by $i\omega_H(\alpha_2)$ and $\mu_{NS}(h, \alpha_2) := e^{ih\omega_{NS}(h,\alpha_2)}$ the critical eigenvalues along the Hopf and Neimark-Sacker curves, respectively, for $(h, \alpha_2) \in (-\rho, \rho) \times (-\delta, \delta)$. Then, by (4.2), it follows that:

$$\begin{aligned}\left|\mu_{NS}(h, \alpha_2) - e^{ih\omega_H(\alpha_2)}\right| &= \left|e^{ih\omega_{NS}(h,\alpha_2)} - e^{ih\omega_H(\alpha_2)}\right|, \\ &\leq L|\omega_{NS}(h, \alpha_2) - \omega_H(\alpha_2)||h|, \\ &\leq C \cdot L|h|^{p+1},\end{aligned}$$

[7]Of course, this result makes sense only if we consider a suitable normalization condition.

where L is a local Lipschitz constant of the exponential function. Therefore, we conclude that

$$\mu_{NS}(h, \alpha_2) = e^{ih\omega_H(\alpha_2)} + O(h^{p+1}),$$

uniformly in α_2, and for h sufficiently small. This relation improves, for this particular case, the general approximation given in [27, Lemma 3.2].

Sofar we have almost completed the analysis of the discretized bifurcation picture shown in Figure 4.1. The effect of discretization methods on the organizing center and the emanating Hopf curve is now understood. However, it remains to determine whether the Hopf branches are discretized "correctly" by one-step methods. Thus, we will now investigate whether the Lyapunov coefficients change sign in the same direction, as they cross the bifurcation points. Let us make this assertion more precise. Let again the assumptions of Theorem 4.1 be fulfilled. Denote by $L_H(\alpha_2)$ and $L_{NS}(h, \alpha_2)$ the first Lyapunov coefficients along the Hopf and Neimark-Sacker curves of systems (1.1) and (2.6), respectively, for $(h, \alpha_2) \in [0, \rho) \times (-\delta, \delta)$. Assume that $L_H(\alpha_2) > 0$ (resp. $L_H(\alpha_2) < 0$) for $\alpha_2 > 0$ (resp. $\alpha_2 < 0$), see Figure 4.1. Denote by $\alpha_{GN}(h) = (\alpha_{1GN}(h), \alpha_{2GN}(h))$ the parameter values at which the one-step map (2.6) undergoes the GN bifurcation, for $h \in [0, \rho)$, cf. Theorem 3.1. In this setting, we will show that

$$L_{NS}(h, \alpha_2) > 0, \quad \text{for all } \alpha_2 > \alpha_{2GN}(h),$$

and

$$L_{NS}(h, \alpha_2) < 0, \quad \text{for all } \alpha_2 < \alpha_{2GN}(h),$$

provided the step-size is sufficiently small. In order to prove this, note first that since the GH point of (1.1) is generic, we have that the Lyapunov coefficient vanishes with nonzero velocity. According to the behavior assumed for L_H, it must be true that[8]

$$L_H(\alpha_2) = L'_H(0)\alpha_2 + \Delta(\alpha_2)\alpha_2^2,$$

for all $\alpha_2 \in (-\delta, \delta)$, where $L'_H(0) > 0$ and Δ is some smooth function. By taking into account the Lyapunov-coefficient-components of (3.5) and the estimate (4.2), it can be shown that

$$L_{NS}(h, \alpha_2) = h(L_H(\alpha_2) + \Lambda(h, \alpha_2)h), \tag{4.5}$$

for all $(h, \alpha_2) \in [0, \rho] \times (-\delta, \delta)$, where Λ is some smooth function. The above relation was first noticed in [1, Theorem 5.1], when dealing with a particular type of one-step methods. Our discussion places (4.5) in a more general context. By combining the last two equations, we arrive at

$$L_{NS}(h, \alpha_2) = h(L'_H(0)\alpha_2 + \Delta(\alpha_2)\alpha_2^2 + \Lambda(h, \alpha_2)h). \tag{4.6}$$

Next, differentiate the above expansion with respect to α_2. We obtain

$$L'_{NS}(h, \alpha_2) = h(L'_H(0) + \Delta'(\alpha_2)\alpha_2^2 + 2\Delta(\alpha_2)\alpha_2 + \Lambda'(h, \alpha_2)h).$$

[8]Throughout this discussion, the symbol $'$ means derivative with respect to α_2.

Now evaluate this expression at the GN point $\alpha_2 = \alpha_{2GN}(h)$. This yields

$$\begin{aligned} L'_{NS}(h, \alpha_{2GN}(h)) =& h(L'_H(0) + \Delta'(\alpha_{2GN}(h))\alpha^2_{2GN}(h) + 2\Delta(\alpha_{2GN}(h))\alpha_{2GN}(h) \\ &+ \Lambda'(h, \alpha_{2GN}(h))h). \end{aligned}$$

Finally, by recalling that $\alpha_{2GN}(0) = 0$ (cf. Theorem 3.1), we can choose a sufficiently small constant $0 < \rho' \leq \rho$, such that

$$L'_{NS}(h, \alpha_{2GN}(h)) > 0,$$

for all $h \in (0, \rho')$. Thereby, the discretized bifurcation picture is completed. Note that in this last discussion we restricted our attention to nonnegative step-sizes. This is not a merely technical assumption, for if we allow negative step-sizes, then the sign of the Lyapunov coefficient will change with the step-size along the Neimark-Sacker-branches shown in Figure 4.1. This fact is due to the leading factor of the right-hand side of (4.6).

5 A Numerical Example

Consider the following continuous-time, dimensionless system:

$$\begin{aligned} \dot{x} &= -\left(\frac{\beta+\alpha}{R}\right)x + \frac{\alpha}{R}y - \frac{C}{R}x^3 + \frac{D}{R}(y-x)^3 - \frac{E}{R}x^5 + \frac{F}{R}(y-x)^5, \\ \dot{y} &= \alpha x - (\alpha + G)y - z - D(y-x)^3 - Hy^3 - F(y-x)^5 - Iy^5, \\ \dot{z} &= y, \end{aligned} \tag{5.1}$$

with state variables $(x, y, z) \in \mathbb{R}^3$ and with parameters $\alpha, \beta, C, D, E, F, G, H, I, R \in \mathbb{R}$, $R > 0$. This system describes the dynamics of a modified Van der Pol-Duffing oscillator. A detailed analysis of this oscillator concerning both local, as well as global phenomena can be found in [2], [8], and a more general discussion concerning the dynamics of this type of circuits can be found in [16, Chapter 7].

In this experiment we assume (α, β) to be our bifurcation parameters, and we let $C = 1$, $D = -5$, $E = 1$, $F = 1$, $G = -1.5$, $H = 1$, $I = 1$, $R = 3$ fixed. Moreover, the numerical computations will be carried out with the continuation software CONTENT, cf. [19].

The purpose of this example is to illustrate the theoretical result obtained in Theorem 3.1, namely, we want to observe whether GH points are shifted and turned into GN points by one-step methods. To achieve this, we need first to find a GH point of system (5.1). Thus, choose $(x_{ini}, y_{ini}, z_{ini}) = (-0.5, 0, 0.2)$, $(\beta_{ini}, \alpha_{ini}) = (-1, 1)$ as initial data for the continuation of equilibria, and let β freely vary. The thus obtained curve is plotted in Figure 5.1. With this procedure we found: two neutral saddles, four Hopf points, two fold, and one branching point, labeled by NTS, H, LP, and BP, respectively. We next switch to the Hopf point located at $(x_H, y_H, z_H) \approx (-0.569, 0, 0.294)$, $(\alpha_H, \beta_H) \approx (1, 0.0876)$. Then, by continuing the Hopf curve with respect to α and β, we find a GH point at $(x_{GH}, y_{GH}, z_{GH}) = (-0.345808037314697, 0, 0.464206984612583)$, $(\alpha_{GH}, \beta_{GH}) = (-0.758767593548692, 1.20850010542548)$.

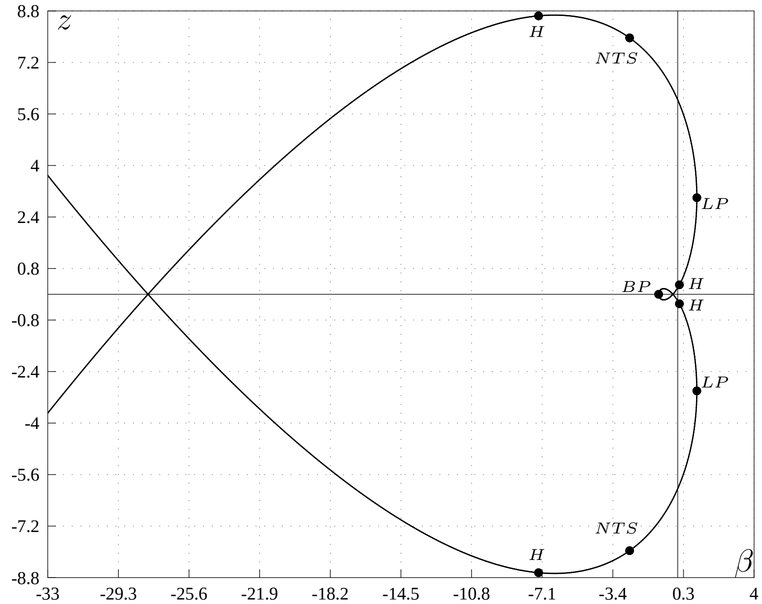

Fig. 5.1: Continuation of equilibria of (5.1) for $\alpha = 1$ fixed.

Next, we discretize system (5.1) by the well-known Runge-Kutta method of fourth order. Define the distance function

$$Dist_{GH}(h) := ||(x_{GN}(h), y_{GN}(h), z_{GN}(h), \beta_{GN}(h), \alpha_{GN}(h)) \\ - (x_{GH}, y_{GH}, z_{GH}, \beta_{GH}, \alpha_{GH})||,$$

for $h > 0$ small, where $|| \cdot ||$ represents the Euclidean norm and $(x_{GN}(h), y_{GN}(h), z_{GN}(h), \beta_{GN}(h), \alpha_{GN}(h))$ stands for a GN point of the Runge-Kutta map. We will then investigate how the distance between GH and GN points of (5.1) and the one-step map, respectively, varies with the step-size. This behavior is shown in Figure 5.2. In this picture, we let h vary from 0.03 to 0.4. For several, fixed values of h in this interval, we computed a GN point of the Runge-Kutta map, and thus we obtained the curve shown in the figure. We plotted the logarithm of the variables, so that we can determine the order of approximation as the slope of the quasi-straight line obtained. This slope is approximately equal to $2.12 \approx 2$, which is of course consistent with Theorem 3.1.

It is important to point out that, although the numerical method applied is of order four, we only obtained a second-order approximation of the GH bifurcation of system (5.1). This experiment allows us to presume that the first-order estimate obtained in Theorem 3.1 is not optimal, and that the optimal order is likely to be $\min(2, p)$, as some other experiments suggest.

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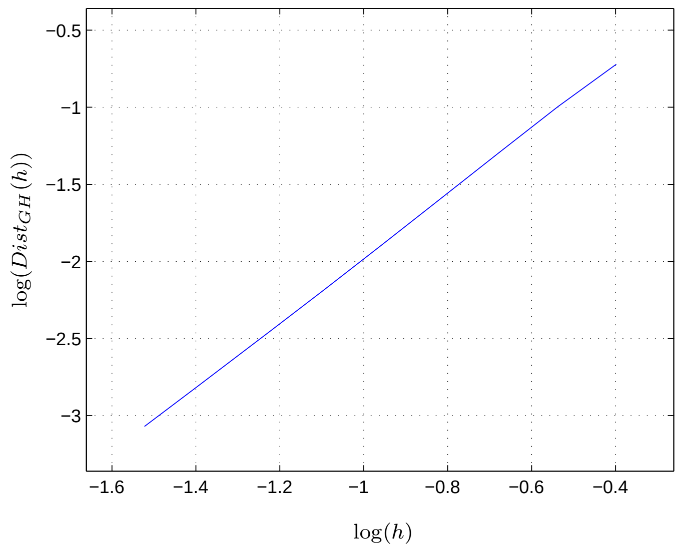

Fig. 5.2: Distance between GH and GN points for different values of step-size.

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