Linear stability of traveling waves in nonstrictly hyperbolic PDEs

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Abstract

In the analysis of traveling waves it is very common that coupled parabolic-hyperbolic problems occur, where the hyperbolic part is not strictly hyperbolic. For example, this happens whenever a reaction diffusion equation with more than one non diffusing component is considered in a co-moving frame. In this paper we analyze the stability of traveling waves in nonstrictly hyperbolic PDEs by reformulating the problem as a partial differential algebraic equation (PDAE). We prove uniform resolvent estimates for the original PDE problem and for the PDAE by using exponential dichotomies. It is shown that the zero eigenvalue of the linearization is removed from the spectrum in the PDAE formulation and, therefore, the PDAE problem is better suited for the stability analysis. This is rigorously done via the vector valued Laplace transform which also leads to optimal rates. The linear stability result presented here is a major step in the proof of nonlinear stability.

Keywords: Hyperbolic partial differential equations, traveling waves, partial differential algebraic equations, linear stability, asymptotic behavior, resolvent estimates

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1 Introduction

An important class of partial differential equations are reaction diffusion equations, where some of the components do not diffuse. Such systems appear in many areas of application and the most prominent model are the spatially extended Hodgkin-Huxley equations which model the signaling of electric pulses along nerve axons. The equations as presented in [17] are of the form

$$u_t = Au_{xx} + f_1(u, v), \quad v_t = f_2(u, v), \tag{1.1}$$

where $u(x, t)$ is a scalar function and $v(x, t)$ is an element of $\mathbb{R}^3$. The functions f_1 and f_2 are typically nonlinear. It is well known that there exist traveling wave solutions of the Hodgkin-Huxley equations. Usually, when analyzing traveling waves in one dimensional systems of

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partial differential equations one considers the problem in a co-moving frame in which the traveling wave becomes a steady state. When (1.1) or equations with a similar structure are considered in a co-moving frame, they become a system of the form

$$\begin{aligned} u_t &= Au_{xx} + \underline{\lambda} u_x + f_1(u, v), \\ v_t &= \underline{\lambda} v_x + f_2(u, v), \end{aligned} \tag{1.2}$$

which consists of parabolic equations for u nonlinearly coupled to hyperbolic equations for v. An immediate observation is that the coefficient matrix of the principal part in the v–equation typically does not have pairwise different eigenvalues, so it is not strictly hyperbolic.

There is a huge amount of literature on the case of (purely) parabolic reaction diffusion equations which begins with the work of [26] and [16] but in spite of its importance only few authors consider the stability of traveling waves in semilinear hyperbolic or hyperbolic-parabolic systems like (1.1). In most cases, for example in [10, 11, 12, 13] or [23], only a specific problem is considered. Only few references consider general problems like in [2], but in this paper the case of fronts must be excluded due to a compactness argument used in the analysis. Another important reference is the paper [19], but there the non strictly hyperbolic case is excluded and the use of the Laplace transform technique used in that reference seems to be rather formal, especially in the case where a 0–eigenvalue appears in the linearization.

In this paper we consider the stability of traveling wave solutions in non-strictly hyperbolic systems of partial differential equations of the form

$$v_t = Bv_x + f(v). \tag{1.3}$$

The stability of such problems is an important step towards the stability analysis of general coupled hyperbolic-parabolic systems of the form

$$u_t = Au_{xx} + g(u, v)_x + f_1(u, v), \quad v_t = Bv_x + f_2(u, v), \tag{1.4}$$

which includes problems of the form (1.2) and will be considered in a forthcoming paper.

In the rest of this paper we denote the traveling wave's profile by $\underline{v}$ and its speed by $\underline{\lambda}$. Furthermore, we always consider the equation in a co-moving frame so that $\underline{v}$ becomes a stationary solution and the speed $\underline{\lambda}$ is zero.

A major problem one has to face when analyzing the stability of traveling waves is the equivariance with respect to the spatial shift. It is well-known, that this leads to non uniqueness and instead of usual Lyapunov stability, stability with asymptotic phase must be considered. It also implies that 0 is an eigenvalue of the linearized right hand side, which in the case of (1.3) reads

$$Pv = Bv_x + f_v(\underline{v})v. \tag{1.5}$$

In most of the literature, where only the case of parabolic reaction diffusion equations is considered, this is dealt with by using spectral projectors to remove the zero eigenvalue from the spectrum of P, which is the generator of an analytic semigroup. Since the spectral mapping theorem does not generalize to arbitrary semigroups, we choose a different road to cope with the asymptotic phase. Our approach is of a more constructive nature and uses ideas from [16]. It is also closely related to the freezing method from [7], which is a numerical method to compute the asymptotic behavior of solutions to equivariant evolution equations.

The approach is as follows: Instead of reducing the linearization of (1.3) to an invariant subspace, our idea is to use nonlinear coordinates $(\tilde{v}, \tilde{\varphi})$ to represent the solution in the form

$$v(x,t) = \underline{v}(x - \tilde{\varphi}(t)) + \tilde{v}(x,t), \tag{1.6}$$

which in fact increases the degrees of freedom by one. To obtain a well-posed problem we then restrict $\tilde{v}$ to a subspace of the function space which is given by the kernel of a linear functional Ψ. To make the ansatz (1.6) unambiguous, we impose the following assumptions on Ψ:

Assumption 1.1. **(A1)** *There is $C_\Psi > 0$ so that the functional $\Psi : H^{-1}(\mathbb{R}, \mathbb{R}^m) \to \mathbb{R}$ satisfies for all $v \in H^{-1}(\mathbb{R}, \mathbb{R}^m)$ the estimate $|\Psi(v)| \le C_\Psi \|v\|_{H^{-1}}$.*

(A2) *The functional satisfies the non-degeneracy condition $\Psi(\underline{v}_x) \neq 0$.*

This leads to the partial differential algebraic equation (short: PDAE)

$$\begin{aligned}
\tilde{\varphi}_t &= \lambda, \\
\tilde{v}_t &= B\Big(\tilde{v}_x + \underline{v}_x(\cdot - \tilde{\varphi})\Big) + f\Big(\underline{v}(\cdot - \tilde{\varphi}) + \tilde{v}\Big) + \underline{v}_x(\cdot - \tilde{\varphi})\lambda, \\
0 &= \Psi(\tilde{v})
\end{aligned} \tag{1.7}$$

for $\tilde{v}$, $\tilde{\varphi}$, and $\tilde{\varphi}_t = \lambda$. Under the assumptions (A1) and (A2), solutions of the Cauchy-problem for (1.3) which are close to the profile $\underline{v}$ lead to solutions of the Cauchy-problem for (1.7) which are close to 0 and vice versa. Therefore, it suffices to analyze the PDAE (1.7).

In this paper we prove linear stability of (1.7) with precise estimates. In a following paper the results obtained here are used to prove nonlinear stability of the PDAE and the original problem and, in particular, also convergence of the freezing method will be proved there.

The rest of the paper is organized as follows. In §2 we derive the linearization of (1.7), which will be the subject of our analysis for the rest of this paper. In §3 we analyze the spectral properties and obtain uniform resolvent estimates for the linearization (1.5) of the original problem, which also appears in the linearization of the PDAE (1.7). These estimates are used in §4 to obtain resolvent estimates and the location of the spectrum for the linearized PDAE. In particular, we will see that the 0–eigenvalue is removed from the spectrum by considering the PDAE problem. A main tool in §3 and §4 are exponential dichotomies (short: ED). The definition of an ED together with several properties are given in Appendix B. In §5 we then present the main stability result of this paper, Theorem 5.3. Its proof is based on the resolvent estimates for the PDAE problem that lead to estimates for the original problem via the Laplace-transform technique. The use of the Laplace-transform for stability proofs in PDE problems has a long history, see for example [20] and the references therein, but to use it in conjunction with a PDAE reformulation seems to be a new approach. This approach also unifies the proof of stability with asymptotic phase. In particular, the convergence of the solution to the asymptotic profile and also the convergence of the asymptotic phase follow at once.

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2 Reformulating the Stability Problem

To make the arguments from above rigorous, impose the assumption $f \in \mathcal{C}^3(\mathbb{R}^m, \mathbb{R}^m)$ for the nonlinearity in (1.3) and also assume $\underline{v} \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^m)$ is the profile of a non constant traveling wave solution with $\underline{v}_x \in H^2(\mathbb{R}, \mathbb{R}^m)$ and traveling with speed 0. Furthermore, assume that the equation (1.3) is hyperbolic, so without loss of generality $B = \mathrm{diag}(B)$. Because of these smoothness assumptions (1.7) can be rewritten in the form

$$\begin{aligned} \tilde{\varphi}_t &= \lambda, \\ \tilde{v}_t &= P\tilde{v} + \lambda \underline{v}_x + F_1(\tilde{\varphi}, \tilde{v}) + F_2(\tilde{\varphi}, \tilde{v}) + R(\tilde{\varphi}, \lambda), \\ 0 &= \Psi(\tilde{v}), \end{aligned} \tag{2.1}$$

where $P\tilde{v} = B\tilde{v}_x + f_v(\underline{v})\tilde{v}$ is the linearization of the right hand side of (1.3) about the wave's profile and the nonlinearities in (2.1) are obtained by a Taylor expansion and read

$$\begin{aligned} F_1(\tilde{\varphi}, \tilde{v}) &= -\int_0^1 f_{vv}\big(\underline{v}(\cdot - s\tilde{\varphi})\big)[\underline{v}_x(\cdot - s\tilde{\varphi}), \tilde{\varphi}\tilde{v}]\, ds, \\ F_2(\tilde{\varphi}, \tilde{v}) &= \int_0^1 (1-s) f_{vv}\big(\underline{v}(\cdot - \tilde{\varphi}) + s\tilde{v}\big)\, ds\, [\tilde{v}, \tilde{v}], \\ R(\tilde{\varphi}, \lambda) &= -\int_0^1 \underline{v}_{xx}(\cdot - s\tilde{\varphi})\, ds\, \tilde{\varphi}\lambda. \end{aligned} \tag{2.2}$$

For the analysis of the nonlinear PDAE we consider the nonlinearities as inhomogeneities for the linear PDAE where F_1, F_2, and R are replaced by some x and t dependent function which does not depend on $\tilde{v}$, $\tilde{\varphi}$, and λ. Since $\tilde{\varphi}$ only appears in the nonlinear terms in the $\tilde{v}$–equation, the first equation in (2.1) then decouples from the rest and can be integrated in an additional step. Hence we obtain the linear PDAE

$$\begin{aligned} v_t &= Pv + \lambda \underline{v}_x + F(x, t), \\ 0 &= \Psi(v), \end{aligned} \tag{2.3}$$

where the $\tilde{}$ is dropped to simplify notation. As is usual for differential algebraic equations, initial conditions for (2.3) cannot be chosen arbitrarily, but there is a hidden constraint. For the PDAE problem (2.3) one easily sees that for given v the value of λ is given by $\lambda = -\Psi(\underline{v}_x)^{-1}(Pv + F)$. Therefore we assume that the problem is subject to consistent initial conditions

$$v(0) = v_0 \in H^1(\mathbb{R}, \mathbb{R}^m), \tag{2.4}$$

where the term *consistent* reflects that $\lambda(0)$ is uniquely defined by the hidden constraint. For the rest of this paper we assume $F \in \mathcal{C}([0, \infty); H^1(\mathbb{R})$ and we make the following definition of a solution for the PDAE (2.3), (2.4):

Definition 2.1. A tuple (v, λ) is called a (classical) solution of (2.3), (2.4) in $[0, T]$ if

$$v \in \mathcal{C}^1\big([0,T]; L^2(\mathbb{R}, \mathbb{R}^m)\big) \cap \mathcal{C}\big([0,T]; H^1(\mathbb{R}, \mathbb{R}^m)\big) \quad \text{and} \quad \lambda \in \mathcal{C}([0,T]; \mathbb{R}),$$

the first equation of (2.3) is an equality in $L^2(\mathbb{R}, \mathbb{R}^m)$, and the second equality holds in $\mathbb{R}$ for all $t \in [0, T]$. Moreover, v satisfies (2.4) and also the hidden constraint is satisfied at $t = 0$. The tuple is called a solution on $[0, \infty)$ if it is a solution on $[0, T]$ for every $T > 0$.

3 Resolvent estimates for PDEs

As already mentioned in the introduction, this section is in a sense preliminary to the following one but the resolvent estimates for linear hyperbolic operators shown here are also of interest in their own right. In particular, the result of Section 3.1, which shows that there is no spectrum in the right half plane of arbitrarily large absolute value is important. For example in numerical Evans function calculations of spectral stability this property can be used to choose smaller contours, see for example [8]. Consider the spectral problem

$$\big(sI_m - P\big)v = F \qquad \text{in } L^2(\mathbb{R}, \mathbb{C}^m), \tag{3.1}$$

for general linear first order operators $P : H^1(\mathbb{R}, \mathbb{C}^m) \to L^2(\mathbb{R}, \mathbb{C}^m)$, given by $Pv = Bv_x + Cv$, which satisfy the following properties:

Assumption 3.1. (H1) *For the functions $B \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{C}^{m,m})$, $C \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{C}^{m,m})$ exist*

$$\lim_{x\to\pm\infty} B(x) = B_\pm, \ \lim_{x\to\pm\infty} B_x(x) = 0, \ \lim_{x\to\pm\infty} C(x) = C_\pm, \ \lim_{x\to\pm\infty} C_x(x) = 0,$$

(H2) $B(x) = \operatorname{diag}(b_1(x)I_{m_1}, \ldots, b_N(x)I_{m_N}) \in \mathbb{R}^{m,m}$ *and for all* $x \in \mathbb{R}$ *and* $i \neq j \in \{1, \ldots, N\}$ *hold* $|b_i(x)| \geq b_0 > 0$, $|b_i(x) - b_j(x)| \geq \gamma > 0$,

(H3) $s \in \sigma(i\omega B_\pm + C_\pm)$ *for some* $\omega \in \mathbb{R}$ *implies* $\operatorname{Re} s < -\delta$.

Here and in the following we use the following notations: As usual I_m denotes the $m \times m$–identity matrix. We write $\operatorname{diag}(D_1, \ldots, D_m)$ for the diagonal matrix with entries $D_1, \ldots, D_m$ on its diagonal, where we allow D_i to be quadratic matrices. For matrix-valued functions $M \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{m,m})$ we define $M(\pm\infty) := M_\pm = \lim_{x\to\pm\infty} M(x)$, if the limits exist, so we write $M(x)$ for all $x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Note that under the smoothness assumptions, imposed in §2 on f and the profile, the operator P from (1.5) obviously is of the form considered above. Moreover, the smoothness assumptions on B and C are immediately satisfied in this case, so that only the invertibility of B (from (H2)) and the dispersion relation (H3) are new assumptions. Also note that in this section no assumptions on the point spectrum of P are needed.

A simple implication of (H3) is stated next.

Lemma 3.2. *Let $C_{j\pm} \in \mathbb{C}^{m_j,m_j}$, $j = 1, \ldots, N$ denote the diagonal blocks of the matrices $C_\pm$, which correspond to the block structure of the matrix B, i.e. $C_\pm = \begin{pmatrix} C_{1\pm} & & * \\ & \cdots & \\ * & & C_{N\pm} \end{pmatrix}$. Then $\lambda \in \sigma(C_{j\pm})$ for some $j \in \{1, \ldots, N\}$ implies* $\operatorname{Re} \lambda \leq -\delta$.

Proof. Let $\lambda \in \sigma(C_{j+})$. By Lemma A.1 there is $\omega_0 > 0$ so that for all $\omega \in \mathbb{R}, \omega \geq \omega_0$ there is an eigenvalue $s(\omega)$ of $i\omega B_+ + C_+$ in the set $\{s \in \mathbb{C} : |s - i\omega b_j - \lambda| \leq C_0|\omega|^{-1/(2m)}\}$, where $C_0 > 0$ is a constant. Assumption 3.1 (H3) implies $\operatorname{Re} s(\omega) < -\delta$, so that considering $\omega \to \infty$ shows $\operatorname{Re} \lambda \leq -\delta$. □

Because of the assumption on B, equation (3.1) can be rewritten as

$$v_x - M(x,s)v = -B^{-1}F, \tag{3.2}$$

where $M(x,s) = \big(sB(x)^{-1} - B(x)^{-1}C(x)\big)$ and $F \in L^2$ (or $F \in H^1$) if and only if $B^{-1}F \in L^2$ (or $B^{-1}F \in H^1$). We first show that the limit matrices $\lim_{x\to\pm\infty} M(x,s)$ are hyperbolic for all s from a right half plane.

Lemma 3.3. *For every $0 < \delta_0 < \delta$ there is $\alpha > 0$, so that $\lambda \in \sigma(M_\pm(s))$ implies $|\operatorname{Re}\lambda| \geq \alpha$ for all $\operatorname{Re} s > -\delta_0$.*

Proof. Assume there is a sequence $(s_n, \lambda_n)_n \in \mathbb{C}^2$ with $\operatorname{Re} s_n \geq -\delta_0$ for all n and $\operatorname{Re}\lambda_n \to 0$ as $n \to \infty$, so that $\det\big(\lambda_n I - s_n B^{-1} + B^{-1}C\big) = 0$ for all n. Because of Assumption 3.1 (H3) this immediately implies the unboundedness of the sequence $(s_n)_n$.

By considering subsequences, Lemma A.1 applied with $D = B^{-1}$ and $E = -B^{-1}C$ shows for suitable $j \in \{1,\dots,N\}$ and $\lambda_{jl} \in \sigma(C_j)$ the convergence $\lim_{n\to\infty} |\lambda_n - s_n b_j^{-1} + b_j^{-1}\lambda_{jl}| = 0$. Therefore, also $\lim_{n\to\infty} \operatorname{Re}\big(\lambda_n - s_n b_j^{-1} + b_j^{-1}\lambda_{jl}\big) = 0$, what contradicts $|\operatorname{Re}(s_n b_j^{-1} - b_j^{-1}\lambda_{jl})| \geq |b_j^{-1}|(\delta - \delta_0) > 0$. □

3.1 Resolvent estimates for large |s|

We first consider the problem for large absolute values of s in (3.1). In this case, Lemma A.2 applies to $B^{-1} - \frac{1}{s}B^{-1}C$ and yields a $\rho_0 > 0$ and for all $s \in \mathbb{C}$ with $|s| \geq \rho_0$ an invertible matrix $T(x, \frac{1}{s}) = I + \frac{1}{s}T_1(x, \frac{1}{s})$, with a uniformly bounded matrix T_1 and $\sup_{\substack{|s|\geq\rho_0 \\ x\in\mathbb{R}}} |T(x,\frac{1}{s})^{-1}| \leq 2$, which transforms $M(x,s) = s\big(B^{-1} - \frac{1}{s}B^{-1}C\big)$ into block diagonal form

$$\widetilde{M}(x,s) := T\big(x,\tfrac{1}{s}\big)^{-1} M(x,s) T\big(x,\tfrac{1}{s}\big) = \begin{pmatrix} M_1(x,s) & & 0 \\ & \ddots & \\ 0 & & M_N(x,s) \end{pmatrix} \quad \forall x \in \overline{\mathbb{R}},\ |s| \geq \rho_0. \tag{3.3}$$

Here $M_j \in \mathbb{C}^{m_j,m_j}$ is the restriction of M to the invariant subspace given by the generalized eigenspace of the $sb_j^{-1}(x)$–group of eigenvalues, i.e. of the set of eigenvalues that cluster around sb_j^{-1}. Thus, using the new variable $w(x) = T(x,\frac{1}{s})^{-1}v(x)$, equation (3.2) becomes

$$L(s)w = w_x - \widetilde{M}(x,s)w + T(x,\tfrac{1}{s})^{-1}T_x(x,\tfrac{1}{s})w = -T(x,\tfrac{1}{s})^{-1}B^{-1}F, \tag{3.4}$$

The nub of the analysis of (3.4) is the proof of exponential dichotomies for the block diagonal system $\widetilde{L}(s)w = w_x - \widetilde{M}(x,s)w$ and to show that for all $0 < \delta_0 < \delta$ the data can be chosen independently of $s \in \mathcal{M}_{\delta_0}(\rho_0) := \{s \in \mathbb{C} : \operatorname{Re} s > -\delta_0, |s| \geq \rho_0\}$, if $\rho_0 > 0$ is chosen sufficiently large. Note that because of the unboundedness of $\mathcal{M}_{\delta_0}(\rho_0)$, well-known results for exponential dichotomies (e.g. given in [9, §3]) do not suffice to prove uniformity of the data. But this property is essential for uniform resolvent estimates.

First we analyze the spectra of the limit matrices $M_{j\pm}(s) = \lim_{x\to\pm\infty} M_j(x,s)$.

Lemma 3.4. *Let M_j be given as above and assume $0 < \delta_0 < \delta$. Then there are $\rho_0, \alpha_0 > 0$ so that for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$,*

$$\text{either} \quad \sigma(M_{j\pm}(s)) \subset \{\operatorname{Re}\lambda \le -\alpha_0\} \quad \text{or} \quad \sigma(M_{j\pm}(s)) \subset \{\operatorname{Re}\lambda \ge \alpha_0\}.$$

Proof. For simplicity drop $\pm$. Let ρ_0 be given as above, so that T transforms $M(s)$ into the block diagonal form (3.3). Recall, that the eigenvalues of $M_j(s)$ cluster around sb_j^{-1}, so that for sufficiently large $s \in \mathbb{R}$, all eigenvalues of $M_j(s)$ have either positive real part, or they all have negative real part, depending on the sign of b_j. Because of the continuity of the eigenvalues the assertion follows from Lemma 3.3. □

Lemma 3.5. *Let the assumptions be as above and assume that $M_j(x,s)$ is given by (3.3). Let $0 < \delta_0 < \delta$. Then there are $\rho_0 > 0$ and $\alpha > 0$, $K_{j,\alpha} > 0$ so that for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$ hold*

$$\left|S_j(x,y)\right| \le K_{j,\alpha} e^{-\alpha(x-y)} \qquad \forall x \ge y, \text{ if } b_{j\pm} < 0, \tag{3.5}$$

$$\left|S_j(x,y)\right| \le K_{j,\alpha} e^{-\alpha(y-x)} \qquad \forall x \le y, \text{ if } b_{j\pm} > 0. \tag{3.6}$$

Moreover, there are positive δ_1, c_0, and K_j, such that for all $s \in \mathcal{M}_{-\delta_1}(\rho_0)$ it is possible to choose $\alpha = c_0 \operatorname{Re} s$ and $K_{j,\alpha} = K_j$.

Proof. Only consider the case $b_{j\pm} < 0$, the case $b_{j\pm} > 0$ then follows by an "inversion of time" argument. Furthermore, it suffices to prove (3.5) only for $x \ge y \ge 0$ because the same reasoning also shows (3.5) for $y \le x \le 0$, so that the estimate for all $x \ge y$ is implied by the semigroup property.

Because of Lemma 3.4 there are $\rho_0, \alpha_0 > 0$ so that for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$ holds $\operatorname{Re}\lambda < -\alpha_0$ for all eigenvalues λ of $M_{j+}(s)$. Let $Q_j = Q_j(s) \in \mathbb{C}^{m_j,m_j}$ be a unitary matrix which transforms $M_{j+}(s) - sB_{j+}^{-1}$ into Schur form

$$Q_j^*\Big(M_{j+}(s) - sB_{j+}^{-1}\Big)Q_j = D_j(s) + N_j(s), \tag{3.7}$$

where $D_j(s) = \operatorname{diag}(d_1(s), \dots, d_{m_j}(s))$ is a diagonal matrix and $N_j(s)$ is a nilpotent upper triangular matrix. By Lemma A.3 (applied with "$D = B^{-1}$" and "$E = B^{-1}C$") and the unitaryness of Q_j, the matrices $D_j(s)$ and $N_j(s)$ are uniformly bounded for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$.

Consider the linear differential operator

$$L_{j+}v = v_x - \big(M_{j+}(s) - sB_{j+}^{-1} + sB_j(x)^{-1}\big)v, \qquad x \ge 0.$$

Because of the structure $B_j = b_j I_{m_j}$, the change of variables $v = Q_j w$ transforms the operator L_{j+} into the form

$$\widetilde{L}_{j+}w := w_x - \Big(Q_j^*\big(M_{j+}(s) - sB_{j+}^{-1}\big)Q_j + sB_j(x)^{-1}\Big)w, \qquad x \ge 0. \tag{3.8}$$

The location of the eigenvalues of $M_{j+}(s)$ implies for the diagonal part

$$\limsup_{x\to\infty} \operatorname{Re}\big(sb_j(x)^{-1} + d_i(s)\big) < -\alpha_0 < 0 \quad \forall i = 1, \dots, m_j,\, \forall s \in \mathcal{M}_{\delta_0}(\rho_0).$$

Since s is restricted to a right half plane also the value

$$I_0(s) := \max_i \max_{0\le y\le x} \int_y^x \operatorname{Re} d_i(s) + sb_j(\xi)^{-1} + \alpha_0 \, d\xi$$

is uniformly bounded for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$. Therefore, Lemma B.7 shows for the solution operator $\widetilde{S}_{j+}$ of $\widetilde{L}_{j+}$ the estimate $|\widetilde{S}_{j+}(x,y)| \le C_{j,\alpha} e^{-\alpha(x-y)}$ for all $x \ge y \ge 0$, where $0 < \alpha < \alpha_0$ and $C_{j,\alpha}$ depends on the choice of α, but can be chosen independently of $s \in \mathcal{M}_{\delta_0}(\rho_0)$. Since the matrix Q_j is unitary, the same estimate holds for the solution operator S_{j+} of the original operator L_{j+}, i.e.

$$|S_{j+}(x,y)| \le C_{j,\alpha} e^{-\alpha(x-y)} \quad \forall x \ge y \ge 0. \tag{3.9}$$

Now consider the operator

$$L_j v = v_x - M_j(x,s)v = L_{j+}v - \Delta_j(x,s)v, \tag{3.10}$$

where $\Delta_j(x,s) = M_j(x,s) - M_{j+}(s) + sB_{j+}^{-1} - sB_j(x)^{-1}$. Lemma A.3 also shows that $\Delta_j(x,s)$ is uniformly bounded for all $x \ge 0$ and $s \in \mathcal{M}_{\delta_0}(\rho_0)$, and the limit

$$\lim_{x\to+\infty} M_j(x,s) - sB_j^{-1}(x) = M_{j+}(s) - sB_{j+}^{-1}$$

exists uniformly in s. Therefore, there is x_{j+} so that $|\Delta_j(x,s)| \le \frac{\alpha}{4C_\alpha}$ for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$ and all $x \ge x_{j+}$. The roughness theorem for exponential dichotomies B.4 shows the existence of $0 < \alpha' < \alpha$ and C'_α which do not depend on $s \in \mathcal{M}_{\delta_0}(\rho_0)$, so that for the solution operator S_j of (3.10) holds

$$|S_j(x,y)| \le C'_\alpha e^{-\alpha'(x-y)} \qquad \forall x \ge y \ge x_{j+}. \tag{3.11}$$

Furthermore, the variation of constant formula implies for all $0 \le y \le x \le x_{j+}$ the estimate

$$\begin{aligned} |S_j(x,y)v_0| &= \left| S_{j+}(x,y)v_0 + \int_y^x S_{j+}(x,\xi)\Delta_j(\xi,s)S_j(\xi,y)v_0 \, d\xi \right| \\ &\le C_\alpha e^{-\alpha(x-y)}|v_0| + \int_y^x C_\alpha e^{-\alpha(x-\xi)} \|\Delta_j(s)\|_\infty |S_j(\xi,y)v_0| \, d\xi, \end{aligned}$$

which shows

$$e^{\alpha(x-y)}|S_j(x,y)v_0| \le C_\alpha|v_0| + \int_y^x C_\alpha \|\Delta_j(s)\| e^{\alpha(\xi-y)} |S_j(\xi,y)v_0| \, d\xi.$$

Therefore, Gronwall's inequality and the arbitraryness of v_0 and $y \in [0, x_{j+}]$ yield

$$|S_j(x,y)| \le C''_\alpha e^{-\alpha(x-y)} \qquad \forall 0 \le y \le x \le x_{j+}, \tag{3.12}$$

where the constant C''_α does not depend on s. Combining (3.11) and (3.12) using the semi-group property proves (3.5).

Finally, the last statement follows from the uniform upper bound of $\operatorname{Re} d_i(s)$ together with Assumption 3.1 (H2) and Lemma B.7, see also the Remark B.8 following the lemma. $\square$

From the block structure of $\widetilde{M}(x,s)$ the above lemma immediately implies that the linear, non-autonomous differential operator $\widetilde{L}(s)$ possess exponential dichotomies. The precise statement is given in the following corollary.

Corollary 3.6. *Let $0 < \delta_0 < \delta$. Then there is $\rho_0 > 0$ so that for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$ the linear system $\widetilde{L}(s)w = w_x - \widetilde{M}(x,s)w$ has an ED on $\mathbb{R}$ with data $(\tilde{K}, \tilde{\beta}, \tilde{\pi})$, which can be chosen independently of s. Furthermore, the projector $\tilde{\pi}$ is given by*

$$\tilde{\pi} = \operatorname{diag}\big(I_{m_1}\mathbb{1}_{b_1<0}, \ldots, I_{m_N}\mathbb{1}_{b_n<0}\big), \text{ where } \mathbb{1}_{b_i<0} = \begin{cases} 1, & b_i < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, there are $\delta_1, c_0 > 0$ such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \delta_1$ it is possible to choose $\tilde{\beta} = \tilde{\beta}(s) = c_0 \operatorname{Re} s$ and $\tilde{\pi}$ and $\tilde{K}$ independently of s.

Together with the Roughness-Theorem B.4 we obtain an ED for the full system (3.4).

Theorem 3.7. *Let $0 < \delta_0 < \delta$. Then there is $\rho_0 > 0$ so that for all $s \in \mathcal{M}_{\delta_0}(\rho_0)$ the operator $L(s)$ from (3.4) has an ED on $\mathbb{R}$ with data (K, β, π), where K and β do not depend on s. Moreover, for the projectors hold with a constant k_0, independent of s, the estimate*

$$\big|\pi(x,s) - \widetilde{\pi}\big| \le \frac{k_0}{|s|}. \tag{3.13}$$

Finally, there are $\delta_1, c_\beta, c_\pi > 0$, such that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \delta_1$ the data (K, β, π) can be chosen with $\beta = \beta(s) = c_\beta \operatorname{Re} s$ and $|\pi(x,s) - \widetilde{\pi}| \le c_\pi \frac{1}{|s| \operatorname{Re} s}$, and K is independent of s.

Proof. With the definition $\Delta(x,s) := T(x, \frac{1}{s})^{-1} T_x(x, \frac{1}{s})$ holds the identity

$$L(s)w = \widetilde{L}(s)w + \Delta(\cdot, s)w$$

and the perturbation satisfies the bound

$$\|\Delta(\cdot, s)\|_\infty \le \frac{1}{s} \|T^{-1}(\cdot, \tfrac{1}{s})\|_\infty \|T_{1,x}(\cdot, \tfrac{1}{s})\|_\infty. \tag{3.14}$$

Because of the uniform bounds of T^{-1} and $T_{1,x}$, the first part of the theorem then follows from the the Roughness-Theorem B.4 by choosing $\rho_0 > 0$, so that for all $|s| \ge \rho_0$ holds $\|\Delta(\cdot, s)\|_\infty \le \frac{\min(1, \widetilde{\beta})}{4\widetilde{K}}$, where $\widetilde{\beta}$ and $\widetilde{K}$ are given in Corollary 3.6.

For the "finally"–part increase δ_1 from Corollary 3.6 so that for all $\operatorname{Re} s > \delta_1$ holds

$$\|\Delta(\cdot, s)\|_\infty \le \frac{c_0 \operatorname{Re} s}{6\widetilde{K}},$$

where c_0 and $\widetilde{K}$ are given in the corollary. By the Roughness-Theorem B.4, for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \delta_1$ the operators $L(s)$ have an ED on $\mathbb{R}$ and the data satisfy

$$
\begin{aligned}
K(s) &= \widetilde{K}\Big(2 + \frac{4\|\Delta(\cdot,s)\|_\infty \widetilde{K}}{\widetilde{\beta}(s) - 3\,\|\Delta(\cdot,s)\|_\infty\,\widetilde{K}}\Big) \le 3\frac{1}{3}\widetilde{K},\\
\beta(s) &= \widetilde{\beta}(s) - 2\,\|\Delta(\cdot,s)\|_\infty\,\widetilde{K} \ge \frac{2}{3}\widetilde{\beta}(s) = \frac{2c_0}{3}\,\mathrm{Re}\,s,\\
|\pi(x,s) - \widetilde{\pi}| &\le K\widetilde{K}\frac{6\|\Delta(\cdot,s)\|_\infty}{5c_0\,\mathrm{Re}\,s}.
\end{aligned}
$$

□

After these preliminaries we are in place to prove solution estimates for (3.1). Let δ_0 and ρ_0 be given as in Theorem 3.7. First assume $F \in L^2(\mathbb{R},\mathbb{C}^{m,m})$, then $v \in H^1(\mathbb{R},\mathbb{C}^{m,m})$ is a solution of (3.1) with $s \in \mathcal{M}_{\delta_0}(\rho_0)$ if and only if $w = T(\cdot,\frac{1}{s})^{-1}v$ is a solution of (3.4). Because of the choice of $s \in \mathcal{M}_{\delta_0}(\rho_0)$, Theorem 3.7 shows that Theorem B.3 applies and a unique solution w exists and satisfies the estimate

$$\beta^2\|w\|^2_{L^2(\mathbb{R},\mathbb{C}^{m,m})} \le 5K^2\|T(\cdot,\tfrac{1}{s})^{-1}B^{-1}F\|^2_{L^2(\mathbb{R},\mathbb{C}^{m,m})}. \tag{3.15}$$

For the original function v this yields the bound

$$\|v\|^2_{L^2} \le c_{L^2}\|F\|^2_{L^2}, \tag{3.16}$$

with a constant c_{L^2} independent of s. Here the uniform bounds of T^{-1} and B^{-1} are used. Moreover, if $F \in H^1(\mathbb{R},\mathbb{C}^{m,m})$, and v solves (3.1), then differentiating (3.1) shows that v_x is a solution of

$$(sI - B\partial_x + CB_x)v_x = s\,(v_x) - B\,(v_x)_x + CB_x\,(v_x) = F_x + C_x v \qquad \text{in } L^2(\mathbb{R},\mathbb{C}^m).$$

Because of Assumption 3.1, this problem has the same structure as the original problem and the same reasoning as above shows the estimate

$$\|v_x\|^2_{L^2} \le c_{H^1}\|F_x\|^2_{L^2}. \tag{3.17}$$

This finishes the proof of solution estimates for equation (3.1) for the case of large $|s|$:

Proposition 3.8. *Let Assumption 3.1 hold and let* $0 < \delta_0 < \delta$. *Then there are* ρ_0 *and* $K > 0$, *so that for all* $s \in \mathcal{M}_{\delta_0}(\rho_0)$ *and every* $F \in \mathcal{H}$, *where* $\mathcal{H} = L^2(\mathbb{R},\mathbb{C}^m)$ *or* $\mathcal{H} = H^1(\mathbb{R},\mathbb{C}^m)$, *the resolvent equation* (3.1) *has a unique solution* $v \in H^1(\mathbb{R},\mathbb{C}^m)$ *which satisfies the estimate*

$$\|v\|^2_{\mathcal{H}} \le K\|F\|^2_{\mathcal{H}}. \tag{3.18}$$

REMARK 3.9. *The proof shows that there is* $\delta_1 > 0$ *so that for all* $s \in \mathbb{C}$ *with* $\mathrm{Re}\,s \ge \delta_1$ *the estimate can be improved to*

$$\mathrm{Re}(s)^2\|v\|^2_{\mathcal{H}} \le K\|F\|^2_{\mathcal{H}}.$$

This is important for the stability analysis of coupled hyperbolic-parabolic PDEs, see [24], but will not be used in this paper.

3.2 Resolvent estimates for bounded |s|

We also need resolvent estimates for spectral values s from compact sets. We begin with the existence of exponential dichotomies on both half lines for (3.2), which already implies Fredholm properties for the operators.

Lemma 3.10. *For every $s \in \{\operatorname{Re} s > -\delta\}$ the operator $L(s)v = v_x - M(x,s)v$ has (ED)s on both half-lines $\mathbb{R}_-$ and $\mathbb{R}_+$ with data $(K_\pm(s), \beta_\pm(s), \pi_\pm(x,s))$ and the projectors satisfy*

$$\dim(\mathcal{R}(\pi_+(x,s))) = \sum_{j:b_{j+}<0} m_j =: m_s \quad \textit{and}$$
$$\dim(\mathcal{R}(I - \pi_-(x,s))) = \sum_{j:b_{j-}>0} m_j =: m_u = m - m_s.$$

Proof. Because of Lemmas 3.3 and 3.4, the limit matrices $M_\pm(s)$ are hyperbolic and have m_s eigenvalues with a negative real part and $m_u = m - m_s$ eigenvalues with a positive real part. Therefore, Corollary B.5 to the Roughness Theorem B.4 shows the claim. □

For bounded s we obtain the following resolvent estimates (also see [24, Theorem 3.13]).

Proposition 3.11. *Let $\Omega \subset \{s \in \mathbb{C} : \operatorname{Re} s > -\delta\} \cap \rho(P)$ be a compact set. Then there is $K > 0$ so that for every $s \in \Omega$ and every $F \in L^2(\mathbb{R}, \mathbb{C}^m)$ there is a unique solution $v \in H^1(\mathbb{R}, \mathbb{C}^m)$ of (3.1). Furthermore, the solution can be estimated by*

$$\|v\|_{H^1}^2 \le K\|F\|^2. \tag{3.19}$$

Proof. Let $s \in \{\operatorname{Re} s > -\delta\} \cap \rho(P)$ be arbitrary. Then $L(s)$, given by $L(s)v = v_x - M(x,s)v$ has (ED)s on $\mathbb{R}_\pm$ with data $(K_\pm, \beta_\pm, \pi_\pm)$. Let $v_0 \in \mathcal{R}(\pi_+(0)) \cap \mathcal{N}(\pi_-(0))$ and let $S(\cdot,\cdot)$ be the solution operator for $L(s)$. It easily follows that $v(x) := S(x,0)v_0$ is a solution of the homogeneous equation $L(s)v = 0$ and also is an element of $H^1(\mathbb{R}, \mathbb{C}^m)$. Since $s \in \rho(P)$, this is only possible if $v \equiv 0$. Therefore, for all such s the operator $L(s)$ has an (ED) on $\mathbb{R}$. Let $(K(s), \beta(s), \pi(s))$ denote the dichotomy data for $L(s)$. Theorem B.3 shows that there is a unique solution $v \in H^1(\mathbb{R}, \mathbb{C}^m)$ of (3.2) and this solution satisfies the estimate $\|v\|_{H^1}^2 \le c_0\|B^{-1}F\|_{L^2}^2$, with a constant c_0 independent of F. A simple perturbation argument and recalling that (3.1) and (3.2) are equivalent finishes the proof. □

4 Resolvent estimates for the PDAE

Now we use the resolvent estimates obtained in the previous section for the linear PDE (3.1) to deduce resolvent estimates for the linearized PDAE (2.3). As usual, the resolvent equation is obtained by a formal application of the Laplace transform. For equation (2.3) this leads to the system

$$\begin{aligned} s\widehat{v} &= P\widehat{w} + \underline{v}_x\widehat{\lambda} + \widehat{F}, && \text{in } L^2(\mathbb{R}, \mathbb{C}^m), \text{ where } s \in \mathbb{C}, \\ 0 &= \Psi(\widehat{v}), && \text{in } \mathbb{C}, \end{aligned} \tag{4.1}$$

where we denote by $\widehat{\ }$ the Laplace transform with respect to t. A rigorous justification of the Laplace transform is given in the next section.

Having in mind the application to traveling wave stability, we assume from now on that the operator P, which is given in (1.5), has no other eigenvalue than 0 in the right half plane. More precisely, we impose the following

Assumption 4.1. *The operator $P : H^1 \subset L^2 \to L^2$, given by* (1.5)*, satisfies*

$$\sigma(P) \cap \{\mathrm{Re}\, s > -\delta\} = \{0\}, \quad P\underline{v}_x = 0,$$

and 0 *is an algebraically simple eigenvalue.*

Rewrite equation (4.1) in operator matrix form

$$\mathcal{A}(s)\begin{pmatrix} v \\ \lambda \end{pmatrix} := \begin{pmatrix} (sI-P) & -\underline{v}_x \\ \Psi(\,\cdot\,) & 0 \end{pmatrix}\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}, \quad \text{in } L^2(\mathbb{R},\mathbb{C}^m)\times\mathbb{C}, \tag{4.2}$$

where we dropped $\hat{}$ for simplicity. We allow for complex valued functions in (4.2) and for the rest of this section since these naturally arise through the Laplace transform. For simplicity we write L^2 and H^1 for $L^2(\mathbb{R};\mathbb{C}^m)$ and $H^1(\mathbb{R};\mathbb{C}^m)$, respectively.

For the analysis of (4.2) we use the projector which projects L^2 onto $\mathrm{span}\{\underline{v}_x\}$ along the subspace given by the kernel of Ψ. Properties of this projector are given in the following lemma.

Lemma 4.2. *The mapping $\Pi : L^2 \to L^2$, $\Pi(v) = \underline{v}_x\Psi(v)\Psi(\underline{v}_x)^{-1}$ is a bounded linear projector in L^2 and satisfies $\|\Pi(v)\|_{H^1} \le C\|v\|_{L^2}$.*
Moreover, the composition $\Pi \circ P : H^1 \to H^1$ continuously extends to a mapping $L^2 \to H^1$.

Proof. The boundedness of Π as a mapping $L^2 \to H^1$ follows from $\|\underline{v}_x\|_{H^1} < \infty$ and Assumption 1.1. The equality $\Pi^2 = \Pi$ is obvious. The other assertion follows because the differential operator $P : L^2 \to H^{-1}$ is continuous and Ψ is a continuous linear functional on H^{-1}. ◻

4.1 Resolvent estimates for large $|s|$

We begin with resolvent estimates for the hyperbolic PDAE (4.2) for large $|s|$. Note that the estimates are uniform in s for all s in a right half plane.

Lemma 4.3. *Let Assumptions 1.1 and 3.1 hold and assume $0 < \delta_0 < \delta$. Then there is a constant $C_0 > 0$, so that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \ge -\delta_0$, $|s| > C_0$, and all $F \in H^1(\mathbb{R},\mathbb{C}^m)$ there exists a unique solution of the resolvent equation* (4.2)*. Furthermore, it holds the estimate*

$$\|w\|_\diamond + |\lambda| \le \mathrm{const}\|F\|_\diamond, \tag{4.3}$$

where $\diamond = L^2$ for $F \in L^2$ and $\diamond = H^1$ for $F \in H^1$. The constant in (4.3) *is independent of F and s.*

Proof. By Proposition 3.8 there are positive constants C_0 and K so that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \ge -\delta_0$, $|s| > C_0$, and all $F \in L^2(\mathbb{R},\mathbb{C}^m)$ there is a unique solution $v^0 \in H^1(\mathbb{R},\mathbb{C}^m)$ of $(sI-P)v = F$, and the solution satisfies the estimates

$$\|v^0\|_{L^2} \le K\|F\|_{L^2} \quad \text{and} \tag{4.4}$$

$$\|v^0\|_{H^1} \le K\|F\|_{H^1} \quad \text{if } F \text{ is an element of } H^1(\mathbb{R},\mathbb{C}^m). \tag{4.5}$$

It is straight forward to verify that (v,λ) given by $v := (I-\Pi)v^0$ and $\lambda := -s\Psi(v^0)\Psi(\underline{v}_x)^{-1}$ yields a solution of the resolvent equation (4.2). In addition, (4.4) and (4.5) together with Lemma 4.2 imply $\|v\|_{L^2} \le \text{const}\|F\|_{L^2}$ and $\|v\|_{H^1} \le \text{const}\|F\|_{H^1}$, respectively. A uniform bound for the algebraic variable λ is obtained by using the identity $sv^0 = Pv^0 + F$:

$$|\lambda| = \frac{\left\|s\underline{v}_x\frac{\langle\psi,v^0\rangle}{\langle\psi,\underline{v}_x\rangle}\right\|_{L^2}}{\|\underline{v}_x\|_{L^2}} = \frac{\|s\Pi v^0\|_{L^2}}{\|\underline{v}_x\|_{L^2}} = \frac{\|\Pi(Pv^0+F)\|_{L^2}}{\|\underline{v}_x\|_{L^2}}$$
$$\le \text{const}\left(\|\Pi(Pv^0)\|_{L^2} + \|\Pi(F)\|_{L^2}\right) \le \text{const}\|F\|_{L^2}.$$

This finishes the proof of estimate (4.3).

For the proof of uniqueness note that the operator $\mathcal{A}(s) : H^1\times\mathbb{C} \to L^2\times\mathbb{C}$ is Fredholm of index zero. This follow from the fact that $(sI-P)$ is Fredholm of index zero by Proposition 3.8 together with the bordering Lemma [4, Lemma 2.3]. Therefore consider $\mathcal{A}(s)(v\ \lambda)^T = (F\ \sigma)^T$ with $F\in L^2$ and $\sigma\in\mathbb{C}$. Let v^0 be given as above. It follows that (v,λ) with $v = (I-\Pi)v^0 + \sigma\underline{v}_x\Psi(\underline{v}_x)^{-1}$, and $\lambda = -s(\Psi(v^0)-\sigma)\Psi(\underline{v}_x)^{-1}$ solves this equation, so that the operator $\mathcal{A}(s) : H^1\times\mathbb{C}\to L^2\times\mathbb{C}$ is onto and hence invertible. □

Remark. *A crucial trick in the proof is the substitution of $Pv^0 + F$ for sv^0 in the estimate of the algebraic variable which a priori does not seem to be uniformly bounded for all s. Note that the same trick is not possible for $\mathcal{A}(s)(v\,\lambda)^T = (F\,\sigma)^T$, which can be seen in the representation of its unique solution.*

4.2 Resolvent estimates in compact subsets of the resolvent set

Now consider bounded spectral values s with a real part larger than $-\delta$. Consider

$$\mathcal{A}(s)\begin{pmatrix}v\\ \lambda\end{pmatrix} = \begin{pmatrix}F\\ \sigma\end{pmatrix}, \quad \text{in } L^2(\mathbb{R},\mathbb{C}^m)\times\mathbb{C} \tag{4.6}$$

with $F\in L^2$ and $\sigma\in\mathbb{C}$. In contrast to the remark following Lemma 4.3, the next Lemma 4.4 states uniform solution estimates of this equation, which also includes a nonzero σ.

Lemma 4.4. *Let Assumptions 1.1, 3.1, and 4.1 hold. Then for every compact set $\Lambda \subset \{s : \text{Re}(s) > -\delta\}\setminus\{0\}$, all $F\in L^2$, and all $\sigma\in\mathbb{C}$ there is a unique solution $(v,\lambda)\in H^1\times\mathbb{C}$ of (4.6). Moreover, there is* const = const(Λ) *independent of s, F, and σ so that it holds the estimate*

$$\|v\|_{H^1} + |\lambda| \le \text{const}(\|F\|_{L^2} + |\sigma|). \tag{4.7}$$

Proof. By Proposition 3.11 there is a unique solution $v^0\in H^1$ of $(sI-P)v = F$ in L^2. The solution can be estimated by

$$\|v^0\|_{H^1} \le K\|F\|, \tag{4.8}$$

where K is independent of F and $s\in\Lambda$. Defining v and λ as in the proof of Lemma 4.3 by $v = (I-\Pi)v^0 + \sigma\underline{v}_x\Psi(\underline{v}_x)^{-1}$ and $\lambda = -s(\Psi(v^0)-\sigma)\Psi(\underline{v}_x)^{-1}$, shows existence and with the same arguments as above, also uniqueness of the solution. The asserted estimate (4.7) easily follows from (4.8) and the definitions of v and λ together with the compactness of Λ. □

4.3 Resolvent estimates for $|s|$ small

We finish this section by proving that the resolvent equation (4.2) is uniquely solvable with uniform solution estimates also in a small neighborhood of $s = 0$. This precisely means that the zero eigenvalue of P is removed from the spectrum by considering the PDAE–problem.

Lemma 4.5. *Under Assumptions 1.1, 3.1, and 4.1 there is $c_0 > 0$ so that for all $s \in \mathbb{C}$ with $|s| \le c_0$ and all $F \in L^2$ there is a unique solution $(v, \lambda) \in H^1 \times \mathbb{C}$ of the resolvent equation* (4.2)*. Furthermore, the estimate*

$$\|v\|_{H^1} + |\lambda| \le \text{const}\|F\|_{L^2} \tag{4.9}$$

holds with const *independent of s and F.*

The principal step is to show that $\mathcal{A}(0)$ is invertible, which is done in the following Lemma 4.6.

Lemma 4.6. *Let the assumptions of Lemma 4.5 hold and assume $s = 0$. Then the equation*

$$\begin{pmatrix} -P & -\underline{v}_x \\ \langle \psi, \cdot \rangle & 0 \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} F \\ \sigma \end{pmatrix} \qquad \text{in } L^2(\mathbb{R}; \mathbb{C}^m) \times \mathbb{C}, \tag{4.10}$$

is uniquely solvable for all $F \in L^2$ and $\sigma \in \mathbb{C}$. Moreover, there is a constant $C_0 > 0$, independent of F and σ, such that the solution $(v, \lambda) \in H^1 \times \mathbb{C}$ can be estimated by

$$\|v\|_{H^1} + |\lambda| \le C_0(\|F\|_{L^2} + |\sigma|). \tag{4.11}$$

For the proof of the estimate we use an estimate for a particular solution of a system which has (ED)s on both half lines. The needed result is given in Appendix B because it does not seem to appear elsewhere in the literature.

Proof. With the operator $L(s)$ from Section 3 rewrite (4.10) in the form

$$\begin{aligned} L_0 v := L(0)v = v_x + B^{-1}C(x)v = -B^{-1}F - B^{-1}\lambda \underline{v}_x, \\ \Psi(v) = \sigma. \end{aligned} \tag{4.12}$$

In Lemma 3.10 it is shown that the operator L_0 has exponential dichotomies on $\mathbb{R}_\pm$ and is a Fredholm operator of index 0. Therefore, Assumption 4.1 implies $\mathcal{N}(L_0) = \text{span}\{\underline{v}_x\}$ and zero is a simple eigenvalue of $P = BL_0$, so that $B^{-1}\underline{v}_x \notin \mathcal{R}(L_0)$. Furthermore, $\mathcal{R}(L_0)$ is a closed linear subspace of L^2 of codimension 1, so that there is a linear, continuous projector $Q : L^2 \to L^2$ onto $\mathcal{R}(L_0)$ along $\text{span}\{B^{-1}\underline{v}_x\}$ (cf. [18, III §3]). Define

$$\begin{aligned} r \in \mathcal{R}(L_0) \subset L^2 \quad \text{by} \quad r := Q(-B^{-1}F), \text{ and} \\ \lambda_0 \in \mathbb{C} \quad \text{by} \quad -B^{-1}F = r + \lambda_0 B^{-1}\underline{v}_x, \end{aligned} \tag{4.13}$$

which is possible because of $L^2 = \mathcal{R}(L_0) \oplus \text{span}\{B^{-1}\underline{v}_x\}$. Moreover, r and λ_0 from (4.13) can be estimated by

$$\|r\|_{L^2} + |\lambda_0| \le C_1\|F\|_{L^2}, \tag{4.14}$$

where C_1 depends on the norms of the projectors Q and $I-Q$. Because of the properties of L_0, Lemma B.6 applies. This proves the existence of a particular solution $v^0 \in H^1$ of

$$Lv^0 = r, \quad \text{which satisfies } \|v^0\|_{H^1} \le C_2 \|r\|_{L^2}, \tag{4.15}$$

for some constant C_2 which does not depend on r. Let $v := (I-\Pi)v_0 + \sigma \frac{\underline{v}_x}{\psi(\underline{v}_x)}$ and $\lambda = \lambda_0$. Together with (4.13) and $P\underline{v}_x = 0$ follow

$$-Pv - \lambda \underline{v}_x = -Pv^0 - \lambda_0 \underline{v}_x = -BLv^0 - \lambda_0 \underline{v}_x = -Br - B\lambda_0 B^{-1} \underline{v}_x = F \qquad \text{and}$$
$$\Psi(v) = \Psi\left((I-\Pi)v^0 + \sigma \frac{\underline{v}_x}{\psi(\underline{v}_x)}\right) = \sigma,$$

which proves the existence of a solution. Estimate (4.11) follows from the definitions of v and λ in combination with the estimates (4.14), (4.15), and Lemma 4.2.

Uniqueness of the solution is shown as in the proofs of Lemmas 4.3 and 4.4 by using the Fredholm property of P and an application of the bordering Lemma 2.3 from [4]. □

Proof of Lemma 4.5. In Lemma 4.6, the operator $\mathcal{A}(0) : H^1 \times \mathbb{C} \to L^2 \times \mathbb{C}$ is shown to be a linear homeomorphism. For general $s \in \mathbb{C}$ rewrite $\mathcal{A}(s)$ as

$$\mathcal{A}(s) = \mathcal{A}(0) + \begin{pmatrix} sI & 0 \\ 0 & 0 \end{pmatrix},$$

where I is considered as the inclusion $H^1 \hookrightarrow L^2$. Hence, $\|\mathcal{A}(s) - \mathcal{A}(0)\|_{H^1 \times \mathbb{C} \to L^2 \times \mathbb{C}} \le |s|$ and a usual perturbation argument proves for $|s| \le \frac{1}{2C_0}$, with C_0 from (4.11), the invertibility of $\mathcal{A}(s)$ and also $\|\mathcal{A}(s)^{-1}\|_{L^2 \times \mathbb{C} \to H^1 \times \mathbb{C}} \le 2C_0$. □

For the extension of the Laplace integral to larger half planes we need the analytic dependence of $\mathcal{A}(s)^{-1}$ on s. This immediately follows from Lemmas 4.4 and 4.6 and we collect the result for reference purposes in the following corollary.

Corollary 4.7. *Let Assumptions 1.1, 3.1, and 4.1 hold. Then for every $s \in \mathbb{C}$ with* $\operatorname{Re} s > -\delta$ *the operator $\mathcal{A}(s) : H^1 \times \mathbb{C} \to L^2 \times \mathbb{C}$ is a linear homeomorphism and its inverse $\mathcal{A}(s)^{-1} : L^2 \times \mathbb{C} \to H^1 \times \mathbb{C}$ depends holomorphically on the parameter s.*

5 Linear stability of the PDAE

In this section we state and prove our main Theorem 5.3 on the exponential stability of the linear partial differential algebraic equation (2.3), which we obtained by linearizing (2.1).

In a first step we rigorously justify the Laplace transform of (2.3) what leads to the resolvent equation (4.1), which was analyzed in §4. In particular, we consider (2.3) subject to consistent initial data (2.4) and the first lemma of this section shows the exponential well-posedness of the linear hyperbolic partial differential algebraic equation (2.3), (2.4).

Lemma 5.1 (Exponential well-posedness). *Let Assumptions 1.1 and 3.1 hold. Then for every* $F \in \mathcal{C}(J; H^1)$, $J = [0,T]$ *with* $T > 0$ *or* $J = \infty$, *and all consistent initial data* $v_0 \in H^1(\mathbb{R}, \mathbb{R}^m)$, *there is a unique (classical) solution* (v, λ) *of (2.3), (2.4) on* J.

Moreover, if $J = [0, \infty)$ *and there are* $C, \kappa \in \mathbb{R}$ *so that the inhomogeneity* F *satisfies an exponential growth bound of the form* $\|F(t)\|_{H^1} \le Ce^{\kappa t}$ *for all* $t \ge 0$, *then there exist constants* $c, \alpha \in \mathbb{R}$ *so that*

$$\|v(t)\|_{H^1} + |\lambda(t)| \le ce^{\alpha t}. \tag{5.1}$$

Proof. It is well-known that the linear operator $P : \mathcal{D}(P) = H^1 \subset L^2 \to L^2$ is the generator of a $\mathcal{C}^0$–semigroup on L^2.

Let Π be the projector defined in Lemma 4.2, then $(I - \Pi)P$ is a bounded linear perturbation of P and hence itself generates a $\mathcal{C}^0$–semigroup $(S(t))_{t\ge 0}$ on L^2 (cf. [22, Chapter 3, Thm 1.1]). Furthermore, Lemma 4.2 also shows $(I - \Pi)F \in \mathcal{C}(J; H^1)$. Thus, following [14, §2 Thm 1.3], there is a unique solution v^0 of the *projected* Cauchy problem

$$v_t = (I - \Pi)Pv + (I - \Pi)F, \qquad v(0) = v_0,$$

which is given by the variation of constants formula

$$v^0(t) = S(t)v_0 + \int_0^t S(t-s)(I - \Pi)F(s)\, ds,\ t \in J, \tag{5.2}$$

and satisfies the smoothness $v^0 \in \mathcal{C}^1(J; L^2) \cap \mathcal{C}^0(J; H^1)$. Let v and λ be given by

$$v(t) := v^0(t) \qquad \text{and} \qquad \lambda(t) := -\frac{\Psi\big(Pv^0(t) + F(t)\big)}{\Psi(\underline{v}_x)}. \tag{5.3}$$

It follows $\lambda \in \mathcal{C}(J; \mathbb{R})$ and (v, λ) thus defined is a solution of the PDAE (2.3), (2.4):

$$v_t = (I - \Pi)Pv + (I - \Pi)F = Pv + F - \big(\Pi Pv + \Pi F\big) = Pv + F + \lambda \underline{v}_x.$$

Moreover, since $v \in \mathcal{C}^1(J; L^2)$ and Ψ is a continuous linear functional on L^2, it follows $\Psi \circ v \in \mathcal{C}^1(J; \mathbb{R})$, so that the chain rule yields

$$\frac{d}{dt}\Psi(v) = \Psi(v_t) = \Psi\big((I - \Pi)Pv + (I - \Pi)F\big) = 0 \quad \forall t \in J. \tag{5.4}$$

Because of $\Psi(v_0) = 0$, this proves $\Psi(v(t)) = 0$ for all $t \in J$ since $\Psi(v_0) = 0$ and the equality (5.4) also implies $\Psi\big(Pv + F + \lambda\underline{v}_x\big) = 0$ for all $t \in J$, so that the hidden constraint is satisfied, too.

For the proof of uniqueness let (v^1, λ^1) be a (classical) solution of the PDAE (2.3),(2.4) and apply $(I - \Pi)$ to the PDE–part. This shows

$$\big((I - \Pi)v^1\big)_t = (I - \Pi)P\left((I - \Pi)v^1\right) + (I - \Pi)F \text{ in } L^2, \quad \left((I - \Pi)v^1\right)(0) = v_0,$$

so that $(I - \Pi)v^1 = v^0$ follows from the unique solvability of the projected equation. Furthermore, the algebraic constraint implies $\Pi(v^1) = 0$, so that $v^1 = v^0$. Then $\lambda^1 = \lambda$ is a direct consequence of the hidden constraint, which holds for all $t \in J$ because of the smoothness $v \in \mathcal{C}^1(J;L^2) \cap \mathcal{C}^0(J;H^1)$.

Now let $J = [0,\infty)$. Note that $S(t)$ is also a $\mathcal{C}^0$–semigroup on $\mathcal{D}(P) = H^1$, so that there are $M, \omega \geq 0$ with $\|S(t)\|_{H^1 \to H^1} \leq Me^{\omega t}$ for all $t \geq 0$. Therefore, the representation (5.2) of the solution v^0 implies together with the exponential growth bound of F the estimate

$$\|v^0(t)\|_{H^1} \leq Me^{\omega t}\|v_0\|_{H^1} + \int_0^t Me^{\omega(t-s)}\|I - \Pi\|_{H^1 \to H^1} Ce^{\kappa s}\, ds \leq c_1 e^{c_2 t} \quad \forall t \geq 0. \tag{5.5}$$

Finally, to obtain the bound for $|\lambda(t)|$, insert the estimate for F and (5.5) into the definition of λ in (5.3). □

The proof of our main theorem makes use of the following basic energy estimate (see for example [25, Lem E.1], similar results are stated in [3, Ch. 2]).

Lemma 5.2. *Let $F \in \mathcal{C}([0,T];H^1)$, $v_0 \in H^1$, B a constant diagonal matrix and $C \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^{m,m})$. Assume that $v \in \mathcal{C}([0,T];H^1) \cap \mathcal{C}^1([0,T];L^2)$ is a classical solution of*

$$v_t = Bv_x + Cv + F, \quad \text{in } \mathbb{R} \times [0,T], \; v(0) = v_0.$$

Then for each $\delta_0 > 0$ there is a constant $K = K(\delta_0, B, C)$ so that

$$e^{2\eta t}\|v(t)\|_{H^1}^2 \leq K \left[\int_0^t e^{2\eta\tau}\|v(\tau)\|_{H^1}^2\, d\tau + \int_0^t e^{2\eta\tau}\, \|F(\tau)\|_{H^1}^2 \, d\tau \right] + \|v_0\|_{H^1}^2 \tag{5.6}$$

for all $\eta \leq \delta_0$ and $t \in [0,T]$.

Now we are ready to prove our main result, Theorem 5.3, which is stated next. The theorem is a major step in the proof of nonlinear stability for traveling waves which will be the subject of a future paper.

The proof draws many of its basic ideas from [19, Section 5]. But there the authors consider a pure Cauchy problem and not a partial differential algebraic equation as we do here. Moreover, in that article the use of the Laplace transform and of Parseval's relation is not completely justified and the proof seems to be on a rather formal level.

The key idea which allows us to deduce solution estimates from the resolvent estimates is as follows: Since the problem is exponentially well-posed, it is possible to Laplace transform the PDAE which leads to the resolvent equation. This transform is a priori only possible for spectral values with a sufficiently large real part. Nevertheless, the results from §4 show that the Laplace transform can be extended holomorphically to the larger half plane $\{\operatorname{Re} s > -\delta\}$. In the proof of Theorem 5.3 it is then shown that this analytic continuation in fact **is** the Laplace transform of the solution, so that we obtain estimates for the solution (v,λ) from the resolvent estimates through Parseval's relation. All the needed results from the vector valued Laplace transform are given in [1] and cited in the proof.

Theorem 5.3 (Linear PDAE–stability)**.** *Let Assumptions 1.1, 3.1, and 4.1 hold. Let $F \in \mathcal{C}(J;H^1)$, $J = [0,T]$ with $T > 0$ or $J = [0,\infty)$, and consider consistent initial data $v_0 \in H^2(\mathbb{R})$. Then there is a unique solution (v,λ) of the linear PDAE (2.3), (2.4) on J.*

Moreover, if $\delta_0 < \delta$ then there is a positive constant $C_l = C_l(\delta_0)$, independent of F and v_0, so that for all $\eta \le \delta_0$ the solution satisfies for all $t \in J$ the estimate

$$\|v(t)\|_{H^1}^2 + e^{-2\eta t}\int_0^t e^{2\eta\tau}\left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right) d\tau \le C_l e^{-2\eta t}\Big[\|v_0\|_{H^2}^2 + \int_0^t e^{2\eta\tau}\|F(\tau)\|_{H^1}^2\, d\tau\Big]. \quad (5.7)$$

Proof. **Step 1:** Let $J = [0,\infty)$, $v_0 = 0$, and assume that there are $C, \kappa \in \mathbb{R}$ so that $\|F(t)\|_{H^1}^2 \le Ce^{\kappa t}$ for all $t \ge 0$ and, furthermore,

$$\int_0^\infty e^{2\delta\tau}\|F(\tau)\|_{H^1}^2\, d\tau < \infty. \quad (5.8)$$

By Lemma 5.1 there is a unique (classical) solution (v,λ) of (2.3), (2.4) which satisfies an exponential growth bound. Therefore, there is $\alpha \in \mathbb{R}$ such that the Laplace transform of (2.3) is justified for all $s \in \mathbb{C}$ with $\operatorname{Re} s \ge \alpha$ (see for example [1, Prop 1.7.6.]) and yields

$$\begin{pmatrix} (sI - P) & -\underline{v}_x \\ \psi(\cdot) & 0 \end{pmatrix}\begin{pmatrix} \hat{v}(s) \\ \hat{\lambda}(s)\end{pmatrix} = \begin{pmatrix} \hat{F}(s) \\ 0 \end{pmatrix} \qquad \text{in } L^2(\mathbb{R},\mathbb{C}^m)\times\mathbb{C}. \quad (5.9)$$

Because of (5.8) and Corollary 4.7 the holomorphic function $s \mapsto (\hat{v}(s),\hat{\lambda}(s)) \in H^1(\mathbb{R},\mathbb{C}^m)\times\mathbb{C}$, which is a priori only defined for $\operatorname{Re} s \ge \alpha$, extends analytically to the half plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\delta\}$ and is given on this larger half plane as the solution of the resolvent equation (5.9). Furthermore, the resolvent estimates from Lemmas 4.3–4.5 imply for all $s \in \mathbb{C}$ with $\operatorname{Re} s \ge -\delta_0 > -\delta$ that there is a constant $K_{\delta_0} > 0$, independent of F and s, so that this extension satisfies

$$\|\hat{v}(s)\|_{H^1}^2 + \left|\hat{\lambda}(s)\right|^2 \le K_{\delta_0}\left\|\widehat{F}(s)\right\|_{H^1}^2 \qquad \forall \operatorname{Re} s \ge -\delta_0. \quad (5.10)$$

Because $(v,\lambda) \in \mathcal{C}([0,\infty); H^1\times\mathbb{R})$ is exponentially bounded and its Laplace transform has a bounded holomorphic extension to the half plane $\{\operatorname{Re} s \ge -\delta_0\}$, [1, Thm 4.4.13.] shows that the Laplace integral of (v,λ) exists for all $s \in \mathbb{C}$ with $\operatorname{Re} s > -\delta$ and is given by the analytic continuation. Using (5.10), Parseval's relation (see for example [1, §1.8]) implies for all $\eta \le \delta_0$ the estimate

$$\begin{aligned}\int_0^\infty e^{2\eta\tau}\left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right) d\tau &= \frac{1}{2\pi}\int_{-\infty}^\infty \|\widehat{v}(-\eta+i\xi)\|_{H^1}^2 + |\widehat{\lambda}(-\eta+i\xi)|^2\, d\xi \\ &\le \frac{K_{\delta_0}}{2\pi}\int_{-\infty}^\infty \left\|\widehat{F}(-\eta+i\xi)\right\|_{H^1}^2 d\xi = K_{\delta_0}\int_0^\infty e^{2\eta\tau}\|F(\tau)\|_{H^1}^2\, d\tau. \end{aligned} \quad (5.11)$$

Step 2: Let $J = [0,\infty)$. Let $F \in \mathcal{C}([0,\infty); H^1)$ be arbitrary and let $v_0 = 0$.

Let (v,λ) be the solution of (2.3), (2.4) to these data. Let $t > 0$ be arbitrary and define

$$\widetilde{F}(\tau) = \begin{cases} F(\tau), & 0 \le \tau \le t, \\ \frac{1}{\varepsilon}(t+\varepsilon-\tau)F(\tau), & t \le \tau \le t+\varepsilon, \\ 0, & \text{otherwise.}\end{cases}$$

The function $\widetilde{F}$ is an element of $\mathcal{C}([0,\infty);H^1)$ and satisfies the properties of F from Step 1. Let $(\widetilde{v},\widetilde{\lambda})$ be the solution of (2.3), (2.4) with the new inhomogeneity $\widetilde{F}$. From the uniqueness of solutions on $[0,t]$ follow the identities $v|_{[0,t]} = \widetilde{v}|_{[0,t]}$ and $\lambda|_{[0,t]} = \widetilde{\lambda}|_{[0,t]}$. The estimate (5.11) from the first step therefore shows for all $\eta \le \delta_0$

$$\begin{aligned}\int_0^t e^{2\eta\tau}\left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right)\,d\tau &\le \int_0^\infty e^{2\eta\tau}\left(\|\widetilde{v}(\tau)\|_{H^1}^2 + |\widetilde{\lambda}(\tau)|^2\right)\,d\tau\\ &\le K_{\delta_0}\int_0^\infty e^{2\eta\tau}\|\widetilde{F}(\tau)\|_{H^1}^2\,d\tau\\ &= K_{\delta_0}\left[\int_0^t e^{2\eta\tau}\|F(\tau)\|_{H^1}^2\,d\tau + \int_t^{t+\varepsilon} e^{2\eta\tau}\left(\frac{t+\varepsilon-\tau}{\varepsilon}\right)^2\|F(\tau)\|_{H^1}^2\,d\tau\right],\end{aligned}$$

so that in the limit $\varepsilon \searrow 0$ we find

$$\int_0^t e^{2\eta\tau}\left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right)\,d\tau \le K_{\delta_0}\int_0^\tau e^{2\eta\tau}\|F(\tau)\|_{H^1}^2\,d\tau. \tag{5.12}$$

Now consider the first equation of (2.3) separately, i.e. consider

$$v_t = Pv + \lambda\underline{v}_x + F, \qquad v(0) = 0,$$

where the term $\lambda\underline{v}_x \in \mathcal{C}([0,\infty);H^1)$ is treated as part of the inhomogeneity. Therefore, the energy estimate from Lemma 5.2 shows that there is $K = K(\delta_0, P)$, which only depends on δ_0 and P, such that v can be estimated by

$$e^{2\eta t}\|v(t)\|_{H^1}^2 \le C_{\delta_0}\int_0^t e^{2\eta\tau}\left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right)\,d\tau + C_{\delta_0}\int_0^t e^{2\eta\tau}\|F(\tau)\|_{H^1}^2\,d\tau.$$

Together with (5.12), this estimate implies

$$\begin{aligned}&e^{2\eta t}\|v(t)\|_{H^1}^2 + \int_0^t e^{2\eta t}\left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right)\,d\tau\\ &\qquad\le [K_{\delta_0} + C_{\delta_0}(1+K_{\delta_0})]\int_0^t e^{2\eta\tau}\|F(\tau)\|_{H^1}^2\,d\tau \quad \forall \eta \le \delta_0.\end{aligned} \tag{5.13}$$

Step 3: Let $J = [0,\infty)$. In the general case $0 \ne v_0 \in H^2$ we use

$$\widetilde{v} = v - e^{-2\delta t}v_0. \tag{5.14}$$

to transform (2.3), (2.4) to homogeneous initial data. The linearity of P and Ψ show

$$\begin{aligned}&\widetilde{v}_t = v_t + 2\delta e^{-2\delta t}v_0 = P\widetilde{v} + \lambda\underline{v}_x + \left\{F + 2\delta e^{-2\delta t}v_0 + e^{-2\delta t}Pv_0\right\} \quad \text{and}\\ &0 = \Psi(\widetilde{v}) = \Psi(v) - \Psi(e^{-2\delta t}v_0) = \Psi(v),\end{aligned}$$

so that the original variables (v, λ) are a solution of the original linear PDAE if and only if the new variables $(\widetilde{v}, \lambda)$ are a solution of the linear PDAE subject to homogeneous initial data and the inhomogeneity F replaced by

$$\widetilde{F} = F + 2\delta e^{-2\delta t} v_0 + e^{-2\delta t} P v_0. \tag{5.15}$$

Because of the smoothness assumption $v_0 \in H^2(\mathbb{R}, \mathbb{R}^m)$, the new inhomogeneity $\widetilde{F}$ is also an element of $\mathcal{C}([0,\infty); H^1)$ and can be estimated by

$$\|\widetilde{F}(t)\|_{H^1} \leq \|F(t)\|_{H^1} + \widetilde{c} e^{-2\delta t} \|v_0\|_{H^2}, \tag{5.16}$$

where $\widetilde{c}$ depends on δ and P. Moreover, the linearity of Ψ implies that the algebraic and hidden constraints are satisfied by the transformed variables $(\widetilde{v}, \lambda)$ if and only if they are satisfied by the original variables (v, λ), so that $(0, \lambda_0)$ is a consistent initial datum for the new PDAE.

Therefore, the estimate (5.13) from Step 2 can be applied to the linear PDAE (2.3), (2.4) with the inhomogeneity F replaced by $\widetilde{F}$ and with the initial condition $v(0) = 0$. Recalling the definitions (5.14) and (5.15) proves for all $\eta \leq \delta_0$

$$\begin{aligned}
&e^{2\eta t} \|v(t)\|_{H^1}^2 + \int_0^t e^{2\eta\tau} \left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2 \right) d\tau \\
&\quad \leq 2 \Big\{ e^{2\eta t} \|\widetilde{v}(t)\|_{H^1}^2 + \int_0^t e^{2\eta\tau} \left(\|\widetilde{v}(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2 \right) d\tau \\
&\qquad\quad + e^{(2\eta - 4\delta)t} \|v_0\|_{H^1}^2 + \int_0^t e^{(2\eta-4\delta)\tau} \|v_0\|_{H^1}^2 \, d\tau \Big\}.
\end{aligned}$$

Now, an application of (5.13) bounds the right hand side by

$$\begin{aligned}
&\leq 2 \Big\{ [K_{\delta_0} + C_{\delta_0}(1 + K_{\delta_0})] \int_0^t e^{2\eta\tau} \|\widetilde{F}(\tau)\|_{H^1}^2 \, d\tau + \left(1 + \frac{1}{4\delta - 2\eta} \right) \|v_0\|_{H^1}^2 \Big\} \\
&\leq 2 \Big\{ 2 \, [K_{\delta_0} + C_{\delta_0}(1 + K_{\delta_0})] \int_0^t \left(e^{2\eta\tau} \|F(\tau)\|_{H^1}^2 + e^{(2\eta - 4\delta)\tau} \widetilde{c}^2 \|v_0\|_{H^2}^2 \right) d\tau \\
&\qquad + \left(1 + \frac{1}{4\delta - 2\eta} \right) \|v_0\|_{H^1}^2 \Big\},
\end{aligned}$$

where we used (5.16) for the last inequality. Therefore, the asserted estimate (5.7) follows.
Step 4: In the case of compact J, considering a continuous continuation of the function F to a function in $\mathcal{C}([0,\infty); H^1)$ finishes the proof. □

Looking at the proof, we get the following, very important remarks.

REMARKS. *1. The H^2 norm of the initial value is introduced in Step 3 of the proof where the problem is transformed to homogeneous initial data. This is also the reason for the appearance of the $\|v_0\|_{H^2}$ on the right hand side of (5.7).*

2. *It should also be noted, that the approach via the Laplace transform essentially depends on the Hilbertspace setting. Nevertheless, we obtain sharp estimates for the exponential convergence rate.*

A Block diagonalization of diagonal dominant matrices

Our first linear algebra lemma is a consequence of Gershgorin's circle theorem. It is used to derive the location of the essential spectrum from Assumption 3.1 (H3).

Lemma A.1. *Let* $D = \operatorname{diag}(d_1 I_{m_1}, \dots, d_N I_{m_N}) \in \mathbb{C}^{m,m}$ *with* $d_i \neq d_j$ *for* $i \neq j$. *Let* $E = \begin{pmatrix} E_1 & & * \\ & \cdots & \\ * & & E_n \end{pmatrix} \in \mathbb{C}^{m,m}$ *with* $E_i \in \mathbb{C}^{m_i,m_i}$. *Let* $\lambda_{j,1}, \dots, \lambda_{j,m_j}$ *denote the eigenvalues of* E_j, *repeated according to their multiplicities.*

Then there are C_0, ρ_0, *so that for all* $s \in \mathbb{C}$ *with* $|s| > \rho_0$ *there are precisely* $n_{j,l}$ *eigenvalues of* $sD + E$ *in the circle*

$$\mathcal{D}^j_{j,l}(s) = \left\{ \lambda \in \mathbb{C} : |\lambda - s d_j - \lambda_{j,l}| \leq C_0 |s|^{-\frac{1}{2m_j}} \right\},$$

where $n_{j,l}$ *denotes the algebraic multiplicity of the eigenvalue* $\lambda_{j,l}$ *of* E_j.

Proof. Let $T = \operatorname{diag}(T_1, \dots, T_N)$, where $T_j \in \mathbb{C}^{m_j,m_j}$ transforms E_j into Jordan canonical form, i.e.

$$T_j^{-1} E_j T_j = J_j = \operatorname{diag}(\lambda_{j,1}, \dots, \lambda_{j,m_j}) + \mathcal{N}_j,$$

where $\mathcal{N}_j \in \mathbb{C}^{m_j,m_j}$ is upper diagonal. Then T transforms $sD + E$

$$T^{-1}(sD + E)\, T = sD + \operatorname{diag}(\lambda_{1,1}, \dots, \lambda_{1,m_1}, \lambda_{2,1}, \dots, \lambda_{N,m_N}) + \mathcal{N}, \tag{A.1}$$

where $\mathcal{N}$ is of the form $\mathcal{N} = \begin{pmatrix} \mathcal{N}_1 & & * \\ & \cdots & \\ * & & \mathcal{N}_N \end{pmatrix}$. Let $S_j(\varepsilon)$ be given by

$$(S_j(\varepsilon))_{ik} = \begin{cases} 0, & i \neq k, \\ 1, & 1 \leq i = k \leq \sum_{r=1}^{j-1} m_r \text{ or } \sum_{r=1}^{j} m_r + 1 \leq i = k, \\ \varepsilon^{-m_j + l}, & i = k = \sum_{r=1}^{j-1} m_r + 1 + l, \end{cases}$$

i.e. $S_j(\varepsilon) = \operatorname{diag}(1, \dots, 1, \varepsilon^{-m_j}, \dots, \varepsilon^{-1}, 1, \dots, 1)$. Therefore one obtains

$$(T S_j(\varepsilon))^{-1}(sD + E)\, T S_j(\varepsilon) = sD + \operatorname{diag}(\lambda_{1_1}, \dots, \lambda_{N_{m_N}}) + S_j(\varepsilon)^{-1} \mathcal{N} S_j(\varepsilon),$$

and for $\varepsilon \leq 1$ the matrix $S_j(\varepsilon)^{-1} \mathcal{N} S_j(\varepsilon)$ is of the form

$$S_j(\varepsilon)^{-1} \mathcal{N} S_j(\varepsilon) = \begin{pmatrix} \mathcal{N}_1 & \mathcal{O}(\varepsilon^{-m_j}) & * \\ & \ddots & \\ \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon) \\ & & \ddots \\ * & \mathcal{O}(\varepsilon^{-m_j}) & \mathcal{N}_N \end{pmatrix}. \tag{A.2}$$

Let $C_0 = \|\mathcal{N}\|_\infty$ and choose $\rho_0 > 1$ so that for every $|s| > \rho_0$ and every $j = 1, \dots, m$, all the circles $D^j_{j,l}(s)$ and $D^j_{j,l'}(s)$ are disjoint if $\lambda_{j,l} \neq \lambda_{j,l'}$ and they are also disjoint from the circles $\mathcal{D}^j_{i,l} = \left\{ \lambda \in \mathbb{C} : |\lambda - s d_i - \lambda_{i,l}| \leq C_0 |s|^{1/2} \right\}$ for $i \neq j$. The assertion then follows from Gershgorin's circle theorem (e.g. [15, p. 320]) by choosing $\varepsilon = |s|^{-(1/(2m_j))}$ in (A.2). ∎

Lemma A.2. *Let $D, E \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{C}^{m,m})$, for which the limits $D(x) \to D_\pm$ and $E(x) \to E_\pm$ as $x \to \pm\infty$ exist. Assume $D = \operatorname{diag}(d_1(x)I_{m_1}, \cdots, d_N(x)I_{m_N})$ satisfies $|d_i(x) - d_j(x)| \geq \gamma > 0$ for all $i \neq j$ and all $x \in \overline{\mathbb{R}}$.*

Then there is $\varepsilon_0 > 0$ such that for all $x \in \overline{\mathbb{R}}$ and $\nu \in \mathbb{C}$ with $|\nu| < \varepsilon_0$ there exists $T_1(x,\nu)$, $T_1 \in \mathcal{C}_b^1(\mathbb{R} \times \{\nu \in \mathbb{C} : |\nu < \varepsilon_0\}, \mathbb{C}^{m,m})$, for which the matrix $T(x,\nu) = I_m + \nu T_1(x,\nu)$ is invertible and transforms the matrix $D - \nu E$ into block-diagonal form

$$T(x,\nu)^{-1}(D(x) - \nu E(x))T(x,\nu) = \operatorname{diag}(M_1(x,\nu), \ldots, M_N(x,\nu)), \qquad \forall x \in \overline{\mathbb{R}},\ |\nu| < \varepsilon_0, \tag{A.3}$$

where the diagonal-blocks $M_j(x,\nu) \in \mathbb{C}^{m_j,m_j}$ correspond to the d_j–group of eigenvalues.

Furthermore, $\|T(x,\nu)^{-1}\| \leq 2$ for all $x \in \overline{\mathbb{R}}$ and all $|\nu| < \varepsilon_0$, and finally, the matrices $T_1(x,\nu)$, $T_{1,x}(x,\nu)$, and $T(x,\nu)^{-1}$ are uniformly in $\{|\nu| < \varepsilon_0\}$ convergent as $x \to \pm\infty$.

Proof. For simplicity we suppress the dependency on x and use the notation $j_l := \sum_{n=1}^{j-1} m_n + l$, where $j \in \{1, \ldots, N\}$ and $l \in \{1, \ldots, m_j\}$.

Since

$$\mathcal{D}_j(\nu) := \{\lambda \in \mathbb{C} : |d_j - \lambda| \leq |\nu| \|E\|_{\infty,\infty}\} \supset \bigcup_{l=1}^{m_j} \Big\{\lambda \in \mathbb{C} : \big|d_j - \nu e_{j_l,j_l} - \lambda\big| \leq |\nu| \sum_{r \neq j_l} |e_{j_l,r}|\Big\},$$

the circle $\mathcal{D}_j(\nu)$ contains the union of the Gershgorin discs which are centered at $d_j - \nu e_{j_l,j_l}$, $l = 1, \ldots, m_j$. Choosing $\varepsilon_0 > 0$ sufficiently small implies $\mathcal{D}_j(\nu) \cap \mathcal{D}_k(\nu) = \emptyset$ for all $j \neq k$, $|\nu| \leq \varepsilon_0$, and $x \in \overline{\mathbb{R}}$. Thus, Gershgorin's circle theorem (e.g. [15, p. 320]) shows that there are precisely m_j eigenvalues (counted with multiplicity) of $D(x) - \nu E(x)$ in $\mathcal{D}_j(x,\nu)$.

Let $\varepsilon > 0$, so that the sets $G_j(x,\varepsilon) = \{z \in \mathbb{C} : |d_j(x) - z| \leq \varepsilon\}$ are pairwise disjoint for all $j = 1, \ldots, N$ and $x \in \overline{\mathbb{R}}$. Furthermore, let $\varepsilon_0 > 0$, so that $\overline{\mathcal{D}_j(x,\nu)} \subset G_j(x, \varepsilon/2)$ for all $x \in \overline{\mathbb{R}}, |\nu| \leq \varepsilon_0$. For every $j = 1, \ldots, m_N$ define for each $x \in \overline{\mathbb{R}}$ and $|\nu| < \varepsilon_0$ the spectral projectors

$$\Pi_j(x,\nu) := \frac{1}{2\pi i} \int_{\Gamma_j(x)} \big(zI - D(x) + \nu E(x)\big)^{-1} dz, \tag{A.4}$$

where the contour is $\Gamma_j(x) = \partial G_j(x,\varepsilon)$. By Cauchy's integral theorem and the continuity of the matrices D and E, the same contour $\Gamma_j(x_0)$ can be chosen for all (x,ν) as above, with x from a small neighborhood of x_0 (with the usual topology on $\mathbb{R} \cup \{\pm\infty\}$. From this easily follows the continuous differentiability of Π_j in $\mathbb{R} \times \{|\nu| < \varepsilon_0\}$ and also the existence of the limits $\lim_{x\to\pm\infty} \Pi_j(x,\nu) = \Pi_{j\pm}(\nu)$. Since

$$(\Pi_j(x,0))_{kl} = \begin{cases} 1 & \text{if } \sum_{n=1}^{j-1} m_n + 1 \leq k = l \leq \sum_{n=1}^{j} m_n, \\ 0 & \text{otherwise,} \end{cases}$$

it is possible to define for all $x \in \overline{\mathbb{R}}$, $|\nu| < \varepsilon_0$

$$\begin{aligned} \nu T_{1,j}(x,\nu) &:= \Pi_j(x,\nu) \begin{pmatrix} 0 & I_{m_j} & 0 \end{pmatrix}^T - \begin{pmatrix} 0 & I_{m_j} & 0 \end{pmatrix}^T \\ &= \frac{1}{2\pi i} \int_{\Gamma_j(x)} \big(zI - D + \nu E\big)^{-1} - \big(zI - D\big)^{-1} dz \begin{pmatrix} 0 & I_{m_j} & 0 \end{pmatrix}^T \\ &= \nu \frac{1}{2\pi i} \int_{\Gamma_j(x)} \big(zI - D + \nu E\big)^{-1} E \big(zI - D\big)^{-1} dz \begin{pmatrix} 0 & I_{m_j} & 0 \end{pmatrix}^T, \end{aligned}$$

where $\begin{pmatrix}0 & I_{m_j} & 0\end{pmatrix}^T$ stands for the $m \times m_j$–matrix with I_{m_j} in the appropriate place. By decreasing ε_0, the terms in the integrand can be estimated uniformly for all $x \in \overline{\mathbb{R}}$, $|\nu| < \varepsilon_0$, $j = 1, \dots, N$ by

$$\begin{aligned} |\big(zI - D\big)^{-1}|_\infty &\le 1/\varepsilon, \quad \text{and} \\ |\big(zI - D + \nu E\big)^{-1}|_\infty \le |\big(zI - D\big)^{-1}|_\infty &|\big(I + \nu E(zI - D)^{-1}\big)^{-1}|_\infty \le 2/\varepsilon. \end{aligned} \tag{A.5}$$

This proves that T, defined by

$$T(x,\nu) := \left[\Pi_1(x,\nu)\begin{pmatrix} I_{m_1} \\ 0 \end{pmatrix}, \dots, \Pi_N(x,\nu)\begin{pmatrix} 0 \\ I_{m_N} \end{pmatrix}\right], \tag{A.6}$$

is of the form $T = I_m + \nu T_1$, where $T_1(x,\nu) = [T_{1,1}(x,\nu), \dots, T_{1,N}(x,\nu)]$. Moreover, $T_1(x,\nu)$ is uniformly bounded for all $x \in \overline{\mathbb{R}}$ and all $|\nu| < \varepsilon_0$ because of (A.5). The usual arguments show that T_1 is continuously differentiable with respect to x and ν. More precisely, for all $x \in \mathbb{R}$ and all $|\nu| < \varepsilon_0$, its derivative is given by

$$\frac{d}{dx}\bigg|_{x=x_0} T_{1,j}(x,\nu) = \frac{1}{2\pi i}\int_{\Gamma_j(x_0)} \frac{d}{dx}\bigg|_{x=x_0} \Big(\big(zI - D + \nu E\big)^{-1} E \big(zI - D\big)^{-1}\Big)\, dz \begin{pmatrix} 0 \\ I_{m_j} \\ 0 \end{pmatrix}. \tag{A.7}$$

This and the properties of D, E, ε, and ε_0, show the existence of $\lim_{x\to\pm\infty} T_{1x}(x,\nu) = T_{1x,\pm}(\nu)$ uniformly in ν and also the uniform bound

$$|T_{1x}(x,\nu)|_\infty \le \frac{\text{const}}{\varepsilon^2} \quad \forall x \in \overline{\mathbb{R}},\ |\nu| < \varepsilon_0.$$

Decreasing ε_0, so that $\varepsilon_0|T_1(x,\nu)| < \frac{1}{2}$ for all $x \in \overline{\mathbb{R}}$ and all $|\nu| < \varepsilon_0$, implies the invertibility of $T(x,\nu)$. Moreover, $\exists \lim_{x\to\pm\infty} T(x,\nu)^{-1} = T_\pm(\nu)^{-1}$ and the convergence is uniform in ν for all $|\nu| < \varepsilon_0$.

Finally, the ranges of the projectors $\Pi_j(x,\nu)$ are invariant subspaces of $\big(D - \nu E\big)$ by construction, so that the block diagonal structure (A.3) follows. □

Lemma A.3. *Let the setting be as above. Let $M(x,s) = sD(x) - E(x)$. Then there is $\rho_0 > 0$ so that for all $s \in \mathbb{C}$ with $|s| \ge \rho_0$ and T from the previous lemma, the matrix*

$$T\big(x, \tfrac{1}{s}\big)^{-1} M(x,s) T\big(x, \tfrac{1}{s}\big) - sD(x)$$

is uniformly bounded for all $s \in \mathbb{C}$, $|s| \ge \rho_0$ and all $x \in \overline{\mathbb{R}}$. Furthermore, the limits

$$\lim_{x\to\pm\infty} T\big(x, \tfrac{1}{s}\big)^{-1} M(x,s)\, T\big(x, \tfrac{1}{s}\big) - sD(x) = T_\pm\big(\tfrac{1}{s}\big)^{-1} M_\pm(s)\, T_\pm\big(\tfrac{1}{s}\big) - sD_\pm$$

exist for all $|s| \ge \rho_0$ and the convergence is uniform in s.

Proof. Let ε_0 be given as in Lemma A.2 and choose $\rho_0 > \frac{1}{\varepsilon_0}$, so that $\frac{1}{\rho_0}|T_1(x, \frac{1}{s})| \le \frac{1}{2}$ for all $x \in \overline{\mathbb{R}}$ and $s \in \mathbb{C}$, $|s| \ge \rho_0$. Then

$$\begin{aligned} s\Big(I + \tfrac{1}{s}T_1\Big)^{-1} D\Big(I + \tfrac{1}{s}T_1\Big) - sD &= s\Big(I - \tfrac{1}{s}T_1\big(I + \tfrac{1}{s}T_1\big)^{-1}\Big)D\big(I + \tfrac{1}{s}T_1\big) - sD \\ &= DT_1 - T_1\big(I + \tfrac{1}{s}T_1\big)^{-1}D\big(I + \tfrac{1}{s}T_1\big). \end{aligned}$$

By Lemma A.2 and the choice of ρ_0 all terms on the right hand side satisfy uniform bounds and are uniformly in s convergent as $x \to \pm\infty$. Since $T(x, \frac{1}{s})^{-1}E(x)T(x, \frac{1}{s})$ is uniformly bounded and also converges uniformly in s for s sufficiently large, the second claim follows from the linearity of the transformation. □

B Exponential Dichotomies

In this appendix we recall the definition and some properties of exponential dichotomies (ED). Basic references are [9] and [21]. We also show a result concerning uniform bounds of particular solutions to systems which possess (ED)s on both half lines. This is essential for the resolvent estimates in §4.3. Let L denote an ordinary differential operator

$$Lz = z_x - M(x)z, \qquad x \in J, \tag{B.1}$$

where $M \in \mathcal{C}(J, \mathbb{C}^{l,l})$ is continuous on the closed interval $J = [x_-, x_+]$, $J = (-\infty, x_+]$, $J = [x_-, \infty)$, or $J = \mathbb{R}$, where $x_- < x_+ \in \mathbb{R}$. The solution-operator for L is denoted by $\mathcal{S}(\cdot, \cdot)$.

Definition B.1. The operator L has an exponential dichotomy on the interval J if there are constants $K, \beta > 0$, and for every $x \in J$ there is a projector $\pi(x) \in \mathbb{C}^{l,l}$ such that

$$\begin{aligned} S(x,y)\pi(y) &= \pi(x)S(x,y) && \forall x, y \in J, \\ |S(x,y)\pi(y)| &\le Ke^{-\beta(x-y)} && \forall x \ge y \in J, \\ |S(x,y)\,(I - \pi(y))| &\le Ke^{-\beta(y-x)} && \forall x < y \in J. \end{aligned}$$

The triple (K, β, π) is called the data of the dichotomy.

Remark B.2 (cf. [6, Appendix A]). *1. Definition B.1 implies for all $x, y \in J$ the equality $\pi(x) = S(x,y)\pi(y)S(y,x)$, so that $\pi(x)$ is uniquely determined for all $x \in J$ if π is known for at least one value $x \in J$.*

2. In general, the projectors $\pi(x)$ are not unique. But if

(a) $J = [x_-, \infty)$, $x_- \in \mathbb{R}$, then their ranges are uniquely given by

$$\mathcal{R}\big(\pi(y)\big) = \Big\{ z \in \mathbb{C}^l : \sup_{x>y} e^{-\beta(x-y)}\big|S(x,y)z\big| < \infty \Big\}.$$

Moreover, if $\hat{\pi}(x_-)$ is another projector with $\mathcal{R}\big(\pi(x_-)\big) = \mathcal{R}\big(\hat{\pi}(x_-)\big)$, then define $\hat{\pi}(x) = S(x, x_-)\hat{\pi}(x_-)S(x_-, x)$, $x \in J$.
It follows $\big|\pi(x) - \hat{\pi}(x)\big| \le Be^{-2\beta(x-x_-)}$ and L has an ED on J with data $(\hat{K}, \beta, \hat{\pi})$, where $B = K^2\big|\pi(x_-) - \hat{\pi}(x_-)\big|$ and $\hat{K} = K(B + K)$.

(b) $J=(-\infty,x_+]$, $x_+\in\mathbb{R}$, *then their kernels are uniquely given by*

$$\mathcal{N}\big(\pi(y)\big)=\mathcal{R}\big(I-\pi(y)\big)=\Big\{z\in\mathbb{C}^l:\sup_{x<y}e^{-\beta(x-y)}\big|S(x,y)z\big|<\infty\Big\}.$$

If there is another projector $\hat{\pi}(x_+)$, *satisfying* $\mathcal{N}\big(\hat{\pi}(x_+)\big)=\mathcal{N}\big(\pi(x_+)\big)$, *then define* $\hat{\pi}(x)=S(x,x_+)\hat{\pi}(x_+)S(x_+,x)$, $x\in J$.
It follows $\big|\pi(x)-\hat{\pi}(x)\big|\le Be^{-2\beta(x_+-x)}$ *and* L *has an ED on* J *with data* $(\hat{K},\beta,\hat{\pi})$, *where* B *and* K *are given as in a).*

Theorem B.3 ([5, Theorem A.1])**.** *Let* L *have an ED on* J *with data* (K,β,π). *Define the Green's function* G *with respect to* π *for all* $x,y\in J$ *by*

$$G(x,y)=\begin{cases}S(x,y)\pi(y), & y\le x,\\ S(x,y)\big(\pi(y)-I\big), & x<y.\end{cases}\tag{B.2}$$

Then for every $r\in L^2(J,\mathbb{C}^l)$, $\gamma_-\in\mathcal{R}\big(\pi(x_-)\big)$, $\gamma_+\in\mathcal{R}\big(I-\pi(x_+)\big)$, *there is a unique solution* $z\in H^1(J,\mathbb{C}^l)$ *of the boundary value problem*

$$\begin{aligned}Lz=r,\quad &\textit{in } L^2(J,\mathbb{C}^l),\\ \pi(x_-)z(x_-)=\gamma_-,\qquad &\big(I-\pi(x_+)\big)z(x_+)=\gamma_+.\end{aligned}\tag{B.3}$$

The solution is given by $z=z_{sp}+z_{hom}$, *where*

$$z_{sp}(x)=\int_J G(x,y)r(y)\,dy\quad \textit{is a solution of (B.3) with } \gamma_\pm=0, \textit{ and}\tag{B.4}$$

$$z_{hom}(x)=S(x,x_-)\gamma_-+S(x,x_+)\gamma_+.\tag{B.5}$$

Moreover, the function z *satisfies the estimates*

$$\beta^2\|z_{sp}\|^2+\beta(|z_{sp}(x_-)|^2+|z_{sp}(x_+)|^2)\le 5K^2\|r\|^2,\tag{B.6}$$

$$\beta\|z_{hom}\|^2+(|z_{hom}(x_-)|^2+|z_{hom}(x_+)|^2)\le(2+3K^2)(|\gamma_-|^2+|\gamma_+|^2).\tag{B.7}$$

In the case of unbounded J, *the boundary conditions at* $\pm\infty$ *are part of the function space and not stated explicitly. In particular, the corresponding* $\gamma_\pm$ *are zero in* (B.5) *and* (B.7).

An important property of EDs is its roughness under perturbations, which is stated in the next Theorem (cf. [5, Thm. A.3]).

Theorem B.4 (Roughness)**.** *Let* L *have an (ED) on* J *with data* (K,β,π). *Assume* $\Delta\in\mathcal{C}(J,\mathbb{C}^{l,l})$ *can be estimated by* $3K\|\Delta\|_\infty<\beta$.

Then the operator $\widetilde{L}z=z_x-(M+\Delta)z$ *has an (ED) on* J, *too. The data* $(\widetilde{K},\widetilde{\beta},\widetilde{\pi})$ *can be chosen so that*

$$\widetilde{K}=K\big(2+\frac{4\|\Delta\|_\infty K}{\beta-3\|\Delta\|_\infty K}\big),\qquad \widetilde{\beta}=\beta-2\|\Delta\|_\infty K,$$

$$\big|\widetilde{\pi}(x)-\pi(x)\big|\le K\widetilde{K}\int_J e^{-(\beta+\widetilde{\beta})|x-y|}\big|\Delta(y)\big|\,dy.$$

A simple corollary to the Theorem is the following result (also see [21, Lemma 3.4] and [5, Theorem A.4]).

Corollary B.5. *Let L have an (ED) on a semi-infinite interval $J = [x_0, \infty)$ with data (K, β, π). Assume $\Delta \in \mathcal{C}(J, \mathbb{C}^{l,l})$ satisfies $|\Delta(x)| \to 0$ as $x \to \infty$.*

Then the operator $\widetilde{L}z = z_x - (M + \Delta)z$ has an (ED) on J, too.

In the next lemma we collect some properties of L for the case where (ED)s hold on both semi-infinite intervals $(-\infty, 0]$ and $[0, \infty)$. The second statement of the lemma states a solution estimate for a special solution of $Lz = r$ on $\mathbb{R}$. Its proof follows ideas from [4, Appendix D.] and [6] but the assertion does not follow from these references and therefore we include a proof.

Lemma B.6. *Let $M \in \mathcal{C}_b(\mathbb{R}, \mathcal{C}^{l,l})$ such that for L from* (B.1) *hold (ED)s $(-\infty, 0]$ and $[0, \infty)$ with data $(K_\pm, \beta_\pm, \pi_\pm)$. Then the following hold:*

1. *$L : H^1(\mathbb{R}, \mathbb{C}^l) \to L^2(\mathbb{R}, \mathbb{C}^l)$ is Fredholm of index $\dim\big(\mathrm{span}(\pi_+(0))\big) + \dim\big(\mathcal{N}(\pi_-(0))\big) - l$;*
2. *if L is Fredholm of index 0, then there is $\mathrm{const} > 0$ such that for every $r \in \mathcal{R}(L) \subset L^2(\mathbb{R})$ there is a solution $z^0 \in H^1(\mathbb{R})$ of $Lz = r$ in $L^2(\mathbb{R}, \mathbb{C}^l)$ which satisfies the estimate*
$$\|z^0\|^2_{H^1(\mathbb{R})} \leq \mathrm{const}\|r\|^2_{L^2(\mathbb{R})}; \tag{B.8}$$
3. *in the case $\dim\big(\mathcal{R}(\pi_+(0))\big) = \dim\big(\mathcal{R}(\pi_-(0))\big)$ and $\mathcal{R}(\pi_+(0)) \cap \mathcal{N}(\pi_-(0)) = \{0\}$, the operator $L : H^1(\mathbb{R}, \mathbb{C}^l) \to L^2(\mathbb{R}, \mathbb{C}^l)$ has an (ED) on $\mathbb{R}$ with data (K, β, π), where the data depends on the data $(K_\pm, \beta_\pm, \pi_\pm)$.*

Proof. Assertion 1. is well-known for the case of L^∞-spaces [4, 21]. A proof for the case of L^p-spaces is presented in [6]. The third point is also well-known and for example stated in [5, Theorem A.5]. In this case the estimate (B.8) follows from (B.6) and by application of the differential equation.

Therefore, assume $\mathcal{R}(\pi_+(0)) \cap \mathcal{N}(\pi_-(0)) = Z \neq \{0\}$. We split the phase space $\mathbb{C}^l$ into
$$\mathbb{C}^l = Z \oplus U_+ \oplus U_- \oplus K, \tag{B.9}$$
where
$$Z = \mathcal{R}(\pi_+(0)) \cap \mathcal{N}(\pi_-(0)),$$
$$Z \oplus U_+ = \mathcal{R}(\pi_+(0)), \qquad Z \oplus U_- = \mathcal{N}(\pi_-(0)),$$
and K is a complementary space of $Z \oplus U_+ \oplus U_-$ in $\mathbb{C}^l$. By counting dimensions follows $\dim(K) = \dim(Z)$. Because of 2. from Remark B.2, we may assume without loss of generality
$$K \oplus U_- = \mathcal{N}(\pi_+(0)), \qquad K \oplus U_+ = \mathcal{R}(\pi_-(0)). \tag{B.10}$$
Let $G_\pm$ denote the Green's function from (B.2) on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Then Theorem B.3 shows that the functions $z_{sp,\pm}$, which are given by
$$z_{sp,\pm}(x) = \int_{\mathbb{R}_\pm} G_\pm(x,y) r(y)\, dy, \qquad x \in \mathbb{R}_\pm,$$

are particular solutions of $Lz = r$ in L^2 on $\mathbb{R}_\pm$. Moreover, they satisfy $\pi_-(0)z_{sp,-}(0) = 0$ and $\big(I - \pi_+(0)\big)z_{sp,+}(0) = 0$. Because of (B.6), these solutions also satisfy the estimates

$$\|z_{sp,\pm}\|^2_{L^2(\mathbb{R}_\pm)} \le \text{const}\,\|r\|^2_{L^2(\mathbb{R}_\pm)}\,, \quad \text{and} \quad |z_{sp,\pm}(0)|^2 \le \text{const}\,\|r\|^2_{L^2(\mathbb{R}_\pm)}\,, \tag{B.11}$$

with a constant that only depends on $\|M\|_\infty$ and the dichotomy data.

Since $z_{sp,+}(0) \in \mathcal{N}(\pi_+(0))$ and $z_{sp,-}(0) \in \mathcal{R}(\pi_-(0))$, the choice of the projectors implies

$$z_{sp,+}(0) = k_+ + u_- \qquad \text{and} \qquad z_{sp,-}(0) = k_- + u_+,$$

with uniquely determined $k_+, k_- \in K$, $u_- \in U_-$, and $u_+ \in U_+$. In particular, (B.11) implies that there is a constant, const > 0, that only depends on the data of the dichotomy, with

$$|k_+| + |k_-| + |u_+| + |u_-| \le \text{const}\|r\|^2_{L^2(\mathbb{R})}. \tag{B.12}$$

Theorem B.3 shows that every solution of $Lz = r$ on $\mathbb{R}_+$ and $\mathbb{R}_-$, is of the form $z_{\alpha,\pm}$ with

$$z_{\alpha,\pm}(x) = z_{sp,\pm}(x) + S(x,0)\alpha_\pm, \;\; \text{where} \begin{cases} \alpha_+ \in \mathcal{R}(\pi_+(0)) & \text{in the + case,} \\ \alpha_- \in \mathcal{N}(\pi_-(0)) & \text{in the − case.} \end{cases} \tag{B.13}$$

To obtain a solution of $Lz = r$ in $L^2(\mathbb{R}, \mathbb{C}^l)$ on the whole real line, the values $z_{\alpha,+}(0)$ and $z_{\alpha,-}(0)$ have to match, which implies the linear problem

$$k_+ + u_- + \alpha_+ = k_- + u_+ + \alpha_-, \tag{B.14}$$

where

$$\begin{gathered} k_+, k_- \in K, \qquad u_+ \in U_+, \qquad u_- \in U_-, \\ \alpha_+ = \widetilde{u}_+ + z_+ \in \mathcal{R}(\pi_+(0)), \quad \text{with} \quad \widetilde{u}_+ \in U_+,\; z_+ \in Z, \\ \alpha_- = \widetilde{u}_- + z_- \in \mathcal{N}(\pi_-(0)), \quad \text{with} \quad \widetilde{u}_- \in U_-,\; z_- \in Z. \end{gathered}$$

Because of the direct sum (B.9) a solution of (B.14) is uniquely determined by

$$k_+ = k_-, \quad z_+ = z_-, \quad u_+ = \widetilde{u}_+, \quad u_- = \widetilde{u}_-. \tag{B.15}$$

Since $r \in \mathcal{R}(L)$, by assumption, and (B.13) parametrizes all solutions on $\mathbb{R}_\pm$, the system (B.15) is solvable. Therefore, all values other than z_+ and z_- are uniquely determined by the values $z_{sp,\pm}(0)$. Thus, choosing $z_+ = z_- = 0$ and define $z^0(x) = z_{sp,+}(x) + S(x,0)(u_+)$ for $x \ge 0$ and $z^0(x) = z_{sp,-}(x) + S(x,0)(u_-)$ for $x \le 0$ yields a solution of the problem on the whole real line. Furthermore, from (B.11), (B.7), and (B.12) follows that there is a constant, const > 0, only depending on the data of the dichotomy, such that

$$\|z^0\|^2_{L^2(\mathbb{R}_\pm,\mathbb{C}^l)} \le \text{const}\|r\|^2_{L^2(\mathbb{R},\mathbb{C}^l)}, \quad \text{which implies} \quad \|z^0\|^2_{L^2(\mathbb{R})} \le \text{const}\|r\|^2_{L^2(\mathbb{R})}.$$

An application of the differential equation and using the boundedness of M proves (B.8). $\square$

The next Lemma shows solution estimates for homogeneous ODEs in upper triangular form.

Lemma B.7. *Let $D, N \in \mathcal{C}_b(\mathbb{R}, \mathbb{C}^{m,m})$, where $D = \operatorname{diag}(d_1(x), \dots, d_m(x))$ and $N = (n_{ij})$ with $n_{ij}(x) = 0\ \forall i \geq j,\ x \in \mathbb{R}$. Assume there is $\alpha_0 > 0$, so that $\limsup_{x\to\pm\infty} \operatorname{Re} d_i(x) < -\alpha_0$ or $\liminf_{x\to\pm\infty} \operatorname{Re} d_i(x) > \alpha_0\ \forall i = 1, \dots, m$.*

Then for every $0 < \alpha < \alpha_0$ there is $C_\alpha > 0$, so that the solution operator S of the linear ODE system $Lv = v_x - (D + N)v$ satisfies

$$|S(x,y)| \leq C_\alpha e^{-\alpha(x-y)} \quad \forall x \geq y \qquad \text{if } \limsup_{x\to\pm\infty} \operatorname{Re} d_i(x) < -\alpha_0 \quad \text{and} \tag{B.16}$$

$$|S(x,y)| \leq C_\alpha e^{-\alpha(y-x)} \quad \forall x \leq y \qquad \text{if } \liminf_{x\to\pm\infty} \operatorname{Re} d_i(x) > \alpha_0, \text{ respectively.} \tag{B.17}$$

The constant C_α only depends on m, $\sup_x |N(x)| = \|N\|_\infty$, $I_0 := \max_i \max_{x\leq y} \int_x^y \operatorname{Re} d_i(\xi) + \alpha_0\, d\xi$, and on $|\alpha - \alpha_0|$.

Proof. Let $0 < \alpha < \alpha_0$. For arbitrary $y \in \mathbb{R}$ and $v_0 \in \mathbb{C}^m$ the function $v(x) = S(x,y)v_0$ solves the homogeneous problem $Lv = 0,\ v(y) = v_0$. Because of the triagonal structure, the components of v are given by the recursive formula

$$v^k(x) = \exp\left(\int_y^x d_k(\xi)\, d\xi\right) v_0^k + \int_y^x \exp\left(\int_\xi^x d_k(\tau)\, d\tau\right) \sum_{j=k+1}^{m} n_{kj}(\xi) v^j(\xi)\, d\xi,$$

where v^k is the k–th component of v. This yields the bound

$$|v^k(x)| \leq e^{I_0} e^{-\alpha_0(x-y)} |v_0^k| + \int_y^x e^{I_0} e^{-\alpha_0(x-\xi)} \|N\|_{\infty,\infty} \max_{j=k+1,\dots,m} |v^j(\xi)|\, d\xi,$$

where $\|N\|_{\infty,\infty} = \sup_{x\in\mathbb{R}} |N(x)|_\infty$. Therefore, by induction one obtains the estimate

$$|v^k(x)| \leq e^{I_0} \sum_{j=0}^{m-k} \left[e^{I_0} \|N\|_{\infty,\infty} (x-y)\right]^j e^{-\alpha_0(x-y)} |v_0|_\infty$$

which is easily seen to be bounded by

$$\leq \left[e^{I_0} \sum_{j=0}^{m} \left(\|N\|_{\infty,\infty} \frac{j}{\alpha_0 - \alpha} e^{I_0 - 1}\right)^j\right] e^{-\alpha(x-y)} |v_0|_\infty. \tag{B.18}$$

This proves (B.16). Estimate (B.17) follows by "inversion of time". □

Remark B.8. *Note that in the case $\operatorname{Re} d_i(x) \leq -\alpha_0$ for all $x \in \mathbb{R}$ and all i, the constant C_α only depends on $\|N\|_\infty$ and the distance $|\alpha - \alpha_0|$ but is independent of D and does not depend on the actual values of α_0 or α. This allows for improved solution estimates for equation (3.2) in the case $\operatorname{Re} s >> 0$.*

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