OPTIMAL REGULARITY FOR SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH MULTIPLICATIVE NOISE

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ABSTRACT. This paper deals with the spatial and temporal regularity of the unique Hilbert space valued mild solution to a semilinear stochastic partial differential equation with nonlinear terms that satisfy global Lipschitz conditions. It is shown that the mild solution has the same optimal regularity properties as the stochastic convolution. The proof is elementary and makes use of existing results on the regularity of the solution, in particular, the Hölder continuity with a non-optimal exponent.

1. INTRODUCTION

Consider the following semilinear stochastic partial differential equation (SPDE)

$$\begin{aligned} \mathrm{d}X(t) + [AX(t) + F(X(t))]\,\mathrm{d}t &= G(X(t))\,\mathrm{d}W(t), \quad \text{for } 0 \le t \le T, \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

where the mild solution X takes values in a Hilbert space H. The linear operator $-A\colon D(A) \subset H \to H$ is self-adjoint and the generator of an analytic semigroup on H. For example, let $-A$ be the Laplacian with homogeneous Dirichlet boundary conditions and $H = L^2(\mathcal{D})$ for some bounded domain $\mathcal{D} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{D}$ or a convex domain with polygonal boundary. The nonlinear operators F and G are assumed to be globally Lipschitz continuous in the appropriate sense and $W\colon [0,T] \times \Omega \to U$ denotes a standard Q-Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ with values in some Hilbert space U.

In a recent paper [5] the authors prove the existence of a unique mild solution $X\colon [0,T]\times\Omega \to H$. Moreover, they show that X enjoys certain spatial and temporal regularity properties.

The spatial regularity is measured in terms of the domains $\dot{H}^r := D(A^{\frac{r}{2}})$, $r \ge 0$, of fractional powers of the operator A. If $-A$ is the Laplacian, these domains coincide with standard Sobolev spaces, for example, $\dot{H}^1 = H^1_0(\mathcal{D})$ or $\dot{H}^2 = H^1_0(\mathcal{D}) \cap H^2(\mathcal{D})$ (c.f. [7, Th. 6.4] or [12, Ch. 3]). The regularity in time is expressed by the Hölder exponent.

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Using only Lipschitz assumptions on F and G the authors of [5] show that for every $\gamma \in [0,1)$ the solution X maps into $\dot{H}^\gamma \subset H$ and is $\frac{\gamma}{2}$-Hölder continuous with respect to the norm $\big(\mathbf{E}\big[\|\cdot\|_H^p\big]\big)^{\frac{1}{p}}$, $p \in [2,\infty)$.

As we will demonstrate in this paper, it turns out that this is also true for the border case $\gamma = 1$. The proof is based on a very careful use of the smoothing property of the corresponding semigroup $E(t) = \mathrm{e}^{-tA}$ (see Lemma 3.2), and on the Hölder continuity of X with a suboptimal exponent (see Lemmas 3.4 and 3.5).

The case $\gamma = 1$ is of special interest in numerical analysis. For example, if one is analysing an approximation scheme based on a finite element method, the spatial regularity determines the order of convergence. Hence, a suboptimal regularity result leads to a suboptimal estimate of the order of convergence (c.f. [12]).

Evolution equations of the form (1.1) are also studied by other authors. We refer to [4, 6, 11, 14] and the references therein. A related result is [16], where conditions for spatial C^∞-regularity are given.

The optimal regularity of stochastic convolutions of the form

$$W_A^\Phi(t) = \int_0^t E(t-\sigma)\Phi(\sigma)\,\mathrm{d}W(\sigma),$$

is studied in [4, Prop. 6.18] and [2]. Here $E(t) = \mathrm{e}^{-tA}$ is an analytic semigroup and Φ is a stochastically square integrable ($p = 2$) process with values in the set of Hilbert-Schmidt operators. If, for $r \geq 0$, the process Φ is regular enough so that the process $t \mapsto A^{\frac{r}{2}}\Phi(t)$ is still stochastically square integrable, then the convolution is a stochastic process, which is square integrable with values in $\dot{H}^{1+r}$. There exist some generalizations of this result, for instance, to Banach space valued integrands [3], to the case $p > 2$ [13], and to Lévy noise [1].

Our regularity result for the mild solution of (1.1) coincides with the optimal regularity property of the stochastic convolution but with the restriction $r < 1$. In this sense we understand our result to be optimal.

This paper consists of four additional sections. In the next section we give a more precise formulation of our assumptions. In Section 3 we are concerned with the spatial regularity of the mild solution. The proof is divided into several lemmas, which contain the key ideas of proof. The lemmas are also useful in the proof of the temporal Hölder continuity in Section 4. The proof of continuity in the border case requires an additional argument in form of Lebesgue's dominated convergence theorem. This technique is also developed in Section 4. The last section briefly reviews our result in the special case of additive noise and gives an example which demonstrates that the spatial regularity results are indeed optimal.

2. Preliminaries

In this section we present the general form of the SPDE we are interested in. After introducing some notation we state our assumptions and cite the result on existence, uniqueness and regularity of a mild solution from [5].

By H we denote a separable Hilbert space $(H, (\cdot,\cdot), \|\cdot\|)$. Further, let $A\colon D(A) \subset H \to H$ be a densely defined, linear, self-adjoint, positive definite operator, which is not necessarily bounded but with compact inverse. Hence, there exists an increasing sequence of real numbers $(\lambda_n)_{n\geq 1}$ and an orthonormal basis of eigenvectors $(e_n)_{n\geq 1}$

in H such that $Ae_n = \lambda_n e_n$ and

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n (\to \infty).$$

The domain of A is characterized by

$$D(A) = \Big\{ x \in H \, : \, \sum_{n=1}^{\infty} \lambda_n^2 (x, e_n)^2 < \infty \Big\}.$$

Thus, $-A$ is the generator of an analytic semigroup of contractions, which is denoted by $E(t) = \mathrm{e}^{-At}$.

By $W \colon [0,T] \times \Omega \to U$ we denote a Q-Wiener process with values in a separable Hilbert space $(U, (\cdot,\cdot)_U, \|\cdot\|_U)$. While our underlying probability space is $(\Omega, \mathcal{F}, P)$, we assume that the Wiener process is adapted to a normal filtration $(\mathcal{F}_t)_{t\in[0,T]} \subset \mathcal{F}$. The covariance operator $Q \colon U \to U$ is linear, bounded, self-adjoint, positive semidefinite but not necessarily of finite trace.

We study the regularity properties of a stochastic process $X \colon [0,T] \times \Omega \to H$, $T > 0$, which is the mild solution to the stochastic partial differential equation (1.1). Thus, X satisfies the equation

$$X(t) = E(t)X_0 - \int_0^t E(t-\sigma)F(X(\sigma))\,\mathrm{d}\sigma + \int_0^t E(t-\sigma)G(X(\sigma))\,\mathrm{d}W(\sigma) \tag{2.1}$$

for all $0 \leq t \leq T$.

In order to formulate our assumptions and main result we introduce the notion of fractional powers of the linear operator A. For any $r \in \mathbb{R}$ the operator $A^{\frac{r}{2}}$ is given by

$$A^{\frac{r}{2}} x = \sum_{n=1}^{\infty} \lambda_n^{\frac{r}{2}} x_n e_n$$

for all

$$x \in D(A^{\frac{r}{2}}) = \Big\{ x = \sum_{n=1}^{\infty} x_n e_n \, : \, (x_n)_{n\geq 1} \subset \mathbb{R} \quad \text{with } \|x\|_r^2 := \|A^{\frac{r}{2}} x\|^2 = \sum_{n=1}^{\infty} \lambda_n^r x_n^2 < \infty \Big\}.$$

By defining $\dot{H}^r := D(A^{\frac{r}{2}})$ together with the norm $\|x\|_r$ for $r \in \mathbb{R}$, $\dot{H}^r$ becomes a Hilbert space.

As usual [4, 9] we introduce the separable Hilbert space $U_0 := Q^{\frac{1}{2}}(U)$ with the inner product $(u_0, v_0)_{U_0} := (Q^{-\frac{1}{2}} u_0, Q^{-\frac{1}{2}} v_0)_U$ with $Q^{-\frac{1}{2}}$ denoting the pseudoinverse. The diffusion operator G maps H into L_2^0, where L_2^0 denotes the space of all Hilbert-Schmidt operators $\Phi \colon U_0 \to H$ with norm

$$\|\Phi\|_{L_2^0}^2 := \sum_{m=1}^{\infty} \|\Phi \psi_m\|^2.$$

Here $(\psi_m)_{m\geq 1}$ is an arbitrary orthonormal basis of U_0 (for details see, for example, Proposition 2.3.4 in [9]). Further, $L_{2,r}^0$ denotes the set of all Hilbert-Schmidt operators $\Phi \colon U_0 \to \dot{H}^r$ together with the norm $\|\Phi\|_{L_{2,r}^0} := \|A^{\frac{r}{2}} \Phi\|_{L_2^0}$.

Let $r \in [0,1)$, $p \in [2,\infty)$ be given. As in [5, 10] we make the following additional assumptions.

Assumption 2.1. *There exists a constant C such that*

$$\|G(x) - G(y)\|_{L_2^0} \leq C\|x - y\| \quad \forall x, y \in H \tag{2.2}$$

and we have that $G(\dot{H}^r) \subset L_{2,r}^0$ and

$$\|G(x)\|_{L_{2,r}^0} \leq C\,(1 + \|x\|_r) \quad \forall x \in \dot{H}^r. \tag{2.3}$$

Assumption 2.2. *The nonlinearity F maps H into $\dot{H}^{-1+r}$. Furthermore, there exists a constant C such that*

$$\|F(x) - F(y)\|_{-1+r} \leq C\|x - y\| \quad \forall x, y \in H. \tag{2.4}$$

Assumption 2.3. *The initial value $X_0 \colon \Omega \to \dot{H}^{r+1}$ is an $\mathcal{F}_0$-measurable random variable with $\mathbf{E}\left[\|X_0\|_{r+1}^p\right] < \infty$.*

Under the above conditions Theorem 1 in [5] states that for every $\gamma \in [r, r+1)$ and $T > 0$ there exists an up to modification unique mild solution $X \colon [0,T] \times \Omega \to \dot{H}^\gamma$ to (1.1) of the form (2.1), which satisfies

$$\sup_{t\in[0,T]} \mathbf{E}\left[\|X(t)\|_\gamma^p\right] < \infty.$$

Moreover, the solution process is continuous with respect to $\left(\mathbf{E}\left[\|\cdot\|_\gamma^p\right]\right)^{\frac{1}{p}}$ and fulfills

$$\sup_{t_1,t_2\in[0,T], t_1\neq t_2} \frac{\left(\mathbf{E}\left[\|X(t_1) - X(t_2)\|_s^p\right]\right)^{\frac{1}{p}}}{|t_1 - t_2|^{\min(\frac{1}{2}, \frac{\gamma - s}{2})}} < \infty$$

for every $s \in [0, \gamma]$.

The aim of this paper is to show that these regularity results also hold with $\gamma = r + 1$.

Remarks. 1. Actually, Theorem 1 in [5] assumes that $F \colon H \to H$ is globally Lipschitz, which is slightly stronger than Assumption 2.2. That Assumption 2.2 is sufficient can be proved by just following the given proof line by line and making the appropriate changes where ever F comes into play.

2. Assumption 2.3 can be relaxed to $X_0 \colon \Omega \to H$ being a $\mathcal{F}_0$-measurable random variable with $\mathbf{E}\left[\|X_0\|^p\right] < \infty$. But, as it is known from deterministic PDE theory, this will lead to a singularity at $t = 0$.

3. The framework is quite general. We refer to the discussion in [10] for some more explicit examples, further references and a related result for temporal regularity. Further examples and a detailed discussion of Assumption 2.1 can be found in [5].

3. Spatial regularity

In this section we deal with the spatial regularity of the mild solution. Our result is given by the following theorem. For a more convenient notation we set $\|\cdot\|_{L^p(\Omega;\mathcal{H})} := \left(\mathbf{E}\left[\|\cdot\|_{\mathcal{H}}^p\right]\right)^{\frac{1}{p}}$ for any Hilbert space $\mathcal{H}$. Also, if applied to an operator, the norm $\|\cdot\|$ is understood as the operator norm for bounded, linear operators from H to H.

Theorem 3.1 (Spatial regularity)**.** *Let $r \in [0,1)$, $p \in [2,\infty)$. Given the assumptions of Section 2 the unique mild solution X in* (2.1) *satisfies*

$$\sup_{t\in[0,T]} \big(\mathbf{E}\big[\|X(t)\|_{r+1}^p\big]\big)^{\frac{1}{p}} \le \big(\mathbf{E}\big[\|X_0\|_{r+1}^p\big]\big)^{\frac{1}{p}} + C\Big(1 + \sup_{t\in[0,T]} \big(\mathbf{E}\big[\|X(t)\|_{r}^p\big]\big)^{\frac{1}{p}}\Big),$$

where the constant C depends on p, r, A, F, G, T and the Hölder continuity constant of X with respect to the norm $\|\cdot\|_{L^p(\Omega;H)}$.

In particular, X maps into $\dot H^{r+1}$ almost surely.

Before we prove the theorem we introduce several useful lemmas. The first states some well known facts on analytic semigroups (c.f. [8]). Since parts *(iii)*, *(iv)* are not readily found in the literature, we provide proofs here.

Lemma 3.2. *For the analytic semigroup $E(t)$ the following properties hold true:*

(i) For any $\mu \ge 0$ there exists a constant $C = C(\mu)$ such that

$$\|A^\mu E(t)\| \le Ct^{-\mu} \text{ for } t > 0.$$

(ii) For any $0 \le \nu \le 1$ there exists a constant $C = C(\nu)$ such that

$$\|A^{-\nu}(E(t) - I)\| \le Ct^\nu \text{ for } t \ge 0.$$

(iii) For any $0 \le \rho \le 1$ there exists a constant $C = C(\rho)$ such that

$$\int_{\tau_1}^{\tau_2} \|A^{\frac{\rho}{2}} E(\tau_2-\sigma)x\|^2 \,\mathrm{d}\sigma \le C(\tau_2-\tau_1)^{1-\rho}\, \|x\|^2 \text{ for all } x \in H,\ 0 \le \tau_1 < \tau_2.$$

(iv) For any $0 \le \rho \le 1$ there exists a constant $C = C(\rho)$ such that

$$\Big\| \int_{\tau_1}^{\tau_2} A^\rho E(\tau_2-\sigma)x \,\mathrm{d}\sigma \Big\| \le C(\tau_2-\tau_1)^{1-\rho}\|x\| \text{ for all } x \in H,\ 0 \le \tau_1 < \tau_2.$$

Proof. *(iii)* We use the expansion of $x \in H$ in terms of the eigenbasis $(e_n)_{n\ge1}$ of the operator A. By Parseval's identity we get

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \big\|A^{\frac{\rho}{2}} E(\tau_2-\sigma)x\big\|^2 \,\mathrm{d}\sigma &= \int_{\tau_1}^{\tau_2} \Big\| \sum_{n=1}^\infty A^{\frac{\rho}{2}} E(\tau_2-\sigma)(x,e_n)e_n \Big\|^2 \,\mathrm{d}\sigma \\ &= \sum_{n=1}^\infty \int_{\tau_1}^{\tau_2} (x,e_n)^2 \lambda_n^\rho \,\mathrm{e}^{-2\lambda_n(\tau_2-\sigma)} \,\mathrm{d}\sigma \\ &= \frac{1}{2}\sum_{n=1}^\infty (x,e_n)^2 \lambda_n^{\rho-1} \Big(1 - \mathrm{e}^{-2\lambda_n(\tau_2-\tau_1)}\Big).\end{aligned}$$

For all $\kappa \in [0,1]$ the function $x \mapsto \frac{1-\mathrm{e}^{-2x}}{x^\kappa}$ is bounded for $x \in [0,\infty)$. Hence,

$$0 \le \frac{1-\mathrm{e}^{-2\lambda_n(\tau_2-\tau_1)}}{(\lambda_n(\tau_2-\tau_1))^{1-\rho}} \le C(\rho)$$

for some constant $C(\rho)$, which depends only on ρ. Therefore,

$$\int_{\tau_1}^{\tau_2} \big\|A^{\frac{\rho}{2}} E(\tau_2-\sigma)x\big\|^2 \,\mathrm{d}\sigma \le \frac{1}{2}C(\rho)(\tau_2-\tau_1)^{1-\rho}\, \|x\|^2.$$

The proof of *(iv)* works in a similar way. We square the left-hand side and use Parseval's identity again. This yields

$$\begin{aligned}\Big\| \int_{\tau_1}^{\tau_2} A^\rho E(\tau_2-\sigma)x \,\mathrm{d}\sigma \Big\|^2 &= \Big\| \int_{\tau_1}^{\tau_2} \sum_{n=1}^\infty A^\rho E(\tau_2-\sigma)(x,e_n)e_n \,\mathrm{d}\sigma \Big\|^2 \\ &= \sum_{n=1}^\infty \Big(\int_{\tau_1}^{\tau_2} (x,e_n)\lambda_n^\rho \,\mathrm{e}^{-\lambda_n(\tau_2-\sigma)} \,\mathrm{d}\sigma \Big)^2 \\ &= \sum_{n=1}^\infty (x,e_n)^2 \Big(\frac{1-\mathrm{e}^{-\lambda_n(\tau_2-\tau_1)}}{\lambda_n^{1-\rho}} \Big)^2.\end{aligned}$$

As above we conclude

$$\Big\| \int_{\tau_1}^{\tau_2} A^\rho E(\tau_2-\sigma)x \,\mathrm{d}\sigma \Big\|^2 \le C(\rho)^2(\tau_2-\tau_1)^{2(1-\rho)}\|x\|^2.$$

The proof is complete. □

The next lemma is a special case of Lemma 7.2 in [4] and will be needed to estimate the stochastic integrals.

Lemma 3.3. *For any $p \ge 2$, $0 \le \tau_1 < \tau_2 \le T$, and for any L_2^0-valued predictable process $\Phi(t)$, $t \in [\tau_1,\tau_2]$, we have*

$$\mathbf{E}\Big[\Big\| \int_{\tau_1}^{\tau_2} \Phi(\sigma)\,\mathrm{d}W(\sigma)\Big\|^p\Big] \le C(p)\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \|\Phi(\sigma)\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big].$$

Here the constant can be chosen to be

$$C(p) = \Big(\frac{p}{2}(p-1)\Big)^{\frac{p}{2}} \Big(\frac{p}{p-1}\Big)^{p(\frac{p}{2}-1)}.$$

The following two lemmas contain our main idea of proof and yield the key estimates.

Lemma 3.4. *Let $s \in [0, r+1]$, $p \ge 2$, and Y be a predictable stochastic process on $[0,T]$ which maps into $\dot{H}^r$ with $\sup_{\sigma\in[0,T]} \|A^{\frac{r}{2}}Y(\sigma)\|_{L^p(\Omega;H)} < \infty$. Then there exists a constant $C = C(p,r,s,A,G)$ such that, for all $\tau_1,\tau_2 \in [0,T]$ with $\tau_1 < \tau_2$,*

$$\begin{aligned}(3.1)\quad &\Big(\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}}E(\tau_2-\sigma)G(Y(\tau_2))\big\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}} \\ &\qquad \le C\Big(1+\sup_{\sigma\in[0,T]} \big\|A^{\frac{r}{2}}Y(\sigma)\big\|_{L^p(\Omega;H)}\Big)(\tau_2-\tau_1)^{\min(\frac{1}{2},\frac{1+r-s}{2})}.\end{aligned}$$

If, in addition, for some $\delta > \frac{r}{2}$ there exists C_δ such that

$$\|Y(t_1)-Y(t_2)\|_{L^p(\Omega;H)} \le C_\delta |t_2-t_1|^\delta \text{ for all } t_1,t_2 \in [0,T],$$

then we also have, with $C = C(p,s,G,C_\delta)$, that

$$\begin{aligned}(3.2)\quad &\Big(\mathbf{E}\Big[\Big(\int_{\tau_1}^{\tau_2} \big\|A^{\frac{s}{2}}E(\tau_2-\sigma)\big(G(Y(\sigma))-G(Y(\tau_2))\big)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma\Big)^{\frac{p}{2}}\Big]\Big)^{\frac{1}{p}} \\ &\qquad \le \frac{C}{\sqrt{1+2\delta-s}}(\tau_2-\tau_1)^{\frac{1+2\delta-s}{2}}.\end{aligned}$$

In particular, with $C = C(T, \delta, p, r, s, A, G, C_\delta)$ it holds that

$$\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\sigma)) \,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \le C\Big(1 + \sup_{\sigma \in [0,T]} \big\| A^{\frac{r}{2}} Y(\sigma) \big\|_{L^p(\Omega;H)}\Big) (\tau_2 - \tau_1)^{\min(\frac{1}{2}, \frac{1+r-s}{2})}. \tag{3.3}$$

Proof. First note that, for $0 \le \tau_1 < \tau_2 \le T$ fixed, the mapping $[\tau_1, \tau_2] \ni \sigma \mapsto A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\sigma))$ is a predictable L_2^0-valued process. Hence, Lemma 3.3 is applicable and gives

$$\begin{aligned}
&\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\sigma)) \,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \\
&\le C(p) \Big\| \Big(\int_{\tau_1}^{\tau_2} \big\| A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\sigma)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\le C(p) \Big\| \Big(\int_{\tau_1}^{\tau_2} \big\| A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\tau_2)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + C(p) \Big\| \Big(\int_{\tau_1}^{\tau_2} \big\| A^{\frac{s}{2}} E(\tau_2 - \sigma) \big(G(Y(\sigma)) - G(Y(\tau_2)) \big) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&=: S_1 + S_2.
\end{aligned}$$

In the second step we just used the triangle inequality. Now we deal with both summands separately. In the first term S_1 the time in $G(Y(\tau_2))$ is fixed. We also notice that $\eta := s - r - \max(0, s - r) \le 0$ and, hence, $A^{\frac{\eta}{2}}$ is a bounded linear operator on H. Furthermore, since $s \in [0, r + 1]$ we have $\rho := \max(0, s - r) \in [0, 1]$ and Lemma 3.2 *(iii)* is applicable. By writing $s = \eta + \rho + r$, we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \big\| A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\tau_2)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \\
&= \int_{\tau_1}^{\tau_2} \sum_{m=1}^{\infty} \big\| A^{\frac{s}{2}} E(\tau_2 - \sigma) G(Y(\tau_2)) \varphi_m \big\|^2 \,\mathrm{d}\sigma \\
&\le \sum_{m=1}^{\infty} \int_{\tau_1}^{\tau_2} \big\| A^{\frac{\eta}{2}} \big\|^2 \big\| A^{\frac{\rho}{2}} E(\tau_2 - \sigma) A^{\frac{r}{2}} G(Y(\tau_2)) \varphi_m \big\|^2 \,\mathrm{d}\sigma \\
&\le C(s, r) \big\| A^{\frac{\eta}{2}} \big\|^2 \big\| A^{\frac{r}{2}} G(Y(\tau_2)) \big\|_{L_2^0}^2 (\tau_2 - \tau_1)^{\min(1, 1+r-s)},
\end{aligned}$$

where $(\varphi_m)_{m \ge 1}$ denotes an orthonormal basis of U_0. We also used that $1 - \rho = 1 - \max(0, s - r) = \min(1, 1 + r - s)$. Finally, by Assumption 2.1 we conclude

$$S_1 \le C(p, r, s, A, G) \Big(1 + \sup_{\sigma \in [0,T]} \big\| A^{\frac{r}{2}} Y(\sigma) \big\|_{L^p(\Omega;H)} \Big) (\tau_2 - \tau_1)^{\min(\frac{1}{2}, \frac{1+r-s}{2})}.$$

This proves (3.1). For S_2 we first make use of the fact that $\|B\Phi\|_{L_2^0} \le \|B\| \|\Phi\|_{L_2^0}$ and then apply Lemma 3.2 *(i)* followed by (2.2) to get

$$\begin{aligned}
S_2 &\le C(p, s, G) \Big\| \Big(\int_{\tau_1}^{\tau_2} (\tau_2 - \sigma)^{-s} \| Y(\sigma) - Y(\tau_2) \|^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&= C(p, s, G) \Big(\Big\| \int_{\tau_1}^{\tau_2} (\tau_2 - \sigma)^{-s} \| Y(\sigma) - Y(\tau_2) \|^2 \,\mathrm{d}\sigma \Big\|_{L^{p/2}(\Omega;\mathbb{R})} \Big)^{\frac{1}{2}} \\
&\le C(p, s, G) \Big(\int_{\tau_1}^{\tau_2} (\tau_2 - \sigma)^{-s} \| Y(\sigma) - Y(\tau_2) \|_{L^p(\Omega;H)}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}}.
\end{aligned}$$

By the Hölder continuity of Y we arrive at

$$\begin{aligned} S_2 &\leq C(p,s,G,C_\delta)\Big(\int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{-s+2\delta}\,\mathrm{d}\sigma\Big)^{\frac{1}{2}} \\ &\leq \frac{C(p,s,G,C_\delta)}{\sqrt{1+2\delta-s}}(\tau_2-\tau_1)^{\frac{1+2\delta-s}{2}}. \end{aligned}$$

This shows (3.2). Combination of the estimates for S_1 and S_2 yields (3.3) by using $(\tau_2-\tau_1)^\delta \leq T^\delta$. □

Lemma 3.5. *Let $s \in [0, r+1]$, $p \geq 2$, and Y be a stochastic process on $[0,T]$ which maps into H with $\sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)} < \infty$. Then there exists a constant $C = C(r,s,F)$ such that, for all $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$,*

$$\begin{aligned} &\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) F(Y(\tau_2))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad \leq C\Big(1 + \sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)}\Big)(\tau_2-\tau_1)^{\frac{1+r-s}{2}}. \end{aligned} \tag{3.4}$$

If, in addition, for some $\delta > 0$ there exists C_δ such that

$$\|Y(t_1) - Y(t_2)\|_{L^p(\Omega;H)} \leq C_\delta |t_2 - t_1|^\delta \text{ for all } t_1, t_2 \in [0,T],$$

then we also have, with $C = C(r,s,F,C_\delta)$, that

$$\begin{aligned} &\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma)\big(F(Y(\tau_2)) - F(Y(\sigma))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad \leq \frac{C}{1+r-s+2\delta}(\tau_2-\tau_1)^{\frac{1+r-s+2\delta}{2}}. \end{aligned} \tag{3.5}$$

In particular, with $C = C(T,\delta,r,s,F,C_\delta)$ it holds that

$$\begin{aligned} &\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) F(Y(\sigma))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad \leq C\Big(1 + \sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)}\Big)(\tau_2-\tau_1)^{\frac{1+r-s}{2}}. \end{aligned} \tag{3.6}$$

Proof. As in the previous lemma the main idea is to use the Hölder continuity of Y to estimate the left-hand side in (3.6). We have

$$\begin{aligned} &\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) F(Y(\sigma))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\quad\leq \Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) F(Y(\tau_2))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\ &\qquad + \Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma)\big(F(Y(\tau_2)) - F(Y(\sigma))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}. \end{aligned}$$

Therefore, if we show (3.4) and (3.5) then (3.6) follows immediately by using $(\tau_2 - \tau_1)^\delta \leq T^\delta$.

For (3.4) first note that the random variable $A^{\frac{r-1}{2}} F(X(\tau_2))$ takes values in H almost surely. Hence, we can apply Lemma 3.2 *(iv)*. Together with Assumption

2.2 this yields

$$
\begin{aligned}
&\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma) F(Y(\tau_2)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad \leq \Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s+1-r}{2}} E(\tau_2-\sigma) A^{\frac{r-1}{2}} F(Y(\tau_2)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad \leq C(r,s)(\tau_2-\tau_1)^{\frac{1+r-s}{2}} \big\| A^{\frac{r-1}{2}} F(Y(\tau_2)) \big\|_{L^p(\Omega;H)} \\
&\quad \leq C(r,s,F)\Big(1+\sup_{\sigma\in[0,T]} \|Y(\sigma)\|_{L^p(\Omega;H)}\Big)(\tau_2-\tau_1)^{\frac{1+r-s}{2}}.
\end{aligned}
$$

Finally, again by Lemma 3.2 and Assumption 2.2, we show (3.5):

$$
\begin{aligned}
&\Big\| \int_{\tau_1}^{\tau_2} A^{\frac{s}{2}} E(\tau_2-\sigma)\big(F(Y(\tau_2))-F(Y(\sigma))\big) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\leq \int_{\tau_1}^{\tau_2} \big\| A^{\frac{s+1-r}{2}} E(\tau_2-\sigma) A^{\frac{r-1}{2}} \left(F(Y(\tau_2))-F(Y(\sigma))\right) \big\|_{L^p(\Omega;H)} \,\mathrm{d}\sigma \\
&\leq C(r,s,F) \int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{\frac{r-s-1}{2}} \|Y(\tau_2)-Y(\sigma)\|_{L^p(\Omega;H)} \,\mathrm{d}\sigma \\
&\leq C(r,s,F,C_\delta) \int_{\tau_1}^{\tau_2} (\tau_2-\sigma)^{\frac{r-s-1+2\delta}{2}} \,\mathrm{d}\sigma = \frac{2C(r,s,F,C_\delta)}{1+r-s+2\delta}(\tau_2-\tau_1)^{\frac{1+r-s+2\delta}{2}}.
\end{aligned}
$$

This completes the proof. □

Now we are well prepared for the proof of Theorem 3.1.

Proof of Theorem 3.1. By taking norms in (2.1) we get, for $t \in [0,T]$,

$$
\begin{aligned}
\big(\mathbf{E}\left[\|X(t)\|_{r+1}^p\right]\big)^{\frac{1}{p}} &= \|A^{\frac{r+1}{2}} X(t)\|_{L^p(\Omega;H)} \\
&\leq \|A^{\frac{r+1}{2}} E(t) X_0\|_{L^p(\Omega;H)} \\
&\quad + \Big\| A^{\frac{r+1}{2}} \int_0^t E(t-\sigma) F(X(\sigma)) \,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad + \Big\| A^{\frac{r+1}{2}} \int_0^t E(t-\sigma) G(X(\sigma)) \,\mathrm{d}W(\sigma) \Big\|_{L^p(\Omega;H)} \\
&=: I + II + III.
\end{aligned}
$$

The first term is well-known from deterministic theory and can be estimated by

$$
\|A^{\frac{r+1}{2}} E(t) X_0\|_{L^p(\Omega;H)} \leq \|A^{\frac{r+1}{2}} X_0\|_{L^p(\Omega;H)} < \infty,
$$

since $X_0 \colon \Omega \to \dot{H}^{r+1}$ by Assumption 2.3.

We recall that, by Theorem 1 in [5], the mild solution X is an $\dot{H}^r$-valued predictable stochastic process which is δ-Hölder continuous for any $0 < \delta < \frac{1}{2}$ with respect to the norm $\|\cdot\|_{L^p(\Omega;H)}$. We choose $\delta := \frac{r+1}{4}$ so that $0 \leq \frac{r}{2} < \delta < \frac{1}{2}$. Hence, we can apply Lemmas 3.4 and 3.5 with $Y = X$.

For the second term we apply (3.6) with $\tau_1 = 0$, $\tau_2 = t$, $s = r+1$ and $Y = X$. This yields

$$
II \leq C\Big(1+\sup_{\sigma\in[0,T]} \|X(\sigma)\|_{L^p(\Omega;H)}\Big) < \infty.
$$

For the last term we apply (3.3) with the same parameters as above:

$$
III \leq C\Big(1+\sup_{\sigma\in[0,T]} \big\|A^{\frac{r}{2}} X(\sigma)\big\|_{L^p(\Omega;H)}\Big) < \infty.
$$

Note that $\sup_{\sigma\in[0,T]} \|X(\sigma)\|_{L^p(\Omega;H)} \le \sup_{\sigma\in[0,T]} \|A^{\frac{r}{2}} X(\sigma)\|_{L^p(\Omega;H)}$ is finite because of Theorem 1 in [5]. □

4. Regularity in time

This section is devoted to the temporal regularity of the mild solution. Our result is summarized in the following theorem.

Theorem 4.1 (Temporal regularity)**.** *Let* $r \in [0,1)$*,* $p \in [2,\infty)$*. Under the assumptions of Section 2 the unique mild solution* X *is continuous with respect to* $\big(\mathbf{E}\,[\|\cdot\|_s^p]\,\big)^{\frac{1}{p}}$ *and satisfies*

$$\sup_{t_1,t_2\in[0,T],t_1\neq t_2} \frac{\big(\mathbf{E}\,[\|X(t_1)-X(t_2)\|_s^p]\,\big)^{\frac{1}{p}}}{|t_1-t_2|^{\min(\frac{1}{2},\frac{1+r-s}{2})}} < \infty \tag{4.1}$$

for every $s \in [0, r+1]$*.*

Before we begin the proof we analyse the continuity properties of the semigroup in the deterministic context. For Hölder continuity the results of Lemma 3.2 will be sufficient. But in order to prove continuity, that is, the case $s = r+1$, we need the following result.

Lemma 4.2. *Let* $0 \le \tau_1 < \tau_2 \le T$*. Then we have*

(i)

$$\lim_{\tau_2-\tau_1\to 0} \int_{\tau_1}^{\tau_2} \big\|A^{\frac{1}{2}} E(\tau_2-\sigma)x\big\|^2 \,\mathrm{d}\sigma = 0 \text{ for all } x\in H,$$

(ii)

$$\lim_{\tau_2-\tau_1\to 0} \Big\| \int_{\tau_1}^{\tau_2} AE(\tau_2-\sigma)x \,\mathrm{d}\sigma\Big\| = 0 \text{ for all } x\in H,$$

(iii)

$$\lim_{\tau_2-\tau_1\to 0} \int_{0}^{\tau_1} \big\|A^{\frac{1}{2}} \big(E(\tau_2-\sigma)-E(\tau_1-\sigma)\big)x\big\|^2 \,\mathrm{d}\sigma = 0 \text{ for all } x\in H,$$

(iv)

$$\lim_{\tau_2-\tau_1\to 0} \Big\| \int_{0}^{\tau_1} A\big(E(\tau_2-\sigma)-E(\tau_1-\sigma)\big)x \,\mathrm{d}\sigma\Big\| = 0 \text{ for all } x\in H.$$

Proof. As in the proof of Lemma 3.2 we use the orthogonal expansion of $x \in H$ with respect to the eigenbasis $(e_n)_{n\ge1}$ of the operator A. Thus, for *(i)* we get as in the proof of Lemma 3.2 *(iii)*

$$\int_{\tau_1}^{\tau_2} \|AE(\tau_2-\sigma)x\|^2 \,\mathrm{d}\sigma = \frac{1}{2}\sum_{n=1}^{\infty}(x,e_n)^2\left(1-\mathrm{e}^{2\lambda_n(\tau_2-\tau_1)}\right).$$

We apply Lebesgue's dominated convergence theorem. Note that the sum is dominated by $\frac{1}{2}\|x\|^2$ for all $\tau_2-\tau_1 \ge 0$. Moreover, for every $n \ge 1$ we have

$$\lim_{\tau_2-\tau_1\to 0}\big(1-\mathrm{e}^{\lambda_n(\tau_2-\tau_1)}\big)(x,e_n)^2 = 0.$$

Hence, Lebesgue's theorem gives us *(i)*. The same argument also yields the remaining cases, since

$$\Big\|\int_{\tau_1}^{\tau_2} AE(\tau_2-\sigma)x \,\mathrm{d}\sigma\Big\|^2 = \sum_{n=1}^{\infty}\big(1-\mathrm{e}^{\lambda_n(\tau_2-\tau_1)}\big)^2(x,e_n)^2,$$

and

$$\begin{aligned}
&\int_0^{\tau_1} \big\|A^{\frac12}\left(E(\tau_2-\tau_1)-I\right)E(\tau_1-\sigma)x\big\|^2\,\mathrm{d}\sigma\\
&\quad= \sum_{n=1}^{\infty}\left(\mathrm{e}^{-\lambda_n(\tau_2-\tau_1)}-1\right)^2\int_0^{\tau_1}\lambda_n\,\mathrm{e}^{-2\lambda_n(\tau_1-\sigma)}\,\mathrm{d}\sigma\,(x,e_n)^2\\
&\quad= \frac12\sum_{n=1}^{\infty}\left(\mathrm{e}^{-\lambda_n(\tau_2-\tau_1)}-1\right)^2\left(1-\mathrm{e}^{-\lambda_n\tau_1}\right)(x,e_n)^2,
\end{aligned}$$

as well as

$$\begin{aligned}
&\Big\|\int_0^{\tau_1} A\left(E(\tau_2-\tau_1)-I\right)E(\tau_1-\sigma)x\,\mathrm{d}\sigma\Big\|^2\\
&\quad= \sum_{n=1}^{\infty}\Big(\mathrm{e}^{-\lambda_n(\tau_2-\tau_1)}-1\Big)^2\Big(\lambda_n\int_0^{\tau_1}\mathrm{e}^{-\lambda(\tau_1-\sigma)}\,\mathrm{d}\sigma\Big)^2(x,e_n)^2\\
&\quad= \sum_{n=1}^{\infty}\left(\mathrm{e}^{-\lambda_n(\tau_2-\tau_1)}-1\right)^2\left(1-\mathrm{e}^{-\lambda_n\tau_1}\right)(x,e_n)^2.
\end{aligned}$$

The proof is complete. □

Proof of Theorem 4.1. First we show (4.1). Let $0\le t_1<t_2\le T$ be arbitrary. By using the mild formulation (2.1) we get

$$\begin{aligned}
&\big(\mathbf{E}\left[\|X(t_1)-X(t_2)\|_s^p\right]\big)^{\frac1p} = \big\|A^{\frac{s}{2}}(X(t_1)-X(t_2))\big\|_{L^p(\Omega;H)}\\
&\quad\le \big\|A^{\frac{s}{2}}(E(t_1)-E(t_2))X_0\big\|_{L^p(\Omega;H)} + \Big\|A^{\frac{s}{2}}\int_{t_1}^{t_2}E(t_2-\sigma)F(X(\sigma))\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)}\\
&\qquad+\Big\|A^{\frac{s}{2}}\int_0^{t_1}\left(E(t_2-\sigma)-E(t_1-\sigma)\right)F(X(\sigma))\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)}\\
&\qquad+\Big\|A^{\frac{s}{2}}\int_{t_1}^{t_2}E(t_2-\sigma)G(X(\sigma))\,\mathrm{d}W(\sigma)\Big\|_{L^p(\Omega;H)}\\
&\qquad+\Big\|A^{\frac{s}{2}}\int_0^{t_1}\left(E(t_2-\sigma)-E(t_1-\sigma)\right)G(X(\sigma))\,\mathrm{d}W(\sigma)\Big\|_{L^p(\Omega;H)}\\
&\quad=: T_1+T_2+T_3+T_4+T_5.
\end{aligned}$$

We estimate the five terms separately. The term T_1 is estimated by

$$\begin{aligned}
T_1 &= \Big\|A^{\frac{s-r-1}{2}}(I-E(t_2-t_1))A^{\frac{r+1}{2}}E(t_1)X_0\Big\|_{L^p(\Omega;H)}\\
&\le C\big\|A^{\frac{r+1}{2}}X_0\big\|_{L^p(\Omega;H)}(t_2-t_1)^{\frac{1+r-s}{2}},
\end{aligned}$$

where we used Lemma 3.2 *(ii)* and Assumption 2.3.

As in the proof of Theorem 3.1 we choose the Hölder exponent $\delta := \frac{r+1}{4}$ so that $\frac{r}{2}<\delta<\frac12$ and we can apply Lemmas 3.4 and 3.5 with $Y=X$.

The term T_2 coincides with (3.6) and we have

$$T_2 \le C\Big(1+\sup_{\sigma\in[0,T]}\|X(\sigma)\|_{L^p(\Omega;H)}\Big)(t_2-t_1)^{\frac{1+r-s}{2}}.$$

For the third term we also apply Lemma 3.2 *(ii)* before we use (3.6):

$$\begin{aligned}T_3 &= \Big\| A^{\frac{s-r-1}{2}} \left(E(t_2-t_1)-I\right) \int_0^{t_1} A^{\frac{r+1}{2}} E(t_1-\sigma)F(X(\sigma))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}\\ &\le C(t_2-t_1)^{\frac{1+r-s}{2}} \Big\| \int_0^{t_1} A^{\frac{r+1}{2}} E(t_1-\sigma)F(X(\sigma))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}\\ &\le C\Big(1+\sup_{\sigma\in[0,T]} \|X(\sigma)\|_{L^p(\Omega;H)}\Big)(t_2-t_1)^{\frac{1+r-s}{2}}.\end{aligned}$$

The fourth term is estimated analogously by using (3.3) instead of (3.6). We get

$$T_4 \le C\Big(1+\sup_{\sigma\in[0,T]} \|A^{\frac{r}{2}}X(\sigma)\|_{L^p(\Omega;H)}\Big)(t_2-t_1)^{\min(\frac12,\frac{1+r-s}{2})}.$$

Finally, for the last term we use Lemma 3.3 first. Since, for $0 \le t_1 < t_2 \le T$ fixed, the function $[0,t_1] \ni \sigma \mapsto A^{\frac{s}{2}}\left(E(t_2-\sigma)-E(t_1-\sigma)\right)G(X(\sigma))$ is a predictable stochastic process Lemma 3.3 can be applied. Then, by using Lemma 3.2 *(ii)* with $\nu = \frac{1+r-s}{2}$ and Lemma 3.4 with $s = r+1$ we get

$$\begin{aligned}T_5 &\le C\Big\|\Big(\int_0^{t_1} \|A^{\frac{s-r-1}{2}}(E(t_2-t_1)-I)A^{\frac{r+1}{2}}E(t_1-\sigma)G(X(\sigma))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}\\ &\le C(t_2-t_1)^{\frac{1+r-s}{2}}\Big(\Big\|\Big(\int_0^{t_1}\|A^{\frac{r+1}{2}}E(t_1-\sigma)G(X(t_1))\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}\\ &\quad+\Big\|\Big(\int_0^{t_1}\|A^{\frac{r+1}{2}}E(t_1-\sigma)\big(G(X(\sigma))-G(X(t_1))\big)\|^2_{L_2^0}\,\mathrm{d}\sigma\Big)^{\frac12}\Big\|_{L^p(\Omega;\mathbb{R})}\Big)\\ &\le C(t_2-t_1)^{\frac{1+r-s}{2}}\Big(1+\sup_{\sigma\in[0,T]}\|A^{\frac{r}{2}}X(\sigma)\|_{L^p(\Omega;H)}\Big).\end{aligned}$$

Altogether, this proves (4.1) and the Hölder continuity of X with respect to the norm $\|A^{\frac{s}{2}}\cdot\|_{L^p(\Omega;H)}$ for all $s \in [0,r+1)$.

It remains to prove continuity in the case $s = r+1$. As already demonstrated in the proof of Lemma 4.2 we use Lebesgue's dominated convergence theorem. We have to discuss all terms T_i, $i = 1,\dots,5$, again.

For T_1 continuity follows immediately: For almost every $\omega \in \Omega$ we get that $X_0(\omega) \in \dot{H}^{r+1}$. Thus, for fixed $\omega \in \Omega$ we have

$$\lim_{t_2-t_1\to 0}\|(E(t_2)-E(t_1))A^{\frac{r+1}{2}}X_0(\omega)\| = 0$$

by the strong continuity of the semigroup. We also have that

$$\|(E(t_2)-E(t_1))A^{\frac{r+1}{2}}X_0(\omega)\| \le \|A^{\frac{r+1}{2}}X_0(\omega)\|,$$

where the latter is an element of $L^p(\Omega;\mathbb{R})$ as a function of $\omega \in \Omega$ by Assumption 2.3. Hence, Lebesgue's theorem is applicable and yields $\lim_{t_2-t_1\to 0} T_1 = 0$.

Since the same argument applies to the remaining terms, it is sufficient to provide the pointwise limit for fixed $\omega \in \Omega$ and a dominating function.

In the case of T_2 we get

$$\begin{aligned}\lim_{t_2-t_1\to 0} T_2 &\le \lim_{t_2-t_1\to 0}\Big\|\int_{t_1}^{t_2} AE(t_2-\sigma)A^{\frac{r-1}{2}}F(X(t_2))\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)}\\ &\quad+\lim_{t_2-t_1\to 0}\Big\|\int_{t_1}^{t_2} A^{\frac{r+1}{2}}E(t_2-\sigma)\big(F(X(t_2))-F(X(\sigma))\big)\,\mathrm{d}\sigma\Big\|_{L^p(\Omega;H)}.\end{aligned}$$

Because of (3.5), where we can choose $s = r+1$ and $\delta = \frac{r+1}{4} > 0$, the limit of the second summand is 0. For the first summand we use Lemma 3.2 *(iv)* with $\rho = 1$

$$\Big\| \int_{t_1}^{t_2} AE(t_2-\sigma)A^{\frac{r-1}{2}} F(X(t_2,\omega))\,\mathrm{d}\sigma \Big\| \le C\|A^{\frac{r-1}{2}} F(X(t_2,\omega))\| \le C\big(1 + \|X(t_2,\omega)\|\big)$$

which belongs to $L^p(\Omega;\mathbb{R})$. It also holds that

$$\lim_{t_2-t_1\to 0} \Big\| \int_{t_1}^{t_2} AE(t_2-\sigma)A^{\frac{r-1}{2}} F(X(t_2,\omega))\,\mathrm{d}\sigma \Big\| = 0$$

by Lemma 4.2 *(ii)*. This completes the proof for T_2.

Next we take care of T_3, which is estimated by

$$T_3 \le \Big\| \int_0^{t_1} A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big) F(X(t_1))\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} + \Big\| \int_0^{t_1} A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big)\big(F(X(t_1)) - F(X(\sigma))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)}.$$

For almost every $\omega \in \Omega$ we have that $A^{\frac{r-1}{2}} F(X(t_1,\omega)) \in H$. Therefore, by Lemma 4.2 *(iv)*,

$$\lim_{t_2-t_1\to 0} \Big\| \int_0^{t_1} A\big(E(t_2-\sigma) - E(t_1-\sigma)\big)A^{\frac{r-1}{2}} F(X(t_1,\omega))\,\mathrm{d}\sigma \Big\| = 0.$$

It is also true that

$$\Big\| \int_0^{t_1} A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big) F(X(t_1,\omega))\,\mathrm{d}\sigma \Big\| \le \Big\| \int_0^{t_1} A^{\frac{r+1}{2}} E(t_1-\sigma) F(X(t_1,\omega))\,\mathrm{d}\sigma \Big\|,$$

which is by (3.4) a random variable in $L^p(\Omega;\mathbb{R})$. Hence, once again, Lebesgue's dominated convergence theorem yields that the limit of this summand is 0 with respect to the norm in $L^p(\Omega;H)$.

For the second summand we get by Lemma 3.2 *(ii)*

$$\begin{aligned}
&\Big\| \int_0^{t_1} A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big)\big(F(X(t_1)) - F(X(\sigma))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\le \int_0^{t_1} \big\|A^{-\frac{\eta}{2}}\big(E(t_2-t_1) - I\big)A^{\frac{r+1+\eta}{2}} E(t_1-\sigma)\big(F(X(t_1)) - F(X(\sigma))\big)\big\|_{L^p(\Omega;H)}\,\mathrm{d}\sigma \\
&\le C(t_2-t_1)^{\frac{\eta}{2}} \int_0^{t_1} (t_1-\sigma)^{-\frac{2+\eta}{2}} \big\|A^{\frac{r-1}{2}}\big(F(X(t_1)) - F(X(\sigma))\big)\big\|_{L^p(\Omega;H)}\,\mathrm{d}\sigma,
\end{aligned}$$

where $\eta \in (0,2]$. We continue the estimate by applying Assumption 2.2 and the Hölder continuity of X with exponent $\frac{1}{2}$ with respect to the norm $\|\cdot\|_{L^p(\Omega;H)}$ as it was shown in (4.1) with $s = 0$. This gives

$$\begin{aligned}
&\Big\| \int_0^{t_1} A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big)\big(F(X(t_1)) - F(X(\sigma))\big)\,\mathrm{d}\sigma \Big\|_{L^p(\Omega;H)} \\
&\quad \le C(t_2-t_1)^{\frac{\eta}{2}} \int_0^{t_1} (t_1-\sigma)^{-\frac{2+\eta-1}{2}}\,\mathrm{d}\sigma = C\frac{2}{1-\eta} t_1^{\frac{1-\eta}{2}} (t_2-t_1)^{\frac{\eta}{2}}.
\end{aligned}$$

Therefore, in the limit this summand also vanishes as long as $\eta \in (0,1)$.

For T_4, one has to use Lemma 3.3 which yields

$$\begin{aligned}
&\lim_{t_2 - t_1 \to 0} T_4 \\
&\leq \lim_{t_2 - t_1 \to 0} C \Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{r+1}{2}} E(t_2 - \sigma) G(X(\sigma)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\leq \lim_{t_2 - t_1 \to 0} C \Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}} E(t_2 - \sigma) A^{\frac{r}{2}} G(X(t_2)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&\quad + \lim_{t_2 - t_1 \to 0} C \Big\| \Big(\int_{t_1}^{t_2} \big\| A^{\frac{r+1}{2}} E(t_2 - \sigma) \big(G(X(\sigma)) - G(X(t_2)) \big) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})}.
\end{aligned}$$

The limit of the second summand is 0 because of (3.2), where we again choose $s = r + 1$ and $\delta = \frac{r+1}{4} > \frac{r}{2}$. By Lemma 3.2 *(iii)* with $\rho = 1$ and Assumption 2.1 the first summand is pointwise dominated by

$$\begin{aligned}
&\Big(\int_{t_1}^{t_2} \big\| A^{\frac{1}{2}} E(t_2 - \sigma) A^{\frac{r}{2}} G(X(t_2, \omega)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \\
&\quad = \Big(\sum_{m=1}^{\infty} \int_{t_1}^{t_2} \big\| A^{\frac{1}{2}} E(t_2 - \sigma) A^{\frac{r}{2}} G(X(t_2, \omega)) \varphi_m \big\|^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \\
&\quad \leq C \Big(\sum_{m=1}^{\infty} \big\| A^{\frac{r}{2}} G(X(t_2, \omega)) \varphi_m \big\|^2 \Big)^{\frac{1}{2}} \leq C \big(1 + \| X(t_2, \omega) \| \big)
\end{aligned}$$

where $(\varphi_m)_{m \geq 1}$ is an arbitrary orthonormal basis of U_0 and the last term belongs to $L^p(\Omega;\mathbb{R})$ by Theorem 1 in [5]. Lemma 4.2 *(i)* yields

$$\begin{aligned}
&\lim_{t_2 - t_1 \to 0} \int_{t_1}^{t_2} \big\| A^{\frac{1}{2}} E(t_2 - \sigma) A^{\frac{r}{2}} G(X(t_2, \omega)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \\
&\quad = \sum_{m=1}^{\infty} \lim_{t_2 - t_1 \to 0} \int_{t_1}^{t_2} \big\| A^{\frac{1}{2}} E(t_2 - \sigma) A^{\frac{r}{2}} G(X(t_2, \omega)) \varphi_m \big\|^2 \,\mathrm{d}\sigma = 0.
\end{aligned}$$

In fact, the interchanging of sum and limit is justified by another application of Lebesgue's Theorem. Altogether this completes the proof for T_4.

Finally, the estimate of the term T_5 is done in a very similar way, since we have

$$\begin{aligned}
T_5 \leq C &\Big\| \Big(\int_0^{t_1} \big\| A^{\frac{r+1}{2}} \big(E(t_2 - \sigma) - E(t_1 - \sigma) \big) G(X(t_1)) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})} \\
&+ C \Big\| \Big(\int_0^{t_1} \big\| A^{\frac{r+1}{2}} \big(E(t_2 - \sigma) - E(t_1 - \sigma) \big) \\
&\qquad \times \big(G(X(t_1)) - G(X(\sigma)) \big) \big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;\mathbb{R})}
\end{aligned}$$

by Lemma 3.3. Using the same technique as for T_4, the desired result for the first summand follows by Lebesgue's dominated convergence theorem together with

Lemma 4.2 *(iii)* and (3.1). The second summand is further estimated by

$$\begin{aligned}
&\Big\| \Big(\int_0^{t_1} \big\| A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big)\big(G(X(t_1)) - G(X(\sigma))\big)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big)^{\frac{1}{2}} \Big\|_{L^p(\Omega;H)}^2 \\
&= \Big\| \int_0^{t_1} \big\| A^{\frac{r+1}{2}} \big(E(t_2-\sigma) - E(t_1-\sigma)\big)\big(G(X(t_1)) - G(X(\sigma))\big)\big\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\le C(t_2-t_1)^{\eta} \Big\| \int_0^{t_1} (t_1-\sigma)^{-1-r-\eta} \, \| G(X(t_1)) - G(X(\sigma))\|_{L_2^0}^2 \,\mathrm{d}\sigma \Big\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\le C(t_2-t_1)^{\eta} \int_0^{t_1} (t_1-\sigma)^{-1-r-\eta} \, \| X(t_1) - X(\sigma)\|_{L^p(\Omega;H)}^2 \,\mathrm{d}\sigma \\
&\le C(t_2-t_1)^{\eta} \frac{1}{1-r-\eta} t_1^{1-r-\eta}.
\end{aligned}$$

For the first inequality we applied Lemma 3.2 *(i)* and *(ii)* with an arbitrary parameter $\eta \in (0, 1-r)$. Then we used (2.2) and the $\frac{1}{2}$-Hölder continuity of X. It follows, as in all previous cases, that the summand vanishes in the limit $t_2 - t_1 \to 0$. This completes the proof. □

5. Additive noise and optimal regularity

In this section we briefly review the assumptions and our results in the case of additive noise, that is, we consider the case where $G \in L_0^2$ is independent of X. Then the SPDE (1.1) has the form

$$\begin{aligned}
\mathrm{d}X(t) + [AX(t) + F(X(t))]\,\mathrm{d}t &= G\,\mathrm{d}W(t), \quad \text{for } 0 \le t \le T, \\
X(0) &= X_0.
\end{aligned} \tag{5.2}$$

For related regularity results in this special case we refer to [4, Ch. 5].

Since now G is a fixed bounded linear operator Assumption 2.1 is simplified to

Assumption 5.1 (Additive noise)**.** *The Hilbert-Schmidt operator G satisfies*

$$\|G\|_{L_{2,r}^0} = \|A^{\frac{r}{2}} G\|_{L_2^0} < \infty. \tag{5.3}$$

Recall that the covariance operator Q of the Wiener process W is incorporated into the norm $\|\cdot\|_{L_2^0}$. If, for example, $H = U$ and G is the identity $I\colon H \to H$, then (5.3) reads as follows

$$\|I\|_{L_{2,r}^0} = \sum_{m=1}^{\infty} \big\| A^{\frac{r}{2}} Q^{\frac{1}{2}} \varphi_m \big\|^2 < \infty,$$

where $(\varphi_m)_{m\ge 1}$ denotes an arbitrary orthonormal basis of the Hilbert space H. This is a common assumption on the covariance operator Q (see [4]). In particular, for $r = 0$ this condition becomes $\|I\|_{L_2^0} = \mathrm{Tr}(Q) < \infty$.

Our result for additive noise is summarized by the following corollary.

Corollary 5.2 (Additive noise)**.** *If the Assumptions 2.2, 2.3 and 5.1 hold for some $r \in [0,1]$, $p \in [2,\infty)$, then the unique mild solution $X\colon [0,T]\times\Omega \to H$ to (5.2) takes values in $\dot{H}^{r+1}$. Moreover, for every $s \in [0, r+1]$, the solution process is continuous with respect to $\big(\mathbf{E}[\|\cdot\|_s^p]\big)^{\frac{1}{p}}$ and fulfills*

$$\sup_{t_1,t_2\in[0,T], t_1\ne t_2} \frac{\big(\mathbf{E}[\|X(t_1)-X(t_2)\|_s^p]\big)^{\frac{1}{p}}}{|t_1-t_2|^{\min(\frac{1}{2}, \frac{r+1-s}{2})}} < \infty.$$

We stress that the case $r = 1$ is now included. In fact, the only place, where $r < 1$ is required, is the estimate (3.2) and its consequences. But in the case of additive noise the left-hand side of this estimate is equal to zero and we avoid this problem. The same is true for the proof of continuity, where the critical terms vanish analogously (c.f. the proof of Theorem 4.1).

We conclude this section by an example, which demonstrates that our spatial regularity results are optimal. Without loss of generality we restrict our discussion to the case $p = 2$. For $p > 2$ one may use the results on the optimal regularity of the stochastic convolution from [13] or [1].

Example 5.3. Let $H = L^2(0,1)$ be the space of all square integrable real functions which are defined on the unit interval $(0,1)$. Further, assume that $-A$ is the Laplacian with Dirichlet boundary conditions. In this situation the orthonormal eigenbasis $(e_k)_{k\geq 1}$ of $-A$ is explicitly known to be

$$\lambda_k = k^2\pi^2 \quad \text{and} \quad e_k(y) = \sqrt{2}\sin(k\pi y) \quad \text{for all } k \geq 1,\ y \in (0,1).$$

Consider the SPDE

$$\begin{aligned} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t &= G\,\mathrm{d}W(t), \quad \text{for } t \in [0,T] \\ X(0) &= 0. \end{aligned} \tag{5.4}$$

We choose W to be a Q-Wiener process on H with $Q = I$ and the operator $G = A^{-\frac{1}{2}}$ so that $Ge_k = \lambda_k^{-\frac{1}{2}} e_k$. Then we have

$$\|G\|_{L^0_{2,r}} = \sum_{k=1}^{\infty} \big\|A^{\frac{r}{2}} G e_k\big\|^2 = \sum_{k=1}^{\infty} \lambda_k^{r-1} = \pi^{2(r-1)} \sum_{k=1}^{\infty} k^{2(r-1)}.$$

Thus, Assumption 5.1 is satisfied for all $r \in [0, \frac{1}{2})$ and Corollary 5.2 yields that the mild solution X to (5.4) takes values in $\dot{H}^{r+1}$ for all $r \in [0, \frac{1}{2})$.

In the following we show that $X(t)$ does not map into $\dot{H}^{\frac{3}{2}}$ almost surely. The mild formulation (2.1) now reads

$$X(t) = \int_0^t E(t-\sigma) G\,\mathrm{d}W(\sigma).$$

Hence, by the Itô-isometry for the stochastic integral we have

$$\begin{aligned} \mathbf{E}\big[\big\|A^{\frac{3}{4}} X(t)\big\|^2\big] &= \int_0^t \big\|A^{\frac{3}{4}} E(t-\sigma) G\big\|^2_{L_2^0}\,\mathrm{d}\sigma \\ &= \int_0^t \sum_{k=1}^{\infty} \lambda_k^{\frac{3}{2}}\,\mathrm{e}^{-2\lambda_k(t-\sigma)} \lambda_k^{-1}\,\mathrm{d}\sigma \\ &= \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \big(1 - \mathrm{e}^{-2\lambda_k t}\big). \end{aligned}$$

Since the eigenvalues λ_k form an increasing sequence we have

$$\big(1 - \mathrm{e}^{-2\lambda_k t}\big) \geq \big(1 - \mathrm{e}^{-2\lambda_1 t}\big)$$

for all $t > 0$. Therefore,

$$\begin{aligned} \mathbf{E}\big[\big\|A^{\frac{3}{4}} X(t)\big\|^2\big] &\geq \frac{1}{2}\big(1 - \mathrm{e}^{-2\lambda_1 t}\big) \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}} \\ &= \frac{1}{2\pi}\big(1 - \mathrm{e}^{-2\pi^2 t}\big) \sum_{k=1}^{\infty} k^{-1} = \infty \quad \text{for all } t > 0. \end{aligned}$$

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