Error analysis of a hybrid method for computing Lyapunov exponents

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Abstract

In a previous paper [6] we suggested a numerical method for computing all Lyapunov exponents of a dynamical system by spatial integration with respect to an ergodic measure. The method extended an earlier approach of Aston and Dellnitz [2] for the largest Lyapunov exponent by integrating the diagonal entries from the QR-decomposition of the Jacobian for an iterated map. In this paper we provide an asymptotic error analysis of the method for the case in which all Lyapunov exponents are simple. We employ Oseledec multiplicative ergodic theorem and impose certain hyperbolicity conditions on the invariant subspaces that belong to neighboring exponents. The resulting error expansion shows that one step of extrapolation is enough to obtain exponential decay of errors.

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1 Introduction

In this paper we analyze a numerical method for computing all Lyapunov exponents of a discrete time dynamical system by spatial integration. This method was proposed in [6] as an extension of earlier work by Aston and Dellnitz [2],[3]. In particular, we will derive an error expansion that justifies the extrapolation procedure applied in [6].

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We consider a dynamical system generated by the iterates of a C^1-diffeomorphism

$$g : M \to M, \tag{1.1}$$

where M is a smooth and compact d-dimensional submanifold of some $\mathbb{R}^k$. The Theorem of Oseledec [24],[28] associates with any invariant ergodic measure μ on the Borel σ-algebra of M a set of Lyapunov exponents

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$$

and a decomposition of the tangent space TM into subspaces invariant under the linearization, for details see Appendix A.

The idea in [6] is to approximate λ_j as follows

$$\lambda_j \approx a_n^j = \frac{1}{n} \int_M \ln(R_{jj}(Dg^n(x)))d\mu(x). \tag{1.2}$$

Here $g^n = g \circ \cdots \circ g$ denotes the n-th iterate of g and Dg^n its Jacobian. The number $R_{jj}(A), A \in \mathbb{R}^{d\times d}$ is the (j, j)-entry of the unique upper triangular matrix R that has positive diagonal entries and satisfies the QR-decomposition $A = QR,\ Q^TQ = I_d$. In [6] this approach was called *hybrid* since it combines a spatial integration method with the well known QR-method for Lyapunov exponents along single trajectories (see [7],[10],[11], [13],[16],[18] and [21],[5] for some general theory).

While one can establish convergence $\lim_{n\to\infty} a_n^j = \lambda_j$ under rather weak conditions (see [6, Theorem 2]) we prove in this paper a more detailed error expansion of the type

$$a_n^j = \lambda_j + \frac{C_j}{n} + \mathcal{O}\left(\frac{e^{-\Delta_j n}}{n}\right). \tag{1.3}$$

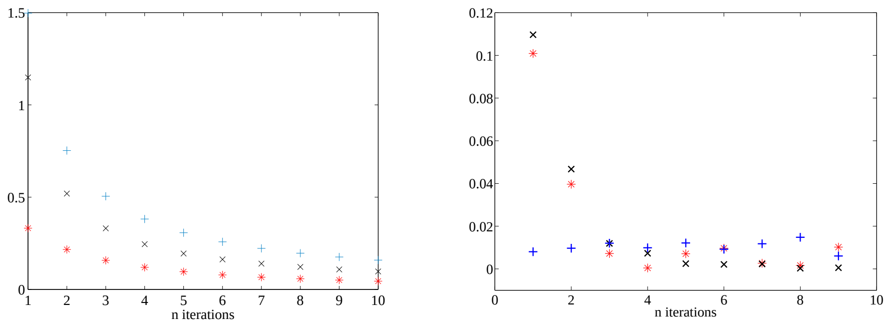
Here C_j is a constant that can be expressed as an appropriate μ-integral (see (2.14)) and Δ_j measures the distance of λ_j to the remaining Lyapunov spectrum (see (2.16)).

The expansion (1.3) immediately suggests to eliminate the slow principal error term by extrapolation (cf. [3]),

$$b_n^j = (n+1)a_{n+1}^j - na_n^j, \quad n = 0, 1, \ldots.$$

In Figure 1 we illustrate the error behavior for the time T-map ($T = 0.2$) of the Lorenz system. For the computations we used the package GAIO by

Figure 1: The Lorenz system : Errors $|\lambda_j - a_n^j|$ of approximate Lyapunov exponents (left) and of extrapolated values $|\lambda_j - b_n^j|$ (right) : $j = 1(*), j = 2(\times), j = 3(+)$. Note the different scales on the vertical axis.



Dellnitz and co-authors (cf. [8], [9]) which allows to approximate invariant measures by elementary measures supported on box collections. Parameter values are as in [6, Example 3] with the exception that the number of rga-steps was raised to 27. This avoids spoiling the convergence rates as $n \to \infty$ by errors in the approximate measure. For general purposes the latter errors should be properly balanced with errors in (1.3), see [6] for some comments on this problem.

Our assumptions and the precise results will be formulated in Section 2 with the proofs given in Section 3. One of our main assumptions will be simplicity of the Oseledec invariant subspaces. Together with some integrability conditions this will suffice to prove an intermediate result of the form $a_n^j = \lambda_j + \frac{C_j}{n} + o(\frac{1}{n})$, see Theorem 1. Then much stronger hyperbolicity conditions will be imposed for the proof of (1.3). The numerical experiments in [6] and [3] suggest that these strong assumptions are actually satisfied for standard examples such as the Henon map or the Lorenz system. But we are not aware of rigorous results in this direction.

Exterior algebra is generally known to be helpful for handling all Lyapunov exponents and has even been used numerically, cf. [1]. It will also be the main tool for deriving (1.3). Therefore we summarize the essentials needed for this paper in Appendix B. However, we emphasize that the hybrid method itself does not make explicit use of exterior products and hence does not suffer from the curse of dimension entailed by spaces of exterior products.

2 Assumptions and main results

We briefly recall the realizaton of the hybrid method from [6] and then discuss the special situation of the Oseledec Theorem that will be considered.

2.1 The hybrid method

Let $A = Q(A)R(A)$ be the *unique* QR-decomposition of a nonsingular matrix $A \in \mathbb{R}^{d\times d}$ such that $Q(A) \in \mathbb{R}^{d\times d}$ is orthogonal and $R(A) \in \mathbb{R}^{d\times d}$ is upper triangular with positive diagonal entries. The hybrid method approximates the $j-$th Liapunov exponent λ_j by the sequence of integrals

$$a_n^j = \frac{1}{n}\int \ln\left(R_{jj}(Dg^n(x)\right)\, d\mu(x), \quad n \in \mathbb{N}_0. \tag{2.4}$$

We note that in case $j = 1$ we have $R_{11}(Dg^n(x)) = ||Dg^n(x)e_1||$ where $e_1 = (1, 0, \dots, 0)^T$. Then the hybrid method coincides with taking $v = e_1$ in the vector method proposed in [4].

The measure μ will be computed approximately by the package GAIO (see [8],[9] in general and [6] for details in our case). The elements $R_{jj}\left(Dg^n(x)\right)$ are computed as in the $QR-$method for single trajectories (see [10],[11],[13],[18]). Take any nonsingular matrix $Z_0 \in \mathbb{R}^{d\times d}$ (e.g. $Z_0 = I_d$) and define the sequence $\{Z_n\}_{n\in\mathbb{N}_0}$ via

$$Z_{n+1} := Dg(g^n(x))\, Q(Z_n)\,, \quad n \in \mathbb{N}_0\,.$$

From the decomposition $Dg^n(x) = Q\left(Dg^n(x)\right) R\left(Dg^n(x)\right)$ one obtains by induction (cf. [10], [18])

$$R\left(Dg^n(x)\right) = \prod_{i=n}^{1} R(Z_i) \quad \text{and} \quad Q\left(Dg^n(x)\right) = Q(Z_n), \quad n \in \mathbb{N}_0.$$

The diagonal values of the upper triangular matrices are given by

$$R_{jj}\left(Dg^n(x)\right) = \prod_{i=1}^{n} R_{jj}(Z_i).$$

2.2 Oseledec spaces and Oseledec minors

In the following we will always assume that there exists some ergodic probability measure μ on the Borel σ-algebra of M. By the theorem of Oseledec (see Appendix A and [20], [24]) there exists a Borel set $M_\mu \subset M$ of full measure, invariant under g such that for all $x \in M_\mu$ and $v \in T_x M$ the limit

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \ln \|Dg^n(x)v\|$$

exists and is independent of x. Moreover, there is a measurable decomposition $T_x M = \bigoplus_{i=1}^{s} W^i(x)$ for some $s \le d$ and there are numbers $\tilde{\lambda}_1 > \ldots > \tilde{\lambda}_s$ such that the following holds for $j = 1, \ldots, s$

$$\tilde{\lambda}_j = \lambda(x, v) \quad \text{for all } x \in M_\mu \text{ and } v \in \bigoplus_{i=j}^{s} W^i(x) \setminus \bigoplus_{i=j+1}^{s} W^i(x).$$

Counting the $\tilde{\lambda}_j$ values according to their multiplicities we obtain the Lyapunov exponents $\lambda_1 \ge \ldots \ge \lambda_d$.

Throughout the paper we will make the simplifying assumption

$$\dim W^i(x) = 1 \quad \text{for } i \in 1, \ldots, s \quad \text{for } \mu - \text{a.e. } x \in M, \tag{2.5}$$

which implies $s = d$ in the decomposition $T_x M = \bigoplus_{i=1}^{d} W^i(x)$. Moreover, there exist vectors $w_i(x)$ of unit length that are measurable with respect to $x \in M$ and span the spaces $W^i(x)$, i.e.

$$W^i(x) = \operatorname{span}\{w_i(x)\}, \quad |w_i(x)| = 1, \quad i = 1, \ldots, d, \quad x \in M_\mu. \tag{2.6}$$

By the invariance of $W^i(x)$ under g the spanning vectors from (2.6) satisfy

$$Dg(x)w_i(x) = a^{(i)}(x)w_i(g(x)), \tag{2.7}$$

for some scalar $a^{(i)}(x)$. For the mappings $a^{(i)} : M_\mu \to \mathbb{R}$ this implies

$$|a^{(i)}(x)| = \|Dg(x)w_i(x)\| \quad i = 1, \ldots, d\,.$$

By the continuity of $Dg(\cdot)$ the mappings $\left|a^{(i)}(\cdot)\right|$ are measurable.

Using induction on equation (2.7) leads to

$$Dg^n(x)w_i(x) = \prod_{\nu=n-1}^{0} a^{(i)}(g^\nu(x))w_i(g^n(x))\,. \tag{2.8}$$

It is convenient to define (cf. [2], [3])

$$A_n^{(i)}(x) = \prod_{\nu=n-1}^{0} a^{(i)}(g^\nu(x)) \quad i = 1, \ldots, d\,. \tag{2.9}$$

For $j \in \{1, \ldots, d\}$ let $Ord(j,d)$ be the set of ordered multiindices [1] and define

$$A_n^i(x) = A_n^{(i_1)}(x) \cdots A_n^{(i_j)}(x), \quad i = (i_1, \ldots, i_j) \in Ord(j,d). \tag{2.10}$$

For the special element $\mathbb{1}_j = (1, \ldots, j) \in Ord(j,d)$ we have

$$A_n^{\mathbb{1}_j}(x) = A_n^{(1)}(x) \cdots A_n^{(j)}(x).$$

Let $W(x) \in \mathbb{R}^{d,d}$ denote the Oseledec matrix with columns $w_1(x), \ldots, w_d(x)$ and introduce its trailing principal minors

$$P_j(x) = \det \begin{pmatrix} W_{j\,j}(x) & \cdots & W_{j\,d}(x) \\ W_{j+1\,j}(x) & \cdots & W_{j+1\,d}(x) \\ \cdots & \cdots & \cdots \\ W_{d\,j}(x) & \cdots & W_{d\,d}(x) \end{pmatrix}. \tag{2.11}$$

One of our main assumptions is positivity of the minors

$$|P_{j+1}(x)| > 0 \text{ and } |P_j(x)| > 0 \text{ for } \mu\text{-a.e. } x \in M, \tag{A1}$$

where $P_j(x) = 1$ in case $j = d+1$. Later on we impose the stronger requirement that for some $\bar{\varepsilon}_j > 0$,

$$|P_{j+1}(x)| > \bar{\varepsilon}_j \quad \text{and} \quad |P_j(x)| > \bar{\varepsilon}_j \quad \mu\text{-a.e..} \tag{A1'}$$

We further require that there exists some $\varepsilon_j > 0$ such that

$$G_j\left(W(x)\right) \geq \varepsilon_j \text{ and } G_{j-1}\left(W(x)\right) \geq \varepsilon_j, \quad \mu\text{-a.e.,} \tag{A2}$$

where

$$G_j(A) = \sqrt{\det\left(A(1:d, 1:j)^T A(1:d, 1:j)\right)}$$

denotes the Gramian volume of the parallelepiped generated by the first j column vectors of a matrix $A \in \mathbb{R}^{d \times d}$ (cf. Appendix B or [17]). In case $j = 1$ we set $G_0(W(x)) = 1$ in (A2).

[1] $Ord(j,d) = \{\delta \in \{1, \ldots, d\}^{\{1,\ldots,j\}} : \delta$ strictly monotone$\}$ see also Appendix B.

Next introduce the inverse matrices $(\alpha_{ij}(x))_{i,j=1}^d = W^{-1}(x) \in \mathbb{R}^{d\times d}$ for $x \in M_\mu$ (cf.[3]) which satisfy

$$e_\nu = \sum_{i=1}^{d} \alpha_{i\nu}(x) w_i(x) \quad \text{for } \nu = 1, \ldots, d. \tag{2.12}$$

For multiindices $i = (i_1, \ldots, i_j) \in Ord(j, d)$ we use minors $\alpha_i(x)$ of the α-matrix defined by

$$\alpha_i(x) = \alpha_{i_1,\ldots,i_j}(x) = \det(\alpha(i, \mathbb{1}_j)(x)) = \det \begin{pmatrix} \alpha_{i_1 1}(x) & \cdots & \alpha_{i_1 j}(x) \\ \alpha_{i_2 1}(x) & \cdots & \alpha_{i_2 j}(x) \\ \cdots & \cdots & \cdots \\ \alpha_{i_j 1}(x) & \cdots & \alpha_{i_j j}(x) \end{pmatrix}.$$

In case $i = \mathbb{1}_j = (1, \ldots, j)$ we have the leading principal minor $\alpha_{\mathbb{1}_j}(x)$.

2.3 Convergence Theorem

In section 3.1 below we show that the Gramian volume of the first j columns of $Dg^n(x)$ may be written as

$$(G_j\,(Dg^n(x)))^2 = \Gamma_{j,n}(x) + \rho_{j,n}(x)\,,$$

where

$$\Gamma_{j,n}(x) = \left(\alpha_{\mathbb{1}_j}(x)\right)^2 \left(A_n^{\mathbb{1}_j}(x)\right)^2 G_j\,(W\,(g^n(x)))^2 \tag{2.13}$$

and $\rho_{j,n}(x)$ is a remainder (see (3.19)) for which we will derive good estimates (see (3.27)).

Theorem 1. *For a fixed index $j \in \{1, \ldots, d\}$ let the system (1.1) satisfy (2.5) and assumptions (A1), (A2). Moreover, assume*

(i) $\ln \dfrac{|P_{j+1}(x)|}{|P_j(x)|}$ *is μ-integrable.*

(ii) The sequences of functions

$$\left(\ln\left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(\ln\left(1 + \frac{\rho_{j-1,n}(x)}{\Gamma_{j-1,n}(x)}\right)\right)_{n\in\mathbb{N}}$$

have a common μ-integrable majorant.

Then the following expansion holds

$$\frac{1}{n}\int \ln R_{jj}(Dg^n(x))d\mu = \lambda_j + \frac{C_j}{n} + o\left(\frac{1}{n}\right),$$

where

$$C_j = \int \ln \frac{|P_{j+1}(x)|\, G_j\,(W(x))}{|P_j(x)|\, G_{j-1}\,(W(x))} d\mu. \tag{2.14}$$

2.4 Hyperbolicity and spectral gaps

The standard definition of a hyperbolic set [25],[21] assumes that 0 is not a Lyapunov exponent and the dynamics can be split into exponentially decaying and growing components. In the following we consider such a splitting at an arbitrary point in the Lyapunov spectrum, similar to the (λ, μ)-splitting in [21] and in analogy to splittings of the Sacker-Sell spectrum in [27],[12].

Definition 2. For some $j \in \{1, \ldots, d-1\}$ the system (1.1) is called *hyperbolic of type* $(\lambda_{j+1}, \lambda_j)$ if there exists a set $M_\mu^j \subset M_\mu$ with $\mu\left(M_\mu^j\right) = 1$ and constants $C \geq 1, \mu_{j+1}^r, \mu_j^l$ such that

(i) $\lambda_{j+1} \leq \mu_{j+1}^r < \mu_j^l \leq \lambda_j$.

(ii) For all $x \in M_\mu^j$,

(a) $\|Dg^n(x)v\| \leq C\, e^{n\mu_{j+1}^r}\|v\|$ if $v \in \bigoplus_{i=j+1}^{d} W^i(x)$ and $n \geq 0$,

(b) $\|Dg^n(x)v\| \geq \frac{1}{C}\, e^{n\mu_j^l}\|v\|$ if $v \in \bigoplus_{i=1}^{j} W^i(x)$ and $n \geq 0$.

The constants C, μ_j^l and μ_{j+1}^r are called the *(hyperbolicity) parameters.*

Remark. *For any fixed $x \in M_\mu$ it is clear that hyperbolicity parameters depending on x exist, see Lemma 5. Our assumption here is that they can be taken independently of x.*

Obviously the set M_μ^j is hyperbolic in the classical sense when the system is hyperbolic of type $(\lambda_{j+1}, \lambda_j)$ with parameters $\mu_{j+1}^r < 0 < \mu_j^l$. The following theorem shows that hyperbolicity of type $(\lambda_{j+1}, \lambda_j)$ leads to exponential estimates of remainders.

Theorem 3. *For some fixed* $j \in \{1, \ldots, d\}$ *let the system (1.1) satisfy (2.5) and assumptions (A1'),(A2), and let it be hyperbolic of type* $(\lambda_{j+1}, \lambda_j)$ *and of type* $(\lambda_j, \lambda_{j-1})$ *with parameters* C, μ^r_{j+1}, μ^l_j *and* C, μ^r_j, μ^l_{j-1}*, respectively. Then the following estimate holds*

$$\frac{1}{n} \int \ln R_{jj}(Dg^n(x))d\mu = \lambda_j + \frac{C_j}{n} + O\left(\frac{e^{-\Delta_j n}}{n}\right), \tag{2.15}$$

where C_j *is defined in* (2.14) *and* Δ_j *is given by*

$$\Delta_j = \min\left\{\mu^l_j - \mu^r_{j+1}, \mu^l_{j-1} - \mu^r_j\right\}. \tag{2.16}$$

Remarks. 1. *Note that the assumptions of Theorem 3 are stronger than those of Theorem 1. Condition (A1') implies both assumptions (A1) and (i) of Theorem 1 and the hyperbolicity condition in Theorem 3 implies the integrability condition (ii) of Theorem 1, see Section 3.2.*

2.*In case* $j = 1$ *the assumption reduces to hyperbolicity of type* (λ_2, λ_1) *with* Δ_1 *given by* $\Delta_1 = \mu^l_1 - \mu^r_2$*. Similarly, in case* $j = d$ *we assume hyperbolicity of type* $(\lambda_d, \lambda_{d-1})$ *and set* $\Delta_d = \mu^l_{d-1} - \mu^r_d$*.*

3. *The proof below suggests that the results of Theorems 1 and 3 remain valid if instead of (2.5) one only assumes simplicity of the* j*-th Lyapunov exponent* λ_j*. However, we have not carried out the details of such a generalization.*

The following corollary shows that error expansions hold if each Lyapunov exponent allows a hyperbolic splitting and if all Oseledec minors behave properly.

Corollary 4. *Let the system (1.1) satisfy (2.5) and the following conditions:*

(i) There exists an $\varepsilon > 0$ *such that*

$$|P_j(x)| > \varepsilon \quad \text{for } j = 2, \ldots, d \quad \text{and} \quad \mu \quad \text{a.e. } x \in M.$$

(ii) The system is hyperbolic of type $(\lambda_{j+1}, \lambda_j)$ *for* $j = 1, \ldots, d-1$*.*

Then the expansion

$$\frac{1}{n} \int \ln R_{jj}(Dg^n(x))d\mu = \lambda_j + \frac{C_j}{n} + O\left(\frac{e^{-\Delta_j n}}{n}\right) \quad \text{for} \quad j = 1, \ldots, d,$$

holds with C_j, Δ_j *given by* (2.14) *and* (2.16)*.*

Note that there are no assumptions on the Gramians G_j. These follow from the hyperbolicity conditions as we will show in Section 3.2.

3 Proof of main results

3.1 Convergence under integrability conditions (Theorem 1)

In the following we will frequently use the following elementary fact. If a real sequence $c_n > 0$ satisfies $\lim_{n\to\infty} \frac{1}{n} \ln c_n = -\Delta < 0$ then $c_n = \mathcal{O}(e^{(-\Delta+\varepsilon)n})$ for every $0 < \varepsilon < \Delta$.

Lemma 5. *Consider $i > j$ and $0 < \varepsilon < \lambda_j - \lambda_i = \Delta$. Then for μ a.e. $x \in M$ there is a constant $C_{x,\varepsilon}$, such that*

$$\left|A_n^{(i)}(x)\right| \le C_{x,\varepsilon} e^{(-\Delta+\varepsilon)n} \left|A_n^{(j)}(x)\right|.$$

In particular, $\lim_{n\to\infty} \frac{\left|A_n^{(i)}(x)\right|}{\left|A_n^{(j)}(x)\right|} = 0.$

Proof. From the Oseledec theorem and (2.8),(2.9) we obtain

$$\begin{aligned}
\frac{1}{n} \ln \frac{\left|A_n^{(i)}(x)\right|}{\left|A_n^{(j)}(x)\right|} &= \frac{1}{n} \ln \frac{\|Dg^n(x) w_i(x)\|}{\|Dg^n(x) w_j(x)\|} \\
&= \frac{1}{n} \ln \|Dg^n(x) w_i(x)\| - \frac{1}{n} \ln \|Dg^n(x) w_j(x)\| \\
&\to \lambda_i - \lambda_j < 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore, for each $x \in M_\mu$ we find some $C_{x,\varepsilon}$ such that

$$\frac{\left|A_n^{(i)}(x)\right|}{\left|A_n^{(j)}(x)\right|} \le C_{x,\varepsilon} e^{(-\Delta+\varepsilon)n}, \quad n \in \mathbb{N}.$$

□

Lemma 6. *Let $i, \ell \in Ord(j,d)$ be arbitrary with $i \neq \mathbb{1}_j = (1,\dots,j)$ and $0 < \varepsilon < \lambda_j - \lambda_{j+1} = \Delta$. Then there exist constants $C_{x,\varepsilon}$ for $x \in M_\mu$ such that*

$$\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|} + \frac{\left|A_n^i(x) A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2} \le C_{x,\varepsilon} e^{(-\Delta+\varepsilon)n}.$$

In particular,

$$\lim_{n\to\infty} \frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|} = \lim_{n\to\infty} \frac{\left|A_n^i(x) A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2} = 0.$$

Proof. As in the proof of Lemma 5 the definition (2.10) of $A_n^{(i)}(x)$ leads to

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|}=\sum_{k=1}^{j}\lambda_{i_k}-\sum_{\nu=1}^{j}\lambda_\nu=-\Delta_i$$

and

$$\lim_{n\to\infty}\frac{1}{n}\ln\frac{\left|A_n^i(x)A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2}=\sum_{k=1}^{j}\lambda_{i_k}+\sum_{k=1}^{j}\lambda_{\ell_k}-2\sum_{\nu=1}^{j}\lambda_\nu=-\Delta_{i,\ell}\,.$$

Note that $0<\Delta\le\Delta_i\le\Delta_{i,\ell}$ follows from $i\neq\mathbb{1}_j$. Hence for some $C_{x,\varepsilon}\ge 1$ and μ a.e. $x\in M$,

$$\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|}\le C_{x,\varepsilon}e^{(-\Delta_i+\varepsilon)n}\quad\text{and}\quad\frac{\left|A_n^i(x)A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2}\le C_{x,\varepsilon}e^{(-\Delta_{i,\ell}+\varepsilon)n}.$$

□

The proof of Theorem 1 proceeds in four steps.

Step 1 (Expansion of Gramian volumes)

We make extensive use of exterior products and their properties as summarized in Appendix B. From the definition (2.12) we obtain

$$\begin{aligned}Dg^n(x)e_k &= Dg^n(x)\sum_{i=1}^{d}\alpha_{ik}(x)w_i(x)\\ &= \sum_{i=1}^{d}\alpha_{ik}(x)Dg^n(x)w_i(x)\\ &= \sum_{i=1}^{d}\alpha_{ik}(x)A_n^{(i)}(x)w_i(g^n(x)),\quad k=1,\ldots,d.\end{aligned}\tag{3.17}$$

We abbreviate $Dg^n(x)_{\cdot k}=Dg^n(x)e_k$ Then the properties of the exterior product (see Appendix B, Lemma 16) and (3.17) yield

$$(Dg^n(x))_{\cdot 1}\wedge\cdots\wedge(Dg^n(x))_{\cdot j}=$$

$$\begin{aligned}&=\sum_{\nu\in Ord(j,d)}\alpha_\nu(x)\left(\wedge_{k=1}^{j}A_n^{(\nu_k)}(x)w_{\nu_k}(g^n(x))\right)\\ &=\sum_{\nu\in Ord(j,d)}\alpha_\nu(x)\prod_{k=1}^{j}A_n^{(\nu_k)}(x)(\wedge_{k=1}^{j}w_{\nu_k}(g^n(x))).\end{aligned}$$

Theorem 11 and Corollary 12 lead to the following expression for the Gramian

$$(G_j\,(Dg^n(x)))^2 = \left\|(Dg^n(x))_{\cdot 1} \wedge \cdots \wedge (Dg^n(x))_{\cdot j}\right\|^2 =$$

$$\begin{aligned}
&= \sum_{\nu,\ell \in Ord(j,d)} \alpha_\nu(x)\alpha_\ell(x) \prod_{k=1}^{j} A_n^{(\nu_k)}(x) A_n^{(\ell_k)}(x) \left\langle \wedge_{k=1}^{j} w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^{j} w_{\ell_k}\left(g^n(x)\right)\right\rangle_D \\
&= \sum_{\nu,\ell \in Ord(j,d)} \alpha_\nu(x)\alpha_\ell(x) A_n^{\nu}(x) A_n^{\ell}(x) \left\langle \wedge_{k=1}^{j} w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^{j} w_{\ell_k}\left(g^n(x)\right)\right\rangle_D \qquad (3.18)\\
&= \Gamma_{j,n}(x) + \rho_{j,n}(x)\,,
\end{aligned}$$

where

$$\Gamma_{j,n}(x) = \left(\alpha_{\mathbb{1}_j}(x)\right)^2 \left(A_n^{\mathbb{1}_j}(x)\right)^2 \left\|w_1(g^n(x)) \wedge \cdots \wedge w_j(g^n(x))\right\|^2$$

and $\rho_{j,n}$ is defined as the remainder term in (3.18), cf. (2.13) in Section 2.2. Therefore,

$$\rho_{j,n}(x) = \qquad (3.19)$$

$$\begin{aligned}
&= \sum_{\substack{\nu,\ell \in Ord(j,d)\\ \ell \neq \mathbb{1}_j}} \alpha_\nu(x)\alpha_\ell(x) A_n^{\nu}(x) A_n^{\ell}(x) \left\langle \wedge_{k=1}^{j} w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^{j} w_{\ell_k}\left(g^n(x)\right)\right\rangle_D \\
&+ \sum_{\substack{\nu \in Ord(j,d)\\ \nu \neq \mathbb{1}_j}} \alpha_\nu(x)\alpha_{\mathbb{1}_j}(x) A_n^{\nu}(x) A_n^{\mathbb{1}_j}(x) \left\langle \wedge_{k=1}^{j} w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^{j} w_{k}\left(g^n(x)\right)\right\rangle_D .
\end{aligned}$$

For the terms $\rho_{j-1,n}(x)$ and $\Gamma_{j-1,n}(x)$ we have analogous expressions with indices running in $Ord(j-1,d)$.

Step 2 (Integral expression of remainder)

We proceed along the lines of [15], [16], use (2.8) and the Oseledec theorem (Appendix A, Theorem 9) to conclude

$$\begin{aligned}
\lambda_j &= \lim_{n\to\infty} \frac{1}{n} \ln \|Dg^n(x) w_j(x)\| \\
&= \lim_{n\to\infty} \frac{1}{n} \ln \prod_{\nu=n-1}^{0} \left|a^{(j)}(g^\nu(x))\right| \underbrace{\|w_j\left(g^n(x)\right)\|}_{=1} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n-1} \ln \left|a^{(j)}(g^\nu(x))\right| .
\end{aligned}$$

By Birkhoff's ergodic theorem (see [26]) we obtain

$$\lambda_j = \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{n-1} \ln \left| a^{(j)}(g^\nu(x)) \right| = \int \ln \left| a^{(j)}(x) \right| d\mu\,.$$

Using the definition (2.7) of $a^{(j)}(x)$ we arrive at

$$\lambda_j = \int \ln \|D\, g(x)\, w_j(x)\| d\mu. \tag{3.20}$$

As in (2.4) we define for $j = 1, \dots, d$

$$a_n^j = \frac{1}{n} \int \ln R_{jj}(Dg^n(x)) d\mu.$$

For the existence of these integrals compare [6, Theorem 2]. We show below that the limit

$$\lim_{n\to\infty} n(a_n^j - \lambda_j)$$

exists and agrees with C_j given by (2.14). Then this proves

$$a_n^j = \lambda_j + \frac{C_j}{n} + o\left(\frac{1}{n}\right).$$

Let us first write

$$\begin{aligned} n a_n^j &= \int \ln R_{jj}(Dg^n(x)) d\mu \\ &= \int \ln \frac{R_{jj}(Dg^n(x))}{\left| A_n^{(j)}(x) \right|} d\mu + \int \ln \left| A_n^{(j)}(x) \right| d\mu. \end{aligned} \tag{3.21}$$

By the definition (2.9) of $A_n^{(j)}(x)$ and the invariance of the μ-integral of $\ln \left| a^{(j)}(x) \right|$ the second term satisfies

$$\begin{aligned} \int \ln \left| A_n^{(j)}(x) \right| d\mu &= \int \ln \prod_{\nu=n-1}^{0} \left| a^{(j)}(g^\nu(x)) \right| d\mu \\ &= \sum_{\nu=0}^{n-1} \int \ln \left| a^{(j)}(g^\nu(x)) \right| d\mu \\ &= n\lambda_j. \end{aligned}$$

Therefore, equation (3.21) leads to

$$n(a_n^j - \lambda_j) = \int \ln \frac{R_{jj}(Dg^n(x))}{\left| A_n^{(j)}(x) \right|} d\mu. \tag{3.22}$$

In the following we discuss the integral on the right-hand side.

Step 3 (Estimates of integrand)

Let us first show that C_j from (2.14) can be written as follows

$$C_j = \int \ln \frac{\left|\alpha_{\mathbb{1}_j}(x)\right| \|w_1(x) \wedge \cdots \wedge w_j(x)\|}{\left|\alpha_{\mathbb{1}_{j-1}}(x)\right| \|w_1(x) \wedge \cdots \wedge w_{j-1}(x)\|} d\mu. \tag{3.23}$$

From [17, §1.4]) we have the following representation of the minors of an inverse matrix for all multi-indices $i, \nu \in Ord(k, d)$ and $k = 1, \dots, d$

$$\det \begin{pmatrix} \alpha_{i_1\,\nu_1}(x) & \cdots & \alpha_{i_1\,\nu_k}(x) \\ \alpha_{i_2\,\nu_1}(x) & \cdots & \alpha_{i_2\,\nu_k}(x) \\ \cdots & \cdots & \cdots \\ \alpha_{i_k\,\nu_1}(x) & \cdots & \alpha_{i_k\,\nu_k}(x) \end{pmatrix} =$$

$$= (-1)^{K_{i,\nu}} \det W(x)^{-1} \det \begin{pmatrix} W_{\hat{\nu}_1\,\hat{i}_1}(x) & \cdots & W_{\hat{\nu}_1\,\hat{i}_{d-k}}(x) \\ W_{\hat{\nu}_2\,\hat{i}_1}(x) & \cdots & W_{\hat{\nu}_2\,\hat{i}_{d-k}}(x) \\ \cdots & \cdots & \cdots \\ W_{\hat{\nu}_{d-k}\,\hat{i}_1}(x) & \cdots & W_{\hat{\nu}_{d-k}\,\hat{i}_{d-k}}(x) \end{pmatrix}$$

where $K_{i,\nu} = \sum_{l=1}^{k} (i_l + \nu_l)$ and the complementary tuples $(\hat{i}_1, \dots, \hat{i}_{d-k}), (\hat{\nu}_1, \dots, \hat{\nu}_{d-k}) \in Ord(d-k, k)$ are defined by

$$\{\hat{i}_1, \dots, \hat{i}_{d-k}\} \cup \{i_1, \dots, i_k\} = \{1, \dots, d\} = \{\hat{\nu}_1, \dots, \hat{\nu}_{d-k}\} \cup \{\nu_1, \dots, \nu_k\}.$$

Writing this in terms of coordinates of exterior products (see Definition 14) yields

$$\left(\bigwedge^k \alpha(x)\right)_{i,\nu} = (-1)^{K_{i,\nu}} \frac{\left(\bigwedge^{d-k} W(x)\right)_{\hat{\nu},\hat{i}}}{\det W(x)}. \tag{3.24}$$

Special cases of this formula are the following (using the notation from (2.11))

$$\left|\alpha_{\mathbb{1}_j}(x)\right| = \left|\frac{P_{j+1}(x)}{\det (W(x))}\right| \quad \text{and} \quad \left|\alpha_{\mathbb{1}_{j-1}}(x)\right| = \left|\frac{P_j(x)}{\det (W(x))}\right|.$$

Hence positivity of $|\alpha_{\mathbb{1}_k}(x)|$ is characterized by positivity of $|P_{k+1}(x)|$ for $k = j-1, j$, and the following equation holds

$$\frac{|P_{j+1}(x)|}{|P_j(x)|} = \frac{|\alpha_{\mathbb{1}_j}(x)|}{|\alpha_{\mathbb{1}_{j-1}}(x)|}.$$

From (3.23) we conclude that for each $n \in \mathbb{N}$

$$C_j = \int \ln \frac{\left|\alpha_{\mathbb{1}_j}(x)\right| \|w_1(g^n(x)) \wedge \cdots \wedge w_j(g^n(x))\|}{\left|\alpha_{\mathbb{1}_{j-1}}(x)\right| \|w_1(g^n(x)) \wedge \cdots \wedge w_{j-1}(g^n(x))\|} d\mu.$$

In order to see this, we write the integrand in (3.23) as a sum of an α and a w term and use the g invariance of the integral for the second summand. By our assumptions (i) and (A2) both integrals in this decomposition exist.

It remains to consider

$$\int \ln \frac{R_{jj}\left(Dg^n(x)\right)}{A_n^{(j)}(x)} - \ln \frac{\left|\alpha_{\mathbb{1}_j}(x)\right| \|w_1(g^n(x)) \wedge \cdots \wedge w_j(g^n(x))\|}{\left|\alpha_{\mathbb{1}_{j-1}}(x)\right| \|w_1(g^n(x)) \wedge \cdots \wedge w_{j-1}(g^n(x))\|} d\mu$$

$$= \int \ln \frac{R_{jj}\left(Dg^n(x)\right) \left|\alpha_{\mathbb{1}_{j-1}}(x)\right| \|w_1(g^n(x)) \wedge \cdots \wedge w_{j-1}(g^n(x))\|}{A_n^{(j)}(x) \left|\alpha_{\mathbb{1}_j}(x)\right| \|w_1(g^n(x)) \wedge \cdots \wedge w_j(g^n(x))\|} d\mu.$$

By the properties of the exterior product and its relation to the QR-decomposition (see Lemma 17 and (B.4) in Appendix B) we obtain

$$\begin{aligned} R_{jj}\left(Dg^n(x)\right) &= \frac{\prod_{i=1}^{j} R_{ii}\left(Dg^n(x)\right)}{\prod_{i=1}^{j-1} R_{ii}\left(Dg^n(x)\right)} = \frac{\left(\bigwedge^j R\left(Dg^n(x)\right)\right)_{\mathbb{1}_j \mathbb{1}_j}}{\left(\bigwedge^{j-1} R\left(Dg^n(x)\right)\right)_{\mathbb{1}_j \mathbb{1}_j}} \\ &= \frac{\left\|\left(\bigwedge^j R\left(Dg^n(x)\right)\right)_{\cdot \mathbb{1}_j}\right\|}{\left\|\left(\bigwedge^{j-1} R\left(Dg^n(x)\right)\right)_{\cdot \mathbb{1}_j}\right\|} \\ &= \frac{\left\|\bigwedge^j R(Dg^n(x))(e_1 \wedge \cdots \wedge e_j)\right\|}{\left\|\bigwedge^{j-1} R(Dg^n(x))(e_1 \wedge \cdots \wedge e_{j-1})\right\|} \\ &= \frac{\left\|\bigwedge^j Dg^n(x)(e_1 \wedge \cdots \wedge e_j)\right\|}{\left\|\bigwedge^{j-1} Dg^n(x)(e_1 \wedge \cdots \wedge e_{j-1})\right\|} \\ &= \frac{\|Dg^n(x)_{\cdot 1} \wedge \cdots \wedge Dg^n(x)_{\cdot j}\|}{\|Dg^n(x)_{\cdot 1} \wedge \cdots \wedge Dg^n(x)_{\cdot j-1}\|}. \end{aligned}$$

From (3.18) we infer

$$\left(R_{jj}\left(Dg^n(x)\right)\right)^2 =$$

$$= \frac{\left\langle (Dg^n(x))_{\cdot 1} \wedge \cdots \wedge (Dg^n(x))_{\cdot j}, (Dg^n(x))_{\cdot 1} \wedge \cdots \wedge (Dg^n(x))_{\cdot j} \right\rangle_D}{\left\langle (Dg^n(x))_{\cdot 1} \wedge \cdots \wedge (Dg^n(x))_{\cdot j-1}, (Dg^n(x))_{\cdot 1} \wedge \cdots \wedge (Dg^n(x))_{\cdot j-1} \right\rangle_D}$$

$$= \frac{\sum_{\nu,\ell \in Ord(j,d)} \alpha_\nu(x)\alpha_\ell(x) A_n^\nu(x) A_n^\ell(x) \left\langle \wedge_{k=1}^j w_{\nu_k}(g^n(x)), \wedge_{k=1}^j w_{\ell_k}(g^n(x)) \right\rangle_D}{\sum_{\nu,\ell \in Ord(j-1,d)} \alpha_\nu(x)\alpha_\ell(x) A_n^\nu(x) A_n^\ell(x) \left\langle \wedge_{k=1}^{j-1} w_{\nu_k}(g^n(x)), \wedge_{k=1}^{j-1} w_{\ell_k}(g^n(x)) \right\rangle_D}.$$

Using $\rho_{j,n}(x)$, $\Gamma_{j,n}(x)$ from (3.19),(2.13) we find from the last equation

$$\left(\frac{R_{jj}(Dg^n(x))}{A_n^{(j)}(x)} \right)^2 = \frac{\Gamma_{j,n}(x) \ + \ \rho_{j,n}(x)}{(\Gamma_{j-1,n}(x) \ + \ \rho_{j-1,n}(x)) \left(A_n^{(j)}(x) \right)^2}$$

$$= \frac{\dfrac{\left(\alpha_{\mathbb{1}_j}(x)\right)^2 \left\| \wedge_{k=1}^j w_k(g^n(x)) \right\|^2}{\left(\alpha_{\mathbb{1}_{j-1}}(x)\right)^2 \left\| \wedge_{k=1}^{j-1} w_k(g^n(x)) \right\|^2} + \dfrac{\rho_{j,n}(x)}{\Gamma_{j-1,n}(x) \left(A_n^{(j)}(x) \right)^2}}{1 + \dfrac{\rho_{j-1,n}(x)}{\Gamma_{j-1,n}(x)}}.$$

Finally, we use

$$\frac{\rho_{j,n}(x)}{\Gamma_{j-1,n}(x) \left(A_n^{(j)}(x) \right)^2} \cdot \frac{\left(\alpha_{\mathbb{1}_{j-1}}(x)\right)^2 \left\| \wedge_{k=1}^{j-1} w_k(g^n(x)) \right\|^2}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2 \left\| \wedge_{k=1}^j w_k(g^n(x)) \right\|^2}$$

$$= \frac{\rho_{j,n}(x)}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2 \prod_{k=1}^j \left(A_n^{(k)}(x) \right)^2 \left\| \wedge_{k=1}^j w_k(g^n(x)) \right\|^2}$$

$$= \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}$$

and arrive at

$$\frac{R_{jj}(Dg^n(x))}{\left| A_n^{(j)}(x) \right|} \cdot \frac{\left| \alpha_{\mathbb{1}_{j-1}}(x) \right| \| w_1(g^n(x)) \wedge \cdots \wedge w_{j-1}(g^n(x)) \|}{\left| \alpha_{\mathbb{1}_j}(x) \right| \| w_1(g^n(x)) \wedge \cdots \wedge w_j(g^n(x)) \|}$$

$$= \left(\left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)} \right) \left(1 + \frac{\rho_{j-1,n}(x)}{\Gamma_{j-1,n}(x)} \right)^{-1} \right)^{\frac{1}{2}}. \qquad (3.25)$$

Step 4 (Limits of integrals)

We now prove

$$\lim_{n \to \infty} \int \ln \left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)} \right)^{\frac{1}{2}} d\mu = 0. \qquad (3.26)$$

For $\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}$ we find the expression

$$\sum_{\substack{\nu,\ell\in Ord(j,d)\\ \ell\neq \mathbb{1}_j}} \frac{\alpha_\nu(x)\alpha_\ell(x)A_n^\nu(x)A_n^\ell(x)\left\langle \wedge_{k=1}^j w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^j w_{\ell_k}\left(g^n(x)\right)\right\rangle_D}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2\left(A_n^{\mathbb{1}_j}(x)\right)^2\left\|\wedge_{k=1}^j w_k\left(g^n(x)\right)\right\|^2} +$$

$$+\sum_{\substack{\nu\in Ord(j,d)\\ \nu\neq \mathbb{1}_j}} \frac{\alpha_\nu(x)\alpha_{\mathbb{1}_j}(x)A_n^\nu(x)A_n^{\mathbb{1}_j}(x)\left\langle \wedge_{k=1}^j w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^j w_k\left(g^n(x)\right)\right\rangle_D}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2\left(A_n^{\mathbb{1}_j}(x)\right)^2\left\|\wedge_{k=1}^j w_k\left(g^n(x)\right)\right\|^2}.$$

We estimate the sequence $\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}$ for points $x\in M$ that satisfy assumptions (A1) and (A2). By Lemma 13 we have for $\nu\in Ord(j,d)$

$$\left\|\wedge_{k=1}^j w_{\nu_k}\left(g^n(x)\right)\right\| \leq \prod_{k=1}^j \left\|w_{\nu_k}\left(g^n(x)\right)\right\| = 1$$

and thus by Cauchy's inequality

$$\left\langle \wedge_{k=1}^j w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^j w_{\ell_k}\left(g^n(x)\right)\right\rangle_D \leq 1 \quad \text{for} \quad \nu,\ell\in Ord(j,d).$$

From our assumption (A2) we then find

$$\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| \leq \tag{3.27}$$

$$\leq \sum_{\substack{\nu,\ell\in Ord(j,d)\\ \ell\neq \mathbb{1}_j}} \left|\frac{\alpha_\nu(x)\alpha_\ell(x)A_n^\nu(x)A_n^\ell(x)\langle \wedge_{k=1}^j w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^j w_{\ell_k}\left(g^n(x)\right)\rangle}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2\left(A_n^{\mathbb{1}_j}(x)\right)^2\left\|\wedge_{k=1}^j w_k\left(g^n(x)\right)\right\|^2}\right|$$

$$+\sum_{\substack{\nu\in Ord(j,d)\\ \nu\neq \mathbb{1}_j}} \left|\frac{\alpha_\nu(x)A_n^\nu(x)\langle \wedge_{k=1}^j w_{\nu_k}\left(g^n(x)\right), \wedge_{k=1}^j w_k\left(g^n(x)\right)\rangle}{\alpha_{\mathbb{1}_j}(x)A_n^{\mathbb{1}_j}(x)\left\|\wedge_{k=1}^j w_k\left(g^n(x)\right)\right\|^2}\right|$$

$$\leq \sum_{\substack{\nu,\ell\in Ord(j,d)\\ \ell\neq \mathbb{1}_j}} \left|\frac{\alpha_\nu(x)\alpha_\ell(x)A_n^\nu(x)A_n^\ell(x)}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2\left(A_n^{\mathbb{1}_j}(x)\right)^2\left\|\wedge_{k=1}^j w_k\left(g^n(x)\right)\right\|^2}\right|$$

$$+\sum_{\substack{\nu\in Ord(j,d)\\ \nu\neq \mathbb{1}_j}} \left|\frac{\alpha_\nu(x)A_n^\nu(x)}{\alpha_{\mathbb{1}_j}(x)A_n^{\mathbb{1}_j}(x)\left\|\wedge_{k=1}^j w_k\left(g^n(x)\right)\right\|^2}\right|$$

$$\leq \frac{1}{\varepsilon_j^2}\left(\sum_{\substack{\nu,\ell\in Ord(j,d)\\ \ell\neq \mathbb{1}_j}} \left|\frac{\alpha_\nu(x)\alpha_\ell(x)A_n^\nu(x)A_n^\ell(x)}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2\left(A_n^{\mathbb{1}_j}(x)\right)^2}\right| + \sum_{\substack{\nu\in Ord(j,d)\\ \nu\neq \mathbb{1}_j}} \left|\frac{\alpha_\nu(x)A_n^\nu(x)}{\alpha_{\mathbb{1}_j}(x)A_n^{\mathbb{1}_j}(x)}\right|\right).$$

The terms $\alpha_\nu(x), \nu \in Ord(m,d)$ are independent of n so that Lemma 6 implies pointwise convergence

$$\lim_{n\to\infty} \ln \left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)^{\frac{1}{2}} = 0$$

for μ a.e. $x \in M$. By Lebesgue's Theorem and assumption (ii) we obtain (3.26) and in a similar way

$$\lim_{n\to\infty} \int \ln \left(1 + \frac{\rho_{j-1,n}(x)}{\Gamma_{j-1,n}(x)}\right)^{-\frac{1}{2}} d\mu = 0.$$

Using (3.22) the proof is finished by taking logarithms in (3.25) and integrating.

3.2 Error expansion in the hyperbolic case (Theorem 3)

The following Lemma gives an estimate of angles between Oseledec spaces. The proof follows an idea of [25] for the hyperbolic case.

Lemma 7. *Let the system (1.1) be hyperbolic of type $(\lambda_{j+1}, \lambda_j)$ with parameters C, μ^r_{j+1} and μ^l_j. Then there exists an $\varepsilon_0 > 0$ such that*

$$\angle \left(\bigoplus_{i=1}^{j} W^i(x)\,,\ \bigoplus_{i=j+1}^{d} W^i(x) \right) \geq \varepsilon_0$$

for all $x \in M^j_\mu$.

Proof. Recall that the angle of two subspaces V, W is given by

$$\sin(\angle(V,W)) = \inf\{||v - w|| : v \in V, w \in W, ||v|| = 1 = ||w||\}.$$

Let $v \in \bigoplus_{i=1}^{j} W^i(x)$ and $w \in \bigoplus_{i=j+1}^{d} W^i(x)$ be vectors of unit norm and define for $x \in M^j_\mu$

$$K_n(x) = Dg^n(x)(v - w).$$

By the compactness of M we have a constant $\beta > 0$ such that

$$\|Dg(x)\| \leq \beta \quad \text{und} \quad \|Dg^{-1}(x)\| \leq \beta$$

for all $x \in M$. This implies

$$\|K_n(x)\| \leq \beta^n \|v - w\|. \tag{3.28}$$

From the hyperbolicity of type $(\lambda_{j+1}, \lambda_j)$ and the triangle inequality we infer

$$\begin{aligned} \|K_n(x)\| &\geq \|Dg^n(x)v\| - \|Dg^n(x)w\| \\ &\geq C^{-1}e^{n\mu_j^l} - Ce^{n\mu_{j+1}^r}. \end{aligned}$$

Since $\mu_{j+1}^r < \mu_j^l$ there exists $\bar{n} \in \mathbb{N}$ such that

$$\bar{\varepsilon} = C^{-1}e^{\bar{n}\mu_j^l} - Ce^{\bar{n}\mu_{j+1}^r} > 0.$$

Using (3.28) we arrive at

$$\|v - w\| \geq \beta^{-\bar{n}} \|K_{\bar{n}}(x)\| \geq \bar{\varepsilon}\beta^{-\bar{n}}$$

for all $x \in M_\mu^j$.

□

Next we improve the estimates of Lemma 6 under the hyperbolicity condition.

Lemma 8. *Let the system (1.1) be hyperbolic of type* $(\lambda_{j+1}, \lambda_j)$ *with parameters* C, μ_j^l *and* μ_{j+1}^r. *Then for any two multiindices* $i, \ell \in Ord(j,d)$ *with* $i \neq \mathbb{1}_j$,

$$\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|} + \frac{\left|A_n^i(x)A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2} \leq C^{4j}e^{-n\Delta_j^{j+1}},$$

where $\Delta_j^{j+1} = \mu_j^l - \mu_{j+1}^r$.

Proof. By the definition of $A_n^i(x)$ and $A_n^{(i_k)}(x)$ we have

$$\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|} = \frac{\|Dg^n(x)w_{i_1}(x)\| \cdots \|Dg^n(x)w_{i_j}(x)\|}{\|Dg^n(x)w_1(x)\| \cdots \|Dg^n(x)w_j(x)\|}.$$

Let us repartition $i, \mathbb{1}_j \in Ord(j,d)$ as follows

$$\begin{aligned} \check{s} &= \#\left(\{i_1, \ldots, i_j\} \cap \{1, \ldots, j\}\right), \\ \hat{s} &= j - \check{s}, \\ \{p_1, \ldots, p_{\check{s}}\} &= \{i_1, \ldots, i_j\} \cap \{1, \ldots, j\}, \\ \{q_1, \ldots, q_{\hat{s}}\} &= \{i_1, \ldots, i_j\} \setminus \{p_1, \ldots, p_{\check{s}}\}, \\ \{r_1, \ldots, r_{\hat{s}}\} &= \{1, \ldots, j\} \setminus \{p_1, \ldots, p_{\check{s}}\}. \end{aligned}$$

Then we can write

$$\begin{aligned}\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|} &= \frac{\prod_{k=1}^{\check{s}}\|Dg^n(x)w_{p_k}(x)\|\prod_{k=1}^{\hat{s}}\|Dg^n(x)w_{q_k}(x)\|}{\prod_{i=1}^{j}\|Dg^n(x)w_i(x)\|}\\ &= \frac{\prod_{k=1}^{\hat{s}}\|Dg^n(x)w_{q_k}(x)\|}{\prod_{k=1}^{\hat{s}}\|Dg^n(x)w_{r_k}(x)\|}.\end{aligned}$$

Note that $1 \le \hat{s} \le j$ by assumption. Since $w_{q_k}(x) \in \bigoplus_{i=j+1}^{d} W^i(x)$ and $w_{r_k}(x) \in \bigoplus_{i=1}^{j} W^i(x)$ for $k = 1, \ldots, \hat{s}$ hyperbolicity leads to

$$\frac{\left|A_n^i(x)\right|}{\left|A_n^{\mathbb{1}_j}(x)\right|} \le \frac{\left(C\,e^{n\mu_{j+1}^r}\right)^{\hat{s}}}{\left(C^{-1}\,e^{n\mu_j^l}\right)^{\hat{s}}} = C^{2\hat{s}}e^{-n\hat{s}\Delta_j^{j+1}} \le C^{2j}e^{-n\hat{s}\Delta_j^{j+1}} \le C^{2j}e^{-n\Delta_j^{j+1}}.$$

In a similar manner one proves

$$\frac{\left|A_n^i(x)A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2} \le C^{4j}e^{-n\Delta_j^{j+1}}.$$

□

Proof. (*Theorem3)* From (3.25) in the proof of Theorem 1 we have

$$n\left(\frac{1}{n}\int \ln R_{jj}(Dg^n(x))d\mu - \lambda_j\right) - C_j =$$

$$= \int \ln\left(\left(1+\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)\left(1+\frac{\rho_{j-1,n}(x)}{\Gamma_{j-1,n}(x)}\right)^{-1}\right)^{\frac{1}{2}} d\mu.$$

Note that the assumptions (A1),(i) of Theorem 1 follow from (A1'). Hence it remains to show

$$e^{n\Delta_j}\int \ln\left(\left(1+\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)\left(1+\frac{\rho_{j-1,n}(x)}{\Gamma_{j-1,n}(x)}\right)^{-1}\right)^{\frac{1}{2}} d \le \hat{C} \tag{3.29}$$

for all $n \ge \bar{n}$.

First consider the term $\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}$ and recall the estimate (3.27)

$$\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| \le$$

$$\le \frac{1}{\varepsilon_j^2}\left(\sum_{\substack{\nu\in Ord(j,d)\\ \ell\in Ord(j,d)\\ \ell\neq\mathbb{1}_j}}\left|\frac{\alpha_\nu(x)\alpha_\ell(x)A_n^\nu(x)A_n^\ell(x)}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2\left(A_n^{\mathbb{1}_j}(x)\right)^2}\right| + \sum_{\substack{\nu\in Ord(j,d)\\ \nu\neq\mathbb{1}_j}}\left|\frac{\alpha_\nu(x)A_n^\nu(x)}{\alpha_{\mathbb{1}_j}(x)A_n^{\mathbb{1}_j}(x)}\right|\right).$$

The formula (3.24) for the minors implies for $\nu \in Ord(j,d)$

$$\left|\frac{\alpha_\nu(x)}{\alpha_{\mathbb{1}_j}(x)}\right| = \left|\frac{\left(\bigwedge^j \alpha(x)\right)_{\nu,\mathbb{1}_j}}{\left(\bigwedge^j \alpha(x)\right)_{\mathbb{1}_j,\mathbb{1}_j}}\right| = \left|\frac{\left(\bigwedge^{d-j} W(x)\right)_{j+1,\dots,d,\,\hat{\nu}}}{\left(\bigwedge^{d-j} W(x)\right)_{j+1,\dots,d\,,j+1,\dots,d}}\right|,$$

where $\hat{\nu} \in Ord(d-j,d)$ satisfies $\{\hat{\nu}_1,\dots,\hat{\nu}_{d-j}\} \cup \{\nu_1,\dots,\nu_j\} = \{1,\dots,d\}$. Since $\left(\bigwedge^{d-j} W(x)\right)_{j+1,\dots,d,\,\hat{\nu}}$ is the $(j+1,\dots,d)$ coordinate of the vector

$$\left(\bigwedge^{d-j} W(x)\right)_{\cdot\hat{\nu}} = w_{\hat{\nu}_1}(x) \wedge \cdots \wedge w_{\hat{\nu}_{d-j}}(x),$$

we obtain

$$\left|\frac{\alpha_\nu(x)}{\alpha_{\mathbb{1}_j}(x)}\right| \le \frac{\left\|w_{\hat{\nu}_1}(x) \wedge \cdots \wedge w_{\hat{\nu}_{d-j}}(x)\right\|}{\left|\left(\bigwedge^{d-j} W(x)\right)_{j+1,\dots,d\,,j+1,\dots,d}\right|} = \frac{\left\|w_{\hat{\nu}_1}(x) \wedge \cdots \wedge w_{\hat{\nu}_{d-j}}(x)\right\|}{|P_{j+1}(x)|}.$$

Using the generalized Hadamard inequality from Lemma 13 and assumption (A1') we find for μ-a.e. $x \in M$

$$\left|\frac{\alpha_\nu(x)}{\alpha_{\mathbb{1}_j}(x)}\right| \le \frac{\prod_{k=1}^{d-j} \|w_{\hat{\nu}_k}(x)\|}{|P_{j+1}(x)|} = \frac{1}{|P_{j+1}(x)|} \le \frac{1}{\bar{\varepsilon}_j}.$$

In a similar way we obtain the following estimate for $\nu, l \in Ord(j,d)$

$$\frac{|\alpha_\nu(x)\alpha_l(x)|}{\left(\alpha_{\mathbb{1}_j}(x)\right)^2} \le \frac{1}{\bar{\varepsilon}_j^2}.$$

Summing up we have for μ-a.e. $x \in M$

$$\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| \le \frac{1}{\varepsilon_j^2\,\bar{\varepsilon}_j^2}\left(\sum_{\substack{\nu\in Ord(j,d)\\ \ell\in Ord(j,d)\\ \ell\neq\mathbb{1}_j}} \frac{\left|A_n^\nu(x)A_n^\ell(x)\right|}{\left(A_n^{\mathbb{1}_j}(x)\right)^2} + \sum_{\substack{\nu\in Ord(j,d)\\ \nu\neq\mathbb{1}_j}} \frac{|A_n^\nu(x)|}{\left|A_n^{\mathbb{1}_j}(x)\right|}\right).$$

The last sum has $\binom{d}{j} - 1$ summands and then Lemma 8 yields

$$\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| \le \frac{1}{\varepsilon_j^2\,\bar{\varepsilon}_j^2}\left(\binom{d}{j} - 1\right) C^{4j} e^{-\Delta_j^{j+1} n} = \tilde{C} e^{-\Delta_j^{j+1} n}. \tag{3.30}$$

In order to prove (3.29) it is sufficient to show for $k = j, j-1$ and $n \ge \bar{n}$

$$e^{n\Delta_k} \int \left|\ln\left(1 + \frac{\rho_{k,n}(x)}{\Gamma_{k,n}(x)}\right)\right| d\mu \le C'. \tag{3.31}$$

We consider $k = j$ since $k = j - 1$ can be handled analogously.

Using (3.30) and estimating $|\ln(1 + x)|$ by $|2x|$ we find an $\bar{n} \in \mathbb{N}$ such that for $n \geq \bar{n}$

$$\left|\ln\left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)\right| \leq 2\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| \quad \text{for } \mu\text{-a.e. } x \in M$$

and hence

$$e^{n\Delta_j} \int \left|\ln\left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)\right| d\mu \leq e^{n\Delta_j} \int 2\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| d\mu.$$

For n sufficiently large equation (3.30) implies

$$2e^{n\Delta_j}\left|\frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right| \leq 2\tilde{C}e^{n\left(\Delta_j - \Delta_j^{j+1}\right)} \leq 2\tilde{C},$$

and then Lebesgue's dominated convergence theorem gives the estimate

$$0 \leq e^{n\Delta_j} \int \left|\ln\left(1 + \frac{\rho_{j,n}(x)}{\Gamma_{j,n}(x)}\right)\right| d\mu \leq \int 2\tilde{C}\, d\mu,$$

which proves (3.31) for $k = j$.

□

3.3 Proof of Corollary 4

It remains to show that the assumptions (A2) of Theorem 3 are satisfied for all $j \in \{1, \ldots, d\}$ and that there exists some $\bar{\epsilon} > 0$ such that

$$|P_1(x)| = |\det(W(x))| > \bar{\epsilon} \quad \text{for } \mu\text{-a.e. } x \in M. \tag{3.32}$$

Setting $\gamma_k^i(x) = |\sin(\angle(w_i(x), \text{span}\{w_{i+1}(x), \ldots, w_k(x)\}))|$ the formula (B.2) leads for $j = 2, \ldots, d$ to the expression

$$\|w_1(x) \wedge \cdots \wedge w_j(x)\| = \gamma_j^1(x) \cdot \gamma_j^2(x) \cdots \gamma_j^{j-1}(x), \quad \text{for } \mu \text{ -a.e. } x \in M.$$

Now Lemma 7 and the hyperbolicity condition (ii) imply

$$\gamma_{i+1,\ldots,d}^i(x) \geq \varepsilon' \quad \text{for } i = 1, \ldots, d-1.$$

Since $\gamma_{i+1,\ldots,j}^i(x) \geq \gamma_{i+1,\ldots,d}^i(x)$ holds for $j \in \{1, \ldots, d\}$ and $i \leq j+1$ we have another $\tilde{\epsilon} > 0$ such that for μ-a.e. x

$$\|w_1(x) \wedge \cdots \wedge w_j(x)\| \geq \tilde{\epsilon}, \quad j \in \{1, \ldots, d\}. \tag{3.33}$$

By Corollary 12 the Gramian determinant coincides with the norm above and hence assumption (A2) is satisfied for all $j \in \{1, \dots, d\}$. Moreover, the estimate (3.33) implies (3.32) since

$$\|w_1(x) \wedge \cdots \wedge w_d(x)\| = |\det(W(x))| .$$

□

A Oseledec's theorem

We state a special version of the multiplicative ergodic theorem of Oseledec (see [24]) which may be found e.g. in [26], [28].

Theorem 9 (Oseledec, 1968)**.** *Let g be a C^1-diffeomorphism of a compact and smooth Riemannian manifold M of dimension d and let μ be an ergodic measure of g on M. Then there exists a Borel set $M_\mu \subset M$ such that $g(M_\mu) = M_\mu$, $\mu(M_\mu) = 1$, and the following properties hold:*

(i) There exist natural numbers $d_1, \dots, d_s$ with $s \leq d$ and $\sum_{j=1}^{s} d_j = d$.

(ii) For every $x \in M_\mu$ there exists a measurable decomposition of the tangent spaces $T_xM = \bigoplus_{j=1}^{s} W^j(x)$ such that $\dim W^j(x) = d_j$ and $D\,g(x)\left(W^j(x)\right) = W^j(g(x))$.

(iii) There are numbers $\lambda_1 > \lambda_2 > \ldots > \lambda_s$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Dg^n(x)v\| = \lambda_j$$

for all $v \in \bigoplus_{i=j}^{s} W^i(x)$ with $v \notin \bigoplus_{i=j+1}^{s} W^i(x)$ and for all $x \in M_\mu$.

Remarks. 1. *The points in M_μ are called (Lyapunov-)regular and the decomposition $T_xM = \bigoplus_{j=1}^{s} W^j(x)$ into invariant subspaces is called the Oseledec decomposition of TM.*

2. *The number λ_j is called the j-th Lyapunov exponent (or characteristic number) with respect to the ergodic measure μ. The number d_j denotes the multiplicity of λ_j.*

3. *The largest Lyapunov exponent* λ_1 *can also be expressed in terms of matrix norms as follows (see [26])*

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \ln \|D\, g^n(x)\| \quad \textit{for } \mu\textit{-a.e.} x \in M.$$

B Exterior products

For the convenience of the reader we summarize in this appendix several results from the theory of exterior products that are used in this paper. Most of the results can be found in [1], [17], [19], but a few details have been added that are important for our estimates.

B.1 Coordinate representation of exterior products

A mapping $\delta : \{1,\ldots,j\} \to \{1,\ldots,d\}$ with $j \le d$ is called *strictly monotone* if $\delta(1) < \delta(2) < \ldots < \delta(j)$ holds. By $Ord(j,d)$ we denote the set of all strictly monotone mappings

$$Ord(j,d) = \{\delta \in \{1,\ldots,d\}^{\{1,\ldots,j\}} \mid \delta \text{ strictly monotone}\}.$$

Let $D = \#Ord(j,d)$ and note that $D = \binom{d}{j}$. We will frequently identify elements $i \in Ord(j,d)$ with tuples $i = (i_1,\ldots,i_j)$ and simply write $i = i_1,\ldots,i_j$, where $1 \le i_1 < \ldots < i_j \le d$. In $Ord(j,d)$ we use lexicographical order written as $\sigma < \delta$ and meaning that for some $\ell \in \{1,\ldots,j\}$ we have

$$\sigma(k) = \delta(k) \;\; \text{for} \;\; k = 1,\ldots,\ell-1 \;\; \text{and} \;\; \sigma(\ell) < \delta(\ell).$$

The smallest element is $\delta_1 = (1,\ldots,j) = \mathbb{1}_j$ and the largest element is $\delta_D = (d+j-1,\ldots,d)$.

Given vectors $x_1,\ldots,x_j \in \mathbb{R}^d$ with coordinates $x_\ell = (x_{1\ell},\ldots,x_{d\ell})^T$ we denote by $X = [x_1,\ldots,x_j]$ the $d \times j$-matrix with columns $x_1,\ldots,x_j$, i.e.

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1j} \\ x_{21} & \cdots & x_{2j} \\ \cdots & \cdots & \cdots \\ x_{d1} & \cdots & x_{dj} \end{pmatrix}.$$

By $X_{i_1,\dots,i_j}$ we denote the minor of X that belongs to the rows with indices $i_1, \dots, i_j$, i.e.

$$X_{i_1 \dots i_j} = \det \begin{pmatrix} x_{i_1 1} & \cdots & x_{i_1 j} \\ x_{i_2 1} & \cdots & x_{i_2 j} \\ \cdots & \cdots & \cdots \\ x_{i_j 1} & \cdots & x_{i_j j} \end{pmatrix}.$$

The *exterior product* of vectors $x_1, \dots, x_j \in \mathbb{R}^d$ is defined as the vector

$$\wedge_{\ell=1}^{j} x_\ell = x_1 \wedge \cdots \wedge x_j \in \mathbb{R}^D.$$

with coordinates

$$(x_1 \wedge \cdots \wedge x_j)_{i_1 \dots i_j} = X_{i_1 \dots i_j}, \quad (i_1, \dots, i_j) \in Ord(j, d). \tag{B.1}$$

From the Cartesian basis $\{e_1, \dots, e_d\}$ in $\mathbb{R}^d$ we obtain

$$\{e_{i_1} \wedge \cdots \wedge e_{i_j} \,|\, (i_1, \dots, i_j) \in Ord(j, d)\}$$

as a basis of $\mathbb{R}^D$.
The coordinate representation of exterior products has standard properties.

Lemma 10. *For all $x_1, \dots, x_j, v \in \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{R}$ the following holds:*

(i) For any permutation $\pi \in S_j$,

$$x_{\pi(1)} \wedge \cdots \wedge x_{\pi(j)} = sign(\pi) x_1 \wedge \cdots \wedge x_j$$

(ii) For $\ell = 1, \dots, j$,

$$x_1 \wedge \cdots \wedge x_{\ell-1} \wedge (\alpha x_\ell + \beta v) \wedge x_{\ell+1} \wedge \cdots \wedge x_j =$$

$$\alpha(x_1 \wedge \cdots \wedge x_\ell \wedge \ldots \wedge x_j) + \beta(x_1 \wedge \cdots \wedge \ v \ \wedge \cdots \wedge x_j)$$

(iii) the vectors $x_1, \dots, x_j$ are linearly dependent if and only if $x_1 \wedge \cdots \wedge x_j = 0$.

Let $\langle \cdot, \cdot \rangle_k$ denote the standard scalar product in $\mathbb{R}^k$ with norm $\|x\| = \sqrt{\langle x \,, \, x \rangle_k}$ for $x \in \mathbb{R}^k$.

Theorem 11. *For two decomposable vectors*

$$x = x_1 \wedge \cdots \wedge x_j,\ y = y_1 \wedge \cdots \wedge y_j \in \mathbb{R}^D \quad \textit{with } x_\ell, y_\ell \in \mathbb{R}^d,\ \ell = 1, \ldots, j,$$

the scalar product can be written as

$$\langle x, y \rangle_D = \langle x_1 \wedge \cdots \wedge x_j, y_1 \wedge \cdots \wedge y_j \rangle_D = \det(Z),$$

where $Z \in \mathbb{R}^{j \times j}$ *is defined by*

$$Z_{i,\ell} = \langle x_i, y_\ell \rangle_d \quad \textit{for } i, \ell \in \{1, \ldots, j\}.$$

Proof. We have $Z = X^T Y$ for the matrices $X = [x_1, \ldots, x_j]$ und $Y = [y_1, \ldots, y_j]$, thus the multiplication theorem for determinants ([14]) shows

$$\det\left(X^T Y\right) = \sum_{i \in Ord(j,d)} X_{i_1, \ldots, i_j} Y_{i_1, \ldots, i_j} = \langle x_1 \wedge \cdots \wedge x_j, y_1 \wedge \cdots \wedge y_j \rangle_D.$$

□

An immediate consequence of this theorem is the following result.

Corollary 12. *For any set of vectors* $x_1, \ldots, x_j \in \mathbb{R}^d$,

$$\|x_1 \wedge \cdots \wedge x_j\| = \sqrt{\det(X^T X)}, \quad X = [x_1, \ldots, x_j],$$

in particular, $\|x_1 \wedge \cdots \wedge x_d\| = |\det(X)|$ *in case* $j = d$.

Recall that the Gramian determinant $\|x_1 \wedge \cdots \wedge x_j\| = \sqrt{\det(X^T X)}$ equals the j-dimensional volume of the parallelepiped spanned by the vectors $x_1, \ldots, x_j$, [17, §9.5].

Angles between vectors and subspaces can also be described in terms of exterior products. Consider, for example, linearly independent vectors $x_1, \ldots, x_j \in \mathbb{R}^d$ and let $x \in \mathbb{R}^d$ be arbitrary. Decompose $x = x^p + x^o$ where $x^p \in \operatorname{span}\{x_1, \ldots, x_j\}$ and $x^o \perp \operatorname{span}\{x_1, \ldots, x_j\}$. Then the volume of the parallelepiped spanned by $x_1, \ldots, x_j, x$ is

$$\|x_1 \wedge \cdots \wedge x_j \wedge x\| = \|x_1 \wedge \cdots \wedge x_j\| \, \|x^o\|.$$

or equivalently

$$\frac{\|x_1 \wedge \cdots \wedge x_j \wedge x\|}{\|x_1 \wedge \cdots \wedge x_j\|} = \|x^o\|.$$

This quotient is the length of the x-component orthogonal to $\operatorname{span}\{x_1,\dots,x_j\}$. In case $\|x\|=1$ we obtain the sine of the angle between x and the subspace

$$\frac{\|x_1\wedge\cdots\wedge x_j\wedge x\|}{\|x_1\wedge\cdots\wedge x_j\|}=\left|\sin\angle\left(x,\operatorname{span}\left\{x_1,\dots,x_j\right\}\right)\right|. \tag{B.2}$$

We also note the obvious estimate

$$\|x_1\wedge\cdots\wedge x_j\wedge x\|\le\|x_1\wedge\cdots\wedge x_j\|\,\|x\|, \tag{B.3}$$

which has the following generalization.

Lemma 13 (Generalized Hadamard inequality)**.**
For vectors $x_1,\dots,x_j\in\mathbb{R}^d$ with $j\le d$ and $k\in\{1,\dots,j\}$,

(i) $\|x_1\wedge\cdots\wedge x_j\|\le\|x_1\wedge\cdots\wedge x_k\|\|x_{k+1}\wedge\cdots\wedge x_j\|$.

(ii) $\|x_1\wedge\cdots\wedge x_j\|\le\prod_{i=1}^{j}\|x_i\|$.

The proof of (i) may be found in [17, §9.5] while (ii) follows from (B.3) by induction.

Definition 14. For a matrix $A\in\mathbb{R}^{d\times d}$ define its *j-th exterior power* $\bigwedge^j A\in\mathbb{R}^{D\times D}$, $D=\binom{d}{j}$ by its action on exterior products

$$\textstyle\bigwedge^j A(x_1\wedge\cdots\wedge x_j)=Ax_1\wedge\cdots\wedge Ax_j\quad\text{for } x_1,\dots,x_j\in\mathbb{R}^d.$$

As an immediate consequence of the definition we obtain that the column of $\bigwedge^j A$ belonging to the index $\ell_1,\dots,\ell_j$ is given by

$$\left(\textstyle\bigwedge^j A\right)_{\cdot\,\ell_1,\dots,\ell_j}=\textstyle\bigwedge^j A\left(e_{\ell_1}\wedge\cdots\wedge e_{\ell_j}\right)=A_{\cdot\ell_1}\wedge\cdots\wedge A_{\cdot\ell_j}. \tag{B.4}$$

By the definition of the exterior product (B.1) we obtain that the element $i_1,\dots,i_j$ of column $\ell_1,\dots,\ell_j$ is

$$\left(\textstyle\bigwedge^j A\right)_{i_1,\dots,i_j,\ell_1,\dots,\ell_j}=\det\begin{pmatrix}A_{i_1\ell_1}&\cdots&A_{i_1\ell_j}\\A_{i_2\ell_1}&\cdots&A_{i_2\ell_j}\\\cdots&\cdots&\cdots\\A_{i_j\ell_1}&\cdots&A_{i_j\ell_j}\end{pmatrix}. \tag{B.5}$$

The exterior power has the following properties:

Lemma 15. *For $A, B \in \mathbb{R}^{d\times d}$ holds,*

(i) $\bigwedge^j (AB) = \bigwedge^j (A) \bigwedge^j (B)$.

(ii) $\bigwedge^j (A^T) = \left(\bigwedge^j A\right)^T$,

(iii) $\bigwedge^j (A^{-1}) = \left(\bigwedge^j A\right)^{-1}$, *if A is nonsingular.*

Finally, we note the transformation rule for exterior products.

Lemma 16. *If $x_1, \dots, x_d$ and $y_1, \dots, y_j$ are vectors in $\mathbb{R}^d$ that satisfy*

$$y_\ell = \sum_{i=1}^{d} A_{i\ell} x_i \quad \textit{for } \ell = 1, \dots, j, \ A_{i\ell} \in \mathbb{R},$$

then

$$y_1 \wedge \cdots \wedge y_j = \sum_{(\ell_1,\dots,\ell_j)\in Ord(j,d)} \left(\bigwedge^j A\right)_{\ell_1,\dots,\ell_j,\mathbb{1}_j} (x_{\ell_1} \wedge \cdots \wedge x_{\ell_j}).$$

Proof. With $X = [x_1, \dots, x_d]$ we obtain

$$y_1 \wedge \cdots \wedge y_j = X\, A_{\cdot 1} \wedge \cdots \wedge X\, A_{\cdot j} = \bigwedge^j X \left(A_{\cdot 1} \wedge \cdots \wedge A_{\cdot j}\right).$$

Using (B.1) and (B.4) we find

$$\begin{aligned} y_1 \wedge \cdots \wedge y_j = & \ \textstyle\sum_{(\ell_1,\dots,\ell_j)\in Ord(j,d)} \left(\bigwedge^j X\right)_{\cdot\, \ell_1,\dots,\ell_j} \left(A_{\cdot 1} \wedge \cdots \wedge A_{\cdot j}\right)_{\ell_1,\dots,\ell_j} \\ = & \ \textstyle\sum_{(\ell_1,\dots,\ell_j)\in Ord(j,d)} \left(\bigwedge^j A\right)_{\ell_1,\dots,\ell_j,\mathbb{1}_j} X_{\cdot \ell_1} \wedge \cdots \wedge X_{\cdot \ell_j}. \end{aligned}$$

This proves the assertion.

□

B.2 Exterior product and QR-decomposition

As noted in [20] without proof the QR-decomposition is consistent with the formation of exterior powers.

Lemma 17. *Let $A \in \mathbb{R}^{d\times d}$ be nonsingular and let $A = QR$ be its unique QR-decomposition (with positive diagonal entries for R). Then $\bigwedge^j A = \left(\bigwedge^j Q\right)\left(\bigwedge^j R\right)$ is the unique QR-decomposition of $\bigwedge^j A$. In particular, the diagonal elements of $R\left(\bigwedge^j A\right)$ are given by*

$$R_{i_1,\dots,i_j,i_1,\dots,i_j}\left(\bigwedge^j A\right) = \prod_{k=1}^{j} R_{i_k i_k}, \quad (i_1, \dots, i_j) \in Ord(j,d). \tag{B.6}$$

Proof. In view of Lemma 15 is is sufficient to show that $\bigwedge^j Q$ is orthogonal and $\bigwedge^j R$ is upper triangular with positive diagonal entries.
Lemma 15 shows the orthogonality of $\bigwedge^j Q$,

$$\left(\bigwedge^j Q\right)^T \bigwedge^j Q = \left(\bigwedge^j Q^T\right) \bigwedge^j Q = \bigwedge^j \left(Q^T Q\right) = \bigwedge^j I_d = I_D.$$

Next note that according to (B.5),

$$\left(\bigwedge^j R\right)_{i_1,\ldots,i_j,\ell_1,\ldots,\ell_j} = \det \begin{pmatrix} R_{i_1\ell_1} & \cdots & R_{i_1\ell_j} \\ R_{i_2\ell_1} & \cdots & R_{i_2\ell_j} \\ \cdots & \cdots & \cdots \\ R_{i_j\ell_1} & \cdots & R_{i_j\ell_j} \end{pmatrix}. \tag{B.7}$$

If $(i_1, \ldots, i_j) > (\ell_1, \ldots, \ell_j)$ then there exists an index $\hat{k}$ such that $i_k = \ell_k$ for $k = 1, \ldots, \hat{k} - 1$ and $i_{\hat{k}} > \ell_{\hat{k}}$. Hence $R_{i_{\hat{k}}\ell_{\hat{k}}} = 0$. Since $\ell_k < \ell_{\hat{k}}$ for $k = 1, \ldots, \hat{k} - 1$ and $i_n > i_{\hat{k}}$ for $n = \hat{k} + 1, \ldots, j$ we arrive at

$$i_n > \ell_k \quad \text{for } k = 1, \ldots, \hat{k} - 1 \text{ and } n = \hat{k} + 1, \ldots, j.$$

and therefore,

$$R_{i_n\ell_k} = 0 \quad \text{for } k = 1, \ldots, \hat{k} - 1 \text{ and } n = \hat{k} + 1, \ldots, j.$$

Thus the first $\hat{k}$ columns of the matrix in (B.7) are of the form

$$(R_{i_1\ell_k}, \ldots, R_{i_{\hat{k}-1}\ell_k}, 0, \ldots, 0)^T \quad \text{for } k = 1, \ldots, \hat{k},$$

and hence linearly dependent. Moreover, the determinant in (B.7) vanishes for $(i_1, \ldots, i_j) > (\ell_1, \ldots, \ell_j)$. Therefore, both equation (B.6) and the positivity of diagonal elements follow from (B.7).

□

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