Stability and freezing of nonlinear waves in first order hyperbolic PDEs

Jens Rottmann-Matthes*

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Abstract

It is a well-known problem to derive nonlinear stability of a traveling wave from the spectral stability of a linearization. In this paper we prove such a result for a large class of hyperbolic systems.

To cope with the unknown asymptotic phase, the problem is reformulated as a partial differential algebraic equation for which asymptotic stability becomes the usual Lyapunov stability. The stability proof is then based on linear estimates from a previous paper and a careful analysis of the nonlinear terms. Moreover, we show that the freezing method [4, 16] is well-suited for the long time simulation and numerical approximation of the asymptotic behavior.

The theory is illustrated by two numerical examples, including a hyperbolic version of the Hodgkin-Huxley equations.

Keywords: Hyperbolic partial differential equations, traveling waves, partial differential algebraic equations, linear stability, asymptotic behavior, resolvent estimates

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1 Introduction

At the latest since the work of Sattinger [17] it is well-known that for many reaction-diffusion systems the nonlinear stability of traveling waves can be derived from their spectral stability. This is usually proved using analytic semigroup theory, see for example [8] and [20, Ch. 5]. Here we consider first order hyperbolic problems which do not generate analytic semigroups.

We analyze the Cauchy problem for hyperbolic PDEs of the form

$$v_t = Bv_x + f(v) \qquad \text{in } \mathbb{R}\times\mathbb{R}_+, \qquad v(0) = v_0,\ v(x,t)\in\mathbb{R}^m, \tag{1.1}$$

that possess a traveling wave solution $V(x,t) = \underline{v}(x-\underline{\lambda}t)$, where $\underline{v}$ is the profile and $\underline{\lambda}$ the speed of the wave. Our assumptions on the matrix B allow (1.1) to be a non-strictly hyperbolic system coupled to a system of ODEs.

*Department of Mathematics, Bielefeld University, P.O. Box 100131, 33501 Bielefeld. This research was supported by CRC 701 "Spectral Analysis and Topological Methods in Mathematics".

Equations of this form have been derived in the modeling of chemical reaction systems. For example, King et al. in [10] consider a chemical process which is basically of the form

$$v_t + q_x = r,$$

where $v(x,t)$ is the concentration of a species, q is the flux, and r a supply rate of the species. The authors observe that for certain supply rates, the usual constitutive assumption on the flux, given by Fick's law (i.e. $q = -Dv_x$), leads to anomalous behavior. Therefore, King et al. propose in [11] a different relation of the form $q + Rq_t = -Dv_x$, which prevents infinite speed of propagation of the species and leads to a system of the form (1.1). In Section 8 we consider several examples where such a constitutive relation is chosen.

In Sections 3–6 we prove asymptotic stability with asymptotic phase of the traveling wave under purely spectral assumptions. In particular, we do not assume any growth bound on the nonlinearity f, but use only local properties. This is very important for applications from reaction diffusion systems, since the reaction term is usually smooth but unbounded.

Our stability result is closely related to the results presented by Kreiss et al. in [12]. We relax the assumption to include also non-strictly hyperbolic problems, which is a severe restriction. Furthermore, we treat the problem of the unknown asymptotic phase by a reformulation as a partial differential algebraic equation (PDAE). This approach seems to be much clearer and more natural then the approach in [12] and by the PDAE approach a rigorous justification of the use of the Laplace transform (see [15]) is possible. This is not given in [12].

From a computational point of view, the asymptotic stability with asymptotic phase is of little use because the speed is unknown. Therefore, in Section 7 we also show asymptotic stability for the PDAE that is used numerically for the freezing method [4, 16]. This generalizes results of [18] to hyperbolic systems. In Section 8 we consider (hyperbolic versions) of the Nagumo equation and the Hodgkin-Huxley equations [9]. These systems satisfy our assumptions and we confirm the theoretically predicted results in numerical experiments.

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2 Assumptions and Main Results

We assume that f is a smooth function and the profile v is asymptotically constant as $|x| \to \infty$.

Assumption 2.1. *The nonlinearity f in (1.1) is an element of $\mathcal{C}^3(\mathbb{R}^m, \mathbb{R}^m)$. There is a non-constant traveling wave solution of (1.1) with profile $\underline{v}$ and speed $\underline{\lambda}$ that satisfies*

$$\underline{v} \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^m), \qquad \underline{v}_x \in H^2(\mathbb{R}, \mathbb{R}^m), \qquad f(\underline{v}) \in L^2(\mathbb{R}, \mathbb{R}^m).$$

As usual, $\mathcal{C}_b^1(\mathbb{R})$ denotes the space of bounded and continuously differentiable functions with bounded derivatives, H^k, $k \geq 1$, are the usual Sobolev spaces. Note that Assumption 2.1 implies $f(\underline{v}) \in H^2$.

We require that equation (1.1) is hyperbolic and, without loss of generality, we assume that $B \in \mathbb{R}^{m,m}$ is a real diagonal matrix with diagonal entries $b_{11} \geq \ldots \geq b_{mm}$. In a co-moving frame the traveling wave becomes a stationary solution of

$$v_t = (\underline{\lambda} I + B)v_x + f(v). \tag{2.1}$$

To fix notation, we denote the linearization of equation (2.1) about $\underline{v}$ by

$$v_t = (\underline{\lambda} I + B)v_x + f_v(\underline{v})v =: (\underline{\lambda} I + B)v_x + C(x)v =: Pv. \tag{2.2}$$

We impose the following conditions on the linear differential operator P

Assumption 2.2. **(H1)** *The matrix $\underline{\lambda} I + B \in \mathbb{R}^{m,m}$ is an invertible, real diagonal matrix with r positive and $m - r$ negative eigenvalues.*

(H2) *The matrix valued function C belongs to $\mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^{m,m})$ and the limits*

$$\lim_{x\to\infty} C(x) = C_{\pm} \qquad \text{and} \qquad \lim_{x\to\infty} C_x(x) = 0 \quad \text{exist.}$$

(H3) *There is $\delta > 0$ so that $s \in \sigma(i\omega(\underline{\lambda} I + B) + C_{\pm})$ for some $\omega \in \mathbb{R}$ implies $\Re s \leq -\delta$, where σ denotes the spectrum.*

Remark. *The existence of the limit matrices $C_{\pm}$ is a consequence of Assumption 2.1. In particular the smoothness of the steady state $\underline{v}$ implies that the matrix valued functions C_x and C_{xx} are uniformly bounded.*

Because of the translational invariance of (1.1), the linear operator P (as a closed operator in L^2) always has an eigenvalue 0 with corresponding eigenfunction $\underline{v}_x$. We assume that no further spectrum of P lies to the right of $-\delta$ in the complex plane:

Assumption 2.3. *The spectrum of the closed linear operator $P : L^2 \to L^2$, satisfies*

$$\sigma(P) \cap \{\mathrm{Re}\, s > -\delta\} = \{0\}$$

and zero is an algebraically simple eigenvalue.

These assumptions lead to our first main result, the asymptotic stability of traveling waves with asymptotic phase, see Theorem 2.5. First let us make precise the notion of a solution.

Definition 2.4. A function v is called a (classical) solution of (1.1) in $[0, T]$ iff

$$v \in \mathcal{C}^1([0, T]; \underline{v} + L^2) \cap \mathcal{C}^0([0, T]; \underline{v} + H^1),$$

$v(0) = v_0$, and $v_t = Bv_x + f(v)$ holds as an equality in $L^2(\mathbb{R}, \mathbb{R}^m)$ for all $t \in [0, T]$. Furthermore, the function v is called a solution on $\mathbb{R}_+$ if it is a solution on $[0, T]$ for all $T > 0$.

Theorem 2.5 (Stability of steady states)**.** *Let Assumptions 2.1, 2.2, 2.3 hold. Then for every $0 < \eta < \delta$ there is $\rho = \rho(\eta) > 0$ such that for all initial data $v_0 \in \underline{v} + H^2$ with $\|v_0 - \underline{v}\|_{H^2} < \rho$ the Cauchy problem (1.1) has a unique global solution.*

Moreover, there is $\varphi_\infty = \varphi_\infty(v_0) \in \mathbb{R}$, depending on v_0, and a constant $C > 0$, depending on η and $\|\underline{v}_x\|_{H^2}$, with

$$|\varphi_\infty| \le C\|v_0 - \underline{v}\|_{H^2}, \text{ and} \tag{2.3}$$

$$\|v(\cdot,t) - \underline{v}(\cdot - \underline{\lambda}t - \varphi_\infty)\|_{H^1} \le C\|v_0 - \underline{v}\|_{H^2} e^{-\eta t} \quad \forall t \ge 0. \tag{2.4}$$

Theorem 2.5 is important from a theoretical point of view, but in applications, one is rather interested in the asymptotic profile itself and its speed. Moreover, long-time simulations of such systems typically face the problem that the solution leaves the computational domain in finite time. One idea to circumvent this is, to separate the evolution of the solution into the evolution of a (time-dependent) profile and a (time-dependent) phase variable. This is the principal idea of the freezing method [4, 16]. The method is suitable for rather general symmetries, but here we consider only traveling waves, where the symmetry is induced by a spatial shift.

Basically, one makes the ansatz

$$v(x,t) = u(x + \Lambda(t), t) \tag{2.5}$$

for the solution v. The new variable u is interpreted as the profile and $\Lambda(t) \in \mathbb{R}$ is interpreted as the position of the solution. If u and Λ are sufficiently smooth with respect to time, differentiating (2.5) and setting $\lambda := \dot{\Lambda}$ leads to

$$u_t = Bu_x + f(u) + \lambda u_x.$$

The new variable λ introduces an additional degree of freedom into the system, and one needs an additional equation to obtain a well-posed problem again. One possibility is to require that $u(\cdot,t)$ always lies in the same hyperplane in L^2. This can be achieved e.g. by a fixed phase condition [5]

$$0 = \Psi(u - \hat{u}),$$

where Ψ is a linear functional in L^2 and $\hat{u}$ is some reference function. The ansatz (2.5) then leads to the system of PDAEs

$$u_t = Bu_x + f(u) + \lambda u_x, \qquad u(0) = u_0, \tag{2.6a}$$

$$0 = \Psi(u - \hat{u}), \tag{2.6b}$$

which can be implemented on a computer.

The functional Ψ and the reference function $\hat{u}$ should satisfy

Assumption 2.6. (P1) *Ψ is linear,*

$$|\Psi(u)| \le C_\Psi \|u\|_{L^2}, \quad \forall u \in L^2(\mathbb{R}, \mathbb{R}^m), \qquad \text{(continuity)}, \tag{2.7}$$

$$\Psi(\underline{v}_x) \ne 0, \qquad \text{(non-degeneracy)}. \tag{2.8}$$

(P2) *The reference function $\hat{u}$ satisfies $\underline{v} - \hat{u} \in H^1$ and $\Psi(\underline{v} - \hat{u}) = 0$.*

By Assumption 2.1, $\underline{v}$ is a stationary solution of $v_t = (B + \underline{\lambda} I)v_x + f(v)$, so that (2.6a) is satisfied. Furthermore, (P2) of Assumption 2.6 implies that $\underline{v}$ satisfies (2.6b) and therefore the profile $\underline{v}$ and its speed $\underline{\lambda}$ is a stationary solution of (2.6).

The following theorem shows that the stationary solution $(\underline{v}, \underline{\lambda})$ is exponentially stable in the sense of Lyapunov and therefore can be approximated by a direct long-time simulation.

Theorem 2.7 (Lyapunov stability for the freezing method)**.** *Let Assumptions 2.1, 2.2, 2.3, and 2.6 hold. Then for all $0 < \eta < \delta$ there is $\rho_0 > 0$ such that for all consistent initial data $u(0) = u_0 \in \underline{v} + H^2$ with $\|u_0 - \underline{v}\|_{H^2} < \rho_0$ there is a unique solution (v, λ) of the PDAE* (2.6) *in $[0, \infty)$. The solution satisfies the smoothness*

$$u \in \mathcal{C}^1([0,\infty); \underline{v} + L^2) \cap C([0,\infty); \underline{v} + H^1),$$
$$\lambda \in \mathcal{C}([0,\infty); \mathbb{R}).$$

Furthermore, there is a constant $C = C(\eta)$, independent of the initial data, so that

$$\|u(t) - \underline{v}\|_{H^1} + |\lambda(t) - \underline{\lambda}| \le C\|u_0 - \underline{v}\|_{H^2} e^{-\eta t} \quad \forall t \ge 0. \tag{2.9}$$

Remarks. *a) For the notion of a solution of the PDAE see Definition 3.7.*

b) When system (2.6) *is solved on a computer, one may choose $\hat{u} = u_0$, so that the algebraic condition* (2.6b) *is immediately satisfied.*

c) A reasonable choice for the phase condition is

$$0 = \Psi(u - \hat{u}) = \int_{\mathbb{R}} \hat{u}_x^T (u - \hat{u})\, dx,$$

see [5]. In this case, Ψ belongs to the dual of L^2 if $\hat{u}$ is an element of $\underline{v} + H^1$. From a computational point of view this is a reasonable assumption. For example it is satisfied if $\hat{u}$ is a continuous, piecewise linear function. (See Section 8.1 for an example.)

d) Usually, the profile $\underline{v}$ is unknown. Because of the non-degeneracy condition (2.8)*, it follows from the inverse function theorem, that if $\hat{u}$ is close enough to some profile $\underline{v}$, there is a shift $\xi \in \mathbb{R}$, such that $\psi(\hat{u} - \underline{v}(\cdot - \xi)) = 0$. In this sense, the second part from Assumption 2.6 can be considered as fixing the properly shifted profile $\underline{v}$, to which the u–variable of the solution of* (2.6) *converges.*

e) In many situations one is not interested in the position Λ, but only in the speed $\lambda = \dot{\Lambda}$ of the wave, which appears in the PDAE. In case the position is needed, one can compute it from λ by integration.

A major difficulty in the proof of Theorem 2.5 is the treatment of the unknown asymptotic phase. For this we reformulate the problem as a partial differential algebraic equation in Section 3.2, which we prove to be (at least locally) equivalent to the original PDE problem. By a careful analysis, exponential stability is shown for this nonlinear PDAE in Sections 4-5. In Section 6 we prove that this implies Theorem 2.5.

In Section 7 we give the proof of Theorem 2.7. Here the main problem is related to the term λu_x, which belongs to the principal symbol. In particular, it cannot be treated as a small perturbation as in the parabolic case considered by Thümmler [19].

3 PDAE reformulation

As a **general assumption** in this whole section assume that Assumption 2.1 holds. Because of hyperbolicity, it is no restriction to assume that B in (1.1) is a real diagonal matrix. Furthermore, we assume without loss of generality $\underline{\lambda} = 0$. If $\underline{\lambda} \neq 0$, the equation is considered in a co-moving frame, i.e. one considers

$$\widetilde{v}_t = (B + \underline{\lambda})\widetilde{v}_x + f(\widetilde{v}), \widetilde{v}(0) = v_0, \tag{1.1'}$$

so that, $\widetilde{v}$ is a solution of (1.1') if and only if $v(x,t) = \widetilde{v}(x - \underline{\lambda}t, t)$ is a solution of (1.1). For this equation the traveling wave is a stationary solution.

3.1 Existence and uniqueness for the PDE

Let us first state a local existence and uniqueness result for the semilinear PDE problem (1.1). When one rewrites the solution as a perturbation w of the steady profile, i.e. $v = \underline{v} + w$, problem (1.1) is equivalent to the following equation for w

$$\begin{aligned} w_t &= Bw_x + f(\underline{v} + w) - f(\underline{v}) =: Bw_x + C(x)w + q(x, w), \\ w(0) &= w_0 := v_0 - \underline{v}, \end{aligned} \tag{3.1}$$

where $C(x) = f_v(\underline{v}(x))$ and $q(\cdot, w) = f(\underline{v} + w) - f(\underline{v}) - f_v(\underline{v})w$. By Taylor's formula holds $q(\cdot, w) = \int_0^1 (1 - s)D^2 f(\underline{v} + sw)ds[w, w]$ as an equality in H^1 for $v \in H^1$. Here and in the following we use the notation $A[u, v]$ for a bilinear mapping A applied to u, v and Au^2 for $A[u, u]$. The following existence and uniqueness result is well-known:

Theorem 3.1 (Existence and uniqueness)**.** *Let the assumptions be as above. For every* $w_0 \in H^1(\mathbb{R}^m)$ *there is* $T = T(\|w_0\|_{H^1}) > 0$*, depending only on the* H^1*-norm of* w_0*, such that the Cauchy problem* (3.1) *possesses a unique solution* $w \in \mathcal{C}^1([0, T]; L^2) \cap \mathcal{C}^0([0, T]; H^1)$*.*

This theorem can easily be proved by the method of characteristics. For example the methods from [6, Ch. 3], where similar results for compact domains are shown, can be adapted to the current setting. Therefore, we omit the details.

Because T in Theorem 3.1 only depends on the H^1-norm of the initial data and (3.1) does not explicitly depend on time, a simple contradiction argument proves the following global continuation result, which we formulate for the original equation:

Theorem 3.2 (Global continuation)**.** *For every* $v_0 \in \underline{v} + H^1(\mathbb{R})$ *there is a unique global solution*

$$v^* \in \mathcal{C}^1([0, T^*); \underline{v} + L^2) \cap \mathcal{C}^0([0, T^*); \underline{v} + H^1)$$

of (1.1)*, so that if* $v \in \mathcal{C}^1([0, T]; \underline{v} + L^2) \cap \mathcal{C}^0([0, T]; \underline{v} + H^1)$ *is a solution of* (1.1) *it follows* $T < T^*$ *and* $v^*|_{[0,T]} = v$*.*
Moreover,

$$\textit{either} \quad T^* = +\infty \quad \textit{or} \quad T^* < \infty \textit{ and } \lim_{t \nearrow T^*} \|v^*(t) - \underline{v}\|_{H^1} = +\infty.$$

A complete proof of this theorem can be found in [14, Ch. 3].

3.2 PDAE reformulation via nonlinear coordinates

By the general assumptions of this section, $\underline{v}$ is a steady state. For the analysis of its stability it is convenient to rewrite the equation using nonlinear coordinates $\widetilde{v}$ and $\widetilde{\varphi}$ in the form:

$$v(x,t) = \underline{v}(x - \widetilde{\varphi}(t)) + \widetilde{v}(x,t). \tag{3.2}$$

This separates the evolution of the position and of the perturbation of the shape. A similar idea is used in [8, Ch. 5.1], where the asymptotic stability of a family of equilibria in parabolic evolution equations is analyzed. A crucial assumption in [8] is that the principal part is a sectorial operator, which is not satisfied here.

The splitting of v in (3.2) is not unique, therefore we impose

$$\psi\big(\widetilde{v}(\cdot,t)\big) = 0, \qquad \forall t \geq 0. \tag{3.3}$$

Here ψ is a linear functional, so that (3.3) restricts $\widetilde{v}$ to some hyperplane. This is very similar to the phase condition (2.6b) of the freezing method.

Assumption 3.3. *The linear functional ψ is bounded as a mapping $H^{-1}(\mathbb{R},\mathbb{R}^m) = (H^1(\mathbb{R},\mathbb{R}^m))' \to \mathbb{R}$ with bound $C_\psi > 0$, i.e.*

$$|\psi(v)| \leq C_\psi \|v\|_{H^{-1}} \qquad \forall v \in H^{-1}(\mathbb{R},\mathbb{R}^m). \tag{3.4}$$

Furthermore, ψ satisfies the non-degeneracy condition

$$\psi(\underline{v}_x) \neq 0. \tag{3.5}$$

The boundedness (3.4) is important in Section 4 and is needed for Theorem 4.2, see also [15, §4, §5].

Lemma 3.4. *Let ψ be given as above and let Assumption 2.1 hold. Then there are open neighborhoods U and V of $0 \in \mathbb{R}$ so that the function*

$$G : U \to V, \quad G(\widetilde{\varphi}) = \psi\big(\underline{v}(\cdot - \widetilde{\varphi}) - \underline{v}\big) \tag{3.6}$$

is a $\mathcal{C}^2$–diffeomorphism.

Proof. The smoothness $\underline{v}_x \in H^2(\mathbb{R},\mathbb{R}^m)$ implies that $\widetilde{\varphi} \mapsto \underline{v}(\cdot - \widetilde{\varphi}) - \underline{v}$ is twice continuously differentiable as a mapping $\mathbb{R} \to L^2$. Therefore, G belongs to the class $\mathcal{C}^2$ by the chain rule. Furthermore, by (3.5)

$$\frac{d}{d\widetilde{\varphi}} G(\widetilde{\varphi})|_{\widetilde{\varphi}=0} = -\psi(\underline{v}_x) \neq 0,$$

and the assertion follows from the inverse function theorem. □

Remark. *Note that the proof does not make use of (3.4), but only uses the L^2–boundedness of ψ. Therefore, the lemma also holds for the functional Ψ from the freezing method.*

Let $\Phi := (G|_U)^{-1} \in \mathcal{C}^2(V,U)$ denote the inverse of the mapping G from Lemma 3.4. For $v \in \underline{v} + L^2$ with $\psi(v-\underline{v}) \in V$ define $\widetilde{v}$ and $\widetilde{\varphi}$ by

$$\Omega(v) := \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{v} \end{pmatrix} := \begin{pmatrix} \Phi\big(\psi(v-\underline{v})\big) \\ v - \underline{v}\big(\cdot - \Phi\big(\psi(v-\underline{v})\big)\big) \end{pmatrix}. \tag{3.7}$$

Conversely, for arbitrary $\widetilde{\varphi} \in \mathbb{R}$ and $\widetilde{v} \in L^2(\mathbb{R},\mathbb{R}^m)$, define

$$\Xi \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{v} \end{pmatrix} := \widetilde{v} + \underline{v}(\cdot - \widetilde{\varphi}). \tag{3.8}$$

These transformations are inverse to each other on the domains

$$\begin{aligned} &\mathcal{D}_\Omega := \big\{ v \in \underline{v} + L^2(\mathbb{R},\mathbb{R}^m) : \psi(v-\underline{v}) \in V \big\}, \quad \text{and} \\ &\mathcal{D}_\Xi := \big\{ (\widetilde{\varphi},\widetilde{v}) \in \mathbb{R} \times L^2(\mathbb{R},\mathbb{R}^m) : \widetilde{\varphi} \in U, \widetilde{v} \in \mathcal{N}(\psi) \big\}, \text{ respectively.} \end{aligned} \tag{3.9}$$

Lemma 3.5. *Under the assumptions of Lemma 3.4, the restrictions $\Omega|_{\mathcal{D}_\Omega}$ of Ω to $\mathcal{D}_\Omega$ and $\Xi|_{\mathcal{D}_\Xi}$ of Ξ to $\mathcal{D}_\Xi$ are inverse to each other.*

Proof. Let $v \in \mathcal{D}_\Omega$, then $(\widetilde{\varphi},\widetilde{v}) = \Omega(v)$ is well defined. By definition of Ω and Lemma 3.4 holds $\widetilde{\varphi} = \Phi\big(\psi(v-\underline{v})\big) \in U$ and therefore

$$\begin{aligned} \psi(\widetilde{v}) &= \psi\left(v - \underline{v}(\cdot - \widetilde{\varphi})\right) = \psi\left(v-\underline{v}\right) - \psi\left(\underline{v}(\cdot-\widetilde{\varphi}) - \underline{v}\right) \\ &= G \circ \Phi\left(\psi(v-\underline{v})\right) - G\circ\Phi\left(\psi\big(\underline{v}(\cdot-\widetilde{\varphi})-\underline{v}\big)\right) = G(\widetilde{\varphi}) - G(\widetilde{\varphi}) = 0, \end{aligned}$$

so that $\Omega\left(\mathcal{D}_\Omega\right) \subset \mathcal{D}_\Xi$. Conversely, for $(\widetilde{\varphi},\widetilde{v}) \in \mathcal{D}_\Xi$ holds

$$\Xi(\widetilde{\varphi},\widetilde{v}) = \widetilde{v} + (\underline{v}(\cdot-\widetilde{\varphi}) - \underline{v}) + \underline{v},$$

so that $\underline{v} \in \mathcal{C}^1_b, \underline{v}_x \in L^2$, implies $\Xi(\widetilde{\varphi},\widetilde{v}) \in \underline{v} + L^2(\mathbb{R},\mathbb{R}^m)$. Moreover, $\psi\left(\widetilde{v} + \underline{v}(\cdot-\widetilde{\varphi}) - \underline{v}\right) = \psi(\underline{v}(\cdot-\widetilde{\varphi})-\underline{v}) \in V$, since $\widetilde{v} \in \mathcal{N}(\psi)$ and $\widetilde{\varphi} \in U$. This shows $\Xi(\mathcal{D}_\Xi) \subset \mathcal{D}_\Omega$.

Finally, the identities $\Xi \circ \Omega|_{\mathcal{D}_\Omega} = \mathrm{id}_{\mathcal{D}_\Omega}$ and $\Omega \circ \Xi|_{\mathcal{D}_\Xi} = \mathrm{id}_{\mathcal{D}_\Xi}$ are easily verified. □

Since these transformations will be used for solutions of (1.1), it is important to analyze how smoothness properties of a time dependent function v relate to smoothness properties of $\widetilde{\varphi}$ and $\widetilde{v}$, which are given by $\big(\widetilde{\varphi}(t),\widetilde{v}(t)\big) = \Omega\big(v(t)\big)$.

Lemma 3.6. *Let Assumptions 2.1 and 3.3 hold. Define*

$$M^T_\Omega := \left\{ v \in \mathcal{C}^1([0,T];\underline{v}+L^2) \cap \mathcal{C}([0,T];\underline{v}+H^1) : v(t) \in \mathcal{D}_\Omega\, \forall t \in [0,T] \right\}.$$

For $v \in M^T_\Omega$ the functions $\widetilde{\varphi}$, $\widetilde{v}$, given by $(\widetilde{\varphi},\widetilde{v}) = \Omega(v)$, satisfy $(\widetilde{\varphi}(t),\widetilde{v}(t)) \in \mathcal{D}_\Xi$ and

$$\widetilde{\varphi} \in \mathcal{C}^1([0,T];\mathbb{R}) \quad \textit{and} \quad \widetilde{v} \in \mathcal{C}^1([0,T];L^2) \cap \mathcal{C}^0([0,T];H^1). \tag{3.10}$$

Conversely, if $\widetilde{\varphi}$ and $\widetilde{v}$ satisfy the smoothness (3.10)*, then*

$$v := \Xi(\widetilde{\varphi},\widetilde{v}) = \widetilde{v} + \underline{v}(\cdot-\widetilde{\varphi}) \in \mathcal{C}^1([0,T];\underline{v}+L^2) \cap \mathcal{C}^0([0,T];\underline{v}+H^1). \tag{3.11}$$

Moreover, let

$$M^T_\Xi := \{(\widetilde{\varphi},\widetilde{v}) \ \textit{with (3.10) holds and} \ (\widetilde{\varphi}(t),\widetilde{v}(t)) \in \mathcal{D}_\Xi\},$$

then $\Omega|_{M^T_\Omega}$ and $\Xi|_{M^T_\Xi}$ are inverse to each other.

Proof. Let $v \in M_\Omega^T$. The smoothness $\widetilde{\varphi} = \Phi\big(\psi(v - \underline{v})\big) \in \mathcal{C}^1([0,T];\mathbb{R})$ follows from $v - \underline{v} \in \mathcal{C}^1([0,T];L^2)$ and Lemma 3.4. For the smoothness of $\widetilde{v} = \big(v - \underline{v}\big) + \big(\underline{v} - \underline{v}(\cdot - \widetilde{\varphi})\big)$ note that $v \in M_\Omega^T$ implies $v - \underline{v} \in \mathcal{C}^1([0,T];L^2) \cap \mathcal{C}^0([0,T];H^1)$ and, by assumption, the mapping

$$\varphi \mapsto \underline{v}(\cdot) - \underline{v}(\cdot - \varphi) \tag{3.12}$$

is differentiable for all $\varphi \in \mathbb{R}$ with derivative at φ given by $\underline{v}_x(\cdot - \varphi) \in H^1$. Since the shift is continuous in L^2, it follows that the mapping in (3.12) is an element of $\mathcal{C}^1([0,T];L^2)$. Moreover, the mapping $\varphi \mapsto \underline{v}_x - \underline{v}_x(\cdot - \varphi)$ is also continuous as a mapping $\mathbb{R} \to L^2$, so that

$$t \mapsto \underline{v} - \underline{v}(\cdot - \widetilde{\varphi}(t)) \in \mathcal{C}^1([0,T];L^2) \cap \mathcal{C}^0([0,T];H^1)$$

by the chain rule. This proves (3.10). The property $(\widetilde{\varphi}(t), \widetilde{v}(t)) \in \mathcal{D}_\Xi$ immediately follows from Lemma 3.5.

The proof of the converse statement follows by applying the same arguments to

$$v = \widetilde{v} + \underline{v}(\cdot - \widetilde{\varphi}) = \widetilde{v} + \big(\underline{v}(\cdot - \widetilde{\varphi}) - \underline{v}\big) + \underline{v}.$$

The last statement is a direct consequence to the above properties and Lemma 3.5. ☐

Now, let v be the solution of (1.1) given by Theorem 3.2 and assume $v \in M_\Omega^T$ with M_Ω^T from Lemma 3.6. For $\widetilde{\varphi}$ and $\widetilde{v}$, given by $(\widetilde{\varphi}, \widetilde{v}) = \Omega(v)$ and $\widetilde{\lambda}(t) := \widetilde{\varphi}'(t) \in \mathcal{C}([0,T])$ we have for all $t \in [0,T]$

$$\widetilde{v}_t = B\widetilde{v}_x + f(\widetilde{v} + \underline{v}(\cdot - \widetilde{\varphi})) - f(\underline{v}(\cdot - \widetilde{\varphi})) + \widetilde{\lambda}\underline{v}_x(\cdot - \widetilde{\varphi}) \tag{3.13}$$

as an equality in $L^2(\mathbb{R},\mathbb{R}^m)$. Because of Assumption 2.1 and the smoothness of $\widetilde{v}$, Taylor's formula and the fundamental theorem of calculus show the identity

$$\begin{aligned} f\big(\widetilde{v} + \underline{v}(\cdot - \widetilde{\varphi})\big) &- f\big(\underline{v}(\cdot - \widetilde{\varphi})\big) \\ &= f_v(\underline{v})\widetilde{v} - \int_0^1 f_{vv}\big(\underline{v}(\cdot - s\widetilde{\varphi})\big)\underline{v}_x(\cdot - s\widetilde{\varphi})\, ds\, \widetilde{\varphi}\widetilde{v} \\ &\qquad + \int_0^1 (1-s) f_{vv}\big(\underline{v}(\cdot - \widetilde{\varphi}) + s\widetilde{v}\big)\, ds\, [\widetilde{v}, \widetilde{v}], \end{aligned} \tag{3.14}$$

as an equality in H^1 for all $t \in [0,T]$. Inserting (3.14) into (3.13) yields

$$\widetilde{v}_t = P\widetilde{v} + \widetilde{\lambda}\underline{v}_x + F_1(\widetilde{\varphi}, \widetilde{v}) + F_2(\widetilde{\varphi}, \widetilde{v}) + R(\widetilde{\varphi}, \widetilde{\lambda}), \tag{3.15}$$

for $\widetilde{v}$, where the equality (3.15) holds in L^2 for all $t \in [0,T]$. The nonlinearities are given by

$$\begin{aligned} F_1(\widetilde{\varphi}, \widetilde{v}) &= -\int_0^1 f_{vv}\big(\underline{v}(\cdot - s\widetilde{\varphi})\big)[\underline{v}_x(\cdot - s\widetilde{\varphi}), \widetilde{\varphi}\widetilde{v}]\, ds, \\ F_2(\widetilde{\varphi}, \widetilde{v}) &= \int_0^1 (1-s) f_{vv}\big(\underline{v}(\cdot - \widetilde{\varphi}) + s\widetilde{v}\big)\, ds\, [\widetilde{v}, \widetilde{v}], \\ R(\widetilde{\varphi}, \widetilde{\lambda}) &= -\int_0^1 \underline{v}_{xx}(\cdot - s\widetilde{\varphi})\, ds\, \widetilde{\varphi}\widetilde{\lambda}. \end{aligned} \tag{3.16}$$

These terms are elements of $\mathcal{C}([0,T];H^1)$, see Lemmas 5.2, 5.3, 5.4. Thus, locally Ω transforms solutions of (1.1) into solutions of the nonlinear partial differential algebraic equation (PDAE)

$$\begin{aligned}\widetilde{v}_t &= P\widetilde{v} + \widetilde{\lambda}\underline{v}_x + F_1(\widetilde{\varphi},\widetilde{v}) + F_2(\widetilde{\varphi},\widetilde{v}) + R(\widetilde{\varphi},\widetilde{\lambda}),\\ \widetilde{\varphi}_t &= \widetilde{\lambda},\\ 0 &= \psi(\widetilde{v}),\end{aligned} \tag{3.17a}$$

for the unknowns $\widetilde{v}$, $\widetilde{\varphi}$, $\widetilde{\lambda}$, which is subject to the consistent initial data

$$\widetilde{v}(0) = \widetilde{v}_0 \quad \text{and} \quad \widetilde{\varphi}(0) = \widetilde{\varphi}_0, \tag{3.17b}$$

given by $(\widetilde{\varphi}_0,\widetilde{v}_0)^T = \Omega(v_0)$. The term *consistent* reflects that, the initial data are not arbitrary, because some of the components are given by hidden constraints. In the PDAE (3.17), for sufficiently small $\widetilde{v}_0$, $\widetilde{\varphi}_0$, the hidden constraints can be solved for $\widetilde{\lambda}_0$. Therefore, whenever we write *consistent initial data*, the hidden constraints are respected implicitly.

Definition 3.7. We call $(\widetilde{v},\widetilde{\varphi},\widetilde{\lambda})$ a (classical) solution of the hyperbolic PDAE (3.17a) in $[0,T]$ subject to consistent initial data $\widetilde{v}(0) = \widetilde{v}_0 \in H^1$, $\widetilde{\varphi}(0) = \widetilde{\varphi}_0 \in \mathbb{R}$, if

$$\begin{aligned}&\widetilde{v} \in \mathcal{C}^1([0,T];L^2) \cap \mathcal{C}^0([0,T];H^1),\\ &\widetilde{\varphi} \in \mathcal{C}^1([0,T];\mathbb{R}),\\ &\widetilde{\lambda} \in \mathcal{C}([0,T];\mathbb{R}),\end{aligned}$$

and the first equation in (3.17a) holds for all $t \in [0,T]$ as an equality in L^2 and the two equations for the algebraic variables hold pointwise for all $t \in [0,T]$.

We call the triple a solution in $[0,\infty)$ if it is a solution in $[0,T]$ for all $T > 0$.

The results of this section are summarized in the following theorem. It shows a one-to-one correspondence of solutions to the original Cauchy problem (3.2) and to the PDAE (3.17), so that the PDAE can be considered as a "*reformulation*" of the original system. In particular, it suffices to analyze the asymptotic behavior of solutions to the PDAE to obtain assertions on the asymptotic behavior of solutions to the original PDE.

Theorem 3.8 (Equivalence of solutions)**.** *Let the setting be as above. Let Assumptions 2.1, 2.2, and 3.3 hold.*

If $v \in \mathcal{C}^1([0,T];\underline{v} + L^2) \cap \mathcal{C}([0,T];\underline{v} + H^1)$ solves (1.1) *with $\psi(v(t) - \underline{v}) \in V$ for all $t \in [0,T]$, then $(\widetilde{v},\widetilde{\varphi},\widetilde{\lambda})$, given by $(\widetilde{\varphi},\widetilde{v}) = \Omega(v)$ and $\widetilde{\lambda} = \widetilde{\varphi}_t$, is a solution of the PDAE* (3.17a) *subject to the consistent initial data*

$$\widetilde{\varphi}(0) = \Phi(\psi(v_0 - \underline{v})) \quad \textit{and} \quad \widetilde{v}(0) = v_0 - \underline{v}(\cdot - \widetilde{\varphi}(0)).$$

Moreover, $\widetilde{\varphi}(t) \in U$ for all $0 \le t \le T$.

Conversely, if $(\widetilde{v},\widetilde{\varphi},\widetilde{\lambda})$ is a solution of the PDAE (3.17a), (3.17b) *in $[0,T]$, it follows that $v = \Xi(\widetilde{\varphi},\widetilde{v}) = \widetilde{v} + \underline{v}(\cdot - \widetilde{\varphi})$ is an element of $\mathcal{C}^1([0,T];\underline{v} + L^2) \cap \mathcal{C}([0,T];\underline{v} + H^1)$ and solves the Cauchy problem* (1.1) *with $v(0) = \widetilde{v}_0 + \underline{v}(\cdot - \widetilde{\varphi}_0)$.*
If $\widetilde{\varphi}(t) \in U$ for all $0 \le t \le T$, the two transformations are inverse to each other.

Proof. The smoothness and the last statement follow from Lemma 3.6.

That solutions of the Cauchy problem (1.1) lead to solutions of the PDAE (3.17) is shown above. For the other implication, i.e. solutions of the PDAE (3.17) lead to solutions of (1.1), the arguments from above can be reversed because of the smoothness assumptions and Lemma 3.6. ☐

4 Linear stability for hyperbolic PDAEs

Let the setting be as above and as a **general hypothesis** impose Assumptions 2.1, 2.2, 2.3, and 3.3. Again consider the problem in a co-moving frame so that $\underline{\lambda} = 0$. Note that the functions F_1, F_2, and R from (3.17a) are at least quadratic functions of their arguments. We replace these higher order terms by time dependent inhomogeneities so that the PDAE becomes linear

$$\begin{aligned} \widetilde{v}_t &= P\widetilde{v} + \underline{v}_x\widetilde{\lambda} + F, && \text{in } L^2, \\ \widetilde{\varphi}_t &= \widetilde{\lambda}, && \text{in } \mathbb{R}, \\ 0 &= \psi(\widetilde{v}), && \text{in } \mathbb{R}. \end{aligned} \tag{4.1}$$

We analyze this linear system in [15], where we impose the following assumption on the inhomogeneity.

Assumption 4.1. *The inhomogeneity F belongs to $F \in \mathcal{C}(J; H^1(\mathbb{R}))$, where $J = [0,T]$ or $J = [0,\infty)$.*

In the linear system (4.1) the equation for $\widetilde{\varphi}$ decouples from the other equations and one can solve for $\widetilde{\varphi}$ in an additional step. Therefore, the system can be reduced to the PDAE

$$\begin{aligned} \widetilde{v}_t &= P\widetilde{v} + \underline{v}_x\widetilde{\lambda} + F, && \text{in } L^2, \\ 0 &= \psi(\widetilde{v}), && \text{in } \mathbb{R}, \end{aligned} \tag{4.2a}$$

for $\widetilde{v}$ and $\widetilde{\lambda}$, which we assume to be subject to consistent initial conditions with

$$\widetilde{v}(0) = \widetilde{v}_0 \in H^1(\mathbb{R}, \mathbb{R}^m). \tag{4.2b}$$

The initial condition $\widetilde{\lambda}(0) = \widetilde{\lambda}_0$ follows from the hidden constraint $0 = \psi\big(P\widetilde{v}_0 + \widetilde{\lambda}_0\underline{v}_x + F(0)\big)$.

Before recalling the main stability result from [15], we adapt the solution-concept from Definition 3.7 to the current case: A pair (v, λ) is called a (*classical*) solution of (4.2) in $[0,T]$ if

$$v \in \mathcal{C}^1\big([0,T]; L^2(\mathbb{R},\mathbb{R}^m)\big) \cap \mathcal{C}\big([0,T]; H^1(\mathbb{R},\mathbb{R}^m)\big) \quad \text{and} \quad \lambda \in \mathcal{C}\big([0,T];\mathbb{R}\big),$$

so that v satisfies (4.2b), the first equation in (4.2a) holds in $L^2(\mathbb{R},\mathbb{R}^m)$, and the second equation holds for all $t \in [0,T]$. Moreover, the hidden constraint is satisfied at $t = 0$. The tuple is called a solution on $[0,\infty)$ if it is a solution on $[0,T]$ for every $T > 0$.

Theorem 4.2 (Linear stability of the PDAE [15, Thm 5.3])**.** *Let Assumptions 2.2, 2.3, 3.3, and 4.1 hold with $J = [0,T]$ for $T > 0$ or $J = [0,\infty)$. Then for every consistent initial data $v_0 \in H^2(\mathbb{R})$ there is a unique solution (v, λ) of the linear PDAE (4.2) on J.*

Moreover, if $\eta_0 < \delta$, with δ from Assumptions 2.2 and 2.3, then there is a positive constant $C_l = C_l(\eta_0)$, independent of F and v_0, so that for all $\eta \le \eta_0$ the solution satisfies for all $t \in J$ the estimate

$$\|v(t)\|_{H^1}^2 + e^{-2\eta t} \int_0^t e^{2\eta\tau} \big(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2 \big)\, d\tau \\ \le C_l e^{-2\eta t} \Big[\|v_0\|_{H^2}^2 + \int_0^t e^{2\eta\tau} \|F(\tau)\|_{H^1}^2\, d\tau \Big]. \quad (4.3)$$

The linear estimates are the basis for the nonlinear analysis (see also the author's PhD thesis [14]). In Section 5, the higher order terms are considered as part of the inhomogeneity. The estimates from here then lead to a priori estimates which are used for the stability proof.

REMARK. *We note that the main tool in the proof of Theorem 4.2 in [15] is the vector-valued Laplace transform (cf. [1]). In the proof, the case of arbitrary initial data is reduced to homogeneous initial data by a homogenization trick, which is a well-known technique for the Laplace transform. This homogenization has the effect that the H^2-norm of the initial data is introduced in the estimates. This is also the reason, why the H^2-norm also appears in our main Theorem 2.5.*

5 Nonlinear stability of the PDAE

With the linear stability result, Theorem 4.2, we are now ready to prove asymptotic stability for the fully nonlinear PDAE (3.17). As **general hypothesis** we impose in this section Assumptions 2.1, 2.2, 2.3, and 3.3.

Theorem 5.1 (Asymptotic stability of the PDAE)**.** *Let the Assumptions be as above.*

Then, for every $0 < \eta < \delta$ there are $\rho_0, \theta_0 > 0$ so that for all consistent initial data $\widetilde{v}_0 \in H^2$, $\widetilde{\varphi}_0 \in \mathbb{R}$ of (3.17) with $\|\widetilde{v}_0\|_{H^2} \le \rho_0$ and $|\widetilde{\varphi}_0| \le \theta_0$ there is a unique (classical) solution $(\widetilde{v}, \widetilde{\varphi}, \widetilde{\lambda})$ on $[0, \infty)$. The solution satisfies $\widetilde{\varphi}(t) \in U$ for all $t \ge 0$, with U the set from Lemma 3.4. Moreover, there is $\widetilde{\varphi}_\infty \in \mathbb{R}$ so that with $C_l = C_l(\eta)$ from Theorem 4.2 the following estimates hold

$$|\widetilde{\varphi}_\infty| \le |\widetilde{\varphi}_0| + \sqrt{\frac{C_l(\eta)}{\eta}} \|\widetilde{v}_0\|_{H^2}, \quad (5.1a)$$

$$|\widetilde{\varphi}(t) - \widetilde{\varphi}_\infty|^2 \le \frac{C_l(\eta)}{\eta} \|\widetilde{v}_0\|_{H^2}^2 e^{-2\eta t}, \quad (5.1b)$$

$$\|\widetilde{v}(t)\|_{H^1}^2 \le C_l(\eta) \|\widetilde{v}_0\|_{H^2}^2 e^{-2\eta t}, \quad (5.1c)$$

$$\int_0^t e^{2\eta\tau} |\widetilde{\lambda}(\tau)|^2\, d\tau \le 2 C_l(\eta) \|\widetilde{v}_0\|_{H^2}^2. \quad (5.1d)$$

Theorem 5.1 is a strictly local result for small $\widetilde{v}$, $\widetilde{\varphi}$, $\widetilde{\lambda}$. We emphasize this in the following proof by rescaling the variables and analyzing the problem in the rescaled form. The proof is given in the next subsection. For completeness, we collect some properties of the nonlinear terms in Subsection 5.2.

5.1 Proof of Theorem 5.1

Let $0 < \eta < \delta$ be given and let $C_l = C_l(\eta)$ be the constant from Theorem 4.2. Choose $\varepsilon_1 > 0$, so that $\varepsilon_1 \leq (2C_l)^{-1/2}$ and $B_{3\varepsilon_1}(0) \subset U$, where $U \subset \mathbb{R}$ is given in Lemma 3.4. Also let V be the set from Lemma 3.4, i.e. $\Phi : V \to U$, $\Phi = (G|_U)^{-1}$ is a $\mathcal{C}^2$–diffeomorphism.

Step 0: Rescaling. Assume that $(\widetilde{v}, \widetilde{\varphi}, \widetilde{\lambda})$ is a (classical) solution in $[0, T]$. For arbitrary $0 < \varepsilon \leq \varepsilon_1$ define the rescaled variables (v, φ, λ) by

$$\widetilde{v} = \varepsilon v, \quad \widetilde{\varphi} = \varepsilon\varphi, \quad \widetilde{\lambda} = \varepsilon\lambda.$$

(For the sake of readability we use v, φ, λ for the new variables, which should not be confused with the variables from Section 1.) In the new variables the PDAE (3.17a) becomes

$$\begin{aligned} v_t &= Pv + \lambda \underline{v}_x + \varepsilon F_1^\varepsilon(\varphi, v) + \varepsilon F_2^\varepsilon(\varphi, v) + \varepsilon R^\varepsilon(\varphi, \lambda), \\ \varphi_t &= \lambda, \\ 0 &= \psi(v), \end{aligned} \tag{5.2a}$$

subject to the consistent initial data

$$\varepsilon v(0) = \varepsilon v_0 := \widetilde{v}_0, \quad \varepsilon\varphi(0) = \varepsilon\varphi_0 := \widetilde{\varphi}_0, \quad \varepsilon\lambda(0) = \varepsilon\lambda_0 := \widetilde{\lambda}_0 \quad \text{at } t = 0. \tag{5.2b}$$

The functions F_j^ε and R^ε in (5.2a) are defined by rescaling the original nonlinearities from (3.16) in the form

$$\varepsilon^2 F_j^\varepsilon(\varphi, v) := F_j(\varepsilon\varphi, \varepsilon v), \ j = 1, 2, \qquad \varepsilon^2 R^\varepsilon(\varphi, \lambda) := R(\varepsilon\varphi, \varepsilon\lambda). \tag{5.3}$$

Step 1: A priori estimates. Let $0 < \varepsilon < \varepsilon_1$ and assume that (v, φ, λ) is a solution of (5.2) on $[0, T]$ for some $T > 0$. Also assume that it satisfies $|\varphi(t)| \leq 2$ and $\|v(t)\|_{H^1} \leq 2$ for all $t \in [0, T]$. Then, the nonlinearities in (5.2a) satisfy for all such t and ε

$$\|F_1^\varepsilon(\varphi, v)\|_{H^1}^2 + \|F_2^\varepsilon(\varphi, v)\|_{H^1}^2 \leq C_n \|v\|_{H^1}^2, \tag{5.4}$$

$$\|R^\varepsilon(\varphi, \lambda)\|_{H^1}^2 \leq C_n |\lambda|^2, \tag{5.5}$$

with a uniform constant $C_n > 0$ by Lemmas 5.2, 5.3, and 5.4. These Lemmas also show that

$$F : t \mapsto F(t) = \varepsilon F_1^\varepsilon(\varphi(t), v(t)) + \varepsilon F_2^\varepsilon(\varphi(t), v(t)) + \varepsilon R^\varepsilon(\varphi(t), \lambda(t)) \tag{5.6}$$

satisfies $F \in \mathcal{C}([0, T]; H^1)$. Thus, considering the nonlinear term $F(t)$ as the inhomogeneity in the linear PDAE (4.2), Theorem 4.2 applies and shows for all $0 \leq t \leq T$ the estimate

$$\begin{aligned} &\|v(t)\|_{H^1}^2 + e^{-2\eta t} \int_0^t e^{2\eta\tau} \left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right) d\tau \\ &\quad \leq C_l e^{-2\eta t} \Big[\|v_0\|_{H^2}^2 + 3\varepsilon^2 C_n \int_0^t e^{2\eta\tau} \Big(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2 \Big) d\tau \Big]. \end{aligned}$$

Here the definition (5.6) of F and estimates (5.4), (5.5) were used. For $0 < \varepsilon \leq \varepsilon_0$, with $0 < \varepsilon_0 \leq \min(\varepsilon_1, (6C_l C_n)^{-1/2})$, this yields for all $0 \leq t \leq T$ the bound

$$\|v(t)\|_{H^1}^2 + \frac{1}{2} e^{-2\eta t} \int_0^t e^{2\eta\tau} \left(\|v(\tau)\|_{H^1}^2 + |\lambda(\tau)|^2\right) d\tau \leq C_l e^{-2\eta t} \|v_0\|_{H^2}^2. \tag{5.7}$$

Moreover, the algebraic variable $\varphi(t) = \varphi_0 + \int_0^t \varphi_t(\tau)\,d\tau$ satisfies for all $0 \le t \le T$

$$\begin{aligned}|\varphi(t)| &\le |\varphi_0| + \int_0^t |\varphi_t(\tau)|\,d\tau = |\varphi_0| + \int_0^t |\lambda(\tau)|\,d\tau \le |\varphi_0| + \int_0^t e^{-\eta\tau}\left(e^{\eta\tau}|\lambda(\tau)|\right)\,d\tau \\ &\le |\varphi_0| + \Big(\int_0^t e^{-2\eta\tau}\,d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t e^{2\eta\tau}|\lambda(\tau)|^2\,d\tau\Big)^{\frac{1}{2}} < |\varphi_0| + \left(\frac{C_l}{\eta}\right)^{\frac{1}{2}}\|v_0\|_{H^2}, \qquad (5.8)\end{aligned}$$

by Cauchy-Schwarz inequality and (5.7).

Step 2: Local Existence and Uniqueness. Let $0 < \varepsilon \le \varepsilon_0$ with ε_0 from Step 1, and assume $\widetilde{\varphi} = \varepsilon\varphi_0$ with some $|\varphi_0| \le 1$. By Step 0 and Theorem 3.8, a triple (v, φ, λ) is a solution of (5.2) in $[0,T]$ with $\varepsilon\varphi(t) \in U$ for all $0 \le t \le T$ if and only if $w \in \mathcal{C}^1([0,T]; \underline{v}+L^2) \cap \mathcal{C}([0,T]; \underline{v}+H^1)$ solves (1.1) with initial data $w(0) = \varepsilon v_0 + \underline{v}(\cdot - \varepsilon\varphi_0)$, satisfies $\psi(w(t) - \underline{v}) \in V$ for all $0 \le t \le T$, and it holds

$$\varphi = \frac{1}{\varepsilon}\Phi(\psi(w - \underline{v})), \quad v = \frac{1}{\varepsilon}(w - \underline{v}(\cdot - \varepsilon\varphi)), \quad \lambda = \varphi', \quad \forall 0 \le t \le T.$$

Therefore, unique solvability of (1.1), proved in Theorem 3.2, implies unique solvability of the PDAE (5.2).

Let w^* be the unique global solution of the Cauchy problem (1.1) subject to initial data $w^*(0) = \varepsilon v_0 + \underline{v}(\cdot - \varepsilon\varphi_0)$, obtained by Theorem 3.2. Let $[0, T^*)$ denote its interval of existence. Assume for the initial datum $|\widetilde{\varphi}_0| < \varepsilon$ so that $|\varphi_0| \le 1$. The consistency of the initial data and linearity of ψ implies $v_0 \in \mathcal{N}(\psi)$, so that $\psi(w^*(0) - \underline{v}) \in V$ follows. By continuity, there is $0 < T_1 < T^*$ so that $\psi(w^*(t) - \underline{v}) \in V$ for all $0 \le t \le T_1$. Theorem 3.8 therefore implies local existence of a classical solution of the PDAE (5.2).

Step 3: Continuation and Global Existence. Choose $\omega_0 = \min(\frac{1}{C_l}, \frac{\eta}{C_l})$ and assume $\widetilde{v}_0 = \varepsilon v_0$ with $\|v_0\|_{H^2}^2 \le \omega_0$. Let w^* and T^* be given as in Step 2. Define

$$\begin{aligned}T_0 := \sup\Big\{T \in [0, T^*) : |\Phi(\psi(w^*(t) - \underline{v}))| < 2\varepsilon \text{ and} \\ \|w^*(t) - \underline{v}\big(\cdot - \Phi(\psi(w^*(t) - \underline{v}))\big)\|_{H^1} < 2\varepsilon\ \forall 0 \le t \le T\Big\}. \qquad (5.9)\end{aligned}$$

First assume $T_0 < T^* = \infty$. By the smoothness $w^* \in \mathcal{C}([0,T]; \underline{v} + H^1)$ for all $T > 0$ and Lemma 3.4, follows continuity of

$$h : t \mapsto \max\Big(|\Phi(\psi(w^*(t) - \underline{v}))|, \|w^*(t) - \underline{v}\big(\cdot - \Phi(\psi(w^*(t) - \underline{v}))\big)\|_{H^1}\Big) \qquad (5.10)$$

as long as $\psi\big(w^*(t) - \underline{v}\big) \in V$. Furthermore, the definition of $0 < \varepsilon \le \varepsilon_1$ yields the implication $(h(t) < 3\varepsilon \Rightarrow \psi\big(w^*(t) - \underline{v}\big) \in V)$, so that by continuity

$$h(T_0) = 2\varepsilon \quad \text{and} \quad h(t) < 2\varepsilon \text{ for all } 0 \le t < T_0. \qquad (5.11)$$

In case $T^* < \infty$, $\lim_{t \nearrow T^*} \|w^*(t) - \underline{v}\|_{H^1} = +\infty$ by Theorem 3.2. If there is $\bar{t} \in (0, T^*)$ with $\psi(w^*(\bar{t}) - \underline{v}) \notin V$, there must exist $T_1 < T^*$ with

$$|\Phi(\psi(w^*(T_1) - \underline{v}))| = 2\varepsilon \quad \text{and} \quad |\Phi(\psi(w^*(t) - \underline{v}))| < 2\varepsilon\ \forall 0 \le t < T_1.$$

The same continuity argument as above then proves (5.11).

If $T^* < \infty$ and $\psi(w^*(t) - \underline{v}) \in V$ for all $0 \le t < T^*$, it follows that $h : [0, T^*) \to \mathbb{R}$ is continuous and, moreover, $\|\underline{v} - \underline{v}\big(\cdot - \Phi(\psi(w^*(t) - \underline{v}))\big)\|_{H^1}$ is uniformly bounded in $0 \le t < T^*$. This shows

$$\lim_{t \nearrow T^*} \|w^*(t) - \underline{v}\big(\cdot - \Phi(\psi(w^*(t) - \underline{v}))\big)\|_{H^1} = \infty,$$

which implies $T_0 < T^*$ and again proves (5.11).

Now let T_0 be given by (5.9) and assume $T_0 < \infty$. Define (v, φ, λ) by

$$\varphi = \frac{1}{\varepsilon}\Phi\big(\psi(w^* - \underline{v})\big), \qquad v = \frac{1}{\varepsilon}\big(w^* - \underline{v}(\cdot - \varepsilon\varphi)\big), \qquad \lambda = \varphi' \qquad \forall 0 \le t \le T_0.$$

As was shown in Step 2, this is a classical solution of the PDAE (5.2) on $[0, T_0]$. By definition of T_0 and the assumption $T_0 < \infty$, the solution satisfies

$$|\varphi(t)| \le 2 \quad \text{and} \quad \|v(t)\|_{H^1} \le 2 \qquad \forall t \in [0, T_0].$$

Therefore, the a priori estimates (5.7) and (5.8) hold in $[0, T_0]$ and the choice of ω_0 shows for all $0 \le t \le T_0$ the bounds

$$\begin{aligned} \|v(t)\|_{H^1}^2 &\le C_l e^{-2\eta t}\|v_0\|_{H^2}^2 \le C_l \omega_0 \le 1, \quad \text{and} \\ |\varphi(t)| &< |\varphi_0| + \left(\frac{C_l}{\eta}\right)^{1/2} \sqrt{\omega_0} \le 2. \end{aligned} \tag{5.12}$$

The strict inequalities contradict $T_0 < \infty$ because of (5.11), and $T_0 = T^* = \infty$ follows.

Step 4: Rate of convergence. Steps 2 and 3 show that for initial data with $\|\widetilde{v}_0\|_{H^2}^2 \le \varepsilon\omega_0 =: \rho_0^2$ and $|\widetilde{\varphi}_0| \le \varepsilon = \theta_0$, the solution of the PDAE (5.2) exists for all positive times and the (scaled) variables satisfy the bounds $|\varphi(t)| < 2$ and $\|v(t)\|_{H^1} < 2$ for all $t \ge 0$. Therefore, the a priori estimates (5.7) and (5.8) from Step 1 apply to every compact interval $[0, T]$, $T < \infty$, and yield

$$\begin{aligned} \|\widetilde{v}(t)\|_{H^1}^2 = \varepsilon^2 \|v(t)\|_{H^1}^2 &\le \varepsilon^2 C_l e^{-2\eta t}\|v_0\|_{H^2}^2 = C_l e^{-2\eta t}\|\widetilde{v}_0\|_{H^2}^2 \\ \int_0^t e^{2\eta\tau}|\widetilde{\lambda}(\tau)|^2\, d\tau = \varepsilon^2 \int_0^t e^{2\eta\tau}|\lambda(\tau)|^2\, d\tau &\le \varepsilon^2 2C_l\|v_0\|_{H^2}^2 = 2C_l\|\widetilde{v}_0\|_{H^2}^2, \end{aligned}$$

i.e. (5.1c) and (5.1d). Finally, define $\widetilde{\varphi}_\infty := \varepsilon\varphi_\infty$, where

$$\varphi_\infty = \lim_{t\to\infty} \varphi_0 + \int_0^t \lambda(\tau)\, d\tau = \varphi_0 + \int_0^\infty \lambda(\tau)\, d\tau.$$

The integral is absolutely convergent by (5.8) which also implies estimate (5.1a). Estimate (5.1b) follows from (5.7) and an application of the Cauchy-Schwarz inequality

$$|\varphi(t) - \varphi_\infty|^2 \le \left(\int_t^\infty |\lambda(\tau)|\, d\tau\right)^2 \le \frac{e^{-2\eta t}}{2\eta} \int_t^\infty e^{2\eta\tau}|\lambda(\tau)|^2\, d\tau \le \frac{C_l}{\eta}\|v_0\|_{H^2}^2 e^{-2\eta t}.$$

□

5.2 Smoothness and estimates of the nonlinear terms

For the analysis of the nonlinear terms in (5.2a), we assume

$$v \in \mathcal{C}([0,T]; H^1(\mathbb{R}, \mathbb{R}^m)), \qquad \varphi \in \mathcal{C}([0,T]; \mathbb{R}), \qquad \lambda \in \mathcal{C}([0,T]; \mathbb{R}),$$

in addition to Assumption 2.1. This is in accordance with the smoothness of the PDAE solution. In this section we derive the estimates for smooth functions, the details for the general case can be found in the appendix.

Lemma 5.2. *For $\varepsilon > 0$ the function $\mathbb{R} \ni t \mapsto F_1^\varepsilon(\varphi(t), v(t))$ is an element of $\mathcal{C}([0,T]; H^1)$ and there is a constant $C = C(f, \underline{v})$, independent of ε, so that*

$$\|F_1^\varepsilon(\varphi(t), v(t))\|_*^2 \le C|\varphi(t)|^2 \|v(t)\|_*^2, \qquad * \in \{L^2(\mathbb{R}, \mathbb{R}^m), H^1(\mathbb{R}, \mathbb{R}^m)\}. \tag{5.13}$$

Proof. We begin with the proof of estimate (5.13). First let $v \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$ and $\varphi \in \mathbb{R}$. From Hölder's inequality and the shift invariance of the L^∞–norm follows the ε-independent estimate

$$\begin{aligned}\|F_1^\varepsilon(\varphi, v)\|^2 &= \int_{\mathbb{R}} \Big| \int_0^1 f_{vv}(\underline{v}(x - s\varepsilon\varphi))\underline{v}_x(x - s\varepsilon\varphi)\, ds \varphi v(x) \Big|^2 dx \\ &\le \|f_{vv}(\underline{v})\underline{v}_x\|_\infty^2 |\varphi|^2 \|v\|^2.\end{aligned}$$

Similarly, one obtains

$$\begin{aligned}\|F_1^\varepsilon(\varphi, v)_x\|^2 = \int_{\mathbb{R}} \Big| &\int_0^1 f_{vvv}(\underline{v}(x - s\varepsilon\varphi))\underline{v}_x(x - s\varepsilon\varphi)^2\, ds\, \varphi v(x) \\ &+ \int_0^1 f_{vv}(\underline{v}(x - s\varepsilon\varphi))\underline{v}_{xx}(x - s\varepsilon\varphi)\, ds\, \varphi v(x) \\ &+ \int_0^1 f_{vv}(\underline{v}(x - s\varepsilon\varphi))\underline{v}_x(x - s\varepsilon\varphi)\, ds\, \varphi v_x(x) \Big|^2 dx \\ \le 3 \Big[&\|f_{vvv}(\underline{v})[\underline{v}_x, \underline{v}_x]\|_\infty^2 + \|f_{vv}(\underline{v})\underline{v}_{xx}\|_\infty^2 + \|f_{vv}(\underline{v})\underline{v}_x\|_\infty^2 \Big] |\varphi|^2 \|v\|_{H^1}^2 \\ \le C &|\varphi|^2 \|v\|_{H^1}^2.\end{aligned}$$

Here we use $\underline{v} \in \mathcal{C}_b^2$, which follows from Sobolev embedding and Assumption 2.1.

For general $v \in H^1(\mathbb{R}, \mathbb{R}^m)$ choose a sequence $v_n \in \mathcal{C}_0^\infty$, $n \in \mathbb{N}$ with $v_n \to v$ in H^1. By considering a subsequence, one may assume $F_1^\varepsilon(\varphi, v_n) \to F_1^\varepsilon(\varphi, v)$ pointwise almost everywhere in $\mathbb{R}$. Because F_1^ε is linear in v, we immediately see that the sequence $(F_1^\varepsilon(\varphi, v_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in H^1 by the first step. Thus $F_1^\varepsilon(\varphi, v) \in H^1(\mathbb{R}, \mathbb{R}^m)$ and estimate (5.13) holds.

Because the techniques of the proof of continuity are similar to the techniques used in the proof of Lemma 5.3, we omit the details (compare [14, App. D]). □

Next consider the rescaled nonlinearity F_2^ε. The analysis is much more involved since F_2^ε depends nonlinearly on the v–term.

Lemma 5.3. *Assume* $0 < \varepsilon \le \varepsilon_0$ *and let* v *be as above with* $\|v\|_{L^\infty([0,T];H^1)} \le K$.

Then the function $\mathbb{R} \ni t \mapsto F_2^\varepsilon\big(\varphi(t), v(t)\big) \in H^1$ *is an element of* $\mathcal{C}([0,T];H^1)$. *Furthermore, there is a constant* $C = C(f, \underline{v}, \varepsilon_0, K)$, *independent of* ε, *such that*

$$\|F_2^\varepsilon(\varphi(t), v(t))\|_*^2 \le C\|v(t)\|_{H^1}^2 \|v(t)\|_*^2, \quad \forall t \in [0,T], \qquad * \in \{L^2, H^1\}. \tag{5.14}$$

Proof. We begin with the proof of (5.14). First consider $\varphi \in \mathbb{R}$ and $v \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$ with $\|v\|_{H^1} \le K_1$, where $K_1 > 0$ is some constant. By Hölder's inequality,

$$\begin{aligned}\|F_2^\varepsilon(\varphi, v)\|^2 &= \int_{\mathbb{R}} \Big| \int_0^1 (1-s) f_{vv}(\underline{v}(x - \varepsilon\varphi) + s\varepsilon v(x))\, ds\, [v(x), v(x)] \Big|^2 dx \\ &\le \sup_{|w| \le \|\underline{v}\|_\infty + \varepsilon\|v\|_\infty} |f_{vv}(w)|^2 \|v\|_\infty^2 \|v\|^2.\end{aligned}$$

Similarly, estimating $F_2^\varepsilon(\varphi, v)_x$ with Hölder's inequality, yields

$$\begin{aligned}\|F_2^\varepsilon(\varphi, v)_x\|^2 \le\ & 3 \sup_{0 \le s \le 1} \|f_{vvv}(\underline{v}(\cdot - \varepsilon\varphi) + s\varepsilon v(\cdot))\underline{v}_x(\cdot - \varepsilon\varphi)\|_\infty^2 \|v\|_\infty^2 \|v\|^2 \\ & + 3 \sup_{0 \le s \le 1} \|f_{vvv}(\underline{v}(\cdot - \varepsilon\varphi) + s\varepsilon v(\cdot))s\|_\infty^2 |\varepsilon|^2 \|v\|_\infty^4 \|v_x\|^2 \\ & + 6 \sup_{0 \le s \le 1} \|f_{vv}(\underline{v}(\cdot - \varepsilon\varphi) + s\varepsilon v(\cdot))\|_\infty^2 \|v\|_\infty^2 \|v_x\|^2 \\ \le\ & \mathrm{const}\|v\|_\infty^2 \|v\|_{H^1}^2.\end{aligned}$$

Estimate (5.14) then follows for smooth functions $v \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$ from Sobolev embedding.

There is $K_1 > 0$, depending on K but not on v, so that there exists an approximating sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$ with $v_n \to v$ in $H^1(\mathbb{R}, \mathbb{R}^n)$ and $\|v_n\|_{H^1} \le K_1$. It remains to prove that $(F_2^\varepsilon(\varphi, v_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^1(\mathbb{R}, \mathbb{R}^m)$. Consider $u, w \in \mathcal{C}_0^\infty$, with $\|u\|_{H^1}, \|w\|_{H^1} \le K_1$, then

$$\|F_2^\varepsilon(\varphi, u) - F_2^\varepsilon(\varphi, w)\|^2 \le \mathrm{const}\|u - w\|_{H^1}^2, \tag{5.15}$$

and, similarly, for the L^2–norm of $\big(F_2^\varepsilon(\varphi, u) - F_2^\varepsilon(\varphi, w)\big)_x$

$$\begin{aligned}\|F_2^\varepsilon(\varphi, u)_x - F_2^\varepsilon(\varphi, w)_x\|^2 \le \mathrm{const}\Big(&\|u - w\|_{H^1}^2 \\ &+ \sup_{\substack{x \in \mathbb{R} \\ s \in [0,1]}} \big|f_{vvv}\big(\underline{v}(x - \varepsilon\varphi) + s\varepsilon u(x)\big) - f_{vvv}\big(\underline{v}(x - \varepsilon\varphi) + s\varepsilon w(x)\big)\big|^2\Big).\end{aligned} \tag{5.16}$$

The constants in (5.15) and (5.16) depend on K_1 and $\sup_{|w| \le \|\underline{v}\|_\infty + \varepsilon_0 K_1} |f^{(j)}(w)|$ for $j = 2, 3$ but are independent of φ. Details are given in Appendix A. This proves that $(F_2^\varepsilon(\varphi, v_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^1(\mathbb{R}, \mathbb{R}^m)$. Here one uses the uniform continuity of f_{vvv} on compact sets for the second summand in (5.16). This finishes the proof of (5.14).

In Appendix A we show that $F_2^\varepsilon : \mathbb{R} \times H^1 \to H^1$ is continuous. Hence, $F_2^\varepsilon\big(\varphi(\cdot), v(\cdot)\big)$ belongs to $\mathcal{C}([0,T];H^1)$. □

Lemma 5.4. *Let the assumptions be as above. Then for every $\varepsilon > 0$ the function $t \mapsto R^\varepsilon(\varphi(t), \lambda(t))$ is an element of $\mathcal{C}([0,T]; H^1)$ and satisfies the estimate*

$$\|R^\varepsilon(\varphi(t), \lambda(t))\|_{H^1}^2 \leq C|\varphi(t)|^2|\lambda(t)|^2, \tag{5.17}$$

with C independent of φ, λ and ε.

Proof. Monotonicity of the Bochner integral and $\underline{v}_x \in H^2$ immediately imply (5.17) and continuity follows from the fact that the shift is a continuous operation in H^1. □

6 Asymptotic stability with asymptotic phase

Now we are ready to prove Theorem 2.5. The notation and setting is as before, in particular, ψ satisfies Assumption 3.3 and U, V, and $\Phi = \left(G\big|_U\right)^{-1} : V \to U$ are constructed in Lemma 3.4.

Proof of Theorem 2.5. First consider the case $\underline{\lambda} = 0$.
Let $0 < \eta < \delta$ and let $\rho_0 = \rho_0(\eta)$ and $\theta_0 = \theta_0(\eta) > 0$ be the constants from Theorem 5.1.

The mapping $v \mapsto \Phi \circ \psi(v - \underline{v})$ is continuously differentiable in a neighborhood of $\underline{v}$ in $\underline{v} + H^1(\mathbb{R}, \mathbb{R}^m)$. Therefore, there are $\rho_1 > 0$ and $C_{lip} > 0$ so that

$$\left|\Phi(\psi(v - \underline{v}))\right| \leq C_{lip}\|v - \underline{v}\|_{H^1} \qquad \forall v \in \underline{v} + H^1(\mathbb{R}, \mathbb{R}^m) \text{ with } \|v - \underline{v}\|_{H^1} \leq \rho_1. \tag{6.1}$$

Let

$$\rho = \min(\rho_1, \frac{\theta_0}{C_{lip}}, \frac{\rho_0}{2}, \frac{\rho_0}{2C_{lip}\|\underline{v}_x\|_{H^2}}).$$

Therefore, for all $v_0 \in \underline{v} + H^2(\mathbb{R}, \mathbb{R}^m)$ with $\|v_0 - \underline{v}\|_{H^2} < \rho$, the initial data $\widetilde{\varphi}(0) = \Phi\big(\psi(v_0 - \underline{v})\big)$ and $\widetilde{v}(0) = v_0 - \underline{v}\big(\cdot - \widetilde{\varphi}(0)\big)$ of the PDAE-reformulation (3.17) satisfy

$$|\widetilde{\varphi}(0)| = |\Phi(\psi(v_0 - \underline{v}))| \leq C_{lip}\|v_0 - \underline{v}\|_{H^1} \leq \theta_0, \tag{6.2}$$

$$\|\widetilde{v}(0)\|_{H^2} \leq \|v_0 - \underline{v}\|_{H^2} + \|\underline{v} - \underline{v}(\cdot - \widetilde{\varphi}(0))\|_{H^2} \leq (1 + C_{lip}\|\underline{v}_x\|_{H^2})\|v_0 - \underline{v}\|_{H^2} \leq \rho_0. \tag{6.3}$$

Theorem 5.1 applies and yields a unique solution $(\widetilde{v}, \widetilde{\varphi}, \widetilde{\lambda})$ of (3.17) on $[0, \infty)$ and, moreover, $\widetilde{\varphi}(t) \in U$ for all $t \geq 0$, where U is the set given in Lemma 3.4. Therefore, by Theorem 3.8, there is a unique solution v of the Cauchy problem (1.1) on $[0, \infty)$, this is given by

$$v(t) = \widetilde{v}(t) + \underline{v}(\cdot - \widetilde{\varphi}(t)) \quad \text{for all } t \geq 0. \tag{6.4}$$

Let $\varphi_\infty = \widetilde{\varphi}_\infty$ be the number obtained in Theorem 5.1. From (5.1a), (6.2), and (6.3) then follows

$$|\varphi_\infty| \leq C_{lip}\|v_0 - \underline{v}\|_{H^1} + \sqrt{\frac{C_l}{\eta}}(1 + C_{lip}\|\underline{v}_x\|_{H^2})\|v_0 - \underline{v}\|_{H^2} \leq C\|v_0 - \underline{v}\|_{H^2}.$$

The constant C depends on C_{lip}, C_l, η and $\|\underline{v}_x\|_{H^2}$, but is independent of v_0. This proves the bound (2.3) for the asymptotic phase.
For the asymptotic behavior of v, representation (6.4) and estimates (5.1b), (5.1c) yield

$$\begin{aligned}\|v(t)-\underline{v}(\cdot-\varphi_\infty)\|_{H^1} &\leq \|\widetilde{v}(t)\|_{H^1}+\|\underline{v}(\cdot-\widetilde{\varphi}(t))-\underline{v}(\cdot-\varphi_\infty)\|_{H^1}\\ &\leq \sqrt{C_l}\|\widetilde{v}_0\|_{H^2}e^{-\eta t}+\|\underline{v}_x\|_{H^1}\sqrt{\frac{C_l}{\eta}}\|\widetilde{v}_0\|_{H^2}e^{-\eta t}=\left(1+\frac{\|\underline{v}_x\|_{H^1}}{\eta^{1/2}}\right)\sqrt{C_l}\|\widetilde{v}_0\|_{H^2}e^{-\eta t}.\end{aligned}$$

Together with estimate (6.3) of the initial condition, this implies

$$\|v(t)-\underline{v}(\cdot-\varphi_\infty)\|_{H^1}\leq C\|v_0-\underline{v}\|_{H^2}e^{-\eta t}\qquad \forall 0\leq t,$$

where the constant C again depends on C_{lip}, C_l, η, and $\|\underline{v}_x\|_{H^2}$, but is independent of v_0. This finishes the proof of (2.4) in the case $\underline{\lambda}=0$.

In the case $\underline{\lambda}\neq 0$, consider the equation in a co-moving frame, moving with speed $\underline{\lambda}$. Then the result for the case $\underline{\lambda}=0$ applies to this new problem. □

7 Stability of the Freezing Method

In this section we prove Theorem 2.7. This justifies the freezing method at least on a continuous level. The proof proceeds in several steps. Its principal idea is the ansatz

$$u(x,t)=v_{pde}(x+\Lambda(t),t),$$

where v_{pde} is a solution of the PDE–problem (1.1), such that $(u,\dot{\Lambda})$ is a solution of the PDAE

$$u_t=Bu_x+f(u)+\dot{\Lambda}u_x,\ 0=\Psi(\hat{u}-u).$$

This ansatz and the hidden constraint yield an ODE for Λ. Because we allow for general Ψ from the dual space of L^2 (see the remarks following Theorem 2.7), we need Peano's Theorem to prove existence. Uniqueness is analyzed in an additional step by using the algebraic equation (2.6b). Asymptotic stability is a consequence of Theorem 2.5. Note that he analysis is a lot easier if Ψ belongs to the dual of H^{-1}, but in applications this is typically not satisfied, see for example Section 8.1.

Proof of Theorem 2.7. Let $0<\eta<\delta$ be given.

Step 0: Without loss of generality assume $\underline{\lambda}=0$. If $\underline{\lambda}\neq 0$, the equation is considered in a co-moving frame and becomes

$$\begin{aligned}u_t&=(B+\underline{\lambda}I)u_x+f(u)+\lambda u_x,\\ 0&=\Psi(\hat{u}-u),\end{aligned}\tag{2.6'}$$

so that (u,λ) solves (2.6') if and only if $(u,\underline{\lambda}+\lambda)$ solves (2.6). Let

$$\begin{aligned}F(u)&:=Bu_x+f(u),\\ R_\rho&:=\{u_0\in\underline{v}+H^2(\mathbb{R},\mathbb{R}^m):\|u_0-\underline{v}\|_{H^2}<\rho\ \text{and}\ \Psi(\hat{u}-u_0)=0\}.\end{aligned}$$

Step 1: [Solution of the PDE] By Theorem 2.5 there exist $\rho = \rho(\eta) > 0$ and $C_{pde} > 0$ so that for all $u_0 \in R_\rho$ there is a unique solution $v_{pde} \in \mathcal{C}^1([0,\infty); \underline{v} + L^2) \cap \mathcal{C}([0,\infty); \underline{v} + H^1)$ of the Cauchy problem

$$v_t = F(v), \qquad v(0) = u_0. \tag{7.1}$$

And there is a number $\varphi_\infty \in \mathbb{R}$ so that

$$\begin{aligned} |\varphi_\infty| \leq C_{pde} \|u_0 - \underline{v}\|_{H^2}, \quad \text{and} \\ \|v_{pde}(\cdot,t) - \underline{v}(\cdot - \varphi_\infty)\|_{H^1} \leq C_{pde} \|u_0 - \underline{v}\|_{H^2} e^{-\eta t}, \quad \forall t \geq 0. \end{aligned} \tag{7.2}$$

To emphasize the dependence on the initial value u_0, we sometimes write $v_{pde}^{[u_0]}$.

Step 2: [Ansatz for a solution of (2.6)]

Lemma 7.1. *Let $u_0 \in R_\rho$ and let $v_{pde}^{[u_0]}$ be given as in Step 1. If $\Lambda \in \mathcal{C}^1([0,T];\mathbb{R})$ solves the ODE*

$$\Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda(t), t)\big)\dot{\Lambda}(t) = -\Psi\big(F(v_{pde}^{[u_0]}(\cdot + \Lambda(t), t))\big), \quad \Lambda(0) = 0, \tag{7.3}$$

in $[0,T]$ for some $T > 0$, then the pair $(u, \dot{\Lambda})$ with $u(\cdot,t) = v_{pde}^{[v_0]}(\cdot + \Lambda(t), t)$ is a solution of (2.6).

Proof. The smoothness of v_{pde} and the assumption on Λ imply $u \in \mathcal{C}^1([0,T]; \underline{v} + L^2)$, and by the chain rule

$$\frac{d}{dt} u(\cdot,t) = F(u)(t) + \dot{\Lambda}(t) u_x(\cdot,t) \quad \text{in } L^2 \text{ for all } t \geq 0. \tag{7.4}$$

Moreover, the smoothness of u and the assumption $\hat{u} - \underline{v} \in H^1(\mathbb{R}, \mathbb{R}^m)$ show $t \mapsto \hat{u} - u(t) \in \mathcal{C}^1([0,T]; L^2)$. Therefore, Assumption 2.6 justifies the use of the chain rule, so that $t \mapsto \Psi(\hat{u} - u(t))$ belongs to $\mathcal{C}^1([0,T];\mathbb{R})$ and by (7.4)

$$\begin{aligned} \frac{d}{dt}\Psi\big(\hat{u} - u(t)\big) &= -\Psi(u_t) = -\Psi\big(F(u) + \dot{\Lambda} u_x\big) \\ &= -\Psi\big(F(v_{pde}(\cdot + \Lambda(t), t))\big) - \dot{\Lambda}(t)\Psi\big(v_{pde,x}(\cdot + \Lambda(t), t)\big) = 0 \ \text{ for all } 0 \leq t \leq T. \end{aligned}$$

Together with consistency of the initial data, i.e. $\Psi\big(\hat{u} - u(\cdot,0)\big) = \Psi\big(\hat{u} - v_{pde}(\cdot + 0, 0)\big) = 0$, the algebraic condition $\Psi\big(\hat{u} - u(t)\big) = 0$ holds for all $0 \leq t \leq T$. ☐

Step 3: [Local existence of a solution] By Step 2 it suffices to find a solution Λ of (7.3) to obtain local existence of a solution to (2.6). The next lemma and its corollary show that this is possible if the initial condition u_0 is sufficiently close to $\underline{v}$.

Lemma 7.2. *There are $\rho_1, \rho_\Lambda > 0$, $\rho_1 \leq \rho$, so that for all $u_0 \in R_{\rho_1}$,*

$$\big|\Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda, t)\big)\big| \geq \frac{|\Psi(\underline{v}_x)|}{2} > 0 \quad \forall |\Lambda| \leq \rho_\Lambda, t \geq 0. \tag{7.5}$$

Moreover, the function $r : B_{\rho_\Lambda}(0) \times [0,\infty) \to \mathbb{R}$, where $B_{\rho_\Lambda}(0) = \{|\Lambda| < \rho_\Lambda\}$, given by

$$r : (\Lambda, t) \mapsto r(\Lambda, t) = -\frac{\Psi\big(F(v_{pde}^{[u_0]}(\cdot + \Lambda, t))\big)}{\Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda, t)\big)}, \tag{7.6}$$

is an element of $\mathcal{C}(B_{\rho_\Lambda}(0) \times [0,\infty); \mathbb{R})$.

The local existence of a solution to the ODE (7.3) then is a simple corollary.

Corollary 7.3. *In the setting of Lemma 7.2 there is a solution* Λ *of* (7.3) *and one has the dichotomy:*

- *Either* Λ *is a solution for all times* $t \geq 0$ *with* $\Lambda \in \mathcal{C}^1([0,\infty); B_{\rho_\Lambda}(0))$,
- *or there is* $0 < T^* < \infty$ *and* $\lim_{t \nearrow T^*} |\Lambda(t)| = \rho_\Lambda$.

Proof of Corollary 7.3. Because of Lemma 7.2, (7.3) can be rewritten in the form $\dot{\Lambda} = r(\Lambda, t)$, $\Lambda(0) = 0$, where $r \in \mathcal{C}(B_{\rho_\Lambda}(0) \times [0,\infty); \mathbb{R})$. Therefore, Peano's Theorem (e.g. [21, II.§6]) proves the assertion. □

Proof of Lemma 7.2. Step 1 shows $v_{pde} = v_{pde}^{[u_0]} \in \mathcal{C}([0,\infty); \underline{v} + H^1)$ for $u_0 \in R_\rho$, so that $F(v_{pde}) \in \mathcal{C}([0,\infty); L^2)$. The continuity of the shift in $L^2(\mathbb{R}, \mathbb{R}^m)$ thus implies

$$
\begin{aligned}
(\Lambda, t) &\mapsto \Psi\big(F(v_{pde}(\cdot + \Lambda, t))\big) \quad \text{belongs to} \quad \mathcal{C}(\mathbb{R} \times [0,\infty); \mathbb{R}), \\
(\Lambda, t) &\mapsto \Psi\big(v_{pde,x}(\cdot + \Lambda, t)\big) \quad \text{belongs to} \quad \mathcal{C}(\mathbb{R} \times [0,\infty); \mathbb{R}),
\end{aligned}
$$

for the nominator and denominator of $r(\Lambda, t)$, respectively. Moreover, it holds

$$
\begin{aligned}
\big|\Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda, t)\big)\big| \geq |\Psi(\underline{v}_x)| - \Big[\big|\Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda, t) - \underline{v}_x(\cdot - \varphi_\infty + \Lambda)\big)\big| + \\
\big|\Psi\big(\underline{v}_x(\cdot - \varphi_\infty + \Lambda) - \underline{v}_x\big)\big|\Big]. \quad (7.7)
\end{aligned}
$$

Therefore, estimate (7.2) and the continuity of Ψ bound the [ ]–term in (7.7) by

$$
\begin{aligned}
&|\Psi(v_{pde,x}^{[u_0]}(\cdot + \Lambda, t) - \underline{v}_x(\cdot - \varphi_\infty + \Lambda))| + |\Psi\big(\underline{v}_x(\cdot - \varphi_\infty + \Lambda) - \underline{v}_x\big)| \\
&\quad \leq C_\Psi\Big(C_{pde}\, \|u_0 - \underline{v}\|_{H^2}\, e^{-\eta t} + \|\underline{v}_x\|_{H^1}(|\varphi_\infty| + |\Lambda|)\Big) \\
&\quad \leq C_\Psi C_{pde} \|u_0 - \underline{v}\|_{H^2} \big(\|\underline{v}_x\|_{H^1} + e^{-\eta t}\big) + C_\Psi \|\underline{v}_x\|_{H^1} |\Lambda| \\
&\quad \leq C_\eta \left(\|u_0 - \underline{v}\|_{H^2} + |\Lambda|\right),
\end{aligned}
$$

where C_η is a constant that is independent of u_0. With this C_η and ρ from Step 1 define

$$
\rho_1 = \min\left(\rho, \frac{|\Psi(\underline{v}_x)|}{4C_\eta}\right) \quad \text{and} \quad \rho_\Lambda = \frac{|\Psi(\underline{v}_x)|}{4C_\eta}.
$$

Then for all $u_0 \in R_{\rho_1}$ and all $\Lambda \in \overline{B_{\rho_\Lambda}}(0) = \{|\Lambda| \leq \rho_\Lambda\}$ the estimate (7.7) yields the lower bound

$$
\big|\Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda, t)\big)\big| \geq \frac{|\Psi(\underline{v}_x)|}{2} > 0
$$

for the denominator. This proves that for every $u_0 \in R_{\rho_1}$ the function r is well-defined on $B_{\rho_\Lambda}(0) \times [0,\infty)$ and, in particular, r is continuous on this set. □

Step 4: [Global existence of a solution] The next lemma shows that for u_0 sufficiently close to $\underline{v}$, the ODE (7.3) has a global solution.

Lemma 7.4. *There is $\rho_0 > 0$, $\rho_0 \le \rho_1$, so that for all $u_0 \in R_{\rho_0}$, every solution Λ of (7.3), satisfying the dichotomy from Corollary 7.3, also satisfies $\Lambda \in \mathcal{C}^1\big([0,\infty); \overline{B}_{\frac{\rho_\Lambda}{2}}(0)\big)$.*

Combining this result with Lemma 7.1, immediately leads to global existence of a solution to the PDAE (2.6).

Corollary 7.5 (Global existence for the freezing method)**.** *Let ρ_0 be given as in Lemma 7.4. Then for every $u_0 \in R_{\rho_0}$ the PDAE (2.6) has a global solution (u, λ) which satisfies*

$$u \in \mathcal{C}^1([0,\infty); \underline{v} + L^2) \cap \mathcal{C}^0([0,\infty); \underline{v} + H^1) \quad \textit{and} \quad \lambda \in \mathcal{C}([0,\infty); \mathbb{R}).$$

Proof of Lemma 7.4. The function $G|_U : U \to V, \widetilde{\varphi} \mapsto G(\widetilde{\varphi}) = \Psi\big(\underline{v}(\cdot - \widetilde{\varphi}) - \underline{v}\big)$ is a $\mathcal{C}^2$ diffeomorphism of the open zero–neighborhoods $U, V \subset \mathbb{R}$ by Lemma 3.4, see also the remark following that lemma. Let $V_0 \subset V$ be a compact neighborhood of 0 and let C_Φ be the Lipschitz constant of $\Phi|_{V_0} = (G|_U)^{-1}|_{V_0}$ (i.e. of Φ restricted to V_0). Choose $0 < \rho_0 \le \rho_1$ with

$$(1 + C_\Phi C_\Psi)\, C_{pde}\rho_0 \le \frac{\rho_\Lambda}{2} \quad \text{and} \quad \overline{B}_{C_\Psi C_{pde}\rho_0}(0) = \{|v| \le C_\Psi C_{pde}\rho_0\} \subset V_0. \tag{7.8}$$

Let $u_0 \in R_{\rho_0}$ and let Λ be a solution of (7.3) with the properties from Corollary 7.3. Define

$$T^* = \sup\{T_0 > 0 : |\Lambda(t)| < \rho_\Lambda \quad \forall 0 \le t \le T_0\}.$$

By the properties of Λ we have for arbitrary $0 < T_0 < T^*$, $0 = \Psi\big(\hat{u} - v_{pde}(\cdot + \Lambda(t), t)\big)$ for all $0 \le t \le T_0$. This is equivalent to

$$G(-\varphi_\infty + \Lambda(t)) = \Psi\big(\underline{v}(\cdot - \varphi_\infty + \Lambda(t)) - v_{pde}^{[u_0]}(\cdot + \Lambda(t), t)\big), \tag{7.9}$$

because of $\Psi(\hat{u} - \underline{v}) = 0$. The bound (2.7) for Ψ and (7.2) show

$$\big|\Psi\big(\underline{v}(\cdot - \varphi_\infty + \Lambda(t)) - v_{pde}^{[u_0]}(\cdot + \Lambda(t), t)\big)\big| \le C_\Psi C_{pde}\|u_0 - \underline{v}\|_{H^2} e^{-\eta t} \le C_\Psi C_{pde}\rho_0,$$

so that the right hand side of (7.9) is an element of V_0. Therefore, Φ can be applied to (7.9) and yields

$$|\Lambda(t) - \varphi_\infty| \le C_\Phi C_\Psi C_{pde}\|u_0 - \underline{v}\|_{H^2} e^{-\eta t} \quad \text{for all } t \in [0, T_0]. \tag{7.10}$$

In particular, it holds

$$|\Lambda(t)| \le |\varphi_\infty| + C_\Phi C_\Psi C_{pde}\|u_0 - \underline{v}\|_{H^2} e^{-\eta t} \le C_{pde}\big(1 + C_\varphi C_\Psi\big)\|u_0 - \underline{v}\|_{H^2} \le \frac{\rho_\Lambda}{2}, \tag{7.11}$$

where the bound $|\varphi_\infty| \le C_{pde}\|u_0 - \underline{v}\|_{H^2}$ from Step 1 was used. Since $T_0 < T^*$ is arbitrary, the dichotomy from Corollary 7.3 implies the existence of the solution Λ for all positive times. Finally, (7.11) shows $\Lambda \in \mathcal{C}^1\big([0,\infty); \overline{B}_{\frac{\rho_\Lambda}{2}}(0)\big)$. □

Step 5: [Unique solvability of the ODE (7.3)] Despite the fact that we used Peano's Theorem for the existence of a solution, we now show that the solution is unique. The principal idea is that the solution must satisfy the algebraic constraint and this is locally uniquely solvable.

Lemma 7.6. *For $u_0 \in R_{\rho_0}$ the ODE (7.3) has a unique global solution $\Lambda_0 \in \mathcal{C}^1([0,\infty); B_{\rho_\Lambda}(0))$.*

In the proof of the lemma we use a contraction argument, which is shown in the next lemma.

Lemma 7.7. *Let $u_0 \in R_{\rho_0}$ and let $\Lambda_0 \in \mathcal{C}^1([0,\infty); \overline{B}_{\frac{\rho_\Lambda}{2}}(0))$ be a solution of (7.3) on $[0,\infty)$. Define the mapping $H : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ by*

$$H : (t,\Lambda) \mapsto H(t,\Lambda) = \Lambda + \Psi\big(v_{pde,x}^{[u_0]}(\cdot + \Lambda_0(t), t)\big)^{-1} \Psi\big(\hat{u} - v_{pde}^{[u_0]}(\cdot + \Lambda, t)\big). \tag{7.12}$$

Then for every $t_0 \in [0,\infty)$ there are $\delta_t = \delta_t(t_0) > 0$ and $\delta_\Lambda = \delta_\Lambda(t_0) > 0$ such that for all $t \in B_{\delta_t}(t_0) \cap [0,\infty)$ and all $\Lambda \in B_{\delta_\Lambda}(\Lambda_0(t))$,

$$\big|H(t,\Lambda) - H(t,\Lambda_0(t))\big| \leq \frac{1}{2}|\Lambda - \Lambda_0(t)|. \tag{7.13}$$

Note that for fixed t the function $H(t,\cdot)$ from (7.12) describes one iteration step of a quasi-Newton method for the equation $F(\Lambda) = 0$, where $F(\Lambda) = \Psi\big(\hat{u} - v_{pde}^{[u_0]}(\cdot + \Lambda, t)\big)$.

Proof of Lemma 7.7. Let $t_0 \in [0,\infty)$ be given.
Since $v_{pde} \in \mathcal{C}^0([0,\infty); \underline{v} + H^1)$ there is $\delta_t > 0$, so that

$$\|v_{pde}(\cdot, t_0) - v_{pde}(\cdot, t)\|_{H^1} \leq \frac{|\Psi(\underline{v}_x)|}{12 C_\Psi} \qquad \forall t \in B_{\delta_t}(t_0) \cap [0,\infty), \tag{7.14}$$

Furthermore, by the continuity of the shift in $H^1(\mathbb{R},\mathbb{R}^m)$ there exists a $\delta_\Lambda > 0$ with

$$\|v_{pde}(\cdot, t_0) - v_{pde}(\cdot + \xi, t_0)\|_{H^1} \leq \frac{|\Psi(\underline{v}_x)|}{12 C_\Psi} \qquad \forall \xi \in B_{\delta_\lambda}(0). \tag{7.15}$$

Now we show that with these δ_t and δ_Λ the assertion holds. For the ease of notation we drop the argument t in Λ_0. The definition of H and estimate (7.5) of Lemma 7.2 imply for all $t \geq 0$ and all $\Lambda \in \mathbb{R}$

$$\begin{aligned}\Big|H(t,\Lambda) - H(t,\Lambda_0)\Big| \leq \frac{2}{|\Psi(\underline{v}_x)|}\Big|&\Psi\big(v_{pde,x}(\cdot + \Lambda_0, t)\big)\big(\Lambda - \Lambda_0\big) \\ &+ \Big[\Psi\big(\hat{u} - v_{pde}(\cdot + \Lambda, t)\big) - \Psi\big(\hat{u} - v_{pde}(\cdot + \Lambda_0, t)\big)\Big]\Big|.\end{aligned} \tag{7.16}$$

Because of the smoothness $v_{pde}(\cdot, t) \in H^1(\mathbb{R},\mathbb{R}^m)$ holds

$$v_{pde}(\cdot + \Lambda_0, t) - v_{pde}(\cdot + \Lambda, t) = -\int_0^1 v_{pde,x}\big(\cdot + \Lambda_0 + s(\Lambda - \Lambda_0), t\big)\, ds\, (\Lambda - \Lambda_0)$$

as an equality in L^2. Inserting this into (7.16) and using the monotonicity of the Bochner-integral, the right hand side of (7.16) is bounded by

$$\frac{2}{|\Psi(\underline{v}_x)|}\int_0^1 \Big|\Psi\big(v_{pde,x}(\cdot + \Lambda_0, t)\big) - \Psi\big(v_{pde,x}(\cdot + \Lambda_0 + s(\Lambda - \Lambda_0), t)\big)\Big|\, ds\, \big|\Lambda - \Lambda_0\big|.$$

Together with the continuity assumption (2.7) on Ψ, this implies the estimate

$$\Big|H(t,\Lambda) - H(t,\Lambda_0(t))\Big| \\ \le \frac{2C_\Psi}{|\Psi(\underline{v}_x)|} \sup_{s\in[0,1]} \big\|v_{pde}(\cdot,t) - v_{pde}\big(\cdot + s(\Lambda-\Lambda_0),t\big)\big\|_{H^1} \big|\Lambda - \Lambda_0\big|. \quad (7.17)$$

If $t' \in B_{\delta_t}(t) \cap [0,\infty)$ and $\Lambda \in B_{\delta_\Lambda}(\Lambda_0(t))$, then inequalities (7.14) and (7.15) show that the $\|\cdot\|_{H^1}$–term in (7.17) satisfies for each $s \in [0,1]$ the estimate

$$\big\|v_{pde}(\cdot,t') - v_{pde}\big(\cdot + s(\Lambda - \Lambda_0(t')),t'\big)\big\|_{H^1} \\ \le 2\big\|v_{pde}(\cdot,t') - v_{pde}(\cdot,t)\big\|_{H^1} + \big\|v_{pde}(\cdot,t) - v_{pde}(\cdot + s(\Lambda-\Lambda_0(t')),t)\big\|_{H^1} \\ \le \frac{|\Psi(\underline{v}_x)|}{6C_\Psi} + \frac{|\Psi(\underline{v}_x)|}{12C_\Psi} = \frac{|\Psi(\underline{v}_x)|}{4C_\Psi}. \quad (7.18)$$

Inserting (7.18) into (7.17) finishes the proof. ☐

Proof of Lemma 7.6. Let Λ_0 and Λ_1 be global solutions of the ODE (7.3). By Lemma 7.4 the functions satisfy $\Lambda_0, \Lambda_1 \in \mathcal{C}^1([0,\infty); B_{\rho_\Lambda}(0))$. For every solution Λ of (7.3) holds $\Psi\big(\hat{u} - v_{pde}(\cdot + \Lambda(t),t)\big) = 0$ for every $t \ge 0$ so that $H(t,\Lambda(t)) = \Lambda(t)$ for every $t \ge 0$.

Let $M := \{t \ge 0 : \Lambda_0(t) = \Lambda_1(t)\}$, which contains 0. Furthermore, M is a closed subset of $[0,\infty)$ by continuity of Λ_0, Λ_1. Moreover, M is open: Assume $t_0 \in M$ and let $\delta_t = \delta_t(t_0)$ and $\delta_\Lambda = \delta_\Lambda(t_0)$ be as in Lemma 7.7. There is $\varepsilon_{t_0} > 0$, $\varepsilon_{t_0} \le \delta_t$, so that $|\Lambda_1(t) - \Lambda_0(t)| < \delta_\Lambda$ for all $t \ge 0$ with $|t - t_0| < \varepsilon_{t_0}$. Then Lemma 7.7 implies

$$|\Lambda_1(t) - \Lambda_0(t)| = |H(t,\Lambda_1(t)) - H(t,\Lambda_0(t))| \le \frac{1}{2}|\Lambda_1(t) - \Lambda_0(t)|,$$

i.e. $\Lambda_1(t) = \Lambda_0(t)$. Then M is also an open set in $[0,\infty)$ and, therefore, $M = [0,\infty)$. ☐

Step 6: [Unique solvability of the PDAE (2.6)]
Let $\widetilde{u} \in \mathcal{C}^1([0,T); \underline{v} + L^2) \cap \mathcal{C}([0,T); \underline{v} + H^1)$, $\widetilde{\lambda} \in \mathcal{C}([0,T);\mathbb{R})$ be a solution of (2.6) and let $\widetilde{\Lambda}(t) := \int_0^t \widetilde{\lambda}(\tau)\,d\tau$ and $\widetilde{v}_{pde}(\cdot,t) := \widetilde{v}(\cdot - \widetilde{\Lambda}(t),t)$. Then

$$\widetilde{v}_{pde} \in \mathcal{C}^1([0,T);\underline{v} + L^2) \cap \mathcal{C}([0,T);\underline{v} + H^1), \quad \text{with} \quad \widetilde{v}_{pde}(0) = \widetilde{v}(0) = u_0$$

and $\widetilde{v}_{pde}$ solves the Cauchy-problem (7.1) in $[0,T)$. Because of uniqueness, $\widetilde{v}_{pde} = v_{pde}^{[u_0]}$. Moreover, the hidden constraint shows that also $\widetilde{\Lambda}$ is a solution of the ODE (7.3). Thus uniqueness of the solution of (2.6) follows.

Step 7: [Exponential convergence] By the previous steps, the unique solution of (2.6) is given by

$$(u,\lambda) = \Big(v_{pde}(\cdot + \Lambda(t),t), \dot{\Lambda}(t)\Big),$$

where Λ solves (7.3). Since the H^1–norm is shift-invariant, this implies

$$\|u(\cdot,t) - \underline{v}\|_{H^1} \le \|v_{pde}(\cdot,t) - \underline{v}(\cdot - \varphi_\infty)\|_{H^1} + \|\underline{v}(\cdot - \varphi_\infty) - \underline{v}(\cdot - \Lambda(t))\|_{H^1}. \quad (7.19)$$

The first summand of (7.19) is bounded by $C_{pde}\,\|u_0 - \underline{v}\|_{H^2}\, e^{-\eta t}$ because of (7.2). Furthermore, inequality (7.10) shows

$$\|\underline{v}(\cdot - \varphi_\infty) - \underline{v}(\cdot - \Lambda(t))\|_{H^1} \le \|\underline{v}_x\|_{H^1} |\varphi_\infty - \Lambda(t)| \le \text{const}\,\|u_0 - \underline{v}\|_{H^2}\, e^{-\eta t}$$

for the second summand. Thus

$$\|u(\cdot,t) - \underline{v}\|_{H^1} \le \text{const}\|u_0 - \underline{v}\|_{H^2} e^{-\eta t}. \tag{7.20}$$

Furthermore, recall the identity $\lambda(t) = \dot{\Lambda}(t)$. Therefore, the ODE (7.3) and estimate (7.5) from Lemma 7.2 imply

$$\begin{aligned} |\lambda(t)| = \frac{|\Psi\big(F(u(t))\big)|}{|\Psi(u(t))|} &\le \frac{2}{|\Psi(\underline{v}_x)|}\,\big|\Psi\big(F(u(t)) - F(\underline{v})\big)\big| \\ &\le \frac{2C_\Psi}{|\Psi(\underline{v}_x)|}\|F(u(t)) - F(\underline{v})\|_{L^2} \le C\|u(t) - \underline{v}\|_{H^1} \le C\|v_0 - \underline{v}\|_{H^2} e^{-\eta t}, \end{aligned}$$

where $F(\underline{v}) = 0$ and the local Lipschitz continuity of F in $\underline{v} + H^1$ was used. This finishes the proof of (2.9). □

Remark. *Note that the global results from Step 1 are needed for the global existence of the solution of the ODE* (7.3) *and in particular for the estimate* (7.5)*, which is important for solving the ODE for* Λ*. But these estimates are not needed for a local existence result of the ODE, which basically relies on the well-definedness of the fraction* (7.6) *and only uses* $|\Psi(u_{0x})| > 0$.
Therefore, the proof shows that the PDAE–system (2.6) *is locally solvable if we have consistent initial conditions for* u_0*, satisfying* $\Psi(u_{0x}) \neq 0$*. If the solution* u_0 *remains bounded but* $\Psi(u_x)$ *becomes singular, a possible idea is to update the phase condition and continue the solution of the PDAE with the new phase condition. One can consider the minimizing phase condition from [5] as one, which is updated in each time-step. But we did not pursue this any further.*

8 Experiments

8.1 Hyperbolic-Nagumo equation

Our first example is a simple hyperbolic test equation with a cubic nonlinearity:

$$u_t + q_x = u(1-u)(u-\beta), \quad q + Tq_t = -Du_x. \tag{8.1}$$

For $T = 0$ this yields a parabolic PDE, also known as the Nagumo equation, which can be derived as a simplification of a population dynamics model (cf. [2]). For $T > 0$ the second equation in (8.1) is a modified Fickian law which was proposed in [11] as a way to prevent the unphysical infinite speed of propagation. It was used in [13] for the generalized Fisher's equation.

Let $T = 1$, $D = 1$ and $\beta = 0.25$, so that (8.1) becomes

$$\begin{pmatrix} u \\ q \end{pmatrix}_t = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ q \end{pmatrix}_x + \begin{pmatrix} u(1-u)(u-0.25) \\ -q \end{pmatrix} =: B \begin{pmatrix} u \\ q \end{pmatrix}_x + f(u,q). \qquad (8.2)$$

Because its parabolic counterpart, i.e. $u_t = u_{xx} + u(1-u)(u-\beta)$, is often called Nagumo equation, we call this system the *hyperbolic Nagumo equation* . The states $(0,0)^T$ and $(1,0)^T$ are rest states of (8.2) and there is a traveling wave solution of (8.2) with profile $(\underline{u}, \underline{q})^T$ and speed $\underline{\lambda}$. The profile is a heteroclinic orbit of the ODE $\frac{d}{dx}(u,q)^T = -(B + \underline{\lambda} I)^{-1} f(u,q) =: F(u,q)$, connecting the two rest states $(0,0)^T$ and $(1,0)^T$. If $|\underline{\lambda}| < 1$ these rest states are both hyperbolic fixed points of the ODE and, therefore, the profile approaches these rest states exponentially fast as $|x| \to \infty$.

In this case the profile satisfies Assumption 2.1. It is also easy to verify (H1) and (H2) from Assumption 2.2. To verify (H3) from Assumption 2.2, one diagonalizes the system and then observes that the assumption is equivalent to $\mathcal{S} \subset \{\Re s < -\delta\}$, where

$$\mathcal{S} = \left\{ s \in \mathbb{C} : \det \left(sI - i\omega \begin{pmatrix} -1 + \underline{\lambda} & 0 \\ 0 & 1 + \underline{\lambda} \end{pmatrix} - C_{\pm} \right) = 0,\ \omega \in \mathbb{R} \right\}, \qquad (8.3)$$

with

$$C_- = \frac{1}{2} \begin{pmatrix} -1.75 & -0.25 \\ -0.25 & -1.75 \end{pmatrix}, \quad C_+ = \frac{1}{2} \begin{pmatrix} -1.25 & -0.75 \\ -0.75 & -1.25 \end{pmatrix}.$$

The equation in the definition of $\mathcal{S}$ can be solved for s and we obtain that the spectral set $\mathcal{S}$ consists of the images of the four curves

$$s_{\pm}^{+}(\omega) = -\frac{1}{2} - \frac{1}{8} + \underline{\lambda}\omega i \pm \frac{1}{2}\sqrt{(1-0.25)^2 - 4\omega^2},\ \omega \in \mathbb{R},$$
$$s_{\pm}^{-}(\omega) = -1 + \frac{1}{8} + \underline{\lambda}\omega i \pm \frac{1}{2}\sqrt{0.25^2 - 4\omega^2},\ \omega \in \mathbb{R}.$$

In particular $\mathcal{S} \subset \{\Re s \le -0.25\}$ so that (H3) holds.

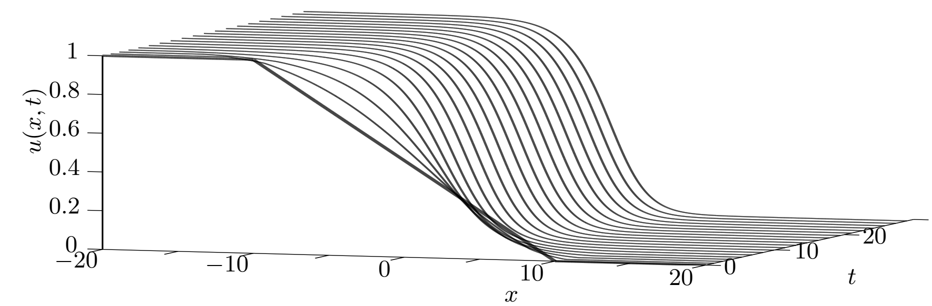
In Figure 1 we present a direct simulation of the freezing system (2.6) for this example. The computational domain is the interval $[-20, 20]$ and we take Neumann boundary conditions. The initial data are

$$u_0(x) = \begin{cases} 1, & x < -10, \\ 0.5 - 0.05x, & -10 \le x \le 10, \\ 0, & x > 10, \end{cases} \qquad q_0(x) = 0,\ \forall x,$$

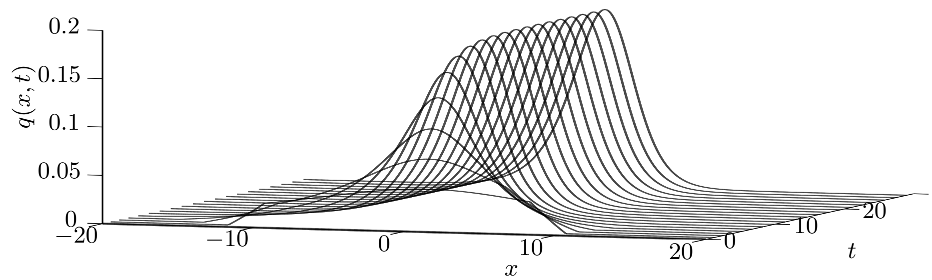
and the reference function is the same as the initial data. Finally, Ψ is chosen as

$$\Psi(u - \hat{u}, q - \hat{q}) = \int_{\mathbb{R}} \hat{u}_x (u - \hat{u}) + \hat{q}_x (q - \hat{q}_x)\, dx,$$

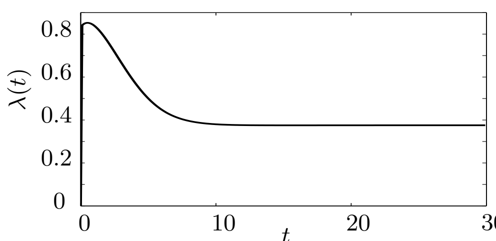
see the remarks following Theorem 2.7. Note that this functional Ψ is an element of $(L^2)'$ but not in the dual of H^{-1}. In the pictures one nicely sees the convergence of (u, q, λ).



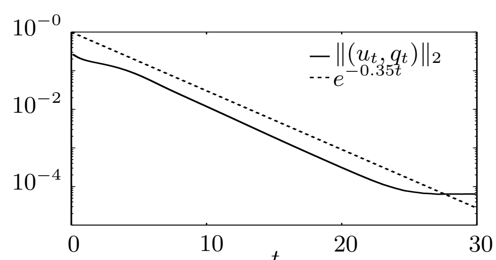
(a) Time evolution of the u variable.



(b) Time evolution of the q variable.



(c) Time evolution of the algebraic variable λ.

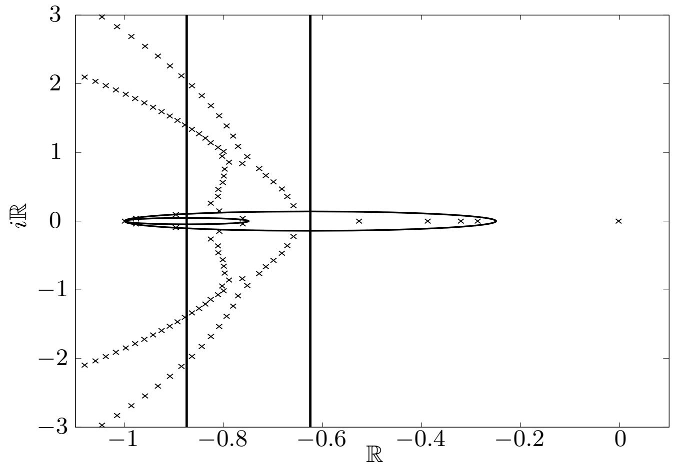


(d) Time evolution of $\|(u_t, q_t)\|_{L^2}$.

Figure 1: Evolution of the frozen hyperbolic Nagumo system. The last plot shows $\|(u_t, q_t)\|_{L^2}$ as an indicator for the convergence to a steady state.

In Figure 1(d) we plot the evolution of $\|(u_t, q_t)\|_{L^2}$ which indicates the convergence to the steady. We observe a rate of convergence of about $e^{-0.35t}$. This is a bit better than the theoretically predicted rate of $e^{-0.25t}$.

The final states of the direct simulation are used as initial data for a numerical computation of the profile and speed with the method presented in [3]. As asymptotic boundary conditions we choose projection boundary conditions and the boundary value problem is solved with `bvp4c` from Matlab. As a result we obtain $\underline{\lambda} \approx 0.3754$. In Figure 2 we show a plot of the spectral set $\mathcal{S}$ and a numerical approximation of the spectrum of the linearized operator. The computation is done with the profile and speed computed with the method described before. The system is discretized on a grid with stepsize $\Delta x = 0.1$ and with up-



First few eigenvalues	
1.	-0.0021
2.	-0.2869
3.	-0.3208
4.	-0.3879
5.	-0.5265
6.	$-0.6591 + 0.2234i$
7.	$-0.6591 - 0.2234i$
8.	$-0.6716 + 0.3575i$
9.	$-0.6716 - 0.3575i$
⋮	⋮

Figure 2: Left: Closeup of the numerical spectrum (crosses) on $[-30, 30]$ with step size $\Delta x = 0.1$ and periodic boundary conditions together with the spectral set $\mathcal{S}$ from (8.3) (continuous lines). Right: The first eigenvalues with largest real part.

winding and periodic boundary conditions on the interval $[-30, 30]$. One observes a simple eigenvalue that is close to zero and then a spectral gap.

8.2 Hyperbolic Hodgkin-Huxley System

Our second example is a hyperbolic version of the Hodgkin-Huxley model. It is formally obtained from the original problem (see [9]) by using a modified Fickian law as above. So we call the following system the *hyperbolic Hodgkin-Huxley system*. It takes the form

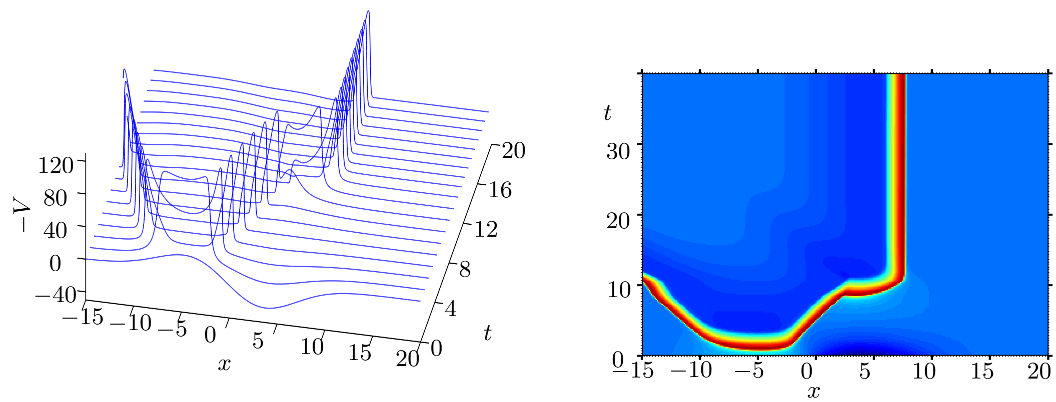
$$\begin{aligned} V_t &= -q_x - \bar{g}_K n^4 (V - V_K) - \bar{g}_{Na} m^3 h (V - V_{Na}) - \bar{g}_l (V - V_l), \\ q_t &= -\tfrac{1}{2} V_x - q, \\ n_t &= \alpha_n \, (1 - n) - \beta_n n, \\ m_t &= \alpha_m \, (1 - m) - \beta_m m, \\ h_t &= \alpha_h \, (1 - h) - \beta_h h, \end{aligned}$$

where

$$\begin{aligned} \alpha_n &= \frac{1}{100}(V + 10)\big(\exp(\tfrac{V+10}{10}) - 1\big)^{-1}, & \beta_n &= \frac{1}{8}\exp(\tfrac{V}{80}), \\ \alpha_m &= \frac{1}{10}(V + 25)\big(\exp(\tfrac{V+25}{10}) - 1\big)^{-1}, & \beta_m &= 4\exp(\tfrac{V}{18}), \\ \alpha_h &= \frac{7}{100}\exp(\tfrac{V}{20}), & \beta_h &= \big(\exp(\tfrac{V+30}{10}) + 1\big)^{-1}. \end{aligned}$$

For the constants we choose the same values as in the original paper [9]

$$\begin{aligned} V_{Na} &= -115, \quad V_K = 12, \quad V_l = -10.613, \\ \bar{g}_{Na} &= 120, \qquad \bar{g}_K = 36, \quad \bar{g}_l = 0.3. \end{aligned}$$



(a) A plot of the V-variable for different times.

(b) The same data, here V is color-coded.

Figure 3: The time-evolution of the V-component of the solution to the hyperbolic-Hodgkin-Huxley equation. The plots show $-V(x,t)$.

It is important to note that the system is hyperbolic, but not strictly hyperbolic, i.e. in the notation of Section 2 the matrix $(\underline{\lambda} I + B)$ has multiple eigenvalues. Note that this case is not covered by the results of Kreiss et al. [12].

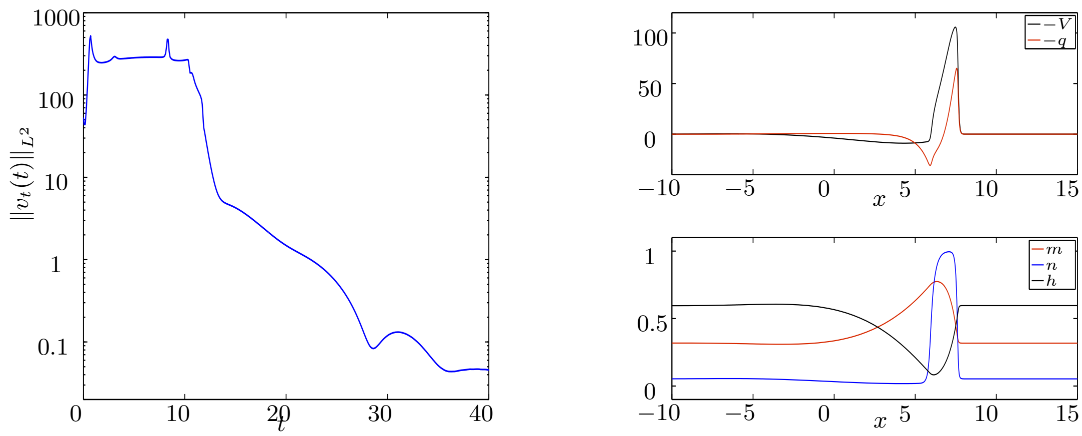
The system has a rest point at $V_\infty = -0.0036$, $q_\infty = 0$, $n_\infty = 0.3177$, $m_\infty = 0.0530$, $h_\infty = 0.5960$, we simply write $v_\infty := (V_\infty, q_\infty, n_\infty, m_\infty, h_\infty)^T$. The numerical computations below, show that there is a traveling wave solution whose profile is a homoclinic connection of the rest point v_∞ to itself.

In Figure 3 we present the results of a numerical simulation of the problem with the freezing method (we only show the V component). One nicely observe the convergence to a steady state. The computation was done by the same method as for the hyperbolic-Nagumo equation, presented above. But here we updated the reference function (and also Ψ, which is derived from the reference function) in each time-step. This leads to the so-called minimizing phase condition from [5]. Note that the authors show how to transform the solutions for different phase conditions into each other.

In Figure 4(a) the value $\|v_t(t)\|_{L^2}$, $v = (V, q, m, n, h)$, is plotted against time, which illustrates the convergence to the steady state. On the right, in Figure 4(b), we show the asymptotic profile of the traveling wave which is again computed using the techniques from [3]. The final values of the forward integration were used as initial data for the boundary value solver. To justify the spectral Assumption 2.2 (H3) for this problem, we resolved the dispersion relation, i.e. we display the set

$$\mathcal{S} = \left\{ s \in \mathbb{C} : \det\bigl(sI - i\omega(\underline{\lambda} I + B) - f_v(v_\infty)\bigr) = 0 \right\},$$

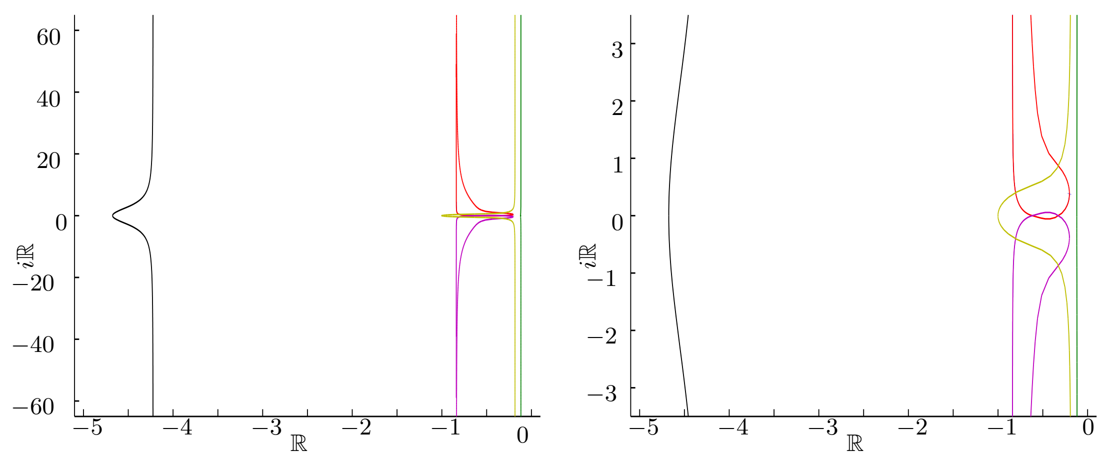
in Figure 5. It is well-known that the set $\mathcal{S}$ belongs to the essential spectrum of the linearized operator. The set is the union of five algebraic curves. For its numerical computation, we



(a) Convergence of the freezing method to a stationary solution, which is indicated by the norm $\|v_t(t)\|_{L^2}$.

(b) The profile of the traveling wave. The corresponding speed of the wave is $\underline{\lambda} \approx 0.6676$.

Figure 4: Convergence to a steady state and the profile of the traveling wave.



(a) The spectral set $\mathcal{S}$.

(b) A zoom into the spectral set $\mathcal{S}$ around 0.

Figure 5: The dispersion relation for the traveling wave solution of the hyperbolic Hodgkin-Huxley equation.

used the continuation toolbox MATCONT [7] and solved for the eigenvalues of $i\omega(\underline{\lambda}I + B) + f_v(v_\infty)$ with respect to ω.

It turns out that the equation

$$G(s,\omega) := \det\bigl(sI - i\omega(\underline{\lambda}I + B) - f_v(v_\infty)\bigr) = 0,$$

is not well suited for the continuation method. Therefore, we use the following real version of a complex eigenvector-eigenvalue system,

$$\begin{aligned}\begin{pmatrix}\operatorname{Re}(s)I - f_v(v_\infty) & \omega(\underline{\lambda} I + B) - \operatorname{Im}(s)I \\ -\omega(\underline{\lambda} I + B) + \operatorname{Im}(s)I & \operatorname{Re}(s)I - f_v(v_\infty)\end{pmatrix}\begin{pmatrix}\operatorname{Re}(v)\\ \operatorname{Im}(v)\end{pmatrix} &= 0,\\ \|\operatorname{Re}(v)\|^2 + \|\operatorname{Im}(v)\|^2 - 1 &= 0,\\ \operatorname{Re}(v)^T \operatorname{Im}(v) &= 0.\end{aligned} \tag{8.4}$$

The last two equations are needed to determine a unique complex eigenvector. The last equation can be derived from the assumption

$$\operatorname{argmin}_{\varphi\in[0,2\pi)}\|\operatorname{Im}(e^{i\varphi}v)\| \in \{0,\pi\},$$

which roughly states that we want the eigenvector to be "as real as possible".

Details of the nonlinear estimates

A Appendix: Details of the nonlinear estimates

Here we provide the details of the estimates used in the proof of Lemma 5.3. For convenience we recall

$$F_2^\varepsilon(\varphi, v) = \int_0^1 (1-s) f_{vv}(\underline{v}(\cdot - \varepsilon\varphi) + s\varepsilon v) ds[v,v].$$

Proof of estimate (5.15)**.** Let $M = \{v \in \mathbb{R}^m : |v| \le \|\underline{v}\|_\infty + \varepsilon_0 K_1\}$. Then for all $u, w \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$, with $\|u\|_{H^1}, \|w\|_{H^1} \le K_1$, one finds

$$\begin{aligned}&\|F_2^\varepsilon(\varphi,u) - F_2^\varepsilon(\varphi,w)\|^2\\ &\quad= \int_{\mathbb{R}}\Big|\int_0^1 (1-s) f_{vv}(\underline{v}(\cdot-\varepsilon\varphi) + s\varepsilon u)\, ds\, u^2\\ &\qquad - \int_0^1 (1-s) f_{vv}(\underline{v}(\cdot-\varepsilon\varphi) + s\varepsilon w)\, ds\, w^2\Big|^2 dx\\ &\quad\le 3\int_{\mathbb{R}}\Big|\int_0^1 (1-s)\left(f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon u) - f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon w)\right) ds\,[u,u]\Big|^2\\ &\qquad + \Big|\int_0^1 (1-s) f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon w)\, ds\,[u-w,u]\Big|^2\\ &\qquad + \Big|\int_0^1 (1-s) f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon w)\, ds\,[w,u-w]\Big|^2 dx\\ &\quad\le 3\Big(\|u\|^2\|u\|_\infty^2 C_1\varepsilon^2\|u-w\|_\infty^2 + \|u\|_\infty^2 C_2\|u-w\|^2 + \|w\|_\infty^2 C_2 \|u-w\|^2\Big),\end{aligned}$$

where C_1 is a Lipschitz constant for f_{vv} on the set M and $C_2 = \sup_{v\in M}|f_{vv}(v)|$. Therefore, Sobolev inequalities show $\|F_2^\varepsilon(\varphi,u) - F_2^\varepsilon(\varphi,w)\|^2 \le C\|u-w\|_{H^1}^2$, where C depends on K_1, but is independent of φ. □

Proof of estimate (5.16)**.** Let M and u, w be as before. Let C_1 be the Lipschitz constant for f_{vv} on M, and let $C_2 = \sup_{v\in M}|f_{vv}(v)|$.

To estimate the L^2–norm of $\big(F_2^\varepsilon(\varphi,u)-F_2^\varepsilon(\varphi,w)\big)_x$, insert the definition to find

$$
\begin{aligned}
&\|F_2^\varepsilon(\varphi,u)_x - F_2^\varepsilon(\varphi,w)_x\|^2 \\
&\le 2\int_{\mathbb{R}}\Big|\Big(\int_0^1 (1-s)f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon u)\,ds\Big)_x [u,u] \\
&\qquad - \Big(\int_0^1 (1-s)f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon w)\,ds\Big)_x [w,w]\Big|^2\,dx \\
&\quad + 4\int_{\mathbb{R}}\Big|\int_0^1 (1-s)f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon u)\,ds\,[u_x,u] \\
&\qquad - \int_0^1 (1-s)f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon w)\,ds\,[w_x,w]\Big|^2\,dx =: 2I_1+4I_2,
\end{aligned}
\tag{A.1}
$$

where I_1 denotes the first and I_2 the second x–integral. In the following, we abbreviate $a_{\varphi,w}(x,s)=\underline{v}(x-\varepsilon\varphi)+s\varepsilon w(x)$.

For the first x–integral we have

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}}\Big|\int_0^1 (1-s)f_{vvv}\big(a_{\varphi,u}(x,s)\big)\big(\underline{v}_x(x-\varepsilon\varphi)+s\varepsilon u_x\big)\,ds\,[u,u] \\
&\qquad - \int_0^1 (1-s)f_{vvv}\big(a_{\varphi,w}(x,s)\big)\big(\underline{v}_x(x-\varepsilon\varphi)+s\varepsilon w_x\big)\,ds\,[w,w]\Big|^2\,dx \\
&\le 4\int_{\mathbb{R}}\Big|\int_0^1 (1-s)\Big(f_{vvv}(a_{\varphi,u}(x,s))-f_{vvv}(a_{\varphi,w}(x,s))\Big)\big(\underline{v}_x(x-\varepsilon\varphi)+s\varepsilon u_x\big)\,ds\,[u,u]\Big|^2 \\
&\quad + \Big|\int_0^1 (1-s)f_{vvv}(a_{\varphi,w}(x,s))\big(s\varepsilon w_x - s\varepsilon u_x\big)\,ds\,[u,u]\Big|^2 \\
&\quad + \Big|\int_0^1 (1-s)f_{vvv}(a_{\varphi,w}(x,s))s\varepsilon w_x\,ds\,[w-u,u]\Big|^2 \\
&\quad + \Big|\int_0^1 (1-s)f_{vvv}(a_{\varphi,w}(x,s))s\varepsilon w_x\,ds\,[w,w-u]\Big|^2\,dx \\
&\le C\Big\{\sup_{\substack{x\in\mathbb{R}\\ s\in[0,1]}}\big|f_{vvv}(a_{\varphi,u}(x,s))-f_{vvv}(a_{\varphi,w}(x,s))\big|^2+\|u-w\|_{H^1}^2\Big\},
\end{aligned}
$$

where Sobolev embedding was used for the last inequality. The constant C depends on $\varepsilon_0\ge\varepsilon>0$, K_1, and $\sup_{v\in M}|f_{vvv}(v)|$, but is independent of φ.

Similarly for the second integral,

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}}\Big|\int_0^1 (1-s)f_{vv}(a_{\varphi,u}(x,s))\,ds\,[u_x,u]-\int_0^1 (1-s)f_{vv}(a_{\varphi,w}(x,s))\,ds\,[w_x,w]\Big|^2\,dx \\
&\le 3\int_{\mathbb{R}}\Big|\int_0^1 (1-s)\big(f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon u)-f_{vv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon w)\big)\,ds\,[u_x,u]\Big|^2 \\
&\quad + \Big|\int_0^1 (1-s)f_{vv}(a_{\varphi,w}(x,s))\,ds\,[w_x-u_x,u]\Big|^2 \\
&\quad + \Big|\int_0^1 (1-s)f_{vv}(a_{\varphi,w}(x,s))\,ds\,[w_x,u-w]\Big|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq 3\Big(\|u_x\|_{L^2}^2\|u\|_{L^\infty}^2 C_1\varepsilon^2\|u-w\|_{L^\infty}^2 + C_2\|w_x-u_x\|_{L^2}^2\|u\|_{L^\infty}^2 + C_2\|w_x\|_{L^2}^2\|w-u\|_{L^\infty}^2\Big)\\
&\leq C\|w-u\|_{H^1}^2.
\end{aligned}
$$

Here the last inequality follows from Sobolev embedding and C depends on K_1 but is independent of φ. Inserting the estimates for I_1 and I_2 into (A.1) proves (5.16). ☐

Proof of continuity of $\mathbf{F_2^\varepsilon}$. We prove that $F_2^\varepsilon : \mathbb{R}\times H^1(\mathbb{R},\mathbb{R}^m) \to H^1(\mathbb{R},\mathbb{R}^m)$ is continuous. Let $(\varphi_n, v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}\times H^1(\mathbb{R},\mathbb{R}^m)$ with $(\varphi_n,v_n)\to(\varphi,v)$ in $\mathbb{R}\times H^1(\mathbb{R},\mathbb{R}^m)$ as $n\to\infty$.

By the proof of (5.14) the convergence $\lim_{n\to\infty}\|F_2^\varepsilon(\varphi,v_n)-F_2^\varepsilon(\varphi,v)\|_{H^1}=0$ is uniform in φ, therefore

$$
\|F_2^\varepsilon(\varphi_n,v_n)-F_2^\varepsilon(\varphi_n,v)\|_{H^1}\to 0 \text{ as } n\to\infty. \tag{A.2}
$$

Let $v\in\mathcal{C}_0^\infty$ and let $M=\{w\in\mathbb{R}^m : |w|\leq \|\underline{v}\|_\infty+\varepsilon_0\|v\|_\infty\}$. For $\varphi,\varphi'\in\mathbb{R}$ holds

$$
\begin{aligned}
&\big\|F_2^\varepsilon(\varphi,v)-F_2^\varepsilon(\varphi',v)\big\|_{H^1}^2\\
&=\int_{\mathbb{R}}\Big|\int_0^1(1-s)\big(f_{vv}\big(\underline{v}(x-\varepsilon\varphi)+s\varepsilon v\big)-f_{vv}\big(\underline{v}(x-\varepsilon\varphi')+s\varepsilon v\big)\big)\,ds\,[v,v]\Big|^2\\
&\quad+3\Big|\int_0^1(1-s)\Big(f_{vvv}\big(\underline{v}(x-\varepsilon\varphi)+s\varepsilon v\big)-f_{vvv}\big(\underline{v}(x-\varepsilon\varphi')+s\varepsilon v\big)\Big)\\
&\qquad\qquad\big(\underline{v}_x(x-\varepsilon\varphi)+s\varepsilon v_x\big)\,ds\,[v,v]\Big|^2\\
&\quad+3\Big|\int_0^1(1-s)f_{vvv}\big(\underline{v}(x-\varepsilon\varphi')+s\varepsilon v\big)\big(\underline{v}_x(x-\varepsilon\varphi')-\underline{v}_x(x-\varepsilon\varphi)\big)\,ds\,[v,v]\Big|^2\\
&\quad+3\Big|\int_0^1(1-s)\big(f_{vv}\big(\underline{v}(x-\varepsilon\varphi)+s\varepsilon v\big)-f_{vv}\big(\underline{v}(x-\varepsilon\varphi')+s\varepsilon v\big)\big)\,ds\,2[v_x,v]\Big|^2\,dx.
\end{aligned}
$$

Therefore, Hölder's inequality, Lipschitz continuity of f_{vv} on M, and Lipschitz continuity of $\underline{v}_x$, yield a constant C_1 so that

$$
\begin{aligned}
\big\|F_2^\varepsilon(\varphi,v)-F_2^\varepsilon(\varphi',v)\big\|_{H^1}^2 &\leq C_1\Big(\big\|\underline{v}(\cdot-\varepsilon\varphi)-\underline{v}(\cdot-\varepsilon\varphi')\|_\infty^2\|v\|_2^2\|v\|_\infty^2\\
&\quad+\sup_{\substack{x\in\mathbb{R}\\ s\in[0,1]}}\big|f_{vvv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon v)-f_{vvv}(\underline{v}(x-\varepsilon\varphi')+s\varepsilon v)\big|^2\\
&\qquad\qquad\big(\|\underline{v}_x\|_{L^2}^2+\varepsilon^2\|v_x\|_{L^2}^2\big)\|v\|_\infty^4\\
&\quad+\sup_{w\in M}|f_{vvv}(w)|^2\varepsilon^2|\varphi-\varphi'|^2\|v\|_\infty^2\|v\|_{L^2}^2\\
&\quad+\big\|\underline{v}(\cdot-\varepsilon\varphi)-\underline{v}(\cdot-\varepsilon\varphi')\|_\infty^2\|v_x\|_2^2\|v\|_\infty^2\Big)\\
&\leq C\big(|\varphi-\varphi'|^2+\sup_{\substack{x\in\mathbb{R}\\ s\in[0,1]}}\big|f_{vvv}(\underline{v}(x-\varepsilon\varphi)+s\varepsilon v)-f_{vvv}(\underline{v}(x-\varepsilon\varphi')+s\varepsilon v)\big|^2\big).
\end{aligned}
$$

Sobolev embedding was used in the last inequality and the constant C depends on $\|v\|_{H^1}$, but not on φ and v itself.

The case of a general $v \in H^1$ follows by approximating v with functions from $\mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^m)$ that satisfy a uniform H^1–bound.

Finally, we have

$$\|F_2^\varepsilon(\varphi_n, v) - F_2^\varepsilon(\varphi, v)\|_{H^1} \to 0 \text{ as } n \to \infty, \tag{A.3}$$

because f_{vvv} is uniformly continuous on compact subsets of $\mathbb{R}^m$ and

$$\sup_{\substack{x\in\mathbb{R}\\ s\in[0,1]}} |\underline{v}(x - \varepsilon\varphi) + s\varepsilon v(x) - \underline{v}(x - \varepsilon\varphi') + s\varepsilon v(x)| \text{ is small for } |\varphi - \varphi'| \text{ small.}$$

Combination of (A.2) and (A.3) proves the continuity of F_2^ε. □

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