Galerkin Finite Element Methods for Semilinear Elliptic Differential Inclusions

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Abstract

Relaxed one-sided Lipschitz conditions play an important role when analyzing ordinary differential inclusions. They allow to derive a-priori estimates of solutions and convergence estimates for explicit and implicit time discretizations. In this paper we consider Galerkin finite element discretizations of semilinear elliptic inclusions that satisfy a relaxed one-sided Lipschitz condition. It is shown that solution sets of both, the continuous and the discrete system, are nonempty closed bounded and connected sets in H^1-norm. Moreover, the solution sets of the Galerkin inclusion converge with respect to the Hausdorff distance measured in L^p-spaces. We also set up a full discretization of the Galerkin inclusion which uses a partitioning of the finite element space into cells and support functionals for measuring the residual of Galerkin approximations. An efficient implementation is developed that utilizes connectedness of the solution set and that is tested on a numerical example.

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1 Introduction

In this paper we analyze and implement Galerkin methods for computing weak solutions of a semilinear elliptic inclusion. Before formulating the set-valued setup, we briefly recall the classical single-valued case.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, let $\mathcal{M}(\Omega)$ denote the (Lebesgue-)measurable functions on Ω and let $L^2 = L^2(\Omega)$, $H_0^1 = H_0^1(\Omega)$, and $H^1 = H^1(\Omega)$ be the standard Hilbert spaces with inner products

$$(u,v)_{L^2} = \int_\Omega u(x)v(x)dx, \quad (u,v)_{H^1} = (Du,Dv)_{L^2} + (u,v)_{L^2}. \tag{1}$$

Consider a bilinear form $a(\cdot,\cdot) : H_0^1 \times H_0^1 \to \mathbb{R}$ that is continuous and coercive, i.e. there exist constants $c, C > 0$ such that

$$\begin{array}{rcl} a(u,v) & \leq & C||u||_{H^1}||v||_{H^1} \quad \text{for all } u,v \in H_0^1, \\ a(u,u) & \geq & c||u||_{H^1}^2 \quad \text{for all } u \in H_0^1. \end{array} \tag{2}$$

Further let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonlinear map such that the associated Nemytskii operator

$$[\tilde{f}(u)](x) = f(x,u(x)), \quad x \in \Omega$$

is defined for $u \in H_0^1$ and satisfies $\tilde{f}(u)v \in L^1$ for all $v \in H_0^1$. A function $u \in H_0^1$ satisfying

$$a(u,v) = \int_\Omega \tilde{f}(u)v\,dx \quad \text{for all } v \in H_0^1 \tag{3}$$

is then called a weak solution of the elliptic problem defined by a and f.

In the set-valued setting we consider a set-valued mapping $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$, where $\mathcal{CC}(\mathbb{R})$ denotes the set of all convex compact subsets of $\mathbb{R}$.

With F we associate a set-valued *Nemytskii operator* $\tilde{F} : \mathcal{M}(\Omega) \rightrightarrows \mathcal{M}(\Omega)$ defined by

$$\tilde{F}(u) := \{v \in \mathcal{M}(\Omega) : \ v(x) \in F(x, u(x)) \text{ a.e.}\}. \tag{4}$$

In Proposition 5 we will show that under suitable conditions on F, the operator $\tilde{F}$ maps elements of $L^p(\Omega)$ to closed, bounded, and convex subsets of its dual space $L^q(\Omega)$. The set-valued analog of (3) then is to find $u \in H_0^1$ such that

$$a(u, v) = \int_\Omega f v dx \quad \text{for all } v \in H_0^1 \quad \text{and some } f \in \tilde{F}(u). \tag{5}$$

For the classical case

$$a(u, v) = (Du, Dv)_{L^2}$$

equation (5) can be regarded as the weak formulation of the semilinear elliptic inclusion

$$-\Delta u \in \tilde{F}(u), \quad u = 0 \quad \text{on } \partial\Omega.$$

For the Galerkin approximation of (5) we consider a family of finite-dimensional subspaces $\mathcal{X}_N \subset H_0^1$ satisfying the approximation property

$$\operatorname{dist}(v, \mathcal{X}_N)_{H^1} = \inf\{||v - v_N||_{H^1} : v_N \in \mathcal{X}_N\} \to 0 \quad \text{as } N \to \infty \tag{6}$$

for all $v \in H_0^1$.

The *weak Galerkin inclusion* then consists in finding $u_N \in \mathcal{X}_N$ such that

$$a(u_N, v) = \int_\Omega f_N v dx \quad \text{for all } v \in \mathcal{X}_N \quad \text{and some } f_N \in \tilde{F}(u_N). \tag{7}$$

Let $S \subset H_0^1$ and $S_N \subset \mathcal{X}_N$ denote the set of weak solutions defined by (5) and (7), respectively. Our aim is to show that the solution sets are nonempty sets that are closed and bounded with respect to the norm $||\cdot||_{H^1}$. Moreover, we study the distances

$$\begin{aligned} \operatorname{dist}(S, S_N) &= \sup_{u \in S} \inf_{v \in S_N} ||u - v||, \quad \operatorname{dist}(S_N, S), \\ \operatorname{dist}_H(S, S_N) &= \max(\operatorname{dist}(S, S_N), \operatorname{dist}(S_N, S)) \end{aligned}$$

as $N \to \infty$ with respect to both norms $||\cdot||_{L^2}$ and $||\cdot||_{H^1}$.

For proving the existence of solutions we employ a global solvability theorem for set-valued inclusions [3] that holds for nonlinear mappings satisfying a *relaxed one-sided Lipschitz condition* with constant $l \in \mathbb{R}$ (see below), which we will impose on F. Under the assumption $l < c$, with c given by (2), we prove in Section 3 that S_N is nonempty and, moreover, that any sequence of solutions in S_N has a subsequence converging weakly in H_0^1 to a weak solution of (5). In the subsequent section we derive error estimates for solution sets in every L^p such that $H^1 \subset\subset L^p$ is compactly embedded, where approximation property (6) is the only assumption imposed on the Galerkin spaces $\mathcal{X}_N$.

In Section 4 we set up and analyze a numerical procedure for solving (7). The method builds on the idea of support functionals [10] and uses finite coverings of spheres in $\mathcal{X}_N$ with respect to the norm $||\cdot||_{H^1}$. The errors introduced by this spatial discretization will be estimated in detail. Moreover we use the path-connectedness of the semi-discretized solution set in order to develop an efficient search algorithm for the computation of the solution of the fully discretized problem.

2 Properties of the Nemytskii operator

As usual, we will denote $\mathrm{Proj}(x, Y) := \{y \in Y : |x - y| = \mathrm{dist}(x, Y)\}$ and $\|Y\| := \sup_{y \in Y} |y|$ for any $x \in \mathbb{R}^d$ and $Y \subset \mathbb{R}^d$. If $\mathbb{R}^d$ is equipped with the Euclidean metric and Y is convex, then $\mathrm{Proj}(x, Y)$ is a singleton.

Let $p \in (1, \infty)$ be such that $H_0^1 \subset L^p$ be continuously embedded, let $C(p)$ be the norm of the embedding $i : H_0^1 \to L^p$, and let q be the dual exponent given by $\frac{1}{p} + \frac{1}{q} = 1$. Recall (for example from [1, Th. 8.9]) that the embedding is continuous for arbitrary p if $d = 1, 2$ and for $p \leq 2d/(d-2)$ if $d \geq 3$ whenever the boundary of Ω is Lipschitz.

Throughout this text, the multivalued nonlinearity will be required to be Caratheodory and relaxed one-sided Lipschitz in the second variable with a uniform constant $l \in \mathbb{R}$.

Definition 1. *A multivalued mapping $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ is called Caratheodory, if for every $s \in \mathbb{R}$, $x \mapsto F(x, s)$ is measurable and for any $x \in \Omega$, $s \mapsto F(x, s)$ is continuous.*

The most important facts about Caratheodory multivalued mappings are displayed in [2]. We also refer to this book for precise definitions of measurability, continuity, and upper and lower semicontinuity of set-valued maps.

Definition 2. *A multivalued mapping $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ is called relaxed one-sided Lipschitz (ROSL) in the second variable with (a uniform) constant*

$l \in \mathbb{R}$, if for any $x \in \Omega$, $s, s' \in \mathbb{R}$, and $t \in F(x,s)$, there exists some $t' \in F(x,s')$ such that

$$(t - t')(s - s') \leq l(s - s')^2.$$

A detailed coverage of the ROSL property can be found in [7] and other work of the same author.

Given an arbitrary multivalued mapping $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$, define the functions $f_+, f_- : \Omega \times \mathbb{R} \to \mathbb{R}$ by $f_+(x,s) := \sup F(x,s)$ and $f_-(x,s) := \inf F(x,s)$.

Lemma 3. *A set-valued mapping $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ is Caratheodory and ROSL in the second variable if and only if f_+ and f_- are Caratheodory and one-sided Lipschitz (in the classical sense) in the second variable with the same Lipschitz constant.*

Proof. Measurability. Assume that for fixed $s \in \mathbb{R}$, the mapping $x \mapsto F(x,s)$ is measurable. By the Characterization Theorem [2, Theorem 8.1.4], there exists a sequence $(f_n)_n$ of measurable selections of $x \mapsto F(x,s)$ such that $F(x,s) = \overline{\cup_{n\in\mathbb{N}} f_n(x)}$ for all $x \in \Omega$. Hence, their pointwise supremum and infimum $f_+(x,s) = \sup_{n\in\mathbb{N}} f_n(x)$ and $f_-(x,s) = \inf_{n\in\mathbb{N}} f_n(x)$ are measurable functions.

Conversely, assume that for fixed $s \in \mathbb{R}$, the functions $x \mapsto f_+(x,s)$ and $x \mapsto f_-(x,s)$ are measurable. But then all functions f_λ given by $f_\lambda(x) := \lambda f_+(x,s) + (1-\lambda) f_-(x,s)$ with $\lambda \in [0,1] \cap \mathbb{Q}$ are measurable, and $F(x) = \overline{\cup_{\lambda\in[0,1]\cap\mathbb{Q}} f_\lambda(x)}$, so that $x \mapsto F(x,s)$ is measurable by the Characterization Theorem.

Continuity. Because of

$$\mathrm{dist}_H(F(x,s), F(x,s')) = \max\{|f_+(x,s) - f_+(x,s')|, |f_-(x,s) - f_-(x,s')|\},$$

the mapping $s \mapsto F(x,s)$ is continuous w.r.t. to the Hausdorff distance if and only if the functions $s \mapsto f_+(x,s)$ and $s \mapsto f_-(x,s)$ are continuous.

ROSL property. Let F be ROSL in the second argument with constant $l \in \mathbb{R}$, and assume that there exist some fixed $x \in \Omega$ and $s, s' \in \mathbb{R}$ with $s > s'$ and

$$(f_+(x,s) - f_+(x,s'))(s - s') > l|s - s'|^2.$$

As F is ROSL, there exists some $t' \in F(x,s')$ satisfying

$$(f_+(x,s) - t')(s - s') \leq l|s - s'|^2 < (f_+(x,s) - f_+(x,s'))(s - s'),$$

but then $f_+(x,s') < t'$, which is a contradiction. Similar computations for the case $s < s'$ and the function $s \mapsto f_-(x,s)$ show that f_+ and f_- are one-sided Lipschitz in the second argument with constant l.

Conversely, fix $x \in \Omega$ and let $s \mapsto f_+(x,s)$ and $s \mapsto f_-(x,s)$ be one-sided Lipschitz with constant l. Then for any $\lambda \in [0,1]$, the function $s \mapsto f_\lambda(x,s) := \lambda f_+(x,s) + (1-\lambda) f_-(x,s)$ is a selection of $s \mapsto F(x,s)$ which is one-sided Lipschitz with constant l. Let $s, s' \in \mathbb{R}$ and and $t \in F(x,s)$ be given. Then $t = f_\lambda(x,s)$ for some $\lambda \in [0,1]$, and hence $t' := f_\lambda(x,s')$ is an element of $F(x,s')$ satisfying

$$(t-t')(s-s') = (f_\lambda(x,s) - f_\lambda(x,s'))(s-s') \leq l|s-s'|^2,$$

so that F is ROSL.

□

Lemma 4. *If $F, G : \Omega \to \mathcal{CC}(\mathbb{R})$ are measurable, then the function*

$$x \mapsto \operatorname{dist}(F(x), G(x))$$

is measurable.

Proof. By the Characterization Theorem [2, Theorem 8.1.4], there exists a sequence $(f_n)_n$ of measurable selections of F such that $F(x) = \overline{\cup_{n\in\mathbb{N}} f_n(x)}$. Hence

$$x \mapsto \operatorname{dist}(F(x), G(x)) = \sup_{t \in F(x)} \operatorname{dist}(t, G(x)) = \sup_{n \in \mathbb{N}} \operatorname{dist}(f_n(x), G(x))$$

is a countable supremum of measurable functions (see [2, Corollary 8.2.13]) and measurable as such. □

The following proposition shows that the notion of a Nemytskii operator is still meaningful in the set-valued context.

Proposition 5. *Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory set-valued mapping which satisfies the growth bound*

$$\|F(x,s)\| \leq \alpha(x) + \beta |s|^{p-1}, \quad x \in \Omega,\ s \in \mathbb{R} \tag{8}$$

for some nonnegative $\alpha \in L^q(\Omega)$ and $\beta \geq 0$. If $\Omega \subset \mathbb{R}^d$ is an open and bounded domain, then the set-valued Nemytskii operator (4) is well-defined as an operator $\tilde{F} : L^p(\Omega) \rightrightarrows L^q(\Omega)$, has closed, convex, and bounded values, and maps bounded sets to bounded sets. Moreover, $\tilde{F}$ is continuous w.r.t. the Hausdorff metric on the closed and bounded subsets of $L^q(\Omega)$.

Remark 6. *i) In view of [9, Theorem 3.4.4], growth estimate* (8) *is a necessary condition for $\tilde{F}$ to map $L^p(\Omega)$ into $L^q(\Omega)$ in the single-valued, and hence also in the set-valued case.*

ii) The single-valued Nemytskii operators $\tilde{f}_+, f_- : L^p \to L^q$ given by $[\tilde{f}_+(u)](x) := f_+(x, u(x))$ and $[\tilde{f}_-(u)](x) := f_-(x, u(x))$ are well-defined and continuous.

Proof. General properties. For any $u \in L^p(\Omega)$, the mapping $x \mapsto F(x, u(x))$ is measurable according to [2, Theorem 8.2.8] and hence admits a measurable selection $v \in \tilde{F}(u)$, which proves that the values of $\tilde{F}$ are nonempty.

If $u \in L^p(\Omega)$ and $v \in \tilde{F}(u)$, then $v(x) \in F(x, u(x))$ a.e. implies

$$|v(x)| \leq \alpha(x) + \beta|u(x)|^{p-1} = \alpha(x) + \beta|u(x)|^{\frac{p}{q}} \text{ a.e.}$$

so that

$$\|v\|_{L^q} \leq \|\alpha + \beta|u|^{\frac{p}{q}}\|_{L^q} \leq \|\alpha\|_{L^q} + \beta\|u^{\frac{p}{q}}\|_{L^q} \leq \|\alpha\|_{L^q} + \beta\|u\|_{L^p}^{\frac{p}{q}}.$$

Consequently, $\tilde{F} : L^p(\Omega) \rightrightarrows L^q(\Omega)$ is well-defined, has bounded images, and maps bounded sets to bounded sets. Moreover, its images are convex, because for any $v_1, v_2 \in \tilde{F}(u)$ and $\lambda \in [0, 1]$, the inclusion

$$[\lambda v_1 + (1 - \lambda)v_2](x) \in \lambda F(x, u(x)) + (1 - \lambda)F(x, u(x)) = F(x, u(x))$$

holds a.e.

For every sequence $(v_n)_n \subset \tilde{F}(u)$ and $v \in L^q(\Omega)$ with $v_n \to v$ in $L^q(\Omega)$, we have $v_n(x) \to v(x)$ a.e. along a subsequence. Since F has closed values, $v(x) \in F(x, u(x))$ a.e. and $\tilde{F}(u)$ is closed.

Continuity properties. Let F be Caratheodory, and let $(u_n)_n \subset L^p(\Omega)$ and $u \in L^p(\Omega)$ such that $u_n \to u$ in $L^p(\Omega)$. If the statement

$$\operatorname{dist}(\tilde{F}(u_n), \tilde{F}(u))_{L^q} \to 0 \text{ as } n \to \infty \tag{9}$$

is false, then there exist $\varepsilon > 0$ and a subsequence $(u_n)_{n \in \mathbb{N}'}$ such that

$$\operatorname{dist}(\tilde{F}(u_n), \tilde{F}(u))_{L^q} > \varepsilon \text{ for all } n \in \mathbb{N}'. \tag{10}$$

There exists a further subsequence $(u_n)_{n \in \mathbb{N}''}$ satisfying $u_n(x) \to u(x)$ a.e. By Lemma 4 and [2, Theorem 8.2.8], the functions

$$\varphi_n(x) := \operatorname{dist}(F(x, u_n(x)), F(x, u(x)))$$

are measurable. Growth condition (8) and [16, (30b), Appendix] ensure that there exists a constant $C(q,3)$ depending only on q such that

$$\begin{aligned}\varphi_n(x)^q &\le \big(\|F(x,u_n(x))\| + \|F(x,u(x))\|\big)^q \\ &\le \big(2\alpha(x) + \beta(|u_n(x)|^{\frac{p}{q}} + |u(x)|^{\frac{p}{q}})\big)^q \\ &\le C(q,3)\big(2^q\alpha(x)^q + \beta^q|u_n(x)|^p + \beta^q|u(x)|^p\big)\end{aligned}$$

and $2^q\alpha(x)^q+\beta^q|u_n(x)|^p+\beta^q|u(x)|^p \to 2^q\alpha(x)^q+2\beta^q|u(x)|^p$ in L^1. Continuity of F in the second argument implies that $\operatorname{dist}(F(x,u_n(x)),F(x,u(x)))^q \to 0$ almost everywhere, and hence

$$\int_\Omega \operatorname{dist}(F(x,u_n(x)),F(x,u(x)))^q dx \to 0 \text{ as } n\to\infty \tag{11}$$

by the Generalized Majorized Convergence Theorem (Theorem 19a in Appendix 2 of [16]).

Now let $f_n \in \tilde{F}(u_n)$, and set $g_n(x) := \operatorname{Proj}(f_n(x), F(x,u(x)))$. By [2, Corollary 8.2.13], the functions $g_n : \Omega \to \mathbb{R}$ are measurable, and growth condition (8) guarantees that $g_n \in L^q(\Omega)$. By construction and because of (11),

$$\|f_n-g_n\|_{L^q}^q = \int_\Omega |f_n(x)-g_n(x)|^q dx \le \int_\Omega \operatorname{dist}(F(x,u_n(x)),F(x,u(x)))^q dx \to 0,$$

so that

$$\operatorname{dist}(\tilde{F}(u_n),\tilde{F}(u))_{L^q} \to 0 \text{ as } \mathbb{N}'' \ni n \to \infty,$$

because f_n were arbitrary. This contradicts (10), and thus (9) holds. As a consequence, the mapping $\tilde{F}$ is upper semicontinuous.

Lower semicontinuity of $\tilde{F}$ can be shown analogously. □

Proposition 7. *Let $F : \Omega\times\mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with constant $l \in \mathbb{R}$ and satisfies growth condition (8). Let $H_0^1 \subset L^p$ be continuously embedded, and let q be the dual exponent. Then the Nemytskii operator $\tilde{F} : H_0^1(\Omega) \rightrightarrows L^q(\Omega)$ is ROSL in the sense that for all $u,u' \in H_0^1$ and $v \in \tilde{F}(u)$, there exists $v' \in \tilde{F}(u')$ such that*

$$\int_\Omega (v-v')(u-u')dx \le l\|u-u'\|_{L^2}^2 \le l^+\|u-u'\|_{H^1}^2,$$

where $l^+ := \max\{0,l\}$.

Proof. Let $u, u' \in H_0^1(\Omega)$ and $v \in \tilde{F}(u)$ be given. As $v(x) \in F(x, u(x))$ a.e., the ROSL property of F implies that the values of the mapping $H : \Omega \rightrightarrows \mathbb{R}$ defined by

$$H(x) := \{t \in F(x, u'(x)) : (t - v(x))(u'(x) - u(x)) \leq l(u'(x) - u(x))^2\}$$

are nonempty for every $x \in \Omega$. The function

$$g(x, t) := (t - v(x))(u'(x) - u(x))$$

is Caratheodory and the values of the mapping

$$G(x) := (-\infty, l(u'(x) - u(x))^2]$$

are closed, so that the intersections

$$H(x) = F(x, u'(x)) \cap g(x, \cdot)^{-1}(G(x))$$

are closed as well. By the Inverse Intersection Lemma [2, Theorem 8.2.9], the mapping H is measurable, and consequently, it admits a measurable selection $v'(\cdot)$. By construction, $v'(x) \in F(x, u'(x))$ a.e., and Proposition 5 ensures that $v' \in L^q(\Omega)$, so that $v' \in \tilde{F}(u')$. As

$$\int_\Omega (v - v')(u - u')dx \leq l \int_\Omega (u' - u)^2 dx = l\|u' - u\|_{L^2}^2 \leq l^+\|u' - u\|_{H^1}^2,$$

$\tilde{F}$ is ROSL with constant l^+. □

Proposition 8. *Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a set-valued mapping which is measurable in the first and L-Lipschitz in the second argument and satisfies growth condition (8). Then the set-valued Nemytskii operator $\tilde{F} : L^p(\Omega) \rightrightarrows L^q(\Omega)$ is well-defined and Lipschitz continuous with Lipschitz constant $L|\Omega|^{1-\frac{2}{p}}$, has closed and convex values, and maps bounded sets to bounded sets.*

Proof. Let $u, u' \in L^p(\Omega)$ and $f \in \tilde{F}(u)$ be given, and define a function $f' : \Omega \to \mathbb{R}$ by $f'(x) := \mathrm{Proj}(f(x), F(x, u'(x)))$. By Corollary 8.2.13 in [2], the function f' is measurable. Growth condition (8) and Lipschitz continuity (which strengthens bound (8)) ensure that $f, f' \in \tilde{F}(u') \cap L^p(\Omega)$. Since

$$\begin{aligned}\|f - f'\|_{L^p}^p &= \int_\Omega |f(x) - f'(x)|^p dx = \int_\Omega \mathrm{dist}(F(x, u(x)), F(x, u'(x)))^p dx \\ &\leq \int_\Omega L^p|u(x) - u'(x)|^p dx = L^p\|u - u'\|_{L^p}^p,\end{aligned}$$

and the interpolation inequalities (see [8, page 623]) imply $\|f-f'\|_{L^q} \leq |\Omega|^{1-\frac{2}{p}}\|f-f'\|_{L^p}$, the Nemytskii operator satisfies

$$\mathrm{dist}(\tilde{F}(u),\tilde{F}(u'))_{L^q} \leq L|\Omega|^{1-\frac{2}{p}}\|u-u'\|_{L^p}.$$

The remaining statements follow from Proposition 5. □

In particular,

$$\mathrm{dist}(\tilde{F}(u),\tilde{F}(u'))_{L^2} \leq L\|u-u'\|_{H^1}$$

for $u, u' \in H_0^1$ and $p = q = 2$.

The following proposition shows that the set-valued Nemytskii operator can be parametrized by a family of single-valued operators with favorable properties. This is the key ingredient for proving connectedness of the solution set of the differential inclusion (see Theorem 14). We can only prove such a result for mappings $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ with $d = 1$, because it is currently unclear under which conditions ROSL multimaps admit parametrizations by OSL single-valued mappings.

Proposition 9. *Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with constant $l \in \mathbb{R}$ and satisfies growth condition (8). Let $H_0^1 \subset L^p$ be continuously embedded, and let q be the dual exponent. Moreover, let the family $\Lambda := \mathcal{M}(\Omega,[0,1])$ of measurable functions from Ω to $[0,1]$ be equipped with the L^∞ norm. Then the nonlinear operator $\tilde{f} : L^p \times \Lambda \to L^q$ given by*

$$\tilde{f}_\lambda(u)(x) := \lambda(x)f_+(x,u(x)) + (1-\lambda(x))f_-(x,u(x)) \tag{12}$$

satisfies

i) $\tilde{F}(u) = \cup_{\lambda\in\Lambda}\tilde{f}_\lambda(u)$ *for all* $u \in H_0^1$*;*

ii) $u \mapsto \tilde{f}_\lambda(u)$ *is continuous and OSL with constant* l^+ *for all* $\lambda \in \Lambda$*;*

iii) $\lambda \mapsto \tilde{f}_\lambda(u)$ *is continuous for every* $u \in H_0^1$*.*

Proof. i) The inclusion $\cup_{\lambda\in\Lambda}\tilde{f}_\lambda(u) \subset \tilde{F}(u)$ for all $u \in H_0^1$ follows directly from (12). Conversely, let $u \in H_0^1$ and $f \in \tilde{F}(u)$ be given. Then

$$\begin{aligned} f(x) \in F(x,u(x)) &= [f_-(x,u(x)), f_+(x,u(x))] \\ &= \{\mu f_+(x,u(x)) + (1-\mu)f_-(x,u(x)) : \mu \in [0,1]\}, \end{aligned}$$

so that the Filippov Theorem [2, Theorem 8.2.10] ensures the existence of a measurable function $\lambda : \Omega \to [0,1]$ satisfying $f = \tilde{f}_\lambda(u)$.

ii) Let $u, u' \in H_0^1$ and $\lambda \in \Lambda$ be given. Then

$$\begin{aligned}
&\|\tilde{f}_\lambda(u) - \tilde{f}_\lambda(u')\|_{L^q} \\
&= \|\big(\lambda\tilde{f}_+(u) + (1-\lambda)\tilde{f}_-(u)\big) - \big(\lambda\tilde{f}_+(u') + (1-\lambda)\tilde{f}_-(u')\big)\|_{L^q} \\
&\le \|\lambda\big(\tilde{f}_+(u) - \tilde{f}_+(u')\big)\|_{L^q} + \|(1-\lambda)\big(\tilde{f}_-(u) - \tilde{f}_-(u')\big)\|_{L^q} \\
&\le \|\tilde{f}_+(u) - \tilde{f}_+(u')\|_{L^q} + \|\tilde{f}_-(u) - \tilde{f}_-(u')\|_{L^q}.
\end{aligned}$$

By Remark 6ii), $\tilde{f}_+$ and $\tilde{f}_-$ are continuous from L^p to L^q and hence in particular from H_0^1 to L^q, so that

$$\|\tilde{f}_\lambda(u) - \tilde{f}_\lambda(u')\|_{L^q} \to 0 \text{ as } \|u - u'\|_{L^p} \to 0$$

and $u \mapsto \tilde{f}_\lambda(u)$ is continuous. The OSL property follows from

$$\begin{aligned}
&\int_\Omega \big(\tilde{f}_\lambda(u) - \tilde{f}_\lambda(u')\big)(u - u')dx \\
= \;& \int_\Omega \{\big(\lambda(x)f_+(x,u(x)) + (1-\lambda(x))f_-(x,u(x))\big) \\
& -\big(\lambda(x)f_+(x,u'(x)) + (1-\lambda(x))f_-(x,u'(x))\big)\} \cdot (u(x) - u'(x))dx \\
= \;& \int_\Omega \lambda(x)\big(f_+(x,u(x)) - f_+(x,u'(x))\big)(u(x) - u'(x))dx \\
& + \int_\Omega (1-\lambda(x))\big(f_+(x,u(x)) - f_+(x,u'(x))\big)(u(x) - u'(x))dx \\
\le \;& \int_\Omega \lambda(x) l |u(x) - u'(x)|^2 dx + \int_\Omega (1-\lambda(x)) l |u(x) - u'(x)|^2 dx \\
= \;& l\|u - u'\|_{L^2}^2 \le l^+\|u - u'\|_{H_0^1}^2.
\end{aligned}$$

iii) Let $u \in H_0^1$ and $\lambda, \lambda' \in \Lambda$. Then

$$\begin{aligned}
&\|\tilde{f}_\lambda(u) - \tilde{f}_{\lambda'}(u)\|_{L^q} \\
&= \|\big(\lambda\tilde{f}_+(u) + (1-\lambda)\tilde{f}_-(u)\big) - \big(\lambda'\tilde{f}_+(u) + (1-\lambda')\tilde{f}_-(u)\big)\|_{L^q} \\
&\le \|\lambda - \lambda'\|_{L^\infty}\big(\|\tilde{f}_+(u)\|_{L^q} + \|\tilde{f}_-(u)\|_{L^q}\big) \to 0 \text{ as } \|\lambda - \lambda'\|_{L^\infty} \to 0.
\end{aligned}$$

□

3 Existence and estimates of solutions

It is useful to reformulate differential inclusion (5) and Galerkin inclusion (7) in operator form. Assume that the boundary of Ω is Lipschitz. We consider $\mathcal{X}_N$ as a Hilbert space endowed with the norm $||\cdot||_{H^1}$ and denote its dual by $\mathcal{X}_N^*$. The Riesz isomorphism $Q_N : \mathcal{X}_N^* \to \mathcal{X}_N$, $N \in \mathbb{N}$ associates with any $\varphi \in \mathcal{X}_N^*$ the unique element $Q_N\varphi \in \mathcal{X}_N$ satisfying

$$\langle \varphi, v\rangle = (Q_N\varphi, v)_{H^1} \text{ for all } v \in \mathcal{X}_N,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. The Ritz projector $R_N : H_0^1 \to \mathcal{X}_N$ is defined by

$$a(R_N u, v) = a(u, v) \text{ for all } v \in \mathcal{X}_N. \tag{13}$$

Let $H_0^1 \subset L^p$ be continuously embedded for some $p > 1$, let $C(p)$ be the norm of the embedding $i : H_0^1 \to L^p$, and let q be the dual exponent. (According to Theorem 8.9 in [1], the embedding is continuous for arbitrary p if $d = 1, 2$ and for $p \le 2d/(d-2)$ if $d \ge 3$, and it is compact for arbitrary p if $d = 1, 2$ and for $p < 2d/(d-2)$ if $d \ge 3$.)

Consider the embedding operators $J : L^q \to H^{-1} = (H_0^1)^*$ and $J_N : L^q \to \mathcal{X}_N^*$ given by $\langle Jf, u\rangle = \int_\Omega uf\,dx$ and $\langle J_N f, u_N\rangle = \int_\Omega u_N f\,dx$ for any $f \in L^q$, $u \in H_0^1$, and $u_N \in \mathcal{X}_N$. Their norms are bounded by $\|J\|, \|J_N\| \le C(p)$. Define $A : H_0^1 \to H^{-1}$ and $A_N : \mathcal{X}_N \to \mathcal{X}_N^*$ by the identities

$$\langle Au, v\rangle = a(u, v) \text{ for all } v \in H_0^1,$$
$$\langle A_N u, v\rangle = a(u, v) \text{ for all } v \in \mathcal{X}_N,$$

and define the nonlinear operators $\mathcal{F} : H_0^1 \rightrightarrows H^{-1}$ and $\mathcal{F}_N : \mathcal{X}_N \to \mathcal{CC}(\mathcal{X}_N^*)$ by $\mathcal{F}(u) = J\tilde{F}(u)$ and $\mathcal{F}_N(u) = J_N\tilde{F}(u)$. Then differential inclusion (5) and Galerkin inclusion (7) can be rewritten as

$$Au \in \mathcal{F}(u), \quad u \in H_0^1 \tag{14}$$
$$A_N u_N \in \mathcal{F}_N(u_N), \quad u_N \in \mathcal{X}_N. \tag{15}$$

The following proposition shows that compactness of the embedding $H_0^1 \subset\subset L^p$ and approximation property (6) imply uniform convergence of the operators R_N on bounded sets. No explicit assumptions on Ω and $\partial\Omega$ are necessary. Textbooks on finite element theory such as [5, Theorem II.7.2], [11, Theorem 5.5], and [15, Lemma 1.1] only treat pointwise convergence, but specify a rate of convergence, which seems to be impossible here.

Proposition 10. *Let $p > 1$ be such that the embedding $H_0^1 \subset\subset L^p$ is compact and let* (6) *hold. Then the Ritz projector satisfies*

$$\sup_{\|u\|_{H^1} \leq 1} \|u - R_N u\|_{L^p} \to 0 \text{ as } N \to \infty. \tag{16}$$

Proof. First note that by the classical Céa Lemma for every $u \in H_0^1$,

$$\|u - R_N u\|_{H^1} \leq \frac{C}{c} \inf\{||u - w||_{H^1} : w \in \mathcal{X}_N\}, \tag{17}$$

and hence by (6) and the continuous embedding

$$||u - R_N u||_{L^p} \leq C(p)\|u - R_N u\|_{H^1} \to 0 \quad \text{as} \quad N \to \infty. \tag{18}$$

If the assertion is false there exists an $\varepsilon > 0$, a subsequence $\mathbb{N}' \subset \mathbb{N}$, and a sequence $u_N \in H_0^1, N \in \mathbb{N}'$ such that

$$||u_N||_{H^1} \leq 1, \quad ||u_N - R_N u_N||_{L^p} \geq \varepsilon, \quad N \in \mathbb{N}'. \tag{19}$$

Then we find a $u \in H_0^1$ and a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that

$$u_N \rightharpoonup u \quad \text{in} \quad H_0^1, \quad \mathbb{N}'' \ni N \to \infty, \tag{20}$$

and by the compact embedding,

$$u_N \to u \quad \text{in} \quad L^p, \quad \mathbb{N}'' \ni N \to \infty. \tag{21}$$

From the triangle inequality we obtain

$$\|u_N - R_N u_N||_{L^p} \leq \|u_N - u\|_{L^p} + \|u - R_N u\|_{L^p} + \|R_N(u - u_N)\|_{L^p}, \tag{22}$$

where the first two terms converge to 0 as $\mathbb{N}'' \ni N \to \infty$ by (18) and (21). We are going to show that the last term also converges to 0 for a suitable subsequence. This contradicts (19) and finishes the proof.

Let $v_N = u - u_N, N \in \mathbb{N}''$ and note that by (19) and (20),

$$\|v_N\|_{H^1} \leq 2 \quad \text{and} \quad v_N \rightharpoonup 0 \text{ in } H_0^1 \text{ as } \mathbb{N}'' \ni N \to \infty. \tag{23}$$

From the uniform bound on R_N in the H^1-norm we infer

$$\|R_N v_N\|_{H^1} \leq \frac{C}{c}\|v_N\|_{H^1} \leq 2\frac{C}{c}.$$

Hence there exists a subsequence $\mathbb{N}''' \subset \mathbb{N}''$ and some $v \in H_0^1$ such that

$$R_N v_N \rightharpoonup v \text{ in } H_0^1 \quad \text{and} \quad R_N v_N \to v \text{ in } L^p \text{ as } \mathbb{N}''' \ni N \to \infty. \tag{24}$$

Now, consider the term

$$\begin{aligned} a(R_N v_N, v) =& a(R_N v_N, R_N v) + a(R_N v_N, v - R_N v) \\ =& a(v_N, R_N v) + a(R_N v_N, v - R_N v) \\ =& a(v_N, v) + a(R_N v_N - v_N, v - R_N v) \\ \leq& a(v_N, v) + 2C\left(\frac{C}{c} + 1\right) \|v - R_N v\|_{H^1}. \end{aligned} \tag{25}$$

Using (18), (23), (24) and the weak continuity of $a(\cdot, v)$ we find that the right-hand side of (25) converges to 0 as $\mathbb{N}''' \ni N \to \infty$ while the left-hand side converges to $a(v, v)$. Therefore, $v = 0$ and (24) shows $\|R_N v_N\|_{L^p} \to 0$ as $\mathbb{N}''' \ni N \to \infty$. $\square$

Lemma 11. *Let $H_0^1 \subset L^p$ be continuously embedded for some $p \geq 2$. Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with constant $l \in \mathbb{R}$ and satisfies growth condition* (8)*. Then the multifunction $Q_N \mathcal{F}_N : \mathcal{X}_N \to \mathcal{CC}(\mathcal{X}_N)$ is continuous and ROSL with the constant $l^+ := \max\{0, l\}$.*

Proof. The images of $Q_N \mathcal{F}_N$ are bounded, because Q_N and J_N are bounded operators and the images of $\tilde{F}$ are bounded according to Proposition 5.

Let $(v_k)_k \subset Q_N \mathcal{F}_N(u)$ and $v \in \mathcal{X}_N$ be such that $v_k \to v$ in $\mathcal{X}_N$. There exists a sequence $(f_k)_k \in \tilde{F}(u)$ with $Q_N J_N f_k = v_k$. As Q_N is an isometric isomorphism, there exists a unique $\varphi \in \mathcal{X}_N^*$ such that $Q_N \varphi = v$, and thus $J_N f_k \to \varphi$ in $\mathcal{X}_N^*$. Since $\tilde{F}(u)$ is closed, bounded, and convex (see Proposition 5), there exists a subsequence $(f_k)_{k \in \mathbb{N}'}$ such that $f_k \rightharpoonup f$ in L^q and $f \in \tilde{F}(u)$ by Mazur's Lemma [13, Theorem I.3.12]. Now

$$\langle \varphi, w \rangle \leftarrow \langle J_N f_k, w \rangle = \int_\Omega w f_k \, dx \to \int_\Omega w f \, dx = \langle J_N f, w \rangle$$

for all $w \in \mathcal{X}_N$ shows that $J_N f = \varphi$, and hence $v = Q_N J_N f \in Q_N \mathcal{F}_N(u)$, so that $\mathcal{F}_N(u)$ is closed.

Continuity of $Q_N \mathcal{F}_N$ follows from

$$\begin{aligned} \operatorname{dist}(Q_N \mathcal{F}_N(u), Q_N \mathcal{F}_N(u'))_{H^1} &= \operatorname{dist}(Q_N J_N \tilde{F}(u), Q_N J_N \tilde{F}(u'))_{H^1} \\ &\leq C(p) \operatorname{dist}(\tilde{F}(u), \tilde{F}(u'))_{L^2} \end{aligned}$$

and continuity of $\tilde{F}$ (see Proposition 5).

If $u, u' \in \mathcal{X}_N$ and $v \in Q_N \mathcal{F}_N(u)$ are given, there exists some $f \in \tilde{F}(u)$ such that $v = Q_N J_N f$. As $\tilde{F}$ is ROSL (see Proposition 7), there exists some $f' \in \tilde{F}(u')$ with

$$\int_\Omega (f - f')(u - u')\, dx \leq l^+ \|u - u'\|_{H^1}^2,$$

and hence the element $v' := Q_N J_N f' \in Q_N \mathcal{F}_N(u')$ satisfies

$$\begin{aligned}(v - v', u - u')_{H^1} &= (Q_N J_N (f - f'), u - u')_{H^1} \\ &= \int_\Omega (f - f')(u - u')\, dx \leq l^+ \|u - u'\|_{H^1}^2\end{aligned}$$

so that $Q_N \mathcal{F}_N$ is ROSL with constant l^+. $\square$

3.1 Existence of solutions and properties of the solution sets

Theorem 12. *Let $H_0^1 \subset\subset L^p$ be compactly embedded for some $p \geq 2$. Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with constant $l \in \mathbb{R}$ such that $l < c$ with c from (2) and which satisfies growth condition (8). Then S_N is nonempty and compact for every $N \in \mathbb{N}$. Moreover, we have the uniform bound*

$$\|S_N\|_{H^1} \leq \kappa^{-1} C(p) \|\alpha\|_{L^q}, \quad \kappa := c - l^+. \tag{26}$$

If, in addition, approximation property (6) holds then any sequence $(u_N)_N$, $u_N \in S_N$, of approximate solutions contains a subsequence converging weakly in $H^1(\Omega)$ to a solution $u \in H_0^1(\Omega)$ of the original inclusion. Moreover, bound (26) holds for the solution set S.

Proof. The set-valued operator $G_N : \mathcal{X}_N \to \mathcal{CC}(\mathcal{X}_N)$ given by

$$G_N(u) = Q_N(\mathcal{F}_N(u) - A_N u) \tag{27}$$

is continuous, because

$$\begin{aligned}\operatorname{dist}(G_N(u), G_N(u'))_{H^1} &= \operatorname{dist}(Q_N[\mathcal{F}_N(u) - A_N u], Q_N[\mathcal{F}_N(u') - A_N u'])_{H^1} \\ &\leq \|Q_N A_N (u - u')\|_{H^1} + \operatorname{dist}(Q_N \mathcal{F}_N(u), Q_N \mathcal{F}_N(u'))_{H^1}\end{aligned}$$

and both $Q_N A_N$ and $Q_N \mathcal{F}_N$ are continuous (see Lemma 11). Because of

$$(-Q_N A_N u + Q_N A_N u', u - u')_{H^1} = -a(u - u', u - u') \leq -c\|u - u'\|^2_{H^1},$$

the operator $-Q_N A_N : \mathcal{X}_N \to \mathcal{X}_N$ is ROSL with constant $-c$. By Lemma 11, the mapping $Q_N \mathcal{F}_N$ is ROSL with constant l^+, and hence the sum $G_N = Q_N \mathcal{F}_N - Q_N A_N$ is ROSL with constant $-\kappa < 0$. Consequently, Theorem 27 guarantees the existence of a solution $u_N \in \mathcal{X}_N$ of inclusion (15) and compactness of S_N.

We derive a bound on the H^1-norm of the solutions that is independent of N. If $u_N \in S_N$, then $Q_N A_N u_N \in Q_N F_N(u_N)$. As $Q_N \mathcal{F}_N$ is ROSL, there exists some $v_N \in Q_N \mathcal{F}_N(0)$ such that

$$(Q_N A_N u_N - v_N, u_N)_{H^1} \leq l^+ \|u_N\|^2_{H^1},$$

and hence

$$\begin{aligned} c\|u_N\|^2_{H^1} &\leq a(u_N, u_N) = \langle A_N u_N, u_N \rangle = (Q_N A_N u_N - v_N, u_N)_{H^1} + (v_N, u_N)_{H^1} \\ &\leq l^+ \|u_N\|^2_{H^1} + \|Q_N \mathcal{F}_N(0)\|_{H^1} \|u_N\|_{H^1}, \end{aligned}$$

so that

$$\|u_N\|_{H^1} \leq \kappa^{-1} \|Q_N \mathcal{F}_N(0)\|_{H^1} \leq \kappa^{-1} C(p) \|\alpha\|_{L^q}.$$

Now consider a sequence $u_N \in \mathcal{X}_N$, $N \in \mathbb{N}$ of solutions for (15), i.e. there exist $f_N \in \tilde{F}(u_N)$ such that

$$a(u_N, v) = \int_\Omega f_N v \, dx \quad \text{for all } v \in \mathcal{X}_N. \tag{28}$$

By (26) the sequence $(u_N)_{N \in \mathbb{N}}$ is bounded in H^1_0 and hence there exist a subsequence $(u_N)_{N \in \mathbb{N}'}$ and a function $u \in H^1_0$ such that $u_N \rightharpoonup u$ in H^1_0 and

$$u_N \to u \text{ in } L^p \text{ as } N \to \infty. \tag{29}$$

By Proposition 5 and (26) the sets $\tilde{F}(u_N)$ are uniformly bounded in L^q and hence we find a subsequence $(f_N)_{N \in \mathbb{N}''}$ and an $f \in L^q$ such that $f_N \rightharpoonup f$ in L^q as $N \to \infty$. Proposition 5 ensures that

$$\operatorname{dist}(f_N, \tilde{F}(u))_{L^q} \leq \operatorname{dist}(\tilde{F}(u_N), \tilde{F}(u))_{L^q} \to 0.$$

Thus for every $\delta > 0$, there exists some $N_\delta \in \mathbb{N}$ such that

$$f_N \in B_\delta(\tilde{F}(u)) \text{ for all } N \geq N_\delta.$$

As $B_\delta(\tilde{F}(u))$ is closed and convex, Mazur's Lemma [13, Theorem I.3.12] implies that $f \in B_\delta(\tilde{F}(u))$. Since δ was arbitrary, $f \in \tilde{F}(u)$ holds.

For an arbitrary $v \in H_0^1$ we have by (6) a sequence $v_N \in \mathcal{X}_N$ such that $||v - v_N||_{H^1} \to 0$ as $N \to \infty$. Then using (28) with $v = v_N$ we obtain

$$\begin{aligned}
-a(u,v) + \int_\Omega fv\,dx &= a(u_N - u, v) - a(u_N, v - v_N) - a(u_N, v_N) + \int_\Omega fv\,dx \\
&= a(u_N - u, v) - a(u_N, v - v_N) - \int_\Omega f_N(v_N - v)\,dx \\
&\quad + \int_\Omega (f - f_N)v\,dx.
\end{aligned}$$

The first term converges to 0 because $a(\cdot, v) : H_0^1 \mapsto \mathbb{R}$ is (weakly) continuous. For the second term we use (2) and the boundedness of $||u_N||_{H^1}$ while the last two terms converge to 0 by the boundedness and the weak convergence of f_N. Therefore, u solves the inclusion (5) with $f \in \tilde{F}(u)$.

The estimate $||S|| \leq \kappa^{-1}C(p)\|\alpha\|_{L^q}$ proceeds along the same lines as in the discrete case. □

Remark 13. *In view of the example in [4, Example 10] one cannot expect the solution sets S_N and S to be convex.*

The following theorem shows that under standard assumptions, the solution sets S_N and S are path-connected. This property allows the use of very efficient search algorithms for the computation of the fully discretized solution sets $S_N^\rho(\delta)$ and $\hat{S}_N^\rho(\delta')$ (see Lemma 26 and the subsequent algorithm).

Theorem 14. *Let $H_0^1 \subset L^p$ be continuously embedded for some $p \geq 2$. Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with constant $l \in \mathbb{R}$ such that $l < c$ and which satisfies growth condition* (8)*. Then the sets $S_N \subset H_0^1$ and $S \subset H_0^1$ are path connected.*

Proof. For $\lambda \in \Lambda = \mathcal{M}(\Omega, [1, 0])$, define

$$f_\lambda(x, s) := \lambda(x)f_+(x, s) + (1 - \lambda(x))f_-(x, s).$$

Clearly, $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ is Caratheodory. By Lemma 3,

$$\begin{aligned}
& (f_\lambda(x,s) - f_\lambda(x,s'))(s-s') \\
= \; & [\lambda(x) f_+(x,s) + (1-\lambda(x)) f_-(x,s)](s-s') \\
& -[\lambda(x) f_+(x,s') + (1-\lambda(x)) f_-(x,s')](s-s') \\
= \; & \lambda(x)[f_+(x,s) - f_+(x,s')](s-s') \\
& +(1-\lambda(x))[f_-(x,s) - f_-(x,s')](s-s') \\
\leq \; & (\lambda(x) + (1-\lambda(x))) l |s-s'|^2 = l|s-s'|^2,
\end{aligned}$$

so that f_λ is OSL with constant l. Theorem 12 applied to the right-hand side $(x,s) \mapsto \{f_\lambda(x,s)\}$ guarantees the existence of a solution $u_{N,\lambda} \in H_0^1$ of

$$a(u_{N,\lambda}, v) = \int_\Omega f_\lambda(x, u_{N,\lambda}(x)) v(x) dx \text{ for all } v \in \mathcal{X}_N. \tag{30}$$

For any two solutions $u_{N,\lambda}$ and $u'_{N,\lambda}$ of (30), inequality

$$\begin{aligned}
c\|u_{N,\lambda} - u'_{N,\lambda}\|_{H^1}^2 &\leq a(u_{N,\lambda} - u'_{N,\lambda}, u_{N,\lambda} - u'_{N,\lambda}) \\
&= \int_\Omega \big(f_\lambda(x, u_{N,\lambda}(x)) - f_\lambda(x, u'_{N,\lambda}(x))\big)(u_{N,\lambda} - u'_{N,\lambda}) dx \\
&\leq l\|u_{N,\lambda} - u'_{N,\lambda}\|_{L^2}^2 \leq l^+\|u_{N,\lambda} - u'_{N,\lambda}\|_{H^1}^2
\end{aligned}$$

implies $\|u_{N,\lambda} - u'_{N,\lambda}\|_{H^1} = 0$. Consequently, problem (30) possesses a unique solution $u_{N,\lambda}$ for any $\lambda \in \Lambda$, and the operator $\psi : \Lambda \to H_0^1$ that maps $\lambda \in \Lambda$ to the unique solution of (30) is well-defined.

Let $u_{N,\lambda} = \psi(\lambda)$ and $u_{N,\lambda'} = \psi(\lambda')$ be given. Then

$$\begin{aligned}
& c\|u_{N,\lambda} - u_{N,\lambda'}\|_{H^1}^2 \\
& \leq a(u_{N,\lambda} - u_{N,\lambda'}, u_{N,\lambda} - u_{N,\lambda'}) \\
& = \int_\Omega \big(\tilde{f}_\lambda(u_{N,\lambda}) - \tilde{f}_{\lambda'}(u_{N,\lambda'})\big)(u_{N,\lambda} - u_{N,\lambda'}) dx \\
& = \int_\Omega \big(\tilde{f}_\lambda(u_{N,\lambda}) - \tilde{f}_\lambda(u_{N,\lambda'})\big)(u_{N,\lambda} - u_{N,\lambda'}) dx \\
& + \int_\Omega \big(\tilde{f}_\lambda(u_{N,\lambda'}) - \tilde{f}_{\lambda'}(u_{N,\lambda'})\big)(u_{N,\lambda} - u_{N,\lambda'}) dx \\
& \leq l^+\|u_{N,\lambda} - u_{N,\lambda'}\|_{H^1}^2 + C(p)\|\tilde{f}_\lambda(u_{N,\lambda'}) - \tilde{f}_{\lambda'}(u_{N,\lambda'})\|_{L^q}\|u_{N,\lambda} - u_{N,\lambda'}\|_{L^p}
\end{aligned}$$

by Proposition 9 ii), so that

$$\|u_{N,\lambda}-u_{N,\lambda'}\|_{H^1} \le \kappa^{-1}C(p)\|\tilde{f}_\lambda(u_{N,\lambda'})-\tilde{f}_{\lambda'}(u_{N,\lambda'})\|_{L^q} \to 0 \text{ as } \|\lambda-\lambda'\|_{L^\infty} \to 0$$

by Proposition 9 iii). In particular, ψ is continuous.

Now let two arbitrary solutions $u_N, u'_N \in S_N$ be given. By Proposition 9 i), there exist $\lambda, \lambda' \in \Lambda$ such that $u_N = \psi(\lambda)$ and $u'_N = \psi(\lambda')$. Define $\mu : [0,1] \to \Lambda$ by $\mu(t) := t\lambda' + (1-t)\lambda$. Then μ is continuous with $\mu(0) = \lambda$ and $\mu(1) = \lambda'$, so that $\psi \circ \mu : [0,1] \to S_N$ is a continuous curve joining $\psi \circ \mu(0) = u_N$ and $\psi \circ \mu(1) = u'_N$.

The path-connectedness of S can be shown by an analogous proof. □

3.2 Estimates of solution sets

Next we estimate the distance of the solution sets.

Theorem 15. *Let $H_0^1 \subset\subset L^p$ be compactly embedded for some $p \ge 2$. Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with some constant $l \in \mathbb{R}$ such that $l < c$ and which satisfies growth condition* (8). *If the approximation property* (6) *holds, then the solution sets S and S_N of* (14) *and* (15) *satisfy*

$$\operatorname{dist}(S_N, S)_{L^p} \to 0 \text{ as } N \to \infty. \tag{31}$$

Proof. If (31) is false, there exist $\varepsilon > 0$ and a sequence $(u_N)_{N\in\mathbb{N}'}$ with $u_N \in S_N$ such that

$$\operatorname{dist}(u_N, S)_{L^p} > \varepsilon \text{ for all } N \in \mathbb{N}'.$$

The proof of Theorem 12 shows that $(u_N)_N$ admits a further subsequence that converges weakly in H_0^1 to a solution u of (5). In particular, equation (29) holds, which contradicts the initial assumption. □

Remark 16. *It is not clear whether convergence with respect to the H^1-norm holds under the conditions of the theorem.*

Theorem 17. *Let $H_0^1 \subset\subset L^p$ be compactly embedded for some $p \ge 2$. Let $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be a Caratheodory mapping which is ROSL in the second argument with some constant $l \in \mathbb{R}$ such that $l < c$ and which satisfies growth condition* (8). *Assume that approximation property* (6) *holds. Then S is relatively compact in $L^p(\Omega)$ and*

$$\operatorname{dist}(R_N S, S_N)_{H^1} \to 0 \text{ as } N \to \infty,$$
$$\operatorname{dist}(S, S_N)_{L^p} \to 0 \text{ as } N \to \infty.$$

Proof. By Theorem 12, $\|S\|_{H^1} \leq \kappa^{-1}C(p)\|\alpha\|_{L^q}$. Because of

$$\begin{aligned} c\|R_N u - u\|_{H^1}^2 &= a(R_N u - u, R_N u - u) \\ &= -a(R_N u - u, u) \leq C\|R_N u - u\|_{H^1} \cdot \|u\|_{H^1}, \end{aligned}$$

we have

$$\|R_N u - u\|_{H^1} \leq \frac{C}{c}\|u\|_{H^1}.$$

Consequently, the set $(\cup_{N\in\mathbb{N}} R_N S) \cup S$ is bounded in $H_0^1(\Omega)$ and hence precompact in $L^p(\Omega)$.

Let $u \in S$, i.e.

$$a(u, v) = \int_\Omega fv\, dx \quad \forall v \in H_0^1(\Omega)$$

for some $f \in \tilde{F}(u)$, and let $N \in \mathbb{N}$ be given. The nonlinear operator $G_N : \mathcal{X}_N \to \mathcal{CC}(\mathcal{X}_N)$ defined in (27) has been shown to be continuous and ROSL with constant $-\kappa < 0$. Hence Theorem 27 guarantees the existence of a solution $u_N \in \mathcal{X}_N$ of the inclusion

$$0 \in G_N(u_N)$$

such that

$$\begin{aligned} \|u_N - R_N u\|_{H^1} &\leq \kappa^{-1} \operatorname{dist}(0, G_N(R_N u))_{H^1} \\ &= \kappa^{-1} \operatorname{dist}(Q_N A_N R_N u, Q_N \mathcal{F}_N(R_N u))_{H^1} \\ &= \kappa^{-1} \inf_{\tilde{f}\in\tilde{F}(R_N u)} \sup_{\substack{v\in\mathcal{X}_N \\ \|v\|_{H^1}=1}} -a(R_N u, v) + \int_\Omega \tilde{f} v\, dx \\ &\leq \kappa^{-1} \inf_{\tilde{f}\in\tilde{F}(R_N u)} \sup_{\substack{v\in\mathcal{X}_N \\ \|v\|_{H^1}=1}} (|a(R_N u - u, v)| \\ &\quad + |a(u, v) - \int_\Omega fv\, dx| + |\int_\Omega (f - \tilde{f})v\, dx|) \\ &\leq C(p)\kappa^{-1} \inf_{\tilde{f}\in\tilde{F}(R_N u)} \|f - \tilde{f}\|_{L^q} \\ &\leq C(p)\kappa^{-1} \operatorname{dist}(\tilde{F}(u), \tilde{F}(R_N u))_{L^q}. \end{aligned}$$

By inequality (16),

$$\sup_{u\in S} \|u - R_N u\|_{L^p} \to 0 \text{ as } N \to \infty. \tag{32}$$

Since $\tilde{F}$ is continuous (from $L^p(\Omega)$ to $L^q(\Omega)$), it is uniformly continuous on the precompact set $(\cup_{N\in\mathbb{N}} R_N S) \cup S$, so that by (32),

$$\sup_{u\in S} \mathrm{dist}(\tilde{F}(u), \tilde{F}(R_N u))_{L^q} \to 0 \text{ as } N \to \infty,$$

and hence

$$\mathrm{dist}(R_N S, S_N)_{H^1} \to 0 \text{ as } N \to \infty.$$

In particular, inequality (32) implies

$$\mathrm{dist}(S, S_N)_{L^p} \leq \mathrm{dist}(S, R_N S)_{L^p} + \mathrm{dist}(R_N S, S_N)_{H^1} \to 0$$

as $N \to \infty$. □

Remark 18. *If instead of* (16) *inequality*

$$\sup_{\|u\|_{H^1}\leq 1} \|u - R_N u\|_{H^1} \to 0 \text{ as } N \to \infty \tag{33}$$

is assumed, then the strengthened version

$$\sup_{u\in S} \|u - R_N u\|_{H^1} \to 0 \text{ as } N \to \infty.$$

of (32) *implies*

$$\mathrm{dist}(S, S_N)_{H^1} \to 0 \text{ as } N \to \infty$$

by the same proof as above. Inequality (33) *holds e.g. whenever* Ω *is a polygonal convex domain and the coefficients of the elliptic operator are sufficiently smooth (see [5, Theorems II.7.2 and II.7.3]).*

4 Implementation

Throughout this section, the mapping $F : \Omega \times \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ will be required to be measurable in the first and Lipschitz continuous with Lipschitz constant $L > 0$ in the second argument. It seems to be necessary to impose this assumption in order to obtain a modulus of continuity of the Nemytskii operator $\tilde{F} : L^2(\Omega) \rightrightarrows L^2(\Omega)$ without disproportionate complications. In view of Proposition 8, it follows that $\tilde{F}$ is Lipschitz continuous with the same Lipschitz constant L. The assumption $\kappa := c - L > 0$ is consistent with the notation employed in Section 3.

It is well-known (see e.g. [11]) that spaces of piecewise linear continuous functions subject to suitable triangulations of the domain Ω satisfy approximation property (6). In what follows, $\mathcal{X}_N$ is assumed to be such a space, and in particular, the value $v(x)$ of some $v \in \mathcal{X}_N$ at $x \in \Omega$ is well-defined.

In the multivalued setting, it is impossible to imitate the classical approach to computing solutions of nonlinear PDEs. The current state of Set-Valued Analysis does not allow to apply an analog of Newton's method to the nonlinear problem (7). Therefore, it seems reasonable to realize the computation of the solution set S_N by a search algorithm after a complete discretization of the problem.

4.1 Full discretization of the elliptic inclusion

There are three levels of discretization. The error $\mathrm{dist}_H(S, S_N)$ caused by the projection of the continuous problem to $\mathcal{X}_N$ has been discussed in Section 3. The space $\mathcal{X}_N$ itself has to be discretized in terms of a grid $\Delta_\rho \subset \mathcal{X}_N$ with the property that for any $u_N \in \mathcal{X}_N$, there exists some $u_N^\rho \in \Delta_\rho$ such that $\|u_N - u_N^\rho\|_{H^1} \leq \rho$. Then bounded subsets of $\mathcal{X}_N$ can be projected to a finite number of grid points. Finally, a discretization of the images of the Nemytskii operator $\mathcal{F}_N$ is needed. Experience from the linear case [12] shows that a direct discretization of an image $\mathcal{F}_N(u) = J_N \tilde{F}(u)$ is problematic, because the sets $\tilde{F}(u) \subset L^2(\Omega)$ are not compact and hence do not admit any straight-forward discretization technique. The indirect discretization of the set $\mathcal{F}_N(u)$ in terms of its support function $\sigma_{\mathcal{F}_N(u)}$ proves much more efficient: The support function $\sigma_E : \{v \in \mathcal{X}_N : \|v\|_{H^1} = 1\} \to \mathbb{R}$ of a subset $E \subset \mathcal{CC}(\mathcal{X}_N^*)$ is defined by

$$\sigma_E(v) := \sup_{e \in E} \langle e, v \rangle. \tag{34}$$

It can be discretized by introducing a δ-net $V_\delta \subset \{v \in \mathcal{X}_N : \|v\|_{H^1} = 1\}$ which is a discrete subset with the property that for every $v \in \mathcal{X}_N$ with $\|v\|_{H^1} = 1$, there exists some $v_\delta \in V_\delta$ such that $\|v - v_\delta\|_{H^1} \leq \delta$. Then the discretized version of σ_E is its restriction to V_δ.

Because of the high complexity of the task, it is important to choose an appropriate setting and to avoid dispensable computations whenever possible. A practical implementation of our algorithm proceeds as follows.

1. Fix a small number of selections f_n, $n = 1, \dots, \bar{n}$, of F such as f_+, f_-, and $\frac{1}{2}(f_+ + f_-)$. Compute the solutions u_n, $n = 1, \dots, \bar{n}$, of the classical problems

$$a(u_n, v) = \int_\Omega f_n(u_n) v \, dx \quad \forall\, v \in \mathcal{X}_N$$

by means of an FEM solver. Obviously, the functions u_n solve (7). Lemma 19 states that

$$S_N \subset U := \big(\cap_{n=1}^{\bar{n}} B(u_n, \kappa^{-1} \operatorname{diam} \tilde{F}(u_n)_{L^2})_{H^1} \big).$$

This area must be searched for solutions.

2. Intersect U with a ρ-grid Δ_ρ. In Sections 4.2 and 4.3 it is shown that the fully discretized solution set

$$S_N^\rho(\delta) := \{u_N^\rho \in U \cap \Delta_\rho : \max_{v \in V_\delta} [a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)] \le (C + L)\rho\}$$

is a good approximation of S_N w.r.t. the H^1 norm, where V_δ is a sufficiently dense covering of the unit sphere in H_0^1 and $\sigma_{\mathcal{F}_N(u_N^\rho)}$ is the support function of the closed, convex, and bounded set $\mathcal{F}_N(u_N^\rho)$. The computation of the defect $a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)$ is simplified considerably by Lemma 24.

3. Redundant computations must be avoided. In principle, it is necessary to check for every $u_N^\rho \in U \cap \Delta_\rho$ whether it is an element of $S_N^\rho(\delta)$. Lemma 25 states that if a defect

$$a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v) \ge (C + L)\rho + \eta$$

with some $\eta > 0$ is computed, then $B(u_N^\rho, \frac{\eta}{C+L})_{H^1} \subset S_N^\rho(\delta)^c$. Consequently, all grid points in this ball can be skipped. A similar strategy can be pursued if the defect is much smaller than allowed and u_N^ρ is surrounded by elements of $S_N^\rho(\delta)$.

   This ball skipping method must be used carefully. For strongly nonlinear right-hand sides, only approximations of the constants L and κ are known. As a consequence, the radii of the balls may be chosen too small, so that the method underperforms, or, which is much worse, too large, when parts of the solution set may be cut off.

Alternatively, it is possible to exploit the path connectedness of S_N. To this end, one approximate solution is computed by means of an FEM solver. Then adjacent grid points of already known approximate solutions are checked successively. This method has the advantage that only grid points very near to S_N need to be checked. In addition, it is more reliable than the ball skipping method described above.

Detailed error estimates for the discretization error are given below.

4.2 Localization of S_N

At first, we fix the dimension of the finite-dimensional subspace $\mathcal{X}_N \subset H_0^1(\Omega)$. The error $\mathrm{dist}_H(S, S_N)$ resulting from the projection of the original problem to $\mathcal{X}_N$ has been estimated in Section 3. A-priori estimate (26) implies that $S_N \subset B_{\kappa^{-1}\|\alpha\|_{L^2}}(0) \subset \mathcal{X}_N$. Alternatively, it is possible to localize S_N by computing one element $u_N^* \in S_N$ by a suitable finite element approach and applying the following lemma.

Lemma 19. *If $u_N, u_N^* \in S_N$, then*

$$\|u_N - u_N^*\|_{H^1} \leq \kappa^{-1} \operatorname{diam} \tilde{F}(u_N^*)_{L^2}.$$

Proof. Since $u_N, u_N^* \in S_N$, there exist $f_N \in \tilde{F}(u_N)$ and $f_N^* \in \tilde{F}(u_N^*)$ such that

$$a(u_N, v) = (f_N, v)_{L^2},$$
$$a(u_N^*, v) = (f_N^*, v)_{L^2}$$

for all $v \in \mathcal{X}_N$. As $\tilde{F}$ is Lipschitz, there exists some $g_N^* \in \tilde{F}(u_N^*)$ such that

$$\|f_N - g_N^*\|_{L^2} \leq L\|u_N - u_N^*\|_{L^2}.$$

Hence

$$\begin{aligned} c\|u_N - u_N^*\|_{H^1}^2 &\leq a(u_N - u_N^*, u_N - u_N^*) = (u_N - u_N^*, f_N - f_N^*) \\ &= (u_N - u_N^*, f_N - g_N^*) + (u_N - u_N^*, g_N^* - f_N^*) \\ &\leq L\|u_N - u_N^*\|_{H^1}^2 + [\operatorname{diam} \tilde{F}(u_N^*)_{L^2}]\|u_N - u_N^*\|_{H^1} \end{aligned}$$

implies the statement of the lemma. □

The diameter

$$\operatorname{diam} \tilde{F}(u_N^*)_{L^2} = \Big(\int_\Omega |f_+(x, u_N^*(x)) - f_-(x, u_N^*(x))|^2 dx \Big)^{\frac{1}{2}}$$

can be computed without any difficulties.

4.3 Projection of S_N to a grid

Let Δ_ρ be a grid in $\mathcal{X}_N$ such that for every $u_N \in \mathcal{X}_N$, there exists some $u_N^\rho \in \Delta_\rho$ with $\|u_N - u_N^\rho\|_{H^1} \leq \rho$. The following proposition shows that the spatially discretized solution set

$$S_N^{\rho,\varepsilon} := \{u_N^\rho \in \Delta_\rho : \mathrm{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \leq (C+L)\rho + \varepsilon\}, \qquad (35)$$

is a good approximation of S_N for small $\varepsilon > 0$. It is useful to allow an inaccuracy of size ε, because in practice, the above distance can only be approximated (see Section 4.4).

Proposition 20. *If $\kappa > 0$, then*

$$\mathrm{dist}_H(S_N, S_N^{\rho,\varepsilon})_{H^1} \leq \max\{\rho, \kappa^{-1}((C+L)\rho) + \varepsilon)\}.$$

Proof. Let $u_N \in S_N$ be given. By definition of Δ_ρ, there exists some $u_N^\rho \in \Delta_\rho$ with $\|u_N - u_N^\rho\|_{H^1} \leq \rho$, and

$$\begin{aligned}\mathrm{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \leq\ & \|Au_N^\rho - Au_N\|_{\mathcal{X}_N^*} \\ &+ \mathrm{dist}(Au_N, \mathcal{F}_N(u_N))_{\mathcal{X}_N^*} \\ &+ \mathrm{dist}(\mathcal{F}_N(u_N), \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*},\end{aligned}$$

where

$$\|Au_N^\rho - Au_N\|_{\mathcal{X}_N^*} = \sup_{\substack{v \in \mathcal{X}_N \\ \|v\|_{H^1} \leq 1}} a(u_N^\rho - u_N, v) \leq C\|u_N^\rho - u_N\|_{H^1} \leq C\rho$$

by definition,

$$\mathrm{dist}(Au_N, \mathcal{F}_N(u_N))_{\mathcal{X}_N^*} = 0,$$

because $u_N \in S_N$, and

$$\begin{aligned}&\mathrm{dist}(\mathcal{F}_N(u_N), \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \\ &= \mathrm{dist}(\{J_N f_N : f_N \in \tilde{F}(u_N)\}, \{J_N f_N^\rho : f_N^\rho \in \tilde{F}(u_N^\rho)\})_{\mathcal{X}_N^*} \\ &= \sup_{f_N \in \tilde{F}(u_N)} \inf_{f_N^\rho \in \tilde{F}(u_N^\rho)} \|J_N(f_N - f_N^\rho)\|_{\mathcal{X}_N^*} \\ &\leq \mathrm{dist}(\tilde{F}(u_N), \tilde{F}(u_N^\rho))_{L^2} \leq L\|u_N - u_N^\rho\|_{L^2} \leq L\rho.\end{aligned}$$

Consequently,

$$\mathrm{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \leq (C+L)\rho,$$

and $u_N^\rho \in S_N^\rho \subset S_N^{\rho,\varepsilon}$, so that $\operatorname{dist}(S_N, S_N^{\rho,\varepsilon})_{H^1} \le \rho$.

Conversely, let $u_N^\rho \in S_N^{\rho,\varepsilon}$ be given, and consider the mapping $G_N : \mathcal{X}_N \to \mathcal{CC}(\mathcal{X}_N)$ defined in the proof of Theorem 12. It was shown that G_N is continuous and ROSL with constant $-\kappa < 0$, and zeroes of G_N are precisely the elements of S_N. By Theorem 27, there exists some $u_N \in S_N$ such that

$$\begin{aligned}\|u_N^\rho - u_N\|_{H^1} &\le \kappa^{-1} \operatorname{dist}(0, G_N(u_N^\rho))_{H^1}\\ &\le \kappa^{-1} \operatorname{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \le \kappa^{-1}((C+L)\rho + \varepsilon),\end{aligned}$$

so that $\operatorname{dist}(S_N^{\rho,\varepsilon}, S_N)_{H^1} \le \kappa^{-1}((C+L)\rho + \varepsilon)$. ☐

4.4 Computation of the defect

Let $\delta \in (0,1]$ be given, and let V_δ be a finite subset of $\mathcal{X}_N$ such that $\|v_\delta\|_{H^1} = 1$ for all $v_\delta \in V_\delta$ and for every $v \in \mathcal{X}_N$ with $\|v\|_{H^1} = 1$, there exists some $v_\delta \in V_\delta$ with $\|v - v_\delta\|_{H^1} \le \delta$. The properties of the support function defined in (34) are listed in [10]. In particular,

$$\operatorname{dist}(E, E')_{\mathcal{X}_N^*} = \max_{\substack{v\in\mathcal{X}_N\\ \|v\|_{H^1}=1}} [\sigma_E(v) - \sigma_{E'}(v)]$$

for any two $E, E' \in \mathcal{CC}(\mathcal{X}_N^*)$. Moreover, σ_E is Lipschitz with constant $\|E\|_{\mathcal{X}_N^*}$ according to [14]. (The statement is given for $\mathbb{R}^d$ equipped with the Euclidean norm, but the proof is correct for any finite-dimensional Hilbert space.)

Lemma 21. *For any two $E, E' \in \mathcal{CC}(\mathcal{X}_N^*)$,*

$$\operatorname{dist}(E, E')_{\mathcal{X}_N^*} \le \max_{v_\delta\in V_\delta}[\sigma_E(v_\delta) - \sigma_{E'}(v_\delta)] + \delta(\|E\|_{\mathcal{X}_N^*} + \|E'\|_{\mathcal{X}_N^*})$$

and

$$\operatorname{dist}(E, E')_{\mathcal{X}_N^*} \ge \max_{v_\delta\in V_\delta}[\sigma_E(v_\delta) - \sigma_{E'}(v_\delta)].$$

Proof. For any $v \in \mathcal{X}_N$ with $\|v\|_{H^1} = 1$, there exists some $v_\delta \in V_\delta$ with $\|v - v_\delta\|_{H^1} \le \delta$, so that

$$\begin{aligned}\sigma_E(v) - \sigma_{E'}(v) &\le |\sigma_E(v) - \sigma_E(v_\delta)| + [\sigma_E(v_\delta) - \sigma_{E'}(v_\delta)] + |\sigma_{E'}(v_\delta) - \sigma_{E'}(v)|\\ &\le [\sigma_E(v_\delta) - \sigma_{E'}(v_\delta)] + \delta(\|E\|_{\mathcal{X}_N^*} + \|E'\|_{\mathcal{X}_N^*}),\end{aligned}$$

which implies the first statement of the lemma. The second statement is obvious. ☐

Define the fully discretized solution set $S_N^\rho(\delta)$ with uniform discretization V_δ, $\delta \in (0,1]$, of the right-hand side by

$$S_N^\rho(\delta) := \{u_N^\rho \in \Delta_\rho : \max_{v \in V_\delta}[a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)] \leq (C+L)\rho\}. \tag{36}$$

Proposition 22. *If $\delta < \frac{c-L}{C+L}$, then the approximation error of the fully discretized solution set $S_N^\rho(\delta)$ is*

$$\operatorname{dist}_H(S_N, S_N^\rho(\delta))_{H^1} \leq \max\{\rho, \kappa^{-1}((C+L)\rho + \mu(\rho,\delta)\delta)\},$$

where $\mu(\rho,\delta) = (C+L)\frac{(C+L)\rho+(1+\delta)\|\alpha\|_{L^2}}{\kappa-(C+L)\delta} + \|\alpha\|_{L^2}\delta$.

Note that $\mu(\rho,\delta) \searrow \kappa^{-1}\|\alpha\|_{L^2}$ as $(\rho,\delta) \to 0$.

Proof. Let $u_N^\rho \in S_N^\rho(\delta)$. By Lemma 21, estimate

$$\begin{aligned}
&\operatorname{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \\
&\leq \max_{v \in V_\delta}[\sigma_{\{Au_N^\rho\}}(v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)] + \delta(\|Au_N^\rho\|_{\mathcal{X}_N^*} + \|\mathcal{F}_N(u_N^\rho)\|_{\mathcal{X}_N^*}) \\
&\leq (C+L)\rho + C\|u_N^\rho\|_{H^1}\delta + \|\tilde{F}(u_N^\rho)\|_{L^2}\delta \\
&\leq (C+L)\rho + C\|u_N^\rho\|_{H^1}\delta + (\|\tilde{F}(0)\|_{L^2} + \operatorname{dist}(\tilde{F}(0), \tilde{F}(u_N^\rho))_{L^2})\delta \qquad (37) \\
&\leq (C+L)\rho + C\|u_N^\rho\|_{H^1}\delta + (\|\alpha\|_{L^2} + L\|u_N^\rho\|_{H^1})\delta \\
&\leq (C+L)\rho + (C+L)\|u_N\|_{H^1}\delta + \|\alpha\|_{L^2}\delta,
\end{aligned}$$

holds. Let $f_N \in \tilde{F}(u_N^\rho)$ be such that $J_N f_N \in \operatorname{Proj}(A_N u_N^\rho, \mathcal{F}(u_N^\rho))$. Then

$$\begin{aligned}
c\|u_N^\rho\|_{H^1}^2 &\leq a(u_N^\rho, u_N^\rho) = (Q_N A_N u_N^\rho, u_N^\rho)_{H^1} \\
&= (Q_N A_N u_N^\rho - Q_N J_N f_N, u_N^\rho)_{H^1} + (Q_N J_N f_N, u_N^\rho)_{H^1} \\
&\leq \operatorname{dist}(A_N u_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*}\|u_N^\rho\|_{H^1} + \|\tilde{F}(u_N^\rho)\|_{L^2}\|u_N^\rho\|_{H^1} \\
&\leq [(C+L)\rho + (C+L)\|u_N\|_{H^1}\delta + \|\alpha\|_{L^2}\delta]\|u_N^\rho\|_{H^1} \\
&+ [\|\alpha\|_{L^2} + L\|u_N^\rho\|_{H^1}]\|u_N^\rho\|_{H^1}
\end{aligned}$$

implies

$$(c - L - (C+L)\delta)\|u_N^\rho\|_{H^1} \leq (C+L)\rho + \|\alpha\|_{L^2}\delta + \|\alpha\|_{L^2}$$

and thus the a-priori estimate

$$\|u_N^\rho\|_{H^1} \leq \frac{(C+L)\rho + (1+\delta)\|\alpha\|_{L^2}}{c - L - (C+L)\delta}. \tag{38}$$

But (38) inserted into (37) yields

$$\begin{aligned}
&\operatorname{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \\
&\leq (C+L)\rho + (C+L)\frac{(C+L)\rho + (1+\delta)\|\alpha\|_{L^2}}{c-L-(C+L)\delta}\delta + \|\alpha\|_{L^2}\delta,
\end{aligned}$$

so that the inclusion

$$S_N^{\rho,0} \subset S_N^\rho(\delta) \subset S_N^{\rho,\mu(\rho,\delta)\delta}$$

implies

$$\operatorname{dist}_H(S_N, S_N^\rho(\delta))_{H^1} \leq \max\{\rho, \kappa^{-1}((C+L)\rho + \mu(\rho,\delta)\delta)\}$$

according to Proposition 20. □

Alternatively, it is possible to define a fully discretized solution set with adaptive discretization $V_{\delta(u_N^\rho)}$ with $\delta : \Delta_\rho \to (0,1]$ by setting

$$\hat{S}_N^\rho(\delta') := \{u_N^\rho \in \Delta_\rho : \max_{v \in V_{\delta(u_N^\rho)}} [a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)] \leq (C+L)\rho\}, \tag{39}$$

where $\delta(u_N^\rho) := \min\{1, \|u_N^\rho\|_{H^1}^{-1}\}\delta'$ with some (uniform) $\delta' \in (0,1]$.

Proposition 23. *If $\kappa > 0$, then the approximation error of the adaptive fully discretized solution set $\hat{S}_N^\rho(\delta')$ is*

$$\operatorname{dist}_H(S_N, \hat{S}_N^\rho(\delta))_{H^1} \leq \max\{\rho, \kappa^{-1}((C+L)\rho + \mu\delta')\},$$

where $\mu = C + L + \|\alpha\|_{L^2}$.

A comparison with Proposition 22 shows that the error estimate given therein is rather pessimistic.

Proof. By estimate (37),

$$\begin{aligned}
&\operatorname{dist}(Au_N^\rho, \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*} \\
&\leq (C+L)\rho + (C+L)\|u_N^\rho\|_{H^1}\delta(u_N^\rho) + \|\alpha\|_{L^2}\delta(u_N^\rho) \\
&\leq (C+L)\rho + (C+L+\|\alpha\|_{L^2})\delta',
\end{aligned}$$

so that

$$S_N^{\rho,0} \subset \hat{S}_N^\rho(\delta') \subset S_N^{\rho,\mu\delta'}$$

Again, Proposition 20 yields

$$\operatorname{dist}_H(S_N, \hat{S}_N^\rho(\delta'))_{H^1} \leq \max\{\rho, \kappa^{-1}((C+L)\rho + \mu\delta')\}.$$

□

For any $E \subset \Omega$, let $\chi_E : \Omega \to \{0, 1\}$ denote the indicator function given by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise.

Lemma 24. *For any $v \in \mathcal{X}_N$ and $u_N^\rho \in \Delta_\rho$,*

$$\sigma_{\mathcal{F}_N(u_N^\rho)} = (f_N^{\rho,v}, v)_{L^2},$$

where

$$f_N^{\rho;v}(x) := \chi_{\{v(x)\geq 0\}}(x) f_+(x, u_N^\rho(x)) + \chi_{\{v(x)<0\}}(x) f_-(x, u_N^\rho(x)).$$

Proof. The function $f_N^{\rho,v}$ is well-defined. Since $v \in \mathcal{X}_N$ is continuous, the evaluation of v at x makes sense and the sets $\Omega_+ := \{v(x) \geq 0\}$ and $\Omega_- := \{v(x) < 0\}$ are measurable. As F is Caratheodory, the functions $x \mapsto f_+(x, u_N^\rho(x))$ and $x \mapsto f_-(x, u_N^\rho(x))$ are measurable by Lemma 3, and by Proposition 5, they are L^2 functions, so that $f_N^{\rho,v} \in L^2(\Omega)$.

Maximality. Let $f \in \mathcal{F}(u_N^\rho)$ be an arbitrary element. Then

$$\begin{aligned}
&\langle J_N f_N^{\rho,v}, v\rangle - \langle J_N f, v\rangle = (f_N^{\rho,v} - f, v)_{L^2} \\
&= (f_+(x, u_N^\rho(x)) - f, v)_{L^2(\Omega_+)} + (f_-(x, u_N^\rho(x)) - f, v)_{L^2(\Omega_-)} \geq 0,
\end{aligned}$$

so that

$$(f_N^{\rho,v}, v)_{L^2} = \langle J_N f_N^{\rho,v}, v\rangle \geq \langle J_N f, v\rangle$$

for all $f \in \mathcal{F}(u_N^\rho)$. □

As a consequence of Lemma 24, the fully discretized solution sets $S_N^\rho(\delta)$ and $\hat{S}_N^\rho(\delta')$ can be rewritten as

$$\begin{aligned}
S_N^\rho(\delta) &:= \{u_N^\rho \in \Delta_\rho : \max_{v\in V_\delta}[a(u_N^\rho, v) - (f_N^{\rho,v}, v)_{L^2}] \leq (C + L)\rho\}, \\
\hat{S}_N^\rho(\delta') &:= \{u_N^\rho \in \Delta_\rho : \max_{v\in V_{\delta(u_N^\rho)}}[a(u_N^\rho, v) - (f_N^{\rho,v}, v)_{L^2}] \leq (C + L)\rho\}.
\end{aligned}$$

4.5 Excluding irrelevant areas

It is intuitively clear that if some $u_N^{\rho,*} \in \Delta_\rho$ yields a large defect in the defining relation (36), then nearby grid points cannot be elements of the solution set $S_N^\rho(\delta)$. If, on the other hand, some $u_N^{\rho,*} \in \Delta_\rho$ yields a defect that is much smaller than allowed, it is obvious that nearby grid points must be contained in $S_N^\rho(\delta)$. The following lemma quantifies this issue.

Lemma 25. *Let $v \in V_\delta$, $u_N^{\rho,*} \in \Delta_\rho$, and $\eta > 0$ be given. If*

$$a(u_N^{\rho,*}, v) - \sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v) > (C+L)\rho + \eta$$

and $\|u_N^\rho - u_N^{\rho,*}\|_{H^1} \le \frac{\eta}{C+L}$, then $u_N^\rho \notin S_N^\rho(\delta)$. If, on the other hand,

$$a(u_N^{\rho,*}, v) - \sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v) \le (C+L)\rho - \eta$$

for all $v \in V_\delta$ and $\|u_N^\rho - u_N^{\rho,*}\|_{H^1} \le \frac{\eta}{C+L}$, then $u_N^\rho \in S_N^\rho(\delta)$.

Note that the maximal defect $\max_{v\in V_\delta}[a(u_N^{\rho,*}, v) - \sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v)]$ can be negative.

Proof. As seen in the proof of Proposition 20, $\mathcal{F}_N$ is Lipschitz with constant L. Hence

$$\begin{aligned}
&a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)\\
&= [a(u_N^\rho, v) - a(u_N^{\rho,*}, v)] + [a(u_N^{\rho,*}, v) - \sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v)] + [\sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)]\\
&\ge [a(u_N^{\rho,*}, v) - \sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v)] - |a(u_N^\rho - u_N^{\rho,*}, v)| - |\sigma_{\mathcal{F}_N(u_N^{\rho,*})}(v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v)|\\
&> (C+L)\rho + \eta - C\|u_N^\rho - u_N^{\rho,*}\|_{H^1} - \mathrm{dist}_H(\mathcal{F}_N(u_N^{\rho,*}), \mathcal{F}_N(u_N^\rho))_{\mathcal{X}_N^*}\\
&\ge (C+L)\rho + \eta - (C+L)\|u_N^\rho - u_N^{\rho,*}\|_{H^1}
\end{aligned}$$

and $(C+L)\|u_N^\rho - u_N^{\rho,*}\|_{H^1} \le \eta$ implies $a(u_N^\rho, v) - \sigma_{\mathcal{F}_N(u_N^\rho)}(v) > (C+L)\rho$ and thus the first statement of the lemma.

The second statement follows from an almost identical computation. □

Obviously, this technique is very sensitive to the constants L and C. If F is not globally Lipschitz, but the solutions of the differential inclusion are bounded by some embedding theorem, then L depends on the upper bound for the solution set and the embedding constant, which can be difficult to determine. It is not recommended to use the ball-skipping method in such a situation, because we observed failure caused by an ill-estimated value of L.

Alternatively, one can rely on the path connectedness of the solution set S_N. Associate with any $v^\rho \in \Delta_\rho$ the Voronoi cell

$$\mathcal{V}_\rho(v^\rho) := \{v \in \mathcal{X}_N : \|v - v^\rho\|_{H^1} \le \inf_{v^{\rho\prime}\in\Delta_\rho} \|v - v^{\rho\prime}\|_{H^1}\}.$$

Clearly, every $\mathcal{V}_\rho(v^\rho)$ is compact, and for every $w \in \mathcal{V}_\rho(v^\rho)$, the estimate $\|w - v^\rho\|_{H^1} \le \rho$ holds. Consequently, if there exists some $w \in \mathcal{V}_\rho(v^\rho) \cap \mathcal{V}_\rho(v^{\rho\prime})$, then $\|v^\rho - v^{\rho\prime}\|_{H^1} \le 2\rho$.

Lemma 26. *Let $u^\rho, u^{\rho\prime} \in B_\rho(S_N) \cap \Delta_\rho := \{v \in \mathcal{X}_N : \operatorname{dist}(v, S_N) \le \rho\} \cap \Delta_\rho$. Then there exists some $n \in \mathbb{N}$ and a sequence*

$$\{u^\rho = u_0^\rho, u_1^\rho, \ldots, u_{n-1}^\rho, u_n^\rho = u^{\rho\prime}\} \in B_\rho(S_N) \cap \Delta_\rho$$

such that $\mathcal{V}_\rho(u_k) \cap \mathcal{V}_\rho(u_{k+1}) \neq \emptyset$ and $\|u_k^\rho - u_{k+1}^\rho\|_{H^1} \le 2\rho$ for $k = 0, \ldots, n-1$.

Proof. Let $u^\rho, u^{\rho\prime} \in B_\rho(S_N) \cap \Delta_\rho$ be given. By definition, there exist points $u_N, u_N' \in S_N$ such that $\|u^\rho - u_N\|_{H^1} \le \rho$ and $\|u^{\rho\prime} - u_N'\|_{H^1} \le \rho$. As S_N is path connected, there exists a continuous curve $w(t) : [0,1] \to S_N$ with $w(0) = u_N$ and $w(1) = u_N'$. Since $w([0,1])$ is compact, the intersection $w([0,1]) \cap \mathcal{V}_\rho(v_k^\rho)$ is nonempty only for finitely many cells $\mathcal{V}_\rho(v_k^\rho)$, $0 \le k \le m$. Set

$$t_k := \min\{t \in [0,1] : w(t) \in \mathcal{V}_\rho(v_k^\rho)\}, \quad k \in \{0, \ldots, m\}.$$

Without loss of generality, it can be assumed that

$$0 = t_0 \le t_1 \le \ldots \le t_m, \quad v_0^\rho = u^\rho.$$

We claim that for every $k \in \{0, \ldots, m\}$, there exist some $n \in \mathbb{N}$ and a subsequence $\{j_0, \ldots, j_n\} \subset \{0, \ldots, k\}$ such that

$$j_0 = 0, \quad j_n = k, \quad \text{and} \quad \mathcal{V}_\rho(v_{j_i}^\rho) \cap \mathcal{V}_\rho(v_{j_{i+1}}^\rho) \neq \emptyset \text{ for } 0 \le i < k. \tag{40}$$

The case $k = 0$ is trivial. Assume that the above statement holds for $0 \le k < m$. Since $w([0, t_{k+1}]) \subset \cup_{j=0}^k V_\rho(v_j^\rho)$ and $w(t_{k+1}) \in \mathcal{V}_\rho(v_{k+1}^\rho)$, there exists some $k' \in \{0, \ldots, k\}$ such that $\mathcal{V}_\rho(v_{k+1}^\rho) \cap \mathcal{V}_\rho(v_{k'}^\rho) \neq \emptyset$. By assumption, there exist some $n \in \mathbb{N}$ and a sequence $\{j_0, \ldots, j_n\} \subset \{0, \ldots, k'\}$ such that (40) is satisfied with k replaced by k'. But then, $\{j_0, \ldots, j_n = k', k+1\} \subset \{0, \ldots, k+1\}$ is a subsequence satisfying (40) with k and n replaced by $k+1$ and $n+1$. By recursion, (40) holds for all $k \in \{0, \ldots, m\}$. As $u^{\rho\prime} = v_k^\rho$ for some $k \in \{0, \ldots, m\}$, the statement of the lemma is proved. □

The above lemma can be used as follows.

Fix $\rho, \delta > 0$ as in Section 4.4.
Choose an arbitrary selection of F such as $f_\pm := \frac{1}{2}(f_+ + f_-)$ and compute a good approximation of the solution $u_N^\pm \in \mathcal{X}_N$ of the single-valued problem

$$a(u_N^\pm, v) = (f_\pm(u_N^\pm), v) \textit{ for all } v \in \mathcal{X}_N.$$

Pick any $u^\rho \in \Delta_\rho$ *with* $\|u^\rho - u_N^\pm\|_{H^1} \le \rho$. *By construction,* $u^\rho \in S_N^\rho(\delta)$.
Initialize a list of grid points and save u^ρ *as its first element. Mark it as unchecked.*
WHILE *there exists an unchecked list element* u^ρ
  IF $u^\rho \in S_N^\rho(\delta)$
    Mark u^ρ *as positive*
    IF *there exist elements* $u^{\rho'} \in B_{2\rho}(u^\rho)$ *such that* $u^{\rho'}$ *not in list*
      Add all such grid points to the list and mark them as unchecked.
    END
  ELSE *Mark* u^ρ *as negative.*
  END
END
Delete all negative elements from the list.

The proof of Proposition 20 shows that $B_\rho(S_N)\cap\Delta_\rho \subset S_N^\rho(\delta)$. The above lemma ensures that when the algorithm terminates, the list of points contains every element of $B_\rho(S_N) \cap \Delta_\rho$, so that

$$B_\rho(S_N) \cap \Delta_\rho \subset \text{ constructed list } \subset S_N^\rho(\delta).$$

As both $\mathrm{dist}_H(B_\rho(S_N)\cap\Delta_\rho, S_N)_{H^1}$ and $\mathrm{dist}_H(S_N^\rho(\delta), S_N)_{H^1}$ are known to be small, the list constructed by the algorithm is an excellent discrete approximation of S_N satisfying

$$\mathrm{dist}_H(\text{constructed list}, S_N) \le \mathrm{dist}_H(S_N^\rho(\delta), S_N)_{H^1}.$$

The advantage of this algorithm is twofold: Only defects at grid points very near to S_N must be computed. The performance gain depends on the shape of S_N, and it seems impossible to prove a general result about it, but in practical computation it was significant. Moreover, the algorithm is stable in contrast to the ball-skipping method, because it is insensitive or at least not more sensitive to the constants C and L than the computation of $S_N^\rho(\delta)$ itself. We did not observe any indication for a failure of the method.

For relatively small N, it may be useful to store the information whether a grid point is marked as checked, unchecked, positive, or negative in an array, because then the information can be accessed without scanning through the list. In higher dimensions, such an array would require too much memory, and searching the list is inevitable. In this case, we recommend the use of a red-black tree (see [6, Chapter 13]) together with a lexicographic order imposed on the coordinates of the representation given in the next section.

It remains to note that the ball-skipping method and the recursive search can both be applied not only for the computation of $S_N^\rho(\delta)$, but also for the computation of the adaptive $\hat{S}_N^\rho(\delta')$.

4.6 Concrete realization

The results above are stated in terms of elements of $\mathcal{X}_N$. For a concrete realization, they have to be transferred to the Euclidean space $\mathbb{R}^N$. Let $\{\varphi_1, \ldots, \varphi_N\} \subset \mathcal{X}_N$ be the standard basis of $\mathcal{X}_N$ and let the matrix $M \in \mathbb{R}^{N\times N}$ be given by

$$M_{ij} = (\varphi_i, \varphi_j)_{H^1}, \quad i, j = 1, \ldots, N.$$

As M is positive definite and symmetric, there exist matrices $Q, \Lambda \in \mathbb{R}^{N\times N}$ with $Q^T Q = \mathrm{id}$, $\Lambda = \mathrm{diag}(\lambda_j)_{j=1}^N$ with $\lambda_N > \ldots > \lambda_1 > 0$, and $M = Q^T \Lambda Q$. The linear isomorphism $K : \mathbb{R}^N \to \mathcal{X}_N$ given by

$$K(z) := \sum_{i=1}^N (Q^T \Lambda^{-\frac{1}{2}} z)_i \varphi_i$$

is an isometry, because

$$\begin{aligned} \|K(z)\|_{H^1}^2 &= (\sum_{i=1}^N (Q^T \Lambda^{-\frac{1}{2}} z)_i \varphi_i, \sum_{i=j}^N (Q^T \Lambda^{-\frac{1}{2}} z)_j \varphi_j)_{H^1} \\ &= (Q^T \Lambda^{-\frac{1}{2}} z)^T M (Q^T \Lambda^{-\frac{1}{2}} z) = |z|_2^2. \end{aligned}$$

In particular, the images of an equidistant grid in $\mathbb{R}^N$ and a δ-net on the unit sphere in $\mathbb{R}^N$ under K are an equidistant grid in $\mathcal{X}_N$ and a δ-net on the unit sphere in $\mathcal{X}_N$. Moreover, the balls in $\mathcal{X}_N$ computed via Lemmas 19 and 25 transfer to Euclidean balls in $\mathbb{R}^N$ with the same radius, and the implementation is straight-forward.

In this setting, the algorithm derived from Lemma 26 is particularly powerful, because if $\mathbb{R}^N$ is decomposed in cubic boxes centered at equidistant grid points, it is enough to check directly adjacent boxes in the course of constructing the list.

Consider the differential inclusion

$$\begin{aligned} -\Delta u \in F(u) = [\frac{1-\varepsilon}{1+u^2}, \frac{1}{1+u^2}] \quad &\text{in} \quad \Omega = [0,1] \\ u = 0 \quad &\text{on} \quad \partial\Omega = \{0,1\} \end{aligned}$$

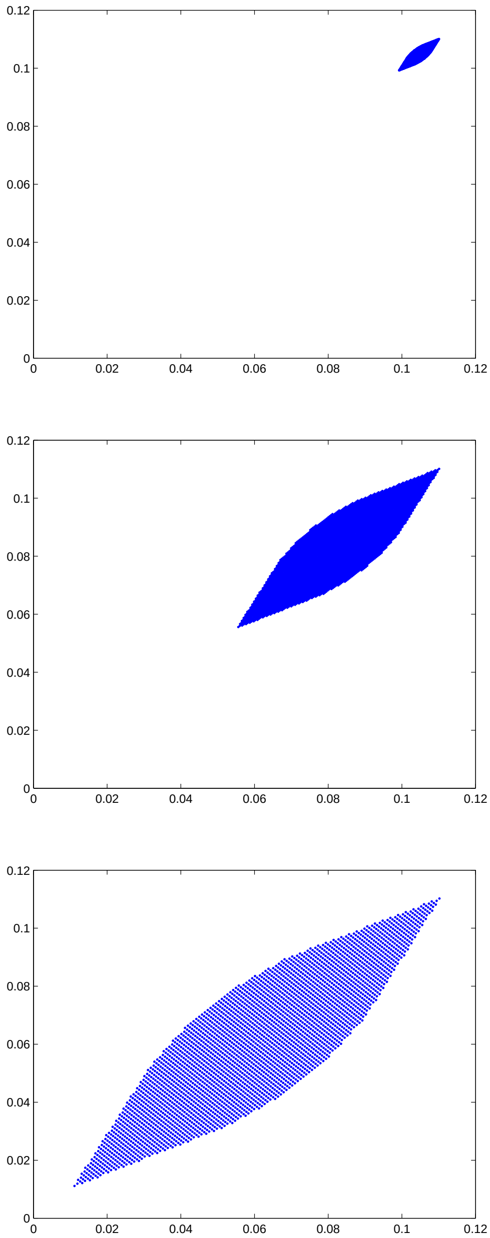

Figure 1: Approximations of the solution set of (41) for $N = 2$ and $\varepsilon = 0.1$, $\varepsilon = 0.5$, and $\varepsilon = 0.9$ in coefficient space.

with weak formulation

$$a(u,v) := \int_\Omega \nabla u \nabla v dx = \int_\Omega f(u)v\; dx$$

for all $v \in H_0^1$ and some $f \in \tilde{F}(u)$. A solution $u_N \in S_N$ satisfies

$$\int_\Omega \nabla u_N \nabla v dx = \int_\Omega f(u_N)v\; dx \quad \text{for all } v \in \mathcal{X}_N. \tag{41}$$

The right-hand side does not depend on the space variable x. As the maximal and minimal selections f_+ and f_- of F have globally bounded derivatives, they are and thus F is globally Lipschitz in the second variable. Suitable constants for this problem are $c = 0.908$, $C = 1$, $L = 0.649$, and $\kappa = 0.259$, and we chose $N = 2$, $\rho = 7.8 \cdot 10^{-5}$ and $\delta = 0.0628$, which corresponds to a discretization of the unit circle by 100 vectors, so that the assumptions for the results of Section 4 are satisfied. For larger N it would have been difficult to visualize the solution set.

Figure 4.6 shows the approximation of the solution sets of (41) for $N = 2$ and three values of ε in the coefficient space. A colored point (x_1, x_2) in the plane means that the function $x_1\varphi_1 + x_2\varphi_2$ is an element of $S_N^\rho(\delta)$, where φ_1, φ_2 are the usual piecewise linear basis functions.

A Solvability of algebraic inclusions

The following theorem summarizes the relevant content of [3, Corollary 1 and Theorem 4]. The original statement is formulated in $\mathbb{R}^d$ equipped with the Euclidean norm, but the proof is valid in an arbitrary finite-dimensional Hilbert space.

Theorem 27. *Let $\mathcal{X}$ be a finite-dimensional Hilbert space, let $u_0 \in \mathcal{X}$, and let $G : B_R(u_0) \subset \mathcal{X} \to \mathcal{CC}(\mathcal{X})$ be continuous and ROSL with constant $l < 0$. If $v_0 \in \mathcal{X}$ satisfies $-\frac{1}{l}\operatorname{dist}(v_0, G(u_0)) \le R$, then there exists a solution $u \in B_R(u_0)$ of the algebraic inclusion $v_0 \in G(u)$ with*

$$\|u - u_0\|_{\mathcal{X}} \le -\frac{1}{l}\operatorname{dist}(v_0, F(u_0))_{\mathcal{X}}, \tag{42}$$

and the set

$$S_G(v_0) := \{u \in \mathcal{X} : v_0 \in G(u)\}$$

is compact.

In the present paper, Theorem 27 is applied with $\mathcal{X} = \mathcal{X}_N$, $R = \infty$, and $G = G_N$ in order to characterize the solution set S_N.

References

[1] H.-W. Alt. *Lineare Funktionalanalysis.* Springer Berlin, Heidelberg, 1999.

[2] J.-P. Aubin and H. Frankowska. *Set-Valued Analysis.* Birkhäuser, Boston, 1990.

[3] W.-J. Beyn and J. Rieger. An implicit function theorem for one-sided Lipschitz mappings. To appear in *Set-Valued and Variational Analysis.*

[4] W.-J. Beyn and J. Rieger. The implicit Euler scheme for one-sided Lipschitz differential inclusions. *Discr. and Cont. Dynamical Systems B*, 14:409–428, 2010.

[5] D. Braess. *Finite Elements.* Cambridge University Press, 1997. Theory, fast solvers, and applications in solid mechanics, Translated from the 1992 German original by Larry L. Schumaker.

[6] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to algorithms.* MIT Press, Cambridge, MA, second edition, 2001.

[7] T. Donchev. Properties of one sided Lipschitz multivalued maps. *Nonlinear Analysis*, 49:13–20, 2002.

[8] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics.* American Mathematical Society, 1998.

[9] L. Gasiński and N. S. Papageorgiou. *Nonlinear analysis*, volume 9 of *Series in Mathematical Analysis and Applications.* Chapman & Hall/CRC, Boca Raton, FL, 2006.

[10] L. Hörmander. Sur la fonction d'appui des ensembles convexes dans un espace localement convexe. *Arkiv för Matematik*, 3:181–186, 1955.

[11] S. Larsson and V. Thomée. *Partielle Differentialgleichungen und numerische Methoden.* Springer Berlin, Heidelberg, 2005.

[12] J. Rieger. Discretizations of linear elliptic partial differential inclusions. To appear in *Numerical Functional Analysis and Optimization*.

[13] W. Rudin. *Functional Analysis*. Mc Graw Hill, Boston, 2003.

[14] G. Smirnov. *Introduction to the Theory of Differential Inclusions*, volume 41 of *Graduate Studies in Mathematics*. AMS, 2002.

[15] V. Thomée. *Galerkin finite element methods for parabolic problems*. Number 25 in Springer Series in Computational Mathematics. Springer, 2006.

[16] E. Zeidler. *Nonlinear Functional Analysis and its Applications*, volume 2B. Springer, Heidelberg, 1985.