Stability and Freezing of Waves in Nonlinear Hyperbolic-Parabolic Systems

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September 12, 2011

Abstract

In this note we consider the application of the freezing method to the approximation of traveling waves in hyperbolic-parabolic systems such as the Hodgkin-Huxley model and the FitzHugh-Nagumo equation. The tuple consisting of the profile and the speed of a traveling wave is a stationary solution for the method and we prove its asymptotic stability with optimal rates. Therefore, the method is suitable for the approximation of traveling waves by time integration. Numerical experiments for the FitzHugh-Nagumo equations confirm our results.

1 Introduction

Traveling wave solutions occur in many problems from different areas of applications from biology, chemistry, and physics. Many of these problems are modeled by reaction diffusion equations for which some of the components do not diffuse. For example we mention the electric signalling in nerve cells which is modelled by the spatial extension of the famous Hodgkin-Huxley equations [5]. These equations read

$$\begin{aligned} V_t &= \frac{a}{2R}V_{xx} - \bar{g}_K n^4(V-V_K) - \bar{g}_{Na} m^3 h(V-V_{Na}) - \bar{g}_l(V-V_l),\\ n_t &= \alpha_n(V)\,(1-n) - \beta_n(V)\,n,\\ m_t &= \alpha_m(V)\,(1-m) - \beta_m(V)\,m,\\ h_t &= \alpha_h(V)\,(1-h) - \beta_h(V)\,h, \end{aligned}$$

with nonlinearities α_n, β_n, ... With the variables $u = V$ and $v = (n, m, h)^T$, the system has the form

$$u_t = Au_{xx} + f_1(u,v), \quad v_t = f_2(u,v). \tag{1.1}$$

A traveling wave solution of (1.1) is a solution (u, v) of the form $u(x,t) = u^o(x - \lambda^o t)$, $v(x,t) = v^o(x - \lambda^o t)$, where (u^o, v^o) is the profile and λ^o is the

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speed of the wave. When (1.1) is considered in a co-moving frame with speed λ^o, i.e. the new spatial variable $\xi = x - \lambda^o t$ is used, the equation becomes

$$u_t = Au_{\xi\xi} + \lambda^o u_\xi + f_1(u,v), \quad v_t = \lambda^o v_\xi + f_2(u,v), \tag{1.2}$$

and (u^o, v^o) is a steady state of this system. Note that (1.2) is parabolic in the u variable and (non-strictly) hyperbolic in the v variable.

The Hodgkin-Huxley equations as a model for nerve signalling motivates that in applications it is of big importance not only to know that there is a traveling wave solution, but also to prove that it is asymptotically stable and to calculate the actual profile (u^o, v^o) and the speed λ^o. To calculate the profile and speed numerically, one approximates the equation by a boundary value problem with asymptotic boundary conditions. Then this nonlinear boundary value problem has to be solved numerically. For this a good initial guess is needed which is usually obtained by a long-time simulation of the original dynamic problem.

The difficulty with this long-time simulation is that a traveling wave solution leaves the computational domain in finite time and one also does not have a direct approximation for the speed of the wave. A method to overcome this problem is the so called "freezing method", introduced in [2], and independently in [12]. Its principal idea is to separate the time evolution of the solution into an evolution of the profile and an evolution of a symmetry part, given by an evolution in a Lie group.

We consider general coupled parabolic-hyperbolic Cauchy problems of the form

$$\begin{aligned} u_t = Au_{xx} + g(u,v)_x + f_1(u,v), \quad v_t = Bv_x + f_2(u,v),\ x \in \mathbb{R}, t \geq 0, \\ u(x,0) = u_0(x) \in \mathbb{R}^n, \quad v(x,0) = v_0(x) \in \mathbb{R}^m. \end{aligned} \tag{1.3}$$

On (1.3) we impose the following **general assumptions:**

- The functions satisfy $f_1, g \in \mathcal{C}^3(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and $f_2 \in \mathcal{C}^3(\mathbb{R}^{n+m}, \mathbb{R}^m)$,
- the matrix $A \in \mathbb{R}^{n,n}$ satisfies $A + A^T \geq \alpha > 0$ in the sense of hermitian matrices,
- the matrix $B \in \mathbb{R}^{m,m}$ is a real diagonal matrix,
- there exists a traveling wave solution with profile (u^o, v^o) and speed λ^o, and $(u^o, v^o) \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}^{n+m})$ and $(u_x^o, v_x^o) \in H^2(\mathbb{R}, \mathbb{R}^{n+m})$.

We denote by C_b^k the k-times differentiable functions with continuous and bounded derivatives, H^k denotes the usual Sobolev space of k-times weakly differentiable functions with L^2 derivatives. In [8] it is shown that these assumptions suffice to show existence and uniqueness for the Cauchy problem (1.3) in a neighborhood of the traveling wave.

In the case $n = 0$ we have a semilinear hyperbolic problem which was analyzed in [9, 11].

The plan for the rest of the paper is as follows. In Section 2 we state a nonlinear stability result for traveling waves in system (1.3). In Section 3 we briefly recall the freezing method and describe how it is applied in the setting of this paper. Our main result is Theorem 2 which we state in Section 3. It shows that under certain *spectral assumptions* and a mild *non-degeneracy assumption* on the *phase condition* the freezing method converges exponentially to the profile and speed of the traveling wave with the rate given by the spectral assumption. Section 4 is devoted to the proof of our main result. We finish the paper with numerical experiments for the FitzHugh-Nagumo equations. The results of these experiments confirm the theoretically predicted rates of convergence.

2 Stability of traveling waves

Let us first consider the existence of solutions to (1.3). In [8] we show that, if the general assumptions hold, the problem (1.3) is well-posed for initial data close to the traveling wave. Note that by considering (1.3) in a co-moving frame (see (1.2)), the assumption $\lambda^o = 0$ is no restriction.

Proposition 1 (Existence and uniqueness, [8, Thm 4.9] and [10])**.** *Consider* (1.3) *and assume that the general assumptions are satisfied with* $\lambda^o = 0$. *Then for every initial data* $u_0 \in u^o + H^1(\mathbb{R}, \mathbb{R}^n)$, $v_0 \in v^o + H^1(\mathbb{R}, \mathbb{R}^m)$, *there is a unique global solution, i.e. there is* $T^* \in (0, \infty]$ *and* (u^*, v^*) *so that for all* $0 < T < T^*$ *holds*

$$\begin{aligned} u^* &\in \mathcal{C}([0,T]; u^o + H^1) \cap H^1(0,T; u^o + L^2) \cap L^2(0,T; u^o + H^2), \\ v^* &\in \mathcal{C}([0,T]; v^o + H^1) \cap H^1(0,T; v^o + L^2), \end{aligned} \tag{2.1}$$

and (u^*, v^*) *is the unique solution of* (1.3) *on* $[0,T]$, *where the equality* (1.3) *holds as an equality in* $L^2(\mathbb{R}, \mathbb{R}^n) \times L^2(\mathbb{R}, \mathbb{R}^m)$ *for almost every* $t \in [0,T]$. *Moreover, it holds the dichotomy that either* $T^* = +\infty$ ***or*** $0 < T^* < +\infty$ *and* $\lim_{t \nearrow T^*} \|u^*(t) - u^o\|_{H^1} + \|v^*(t) - v^o\|_{H^1} = +\infty$.

Of course the general assumptions do not suffice to prove stability. To state the needed assumptions, we consider the equation (1.3) in the co-moving frame $\xi = x - \lambda^o t$ and obtain (we again write x instead of ξ)

$$u_t = A u_{xx} + \big(g(u,v) + \lambda^o u\big)_x + f_1(u,v), \quad v_t = \big(B + \lambda^o\big) v_x + f_2(u,v). \tag{2.2}$$

The profile (u^o, v^o) is a steady state of (2.2). Linearization of (2.2) about the profile (u^o, v^o) leads to the linear PDE

$$\begin{aligned} u_t &= A u_{xx} + \big(\partial_1 g^o + \lambda^o\big) u_x + \partial_2 g^o v_x + \big(\partial_1 g^o_x + \partial_1 f^o_1\big) u + \big(\partial_2 g^o_x + \partial_1 f^o_2\big) v, \\ v_t &= \big(B + \lambda^o\big) v_x + \partial_1 f^o_2 u + \partial_2 f^o_2 v, \end{aligned} \tag{2.3}$$

where we abbreviate $g^o(x) = g(u^o(x), v^o(x))$, $\partial_1 g^o(x) = g_u(u^o(x), v^o(x))$, etc. We define P to be the operator on the right hand side of (2.3) that is applied to (u,v). It can be written in the form

$$P \begin{pmatrix} u \\ v \end{pmatrix} = \widetilde{A} \begin{pmatrix} u \\ v \end{pmatrix}_{xx} + \widetilde{B} \begin{pmatrix} u \\ v \end{pmatrix}_x + \widetilde{C} \begin{pmatrix} u \\ v \end{pmatrix}$$

and we consider P as a closed operator on L^2. Note that although in general the solution (u^*, v^*) from Proposition 1 belongs to an affine space, the variables (u,v) in the linearized equation belong to the standard L^2 space. The general assumption imply that the coefficients $\widetilde{B}, \widetilde{C}$ belong to $\mathcal{C}^1_b(\mathbb{R}, \mathbb{R}^{n+m,n+m})$ and are asymptotically constant. Let $\widetilde{B}_\pm := \lim_{x \to \pm\infty} \widetilde{B}(x)$ and $\widetilde{C}_\pm := \lim_{x \to \pm\infty} \widetilde{C}(x)$.

It is well-known that the spatial equivariance of equation (2.2) implies that the spatial derivative (u^o_x, v^o_x) of the profile (u^o, v^o) belongs to the kernel $\mathcal{N}(P)$ of the operator P.

For the stability theorem we require the following **spectral assumptions**:

- The matrix $B_{22} = B + \lambda^o$ is invertible,
- there is $\delta > 0$ so that for all $\omega \in \mathbb{R}$, $s \in \sigma(-\omega^2 \widetilde{A} + i\omega \widetilde{B}_- + \widetilde{C}_-)$ or $s \in \sigma(-\omega^2 \widetilde{A} + i\omega \widetilde{B}_+ + \widetilde{C}_+)$ implies $\operatorname{Re} s \leq -\delta < 0$,
- for the point spectrum $\sigma_{pt}(P)$ of the operator P on L^2 holds $\sigma_{pt}(P) \cap \{\operatorname{Re} s > -\delta\} = \{0\}$ and 0 is an algebraically simple eigenvalue of P.

The stability result, proved in [10], also see [8, Cor. 4.42], is the following:

Theorem 1 (Asymptotic stability of traveling waves [10])**.** *Let the general assumptions and the spectral assumptions hold. Then for every $0 < \eta < \delta$ there is $\rho > 0$ so that for all initial data $u_0 \in u^o + H^2$ and $v_0 \in v^o + H^2$, with $\|u_0 - u^o\|^2_{H^2} + \|v_0 - v^o\|^2_{H^2} < \rho^2$, exists a unique solution (u, v) of (1.3) on $[0, \infty)$. The solution satisfies the smoothness properties from Proposition 1. Moreover, there is $\varphi_\infty = \varphi_\infty(u_0, v_0) \in \mathbb{R}$ and a constant $C_{pde} = C_{pde}(\eta) > 0$, independent of (u_0, v_0), so that*

$$\begin{aligned} |\varphi_\infty| &\leq C_{pde}\big(\|u_0 - u^o\|_{H^2} + \|v_0 - v^o\|_{H^2}\big), \qquad \textit{and} \\ \|u(t) - u^o(\cdot - \lambda^o t - \varphi_\infty)\|_{H^1} &+ \|v(t) - v^o(\cdot - \lambda^o t - \varphi_\infty)\|_{H^1} \\ &\leq C_{pde}\big(\|u_0 - u^o\|_{H^2} + \|v_0 - v^o\|_{H^2}\big)e^{-\eta t} \quad \forall t \geq 0. \end{aligned} \tag{2.4}$$

Due to its importance in applications, the stability of traveling waves in parabolic-hyperbolic PDEs has been analyzed by many authors. Our result is closely related to the result of Kreiss, Kreiss, and Petersson [6] who do not allow a non-strictly hyperbolic part and also do not state precise rates of convergence. Other important results are for example obtained by Evans [3], who analyzed the nerve axon equations. His analysis does not include a nonlinear advection term in the "u"–equation. We also mention the paper [1] by Bates and Jones. Their analysis uses a compactness argument which prohibits the analysis of front solutions, which are included in our result. Finally, recent results are due to Ghazaryan, Latushkin and Schecter [4]. They consider a system from gasless combustion, where the essential spectrum touches the imaginary axis which they overcome by considering semigroups in different weighted spaces, but they also do not allow for a nonlinear advection term.

3 Approximation by the freezing method

We do not give a full introduction to the freezing method [2] and [12] here, but only give the central ideas needed in the special case of traveling waves. For the general formulation we refer to the cited references.

Theorem 1 shows the asymptotic stability of a traveling wave under spectral assumptions. So, in principal, it is possible to approximate the profile by a long-time simulation, but since λ^o is unknown, we cannot use the correct co-moving frame and the solution eventually leaves the computational domain. The idea is to calculate a proper speed –and with this a proper reference frame–

simultaneously. We make the following
Ansatz: Write the solution $\big(u(x,t), v(x,t)\big)$ of the Cauchy problem (1.3) in the form

$$u(x,t) = U(x - \Lambda(t), t), \quad v(x,t) = V(x - \Lambda(t), t). \tag{3.1}$$

We do not make this more precise here, but the following formal calculations lead to the freezing method, which can be implemented on a computer and for which we show in Theorem 2 that it really approximates the profile and the speed of the traveling wave we look for.

Differentiate $u(x,t) = U(x-\Lambda(t), t)$ and $v(x,t) = V(x-\Lambda(t), t)$ with respect to time and obtain with the chain rule and the shift equivariance of (1.3) (we set $\lambda := \dot{\Lambda}$)

$$\begin{gathered} U_t = AU_{xx} + g(U,V)_x + f_1(U,V) + \lambda U_x, \quad V_t = BV_x + f_2(U,V) + \lambda V_x, \\ U(x,0) = u_0(x), \quad V(x,0) = v_0(x), \end{gathered} \tag{3.2}$$

for which $U \equiv u^o$, $V \equiv v^o$, $\lambda \equiv \lambda^o$ is a stationary solution. Because, compared to (1.3), (3.2) has the additional unknown λ, one needs a suitable additional equation, to obtain a well-posed problem again. In this note we only consider the so called *fixed phase condition* (see [2]) which we generalize to have the following form:

$$0 = \Psi(\hat{U} - U, \hat{V} - V), \tag{3.3}$$

where Ψ is a linear functional and $(\hat{U}, \hat{V})$ is a suitable reference function. The full system, consisting of (3.2) and (3.3), is a partial differential algebraic equation (PDAE) for the unknowns (U, V, λ). Of course Ψ and $\hat{U}$, $\hat{V}$, in (3.3) cannot be chosen arbitrarily and we impose the following **phase assumptions**:

- The linear functional Ψ is of the form $\Psi(u,v) = \psi_1(u) + \psi_2(v)$ for $(u,v) \in L^2(\mathbb{R}, \mathbb{R}^{n+m})$, where $\psi_1 \in H^{-1}$ is given by $\psi_1(u) = \int_{\mathbb{R}} h(x)^T u(x)\, dx$ with $h \in H^1(\mathbb{R}, \mathbb{R}^n)$, and $\psi_2 \in (L^2)'$,
- the non-degeneracy condition $\Psi(u^o_x, v^o_x) \neq 0$ holds,
- the reference functions $\hat{U}$ and $\hat{V}$ are elements of the affine spaces $u^o + H^1$ and $v^o + H^1$, respectively, and $\Psi(\hat{U} - u^o, \hat{V} - v^o) = 0$.

Note that the system (3.2) and (3.3) has no initial condition for λ. But using the phase assumptions, (u_0, v_0) close to (u^o, v^o) uniquely determines $\lambda(0)$ by differentiating (3.3) with respect to time. It is a common property of (P)DAEs that for some dependent variables initial data cannot be prescribed but are given by *hidden constraints.* Our main result is

Theorem 2 (Stability of the freezing method)**.** *Impose the* ***general, spectral,*** *and* ***phase assumptions****. Then for every $0 < \eta < \delta$, there is $\rho_0 > 0$ so that for all $u_0 \in u^o + H^2$, $v_0 \in v^o + H^2$ with $\Psi(\hat{U} - u_0, \hat{V} - v_0) = 0$ and $\|u_0 - u^o\|^2_{H^2} + \|v_0 - v^o\|^2_{H^2} < \rho_0$, there is a unique solution (U, V, λ) of the freezing system* (3.2)*,* (3.3)*, i.e. for all $T > 0$, U and V belong to the spaces in* (2.1)*, $\lambda \in \mathcal{C}([0,T]; \mathbb{R})$, and the equalities in* (3.2) *hold in L^2 and equality* (3.3) *holds pointwise. Furthermore, there is $C = C(\eta) > 0$, so that for all $t \geq 0$ holds*

$$\|U(t) - u^o\|_{H^1} + \|V(t) - v^o\|_{H^1} + |\lambda(t) - \lambda^o| \leq C\big(\|u_0 - u^o\|_{H^2} + \|v_0 - v^o\|_{H^2}\big)e^{-\eta t}. \tag{3.4}$$

4 Proof of Theorem 2

Let $0 < \eta < \delta$ be given. By going into a co-moving frame it is no restriction to assume $\lambda^o = 0$. To simplify notation, we define

$$F_p(u,v) := A_{11}u_{xx} + g(u,v)_x + f_1(u,v) \quad \text{and} \quad F_h(u,v) := B_{22}v_x + f_2(u,v),$$

$$R_\rho = \Big\{ (u_0,v_0) : u_0 \in u^o + H^2, v_0 \in v^o + H^2, \\ \|u_0 - u^o\|_{H^2}^2 + \|v_0 - v^o\|_{H^2}^2 \le \rho^2,\ \Psi(\hat{u} - u_0, \hat{v} - v_0) = 0 \Big\}.$$

Step 1: [Solution of the PDE] By Theorem 1 exist $\rho > 0$ and $C_{pde} > 0$, so that for $(u_0,v_0) \in R_\rho$ equation (1.3) has a unique solution (u,v) on $[0,\infty)$ and there is $\varphi_\infty \in \mathbb{R}$ so that (2.4) holds. In the following $(\mathbf{u},\mathbf{v})$ always denotes this solution of (1.3).

Step 2: [Solution ansatz] Ansatz (3.1) is now used to obtain a solution of (3.2), (3.3) from the PDE solution.

Lemma 1. *Let $(u_0,v_0) \in R_\rho$, ρ from Step 1. If $\Lambda \in \mathcal{C}^1([0,T];\mathbb{R})$ satisfies*

$$\Psi\Big(\mathbf{u}_x(\cdot+\Lambda,t), \mathbf{v}_x(\cdot+\Lambda,t)\Big)\dot{\Lambda} \\ = -\Psi\Big(F_p\big(\mathbf{u}(\cdot+\Lambda,t),\mathbf{v}(\cdot+\Lambda,t)\big), F_h\big(\mathbf{u}(\cdot+\Lambda,t),\mathbf{v}(\cdot+\Lambda,t)\big)\Big), \quad (4.1)$$

for all $t \in [0,T]$ and $\Lambda(0) = 0$. Then the triple $(U,V,\dot{\Lambda})$, given by $U(\cdot,t) = \mathbf{u}(\cdot+\Lambda(t),t)$ and $V(\cdot,t) = \mathbf{v}(\cdot+\Lambda(t),t)$ is a solution of (3.2), (3.3) on $[0,T]$.

Proof. By assumption and Step 1, the functions (U,V,λ), $\lambda = \dot{\Lambda}$, belong to the smoothness classes asserted in Theorem 2. Moreover, for the distributional derivatives hold

$$\frac{d}{dt}U(t) = F_p(U,V)(t) + \dot{\Lambda}(t)U_x(t), \quad \frac{d}{dt}V(t) = F_h(U,V)(t) + \dot{\Lambda}(t)V_x(t), \quad (4.2)$$

as equalities in L^2 for a.e. $t \in [0,T]$. The assumptions $\hat{U} \in u^o + H^1$ and $\hat{V} \in v^o + H^1$ imply $\hat{U} - U \in H^1([0,T];L^2)$ and $\hat{V} - V \in H^1([0,T];L^2)$. It follows (see [8, Lem. 4.18]) $\Psi(\hat{U}-U,\hat{V}-V) \in H^1([0,T];\mathbb{R})$ and its distributional derivative is given by

$$\frac{d}{dt}\Psi(\hat{U}-U,\hat{V}-V) = -\Psi(F_p(U,V),F_h(U,V)) - \dot{\Lambda}\Psi(U_x,V_x) \quad \text{for a.e. } t \in [0,T].$$

But because of (4.1) the right hand side is equal to zero and from continuity and $\Psi(\hat{U}-u_0,\hat{V}-v_0) = 0$ follows $\Psi(\hat{U}-U,\hat{V}-V) \equiv 0$, so that (3.2) and (3.3) hold. □

Step 3: [Solvability] By Lemma 1 a solution of (3.2), (3.3) can be obtained by solving the ODE (4.1). Because of (2.4) there are $\rho_1, \rho_\Lambda > 0$, so that for all $(u_0,v_0) \in R_{\rho_1}$ holds for all $|\Lambda| \le \rho_\Lambda$ and all $t \ge 0$ the lower bound

$$\Big|\Psi\big(\mathbf{u}_x(\cdot+\Lambda,t),\mathbf{v}_x(\cdot+\Lambda,t)\big)\Big| \ge \frac{|\Psi(u_x^o,v_x^o)|}{2}. \quad (4.3)$$

Therefore, $r : B_{\rho_\Lambda}(0) \times [0,\infty) \to \mathbb{R}$, given by

$$r(\Lambda,t) = -\frac{\Psi\big(F_p\big(\mathbf{u},\mathbf{v}\big)(\cdot+\Lambda,t), F_h\big(\mathbf{u},\mathbf{v}\big)(\cdot+\Lambda,t)\big)}{\Psi\big(\mathbf{u}_x(\cdot+\Lambda,t), \mathbf{v}_x(\cdot+\Lambda,t)\big)},$$

is well defined and it is not difficult to see that the assumptions on Ψ are precisely what is needed to prove continuity of r if Λ is sufficiently small. Thus, the Theorem of Peano applies and yields

Lemma 2. *If (u_0, v_0) is chosen as above, there is a solution Λ of the ODE* (4.1) *and either Λ is a solution for all $t \geq 0$ with $\Lambda \in \mathcal{C}^1([0,\infty); B_{\rho_\Lambda}(0))$, or there is $0 < T^* < \infty$ and $\Lambda \in \mathcal{C}^1([0,T^*); B_{\rho_\Lambda}(0))$, with $\lim_{t \nearrow T^*} |\Lambda(t)| = \rho_\Lambda$.*

In [8] (also see [11] for the purely hyperbolic case) it is shown that if ρ_1 is sufficiently small, the second case actually never occurs, so that one obtains global existence. Moreover, despite the fact that we have utilized the Theorem of Peano to obtain a solution, also possible to proved that the solution in fact is unique (for the details we refer to [8]). More precisely, it holds

Lemma 3 ([8, Lem. 4.48 and Lem. 4.50])**.** *There is $\rho_0 > 0$, $\rho_0 \leq \rho_1$ so that for all $(u_0, v_0) \in R_{\rho_0}$, the ODE* (4.1)*, has a unique global solution $\Lambda \in \mathcal{C}^1([0,\infty); B_{\rho_\Lambda}(0))$. There is $C_\varphi > 0$, independent of $(u_0,v_0) \in R_{\rho_0}$, so that for all $t \geq 0$ holds with φ_∞ from Step 1*

$$|\Lambda(t) - \varphi_\infty| \leq C_\varphi\big(\|u_0 - u^o\|_{H^2} + \|v_0 - v^o\|_{H^2}\big)e^{-\eta t}. \tag{4.4}$$

Step 4: [Unique solvability of the freezing system (3.2)**,** (3.3)] Solvability follows from the previous steps and uniqueness follows from the unique solvability of the Cauchy problem (1.3) and of the ODE (4.1). For the details we refer to [8].

Step 5: [Exponential convergence] By Steps 1–4, the unique solution of the freezing system (3.2), (3.3) is given by the triple (U,V,λ), where $U = \mathbf{u}(\cdot+\Lambda(t),t)$, $V = \mathbf{v}(\cdot+\Lambda(t),t)$, and $\lambda = \dot\Lambda$ with Λ the unique solution of (4.1). Therefore, we estimate

$$\begin{aligned}&\|U(t) - u^o\|_{H^1}^2 + \|V(t) - v^o\|_{H^1}^2 \\ &\quad\leq 2\Big\{\big\|\mathbf{u}(\cdot,t) - u^o(\cdot-\varphi_\infty)\big\|_{H^1}^2 + \big\|u^o(\cdot-\varphi_\infty) - u^o(\cdot-\Lambda(t))\big\|_{H^1}^2 \\ &\qquad + \big\|\mathbf{v}(\cdot,t) - v^o(\cdot-\varphi_\infty)\big\|_{H^1}^2 + \big\|v^o(\cdot-\varphi_\infty) - v^o(\cdot-\Lambda(t))\big\|_{H^1}^2\Big\}\end{aligned}$$

and find with (2.4), the assumptions on (u^o, v^o), and (4.4), that this can be bounded by

$$\leq 2\Big(C_{pde}^2 + C_\varphi^2\big(\|u_x^o\|_{H^1} + \|v_x^o\|_{H^1}\big)^2\Big)\big(\big\|u_0 - u^o\big\|_{H^2} + \big\|v_0 - v^o\big\|_{H^2}\big)^2 e^{-2\eta t}. \tag{4.5}$$

To prove the claimed estimate for the convergence of λ we use the identities $\lambda(t) = \dot\Lambda(t)$ and $\dot\Lambda(t) = r(\Lambda(t),t)$, so that (4.3) and $F_p(u^o,v^o) = 0$ and $F_h(u^o,v^o) = 0$ show

$$|\lambda(t)| \leq 2\frac{\big|\psi_1\big(F_p(U,V) - F_p(u^o,v^o)\big)\big| + \big|\psi_2\big(F_h(U,V) - F_h(u^o,v^o)\big)\big|}{|\Psi(u_x^o, v_x^o)|}. \tag{4.6}$$

For the ψ_1–term we obtain

$$\begin{aligned}&\big|\psi_1\big(F_p(U(t),V(t)) - F_p(u^o,v^o)\big)\big| \le \big|\psi_1\big(A_{11}(U(t)-u^o)_{xx}\big)\big| \\ &\quad + \Big|\psi_1\big(\big(g(U(t),V(t)) - g(u^o,v^o)\big)_x\big) + \psi_1\big(f_1(U(t),V(t)) - f_1(u^o,v^o)\big)\Big|.\end{aligned}$$

Because of the assumption on ψ_1, the first summand on the right hand side can be bounded by

$$\big|\psi_1\big(A_{11}(U(t)-u^o)_{xx}\big)\big| \le \mathrm{const}\|U(t)-u^o\|_{H^1}.$$

Similar estimates also hold for the other summands because of the continuity of ψ_1 on L^2 and the smoothness of g and f_1 together with Sobolev embedding. Therefore,

$$\big|\psi_1\big(F_p(U,V)) - F_p(u^o,v^o)\big)\big| \le \mathrm{const}\big(\|U(t)-u^o\|_{H^1}^2 + \|V(t)-v^o\|_{H^1}^2\big)^{1/2}.$$

Since $\psi_2 \in (L^2)'$ a similar estimate holds for the ψ_2 term. Combination of these estimates yields

$$|\lambda(t)| \le \mathrm{const}\big(\|U(t)-u^o\|_{H^1}^2 + \|V(t)-v^o\|_{H^1}^2\big)^{1/2}, \tag{4.7}$$

so that together with (4.5)–(4.7) follows asserted estimate (3.4). This finishes the proof of Theorem 2. □

5 Results for the FitzHugh-Nagumo equation

We apply the freezing method (3.2), (3.3) in the following way: The system (3.2), (3.3) is a PDAE of differentiation index 2 with respect to time (see [7]). We differentiate equation (3.3) with respect to time in order to reduce the index of the system to 1, this yields

$$\begin{aligned}U_t &= AU_{xx} + g(U,V)_x + f_1(U,V) + \lambda U_x,\\ V_t &= BV_x + f_1(U,V) + \lambda V_x,\\ 0 &= \Psi(U_t,V_t).\end{aligned} \tag{5.1}$$

Inserting the first two equalities of (5.1) into the last expression yields an equation that can be solved for λ if U_x is close to u^o_x and V_x is close to v^o_x:

$$\lambda = -\frac{\Psi\big(AU_{xx} + g(U,V)_x + f_1(U,V), BV_x + f_1(U,V)\big)}{\Psi\big(U_x,V_x\big)}.$$

We now apply our results to the FitzHugh-Nagumo equations, which read

$$\begin{gathered}u_t = u_{xx} + u - \tfrac{1}{3}u^3 - v, \quad v_t = \phi(u + a - bv),\ x \in \mathbb{R}, t \ge 0,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x).\end{gathered} \tag{5.2}$$

We choose the parameter values $a = 0.7$, $b = 3$, $\phi = 0.08$. For this choice there exists a traveling front solution connecting the two rest states $(u_{-\infty}, v_{-\infty})$ and $(u_{+\infty}, v_{+\infty})$ which approximately are $(1.188, 0.629)$ and $(-1.564, -0.288)$, respectively. The general assumptions and also first of the spectral assumptions

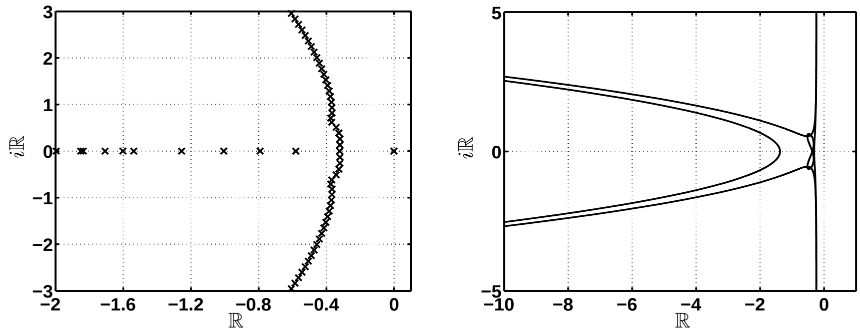

Figure 1: Left: Approximated pointspectrum. Right: Dispersion curves.

are easily verified. To verify the rest of the spectral assumptions, assume $v \in \mathbb{C}^2$ is an eigenvector of $(-\omega^2 \widetilde{A} + i\omega \widetilde{B}_- + \widetilde{C}_-)$ for some $\omega \in \mathbb{R}$, i.e.

$$Mv := \begin{pmatrix} -\omega^2 + i\lambda^o \omega + 1 - u_{-\infty}^2 & -1 \\ \phi & i\lambda^o\omega - b\phi \end{pmatrix} v = sv.$$

Let $H = \operatorname{diag}(1, \frac{1}{\phi})$ and it follows

$$2\operatorname{Re}(s) v^* \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\phi} \end{pmatrix} v = v^*\big(HM + M^*H\big)v = 2v^* \begin{pmatrix} -\omega^2 + 1 - u_{-\infty}^2 & 0 \\ 0 & -b \end{pmatrix} v,$$

what immediately implies $\operatorname{Re}(s) \le \max(-\omega^2 + 1 - u_{-\infty}^2, -b\phi) = -0.24$. The same argument also holds for $+\infty$. For the location of the point spectrum we use a numerical approximation whose result is given in Figure 1. For the approximation we used a good calculation of the traveling front and calculate the spectrum of the equation, linearized about this front and discretized by finite differences with upwinding for the first order derivatives and periodic boundary conditions. In summary, we find that the **spectral assumptions** hold with $\delta = 0.24$. As phase condition we choose $\hat{U} = u_0$, $\hat{V} = v_0$, $\psi_1 = 0$, and $\psi_2(v) = \int_{-5}^{5} v(x)\,dx$, so that the phase assumptions hold.

The first two pictures of Figure 2 show a numerical simulation of the u–variable for the original and for the frozen system. One can nicely see that the solution of the original system leaves the computational domain very quickly and a much larger domain would be needed. In contrast to this the solution

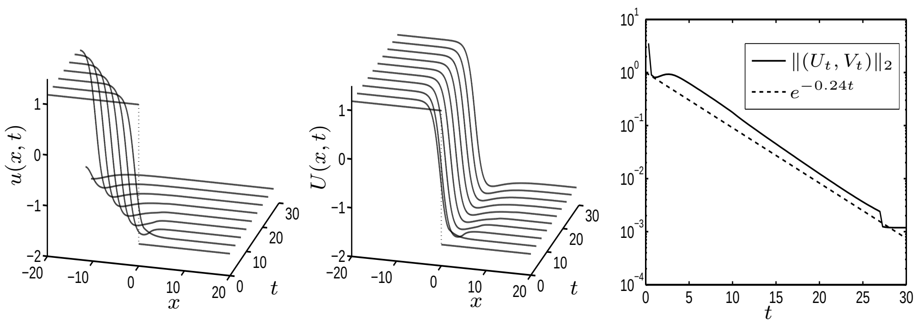

Figure 2: Left: Evolution of u for the original system. Middle: Evolution of U for the freezing system. Right: Convergence of $\|U_t\|_{L^2} + \|V_t\|_{L^2} \to 0$.

of the frozen system remains in the computational domain and converges to a steady state. The last picture shows the L^2–convergence of the frozen solution to this steady state. The observed rate seems to be slightly better than the rate $e^{-0.24t}$ predicted by Theorem 2. Also note, that the numerically approximated spectrum shows a gap of size $\delta = 0.32$. Solely relying on this spectral gap largely overestimates the rate of convergence.

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