A numerical method for the solution of relaxed one-sided Lipschitz algebraic inclusions

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Abstract

An existing solvability result for relaxed one-sided Lipschitz algebraic inclusions is substantially improved. This enhanced solvability result allows the design of a robust iterative method for the numerical solution of the algebraic inclusion. Sharp error estimates for this method, illustrative analytic examples and a numerical example are provided.

Key words. Set-valued analysis, algebraic inclusions, relaxed one-sided Lipschitz property, numerical method.
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1 Introduction and notation

The solution of nonlinear equations and inclusions is one of the fundamental problems in pure and applied mathematics. A multitude of analytical concepts for the identification and localization of solutions as well as numerical methods for their approximation have been developed that exploit characteristic features of particular types of mappings. In this paper, solutions of the algebraic inclusion

$$\bar{y} \in F(x) \tag{1}$$

with given $\bar{y} \in \mathbb{R}^d$ are considered for the class of relaxed one-sided Lipschitz (ROSL, see below) multivalued mappings F with negative one-sided Lipschitz bound. The relatively modern ROSL property was introduced and investigated in [5] and other works of the same author. It generalizes the classical one-sided Lipschitz property and is a key criterion for the analysis of differential inclusions and numerical approximations of their solution sets (see e.g. [6]), where algebraic inclusions of type (1) with ROSL multifunctions F arise in a natural way. Moreover, the ROSL property is intimately related to the notion of metric regularity, which is discussed in [7, Chapter 3].

A solvability result for the class of multivalued mappings satisfying the ROSL property was proved in [3, Corollary 3]. It states that given an initial guess $\tilde{x}$, there exists a solution $\bar{x}$ of (1) in a closed ball centered at $\tilde{x}$ with radius depending on the residual $\operatorname{dist}(\bar{y}, F(\tilde{x}))$. A substantially improved version of this result is given in Theorem 2 below, which allows to localize a solution of (1) in a smaller ball B with $\tilde{x} \in \partial B$ and thus specifies not only a distance but also a direction in which a solution is to be found (see Figure 1). Moreover, we generalize the ROSL constant to an ROSL matrix bound which leads to a-priori balls with respect to an adapted inner product. f the mapping F is in addition Lipschitz continuous, then the localization of the solution can once again be strengthened.

This information can be used to design a very robust numerical algorithm for the approximation of a solution of (1) that uses the current state as initial guess for the improved solvability theorem and defines the next iterate as the center of the ball B. Proposition 7 provides error estimates for this numerical scheme, and Example 9 shows that they are sharp for dimension $d > 1$. The one-dimensional case is treated separately in Proposition 11. Enhancements of the numerical method for L-Lipschitz multimaps F are briefly analyzed in

Propositions 12 and 13, and a numerical example is provided.

With every symmetric positive definite matrix $P \in \mathbb{R}^{d\times d}$ we associate the inner product and norm

$$\langle x, x'\rangle_P = \langle Px, x'\rangle, \quad |x|_P^2 = \langle x, x\rangle_P, \quad x, x' \in \mathbb{R}^d,$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the standard Euclidean inner product and norm, respectively. Closed balls with respect to $|\cdot|_P$ will be denoted by $B_P(x, R) = \{x' \in \mathbb{R}^d : |x' - x|_P \le R\}$. The family of nonempty compact and convex subsets of $\mathbb{R}^d$ is denoted by $\mathcal{CC}(\mathbb{R}^d)$, the one-sided Hausdorff-distance of two sets $A, B \in \mathcal{CC}(\mathbb{R}^d)$ is defined by

$$\mathrm{dist}_P(A, B) := \sup_{a\in A} \inf_{b\in B} |a - b|_P,$$

and the so-called norm of a set $A \in \mathcal{CC}(\mathbb{R}^d)$ is $\|A\|_P := \max_{a\in A} |a|_P$. The metric projection of a point $y \in \mathbb{R}^d$ to a set $A \in \mathcal{CC}(\mathbb{R}^d)$ is the unique point $\mathrm{Proj}_P(y, A) \in A$ satisfying $|y - \mathrm{Proj}_P(y, A)|_P = \mathrm{dist}_P(y, A)$. For all these notions we drop the index P in case of the standard Euclidean norm, i.e. when P is the identity.

Consider a multivalued mapping $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$. It is called upper semicontinuous (usc) at $x \in \mathbb{R}^d$ if

$$\mathrm{dist}(F(x'), F(x)) \to 0 \text{ as } x' \to x,$$

usc if it is usc at every $x \in \mathbb{R}^d$, and L_P-Lipschitz with respect to $|\cdot|_P$ and $|\cdot|_{P^{-1}}$ if

$$\mathrm{dist}_{P^{-1}}(F(x), F(x')) \le L_P|x - x'|_P \text{ for all } x, x' \in \mathbb{R}^d. \tag{2}$$

We call the mapping F relaxed one-sided Lipschitz with matrix $\Lambda \in \mathbb{R}^{d\times d}$ (or Λ-ROSL) if for any $x, x' \in \mathbb{R}^d$ and $y \in F(x)$, there exists some $y' \in F(x')$ such that

$$\langle y - y', x - x'\rangle \le \langle \Lambda(x - x'), x - x'\rangle.$$

Note that this definition generalizes the standard notion of an ROSL constant l (see [5],[4]) which corresponds to the case $\Lambda = lI_d$. In the following the mapping F will be assumed to be $(-P)$-ROSL where $P \in \mathbb{R}^{d\times d}$ is positive definite and symmetric. In some examples and in the one-dimensional case, it makes sense to return to the standard ROSL notion.

The following lemma generalizes the well-known fact that for classical l-ROSL and L-Lipschitz multifunctions with $l < 0$, the Lipschitz constants satisfy the relation $-l \leq L$.

Lemma 1. *If F is $(-P)$-ROSL and L_P-Lipschitz in the above sense, then $L_P \geq 1$.*

Proof. As P is positive definite and symmetric, its Cholesky decomposition $P = CC^T$ exists. Take any $x, x' \in \mathbb{R}^d$ with $x \neq x'$ and $y \in F(x)$. By the ROSL property, there exists some $y' \in F(x')$ such that

$$\begin{aligned}\langle y - y', x - x' \rangle &\leq -\langle P(x - x'), x - x' \rangle \\ &= -\langle C^T(x - x'), C^T(x - x') \rangle = -|C^T(x - x')|^2.\end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned}|y - y'|^2_{P^{-1}} &= \langle (CC^T)^{-1}(y - y'), y - y' \rangle \\ &= \langle C^{-1}(y - y'), C^{-1}(y - y') \rangle = |C^{-1}(y - y')|^2,\end{aligned}$$

and hence

$$\begin{aligned}\langle y - y', x - x' \rangle &= \langle CC^{-1}(y - y'), x - x' \rangle = \langle C^{-1}(y - y'), C^T(x - x') \rangle \\ &\geq -|C^{-1}(y - y')| \cdot |C^T(x - x')| = -|y - y'|_{P^{-1}} \cdot |C^T(x - x')|.\end{aligned} \tag{4}$$

Combining (3) and (4) yields

$$|y - y'|_{P^{-1}} \geq |C^T(x - x')| = |x - x'|_P,$$

so that $L_P \geq 1$ is forced. $\square$

2 Solvability of ROSL algebraic inclusions

The following theorem is the core of this paper. It is a strongly improved version of the solvability theorem given in [3, Corollary 3], and its assumptions on the mapping F can still be weakened (see Remark 5). Its statement is illustrated in Figure 1.

Theorem 2. *Let $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be usc and $(-P)$-ROSL with a positive definite matrix $P \in \mathbb{R}^{d \times d}$, and let $\tilde{x} \in \mathbb{R}^d$ and $\bar{y} \in \mathbb{R}^d$ be given. Then there exists a solution*

$$\bar{x} \in S_F(\bar{y}) := \{x \in \mathbb{R}^d : \bar{y} \in F(x)\}$$

satisfying

$$|\bar{x} - x_c|_P \leq \frac{1}{2} \operatorname{dist}_{P^{-1}}(\bar{y}, F(\tilde{x})), \tag{5}$$

where

$$x_c = \tilde{x} - \frac{1}{2} P^{-1}(\bar{y} - \operatorname{Proj}_{P^{-1}}(\bar{y}, F(\tilde{x}))) \tag{6}$$

and the set $S_F(\bar{y})$ is closed. If F is in addition L_P-Lipschitz w.r.t. $|\cdot|_P$ and $|\cdot|_{P^{-1}}$, then for any $\bar{x} \in S_F(\bar{y})$,

$$|\bar{x} - \tilde{x}|_P \geq \frac{1}{L_P} \operatorname{dist}_{P^{-1}}(\bar{y}, F(\tilde{x})). \tag{7}$$

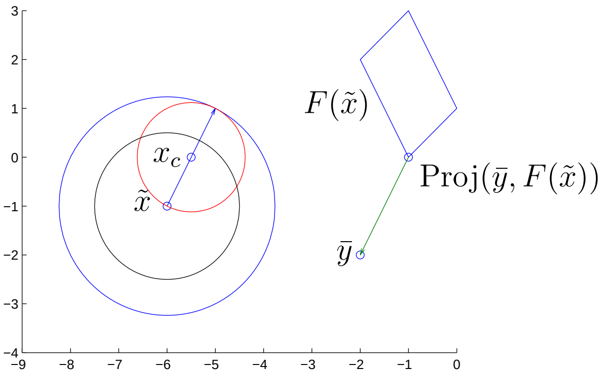

Figure 1: Schematic illustration of Theorem 2 in the classical case $-P = lI_d$, $l < 0$. The solvability theorem given in [3, Corollary 3] only guarantees the existence of a solution $\bar{x}$ of $\bar{y} \in F(x)$ in the (blue) ball of radius $-\frac{1}{l} \operatorname{dist}(\bar{y}, F(\tilde{x}))$ centered at $\tilde{x}$. Theorem 2 guarantees such a solution in the (red) ball with radius $-\frac{1}{2l} \operatorname{dist}(\bar{y}, F(\tilde{x}))$ centered at $x_c = \tilde{x} + \frac{1}{2l}(\bar{y} - \operatorname{Proj}(\bar{y}, F(\tilde{x})))$, and if F is L-Lipschitz, it states that no solution is contained in the (black) ball of radius $\frac{1}{L} \operatorname{dist}(\bar{y}, F(\tilde{x}))$ centered at $\tilde{x}$.

The following Lemma shows the first assertion of Theorem 2 for the special case $\tilde{x} = \bar{y} = 0$.

Lemma 3. *Let $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ be usc and $(-P)$-ROSL with positive definite P. Then the inclusion $0 \in F(x)$ has a solution $\bar{x}$ with*

$$\langle P\bar{x}, \bar{x} \rangle \leq \langle y_0, \bar{x} \rangle, \quad \textit{where} \quad y_0 = \operatorname{Proj}_{P^{-1}}(0, F(0)) \tag{8}$$

that satisfies the property

$$|\bar{x} - x_c|_P \leq |x_c|_P = \frac{1}{2}\,\mathrm{dist}_{P^{-1}}(0, F(0)), \quad \textit{for} \quad x_c = \frac{1}{2}P^{-1}y_0. \tag{9}$$

Proof. By definition the element $y_0 = \mathrm{Proj}_{P^{-1}}(0, F(0))$ minimizes the value of $|P^{-1}y|_P = |y|_{P^{-1}}, y \in F(0)$ and we set $R_0 = |P^{-1}y_0|_P = \mathrm{dist}_{P^{-1}}(0, F(0))$. By the ROSL property of F, the mapping $\Psi : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ given by

$$\Psi(x) := F(x) \cap \{y \in \mathbb{R}^d : \langle y - y_0, x\rangle \leq -\langle Px, x\rangle\}$$

has nonempty images. By [1, Theorem 1.1.1], it is usc. Define the usc mapping $G : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ by

$$G(x) := x + \alpha P^{-1}\Psi(x)$$

where $\alpha > 0$ will be chosen below. For $R > R_0$ we consider $x \in B_P(0, R)$, $y \in \Psi(x)$ and set $z := x + \alpha P^{-1}y$. Then, for $\alpha \leq \frac{1}{2}$,

$$\begin{aligned} |z|_P^2 &= |x|_P^2 + 2\alpha\langle P^{-1}y, x\rangle_P + \alpha^2|P^{-1}y|_P^2 \\ &= |x|_P^2 + 2\alpha\langle y - y_0, x\rangle + 2\alpha\langle P^{-1}y_0, x\rangle_P + \alpha^2|P^{-1}y|_P^2 \\ &\leq |x|_P^2 + 2\alpha\langle -Px, x\rangle + 2\alpha|P^{-1}y_0|_P|x|_P + \alpha^2|P^{-1}y|_P^2 \\ &\leq R^2 + 2\alpha R(-R + R_0 + \frac{\alpha}{2R}|P^{-1}y|_P^2). \end{aligned} \tag{10}$$

As F is usc,

$$M_R := \sup_{x \in B_P(0,R)} \|F(x)\|_{P^{-1}} < \infty,$$

and there exists an $\alpha > 0$ such that $|z|_P^2 \leq R^2$ follows from (10) and $R > R_0$. This means that for this fixed α,

$$H(x) := G(x) \cap B_P(0, R) \neq \emptyset \text{ for all } x \in B_P(0, R),$$

and $H(\cdot)$ is also usc. By the Kakutani Theorem (see [2, Theorem 3.2.3]), H and thus also G have a fixed point x_R in $B_P(0, R)$, which implies that $0 \in \Psi(x_R)$.

In particular, we find elements $x_n \in B_P(0, R_0 + 1/n)$ for all $n \in \mathbb{N}$ such that $0 \in \Psi(x_n)$. As $B_P(0, R_0 + 1)$ is compact, there exists a convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ with limit

$$\bar{x} \in B_P(0, R_0).$$

Since Ψ is usc,

$$0 \in \Psi(\bar{x}) \subset F(\bar{x}).$$

Property (8) follows from the construction of Ψ. Finally, taking squares in (9) we obtain that (8) and (9) are equivalent. $\square$

Remarks 4. a) *The proof shows that statements* (8),(9) *hold for any vector* $y_0 \in F(0)$. *The particular choice of* y_0 *minimizes the radius of the a-priori ball* $B_P(0, R_0)$. *Note, however, that* (9) *implies* $|\bar{x}|_P \leq 2|x_c|_P = R_0$, *hence the a-priori ball* $B_P(x_c, |x_c|_P)$ *from* (9) *is always contained in* $B_P(0, R_0)$.
b) *For the special* $-P = lI_d$ *with* $l < 0$ *equation* (9) *yields the estimate*

$$|\bar{x}| = \frac{1}{\sqrt{|l|}}|\bar{x}|_P \leq \frac{1}{\sqrt{|l|}} \operatorname{dist}_{P^{-1}}(0, F(0)) = \frac{1}{|l|} \operatorname{dist}(0, F(0)),$$

which agrees with the result from [4, Theorem 1].

Proof of Theorem 2. Consider the set-valued mapping

$$G(z) := F(z + \tilde{x}) - \bar{y},$$

which is $(-P)$-ROSL. By Lemma 3 there exists some $\bar{z}$ with $0 \in G(\bar{z})$ and

$$|\bar{z} - \frac{1}{2}P^{-1}y_0|_P \leq |\frac{1}{2}P^{-1}y_0|_P$$

for

$$y_0 = \operatorname{Proj}_{P^{-1}}(0, G(0)) = \operatorname{Proj}_{P^{-1}}(\bar{y}, F(\tilde{x})) - \bar{y}.$$

Defining $\bar{x} = \tilde{x} + \bar{z}$ and $x_c = \tilde{x} + \frac{1}{2}P^{-1}y_0$ we obtain $\bar{y} \in F(\bar{x})$ and the assertion (5).

The fact that $S_F(\bar{y})$ is closed follows directly from the usc property of F.

If F is in addition L_P-Lipschitz w.r.t. $|\cdot|_P$ and $|\cdot|_{P^{-1}}$ and $\bar{x} \in S_F(\bar{y})$, then

$$\operatorname{dist}_{P^{-1}}(\bar{y}, F(\tilde{x})) \leq \operatorname{dist}_{P^{-1}}(F(\bar{x}), F(\tilde{x})) \leq L_P|\bar{x} - \tilde{x}|_P$$

implies

$$|\bar{x} - \tilde{x}|_P \geq \frac{1}{L_P} \operatorname{dist}_{P^{-1}}(\bar{y}, F(\tilde{x})).$$

$\square$

Remark 5. *The assumptions of Theorem 2 can be weakened. In particular, the set-valued mapping* F *may be defined only on* $B := B_P(\tilde{x}, \operatorname{dist}_{P^{-1}}(\bar{y}, F(\tilde{x})))$.

a) In order to obtain the existence of a solution and estimate (5)*, it is sufficient to require that* $F : B \to \mathcal{CC}(\mathbb{R}^d)$ *is usc and that for all* $x \in B$ *there exists a* $y \in F(x)$ *satisfying*

$$\langle y - \mathrm{Proj}_{P^{-1}}(\bar{y}, F(\tilde{x})), x - \tilde{x}\rangle \leq -\langle P(x - \tilde{x}), x - \tilde{x}\rangle. \qquad (11)$$

The mapping F *can then be extended by the same construction as in [4, proof of Theorem 2] to a set-valued function* $\tilde{F} : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ *that coincides with* F *on* B*, is usc, and satisfies property* (11) *for all* $x \in \mathbb{R}^d$*. The proof of Theorem 2 can be applied to the mapping* $\tilde{F}$ *without changes.*

b) To show estimate (7)*, it is enough for* $F : B \to \mathcal{CC}(\mathbb{R}^d)$ *to be L-Lipschitz relative to* $\tilde{x}$ *in the sense that*

$$\mathrm{dist}_{P^{-1}}(F(x), F(\tilde{x})) \leq L_P|x - \tilde{x}|_P \text{ for all } x \in B.$$

It follows directly that for any $\bar{x} \in S_F(\bar{y}) \cap B$*,*

$$|\bar{x} - \tilde{x}|_P \geq \frac{1}{L_P}\,\mathrm{dist}_{P^{-1}}(\bar{y}, F(\tilde{x})).$$

In fact, this estimate holds for all $\bar{x} \in S_F(\bar{y})$*, because* $L_P \geq 1$ *according to Lemma 1.*

Remark 6. *It is unclear if additional assumptions are needed to guarantee the connectedness of* $S_F(\bar{y})$*. This question is linked with the parametrization problem for ROSL multifunctions (see Lemma 12 in [3]).*

3 A numerical solver for ROSL algebraic inclusions

A numerical method for finding a solution $\bar{x}$ of the inclusion $\bar{y} \in F(x)$ can be deduced directly from Theorem 2 by defining the next iterate of the scheme as the center of the ball specified by (5).

Throughout this section, the mapping $F : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)$ will be assumed to be $(-P)$-ROSL with symmetric positive definite P and L_P-Lipschitz w.r.t. $|\cdot|_P$ and $|\cdot|_{P^{-1}}$. For a scalar ROSL bound $-P = lI_d$ with $l < 0$, this condition holds with $L_P = \frac{L}{|l|}$ provided F is Lipschitz with constant L with respect to the standard Hausdorff distance dist induced by the Euclidean norm.

Proposition 7. *Assume* (2) *with* $\kappa = \frac{L_P}{2} < 1$, *and let* $x_0 \in \mathbb{R}^d$ *and* $\bar{y} \in \mathbb{R}^d$ *be given. Then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by*

$$x_{n+1} := \Phi(x_n) := x_n - \frac{1}{2}P^{-1}(\bar{y} - \mathrm{Proj}_{P^{-1}}(\bar{y}, F(x_n))) \tag{12}$$

converges to a solution $\bar{x}$ *of the inclusion* $\bar{y} \in F(x)$ *and satisfies the estimates*

$$\mathrm{dist}_P(x_n, S_F(\bar{y})) \leq \frac{1}{2}\kappa^{n-1}\,\mathrm{dist}_{P^{-1}}(\bar{y}, F(x_0)) \tag{13}$$

and

$$|x_n - \bar{x}|_P \leq \frac{1}{2}\frac{\kappa^n}{1-\kappa}\,\mathrm{dist}_{P^{-1}}(\bar{y}, F(x_0)) \tag{14}$$

for $n \geq 1$.

Proof. Set $v_n := \bar{y} - \mathrm{Proj}_{P^{-1}}(\bar{y}, F(x_n))$ for $n \in \mathbb{N}$. Then (5) implies that there exists some $\bar{x}_n \in S_F(\bar{y})$ such that

$$\mathrm{dist}_P(x_{n+1}, S_F(\bar{y})) \leq |\bar{x}_n - (x_n - \frac{1}{2}P^{-1}v_n)|_P \leq \frac{1}{2}|P^{-1}v_n|_P. \tag{15}$$

Now

$$\begin{aligned} |P^{-1}v_{n+1}|_P &= \mathrm{dist}_{P^{-1}}(\bar{y}, F(x_{n+1})) \leq \mathrm{dist}_{P^{-1}}(F(\bar{x}_n), F(x_{n+1})) \\ &\leq L_P|\bar{x}_n - x_{n+1}|_P \leq \frac{L_P}{2}|P^{-1}v_n|_P \end{aligned} \tag{16}$$

by (15) for $n \in \mathbb{N}$, so that

$$|P^{-1}v_n|_P \leq \kappa^n|P^{-1}v_0|_P,$$

and again by (15), we have

$$\mathrm{dist}_P(x_n, S_F(y)) \leq \frac{1}{2}|P^{-1}v_{n-1}|_P \leq \frac{1}{2}\kappa^{n-1}|P^{-1}v_0|_P \tag{17}$$

for $n \geq 1$, which shows (13). Since

$$|x_{n+1} - x_n|_P \leq \frac{1}{2}|P^{-1}v_n|_P \leq \frac{1}{2}\kappa^n|P^{-1}v_0|_P \tag{18}$$

for all $n \in \mathbb{N}$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and converges to some $\bar{x} \in \mathbb{R}^d$. As $S_F(\bar{y})$ is closed, estimate (17) shows that $\bar{x} \in S_F(\bar{y})$. Finally, for all $n, N \in \mathbb{N}$ with $N > n$, it follows from (18) that

$$\begin{aligned}|x_N - x_n|_P &\leq \sum_{j=n}^{N-1} |x_{j+1} - x_j|_P \leq \frac{1}{2}|P^{-1}v_0|_P \sum_{j=n}^{N-1} \kappa^j \\ &\leq \frac{1}{2}|P^{-1}v_0|_P \frac{\kappa^n}{1-\kappa}.\end{aligned}$$

Passing to the limit as $N \to \infty$ yields (14). □

Remark 8. *By Theorem 2, any numerical iteration $\{x_n\}_{n\in\mathbb{N}}$ will converge to $S_F(\bar{y})$ provided that the sequence $\{v_n\}_{n\in\mathbb{N}}$ of residuals converges to zero. Let $0 < \varepsilon < 1 - \frac{L_P}{2}$. If $x \in \mathbb{R}^d$ satisfies*

$$|(x_n - \frac{1}{2}P^{-1}v_n) - x|_P \leq \frac{1 - \varepsilon - \frac{L_P}{2}}{L_P} \operatorname{dist}_{P^{-1}}(\bar{y}, F(x_n)),$$

then estimate (16) *yields*

$$\begin{aligned}&\operatorname{dist}_{P^{-1}}(\bar{y}, F(x)) \\ &\leq \operatorname{dist}_{P^{-1}}(\bar{y}, F(x_n - \frac{1}{2}P^{-1}v_n)) + \operatorname{dist}_{P^{-1}}(F(x_n - \frac{1}{2}P^{-1}v_n), F(x)) \\ &\leq \frac{L_P}{2} \operatorname{dist}_{P^{-1}}(\bar{y}, F(x_n)) + L_P|(x_n - \frac{1}{2}P^{-1}v_n) - x|_P \\ &\leq (1-\varepsilon) \operatorname{dist}_{P^{-1}}(\bar{y}, F(x_n)),\end{aligned}$$

so that for all admissible x the residual decreases by a factor $1 - \varepsilon$ and the algorithm still converges linearly with reduced speed if x_{n+1} is chosen from that region. This means that even if P is not known precisely, it is still possible to find a next iterate with smaller residual.

The following example shows that Proposition 7 is sharp (apart from statement (14)).

Example 9. *Let $l < 0$ and $L \geq -l$, and set $F(x) := lx + \alpha x^{\perp}$, where $\alpha := \sqrt{L^2 - l^2}$ and $x^{\perp} := (x^{(2)}, -x^{(1)})$ is the image of x under the rotation with angle $-\pi/2$ around the origin. The single-valued mapping F is l-OSL* *and L-Lipschitz. If the numerical method* (12) *is applied to the problem* $0 = F(x)$*, we have*

$$\Phi(x) = x - \frac{1}{2l}F(x) = \frac{1}{2}\begin{pmatrix} 1 & -\alpha/l \\ \alpha/l & 1 \end{pmatrix} x.$$

The eigenvalues of the above matrix are $\lambda_{1/2} = \frac{1}{2} \pm \frac{\alpha}{2l}i$*, i.e. the iteration converges if and only if* $L < -2l$*. Moreover,*

$$\|\frac{1}{2}\begin{pmatrix} 1 & -\alpha/l \\ \alpha/l & 1 \end{pmatrix}\|_2 = -\frac{L}{2l},$$

so that the iteration converges with rate $-\frac{L}{2l}$ *whenever* $L < -2l$*. In fact, it can be shown easily by using rotational symmetry of* F *that estimate* (13) *is sharp for every initial state* $x_0 \in \mathbb{R}^2$*.*

The following example shows that the condition $L < -2l$ is not sharp for convergence of the method (12) in $d = 1$.

Example 10. *Consider the function* $F : \mathbb{R} \to \mathbb{R}$ *given by*

$$F(x) = \begin{cases} -L + l(x-1), & 1 \leq x \\ -Lx, & -1 \leq x \leq 1 \\ +L + l(x+1), & x \leq -1 \end{cases}$$

with $l < 0$ *and* $L \geq -l$*. Clearly,* F *is* l*-OSL and* L*-Lipschitz. Let* $x_n \in [-1, 1]$ *be a state of the root finding method that is supposed to solve* $0 = F(x)$*. Then*

$$x_{n+1} = x_n - \frac{F(x_n)}{2l} = x_n + \frac{Lx_n}{2l} = (1 + \frac{L}{2l})x_n,$$

so that $|x_{n+1}| < |x_n|$ *if and only if* $L < -4l$*. Figure 2 illustrates the global behavior of the function* F *and the numerical method* Φ *for characteristic ratios* $-L/l$*.*

The gap between the condition $L < -2l$ required for convergence in Proposition 7 and the condition $L < -4l$ observed in Example 10 is due to the fact that for multifunctions $F : \mathbb{R} \to \mathcal{CC}(\mathbb{R})$, the ROSL property is much stronger than in $\mathbb{R}^d$ with $d > 1$. In this particular context, it is possible to derive estimates for some of the defects (see Case 1a in the following proof) that only depend on the one-sided Lipschitz constant l and not on the Lipschitz constant L.

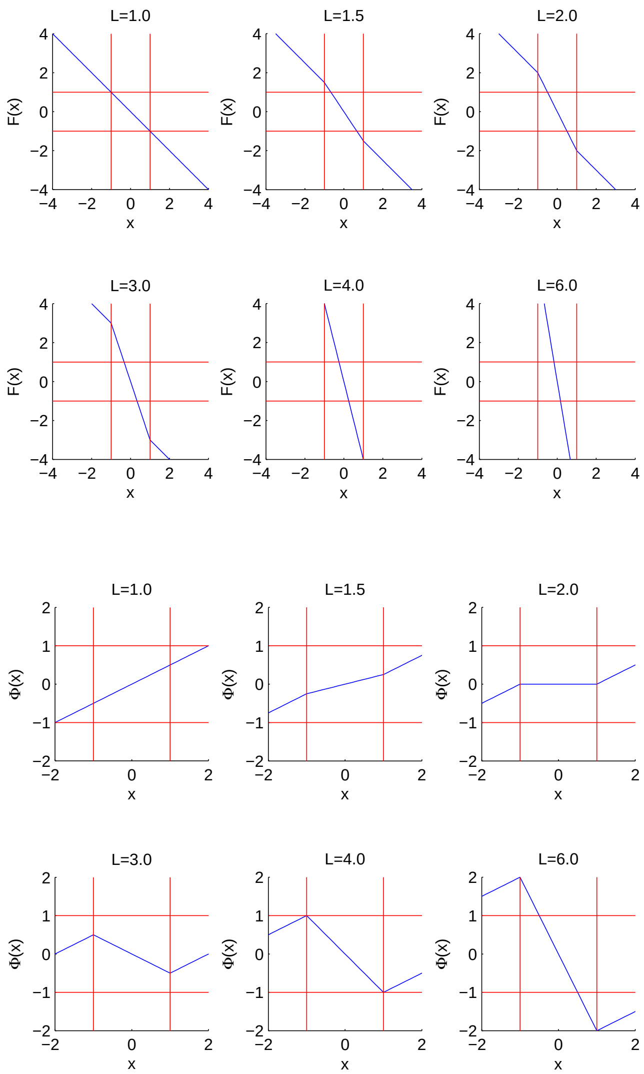

Figure 2: Behavior of the function F from Example 10 and the corresponding numerical method Φ for $l = -1$ and characteristic values of L. The red lines limit the central interval $[-1, 1]$ in space and image. The value $L = -4l$ is the critical threshold.

Proposition 11. *Let $F : \mathbb{R} \to \mathcal{CC}(\mathbb{R})$ be l-ROSL and L-Lipschitz with $l < 0$ and $L < -4l$, and let $x_0 \in \mathbb{R}$ and $\bar{y} \in \mathbb{R}$ be given. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by*

$$x_{n+1} := x_n + \frac{1}{2l}(\bar{y} - \mathrm{Proj}(\bar{y}, F(x_n)))$$

converges to a solution $\bar{x}$ of the inclusion $\bar{y} \in F(x)$ and satisfies the estimates

$$\mathrm{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l}\kappa^{n-1}\,\mathrm{dist}(\bar{y}, F(x_0)) \tag{19}$$

and

$$|x_n - \bar{x}| \leq -\frac{1}{2l}\frac{\kappa^n}{1-\kappa}\,\mathrm{dist}(\bar{y}, F(x_0)) \tag{20}$$

for $n \geq 1$, where $\kappa := \max\{\frac{1}{2}, |1 + \frac{L}{2l}|\}$.

Proof. Let $-2l \leq L < -4l$ and set $v_n := \bar{y} - \mathrm{Proj}(\bar{y}, F(x_n))$ for $n \in \mathbb{N}$. Without loss of generality, $\bar{y} \notin F(x_n)$ and $\bar{y} \notin F(x_{n+1})$, because otherwise the sequences $\{v_n\}$ and $\{x_n\}$ become constant and all estimates are trivially satisfied. As $F(x_n)$ is an interval, there are only two cases.

Case 1: $\bar{y} > y$ for all $y \in F(x_n)$.
In particular, $v_n > 0$. If $\bar{x} \in S_F(\bar{y})$, then the ROSL property yields some $y \in F(x_n)$ such that

$$(\bar{y} - y)(\bar{x} - x_n) \leq l|\bar{x} - x_n|^2,$$

which implies $\bar{x} \leq x_n$. By Theorem 2,

$$S_n := S_F(\bar{y}) \cap [x_n + \frac{1}{l}v_n, x_n - \frac{1}{L}v_n] \neq \emptyset.$$

Let $\bar{x}_n := \max S_n$. Without loss of generality, $x_n \neq \bar{x}_n \neq x_{n+1}$, because otherwise the sequences $\{v_n\}$ and $\{x_n\}$ become constant. There are two subcases.

Subcase 1a: $\bar{x}_n \in [x_n + \frac{1}{l}v_n, x_n + \frac{1}{2l}v_n)$.
Assume that there exists some $y^* \in F(x_{n+1})$ with $\bar{y} < y^*$. Since $y < \bar{y}$ for all $y \in F(x_n)$, there exists some $x^* \in (x_{n+1}, x_n)$ with $\bar{y} \in F(x^*)$ by the set-valued intermediate value theorem (see Appendix). But then $x^* \in S_F(\bar{y})$, which contradicts the maximality of $\bar{x}_n$. Therefore,

$$\bar{y} > y \text{ for all } y \in F(x_{n+1}), \tag{21}$$

and

$$\text{Proj}(\bar{y}, F(x_n)) = \max F(x_n), \text{Proj}(\bar{y}, F(x_{n+1})) = \max F(x_{n+1}).$$

It is easy to see that if F is l-ROSL, then the single-valued function $\max F$ is l-OSL, and hence

$$\begin{aligned}
&\frac{1}{2l} v_n [\text{Proj}(\bar{y}, F(x_{n+1})) - \text{Proj}(\bar{y}, F(x_n))] \\
&= [\text{Proj}(\bar{y}, F(x_{n+1})) - \text{Proj}(\bar{y}, F(x_n))](x_{n+1} - x_n) \\
&= (\max F(x_{n+1}) - \max F(x_n)) \cdot (x_{n+1} - x_n) \\
&\leq l|x_{n+1} - x_n|^2 \leq \frac{1}{4l} v_n^2,
\end{aligned}$$

which implies

$$\text{Proj}(\bar{y}, F(x_{n+1})) - \text{Proj}(\bar{y}, F(x_n)) \geq \frac{1}{2} v_n$$

and thus

$$\bar{y} - \text{Proj}(\bar{y}, F(x_n)) - \frac{1}{2} v_n \geq \bar{y} - \text{Proj}(\bar{y}, F(x_{n+1}))$$

and

$$\frac{1}{2} v_n \geq v_{n+1}.$$

Since $v_{n+1} > 0$ by inequality (21),

$$|v_{n+1}| \leq \frac{1}{2} |v_n|.$$

Subcase 1b: $\bar{x}_n \in (x_n + \frac{1}{2l} v_n, x_n - \frac{1}{L} v_n]$.
In this case,

$$\begin{aligned}
|v_{n+1}| &= \text{dist}(\bar{y}, F(x_{n+1})) \leq \text{dist}(F(\bar{x}_n), F(x_{n+1})) \leq L|\bar{x}_n - x_{n+1}| \\
&\leq L|(x_n - \frac{1}{L} v_n) - (x_n + \frac{1}{2l} v_n)| \leq L|\frac{1}{2l} + \frac{1}{L}| \cdot |v_n| = |1 + \frac{L}{2l}| \cdot |v_n|.
\end{aligned}$$

Case 2: $\bar{y} < y$ for all $y \in F(x_n)$.
All arguments and estimates are symmetric to those in Case 1.

Summarizing Cases 1 and 2,

$$|v_{n+1}| \leq \max\{\frac{1}{2}, |1 + \frac{L}{2l}|\} |v_n| =: \kappa |v_n|,$$

so that by induction,

$$|v_n| \leq \kappa^n |v_0|.$$

By estimate (5), we have

$$\operatorname{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l}|v_{n-1}| \leq -\frac{1}{2l}\kappa^{n-1}|v_0| \tag{22}$$

for $n \geq 1$, which shows (19). Since

$$|x_{n+1} - x_n| \leq -\frac{1}{2l}|v_n| \leq -\frac{1}{2l}\kappa^n |v_0| \tag{23}$$

for all $n \in \mathbb{N}$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy and converges to some $\bar{x} \in \mathbb{R}$. As $S_F(y)$ is closed, estimate (22) shows that $\bar{x} \in S_F(\bar{y})$. Finally, for all $N, n \in \mathbb{N}$ with $N > n$, it follows from (23) that

$$\begin{aligned}
|x_N - x_n| &\leq \sum_{j=n}^{N-1} |x_{j+1} - x_j| \leq -\frac{1}{2l}|v_0| \sum_{j=n}^{N-1} \kappa^j \\
&= -\frac{1}{2l}|v_0|\kappa^n \sum_{j=0}^{N-n-1} \kappa^j = -\frac{1}{2l}|v_0|\kappa^n \frac{1-\kappa^{N-n}}{1-\kappa} \\
&\leq -\frac{1}{2l}|v_0| \frac{\kappa^n}{1-\kappa}.
\end{aligned}$$

Passing to the limit as $N \to \infty$ yields (20).

If $L < -2l$, then Cases 1b and 2b cannot occur, so that all estimates hold with the optimal rate $\kappa = \frac{1}{2}$. □

If the Lipschitz constant L of the mapping F is known explicitly, the numerical method (12) can be refined using estimate (7) from Theorem 2. The proofs will only be sketched, because they coincide in large parts with those of the above propositions. The following proposition will only be given for the classical ROSL condition, because a general formulation with P-norms tends to conceal the idea of the improvement.

Proposition 12. *If $d > 1$ and $L \leq -\sqrt{2}l$, then the iteration*

$$x_{n+1} := x_n + \frac{l}{L^2}(\bar{y} - \operatorname{Proj}(\bar{y}, F(x_n)))$$

converges to a solution $\bar{x} \in S_F(\bar{y})$ *and satisfies*

$$\operatorname{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l}\kappa^{n-1} \operatorname{dist}(\bar{y}, F(x_0))$$

and

$$|x_n - \bar{x}| \leq -\frac{l}{L^2}\frac{\kappa^n}{1-\kappa} \operatorname{dist}(\bar{y}, F(x_0)),$$

where $\kappa := \frac{\sqrt{L^2-l^2}}{L}$.

Sketch of proof. Define $S_n := B(x_n + \frac{1}{2l}v_n, -\frac{1}{2l}|v_n|) \setminus B(x_n, \frac{1}{L}|v_n|)$. By Theorem 2, there exists some $\bar{x}_n \in S_F(\bar{y}) \cap S_n$. By simple geometric arguments,

$$|\bar{x}_n - x_{n+1}| \leq \operatorname{dist}(S_n, x_{n+1}) \leq \frac{\sqrt{L^2-l^2}}{L^2}|v_n|,$$

so that

$$\begin{aligned}|v_{n+1}| &= \operatorname{dist}(\bar{y}, F(x_{n+1})) \leq \operatorname{dist}(\bar{y}, F(\bar{x}_n)) + \operatorname{dist}(F(\bar{x}_n), F(x_{n+1})) \\ &\leq L|\bar{x}_n - x_{n+1}| \leq \frac{\sqrt{L^2-l^2}}{L}|v_n| =: \kappa|v_n|.\end{aligned}$$

□

The case $d = 1$ allows more effective estimates.

Proposition 13. *If* $d = 1$ *and* $L \leq -2l$, *then the iteration*

$$x_{n+1} := x_n + \frac{1}{2}(\frac{1}{l} - \frac{1}{L})(\bar{y} - \operatorname{Proj}(\bar{y}, F(x_n)))$$

converges to a solution $\bar{x} \in S_F(\bar{y})$ *and satisfies*

$$\operatorname{dist}(x_n, S_F(\bar{y})) \leq -\frac{1}{2l}\kappa^{n-1} \operatorname{dist}(\bar{y}, F(x_0))$$

and

$$|x_n - \bar{x}| \leq \frac{1}{2}(\frac{1}{L} - \frac{1}{l})\frac{\kappa^n}{1-\kappa} \operatorname{dist}(\bar{y}, F(x_0))$$

for $n \geq 1$, *where* $\kappa := \frac{1}{2}(1 - \frac{L}{l})$.

Sketch of proof. By Theorem 2, there exists some $\bar{x}_n \in S_F(\bar{y}) \cap S_n$, where

$$S_n := [x_n + \frac{1}{l}v_n, x_n] \setminus [x_n - \frac{1}{L}v_n, x_n + \frac{1}{L}v_n] = [x_n + \frac{1}{l}v_n, x_n - \frac{1}{L}v_n).$$

Therefore,

$$\begin{aligned}|v_{n+1}| &= \operatorname{dist}(\bar{y}, F(x_{n+1})) \le \operatorname{dist}(\bar{y}, F(\bar{x}_n)) + \operatorname{dist}(F(\bar{x}_n), F(x_{n+1})) \\ &\le L|\bar{x}_n - x_{n+1}| \le \frac{L}{2}|\frac{1}{L} - \frac{1}{l}| \cdot |v_n| =: \kappa|v_n|.\end{aligned}$$

□

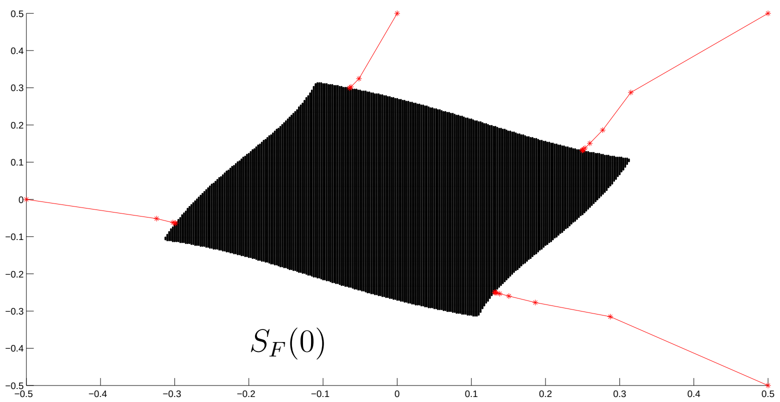

Figure 3: Solution set $S_F(0)$ of inclusion (24) and some typical trajectories of the numerical scheme (12).

The following numerical example illustrates that the algorithm (12) indeed approximates an element of the solution set $S_F(\bar{y})$ successfully for any given initial value.

Example 14. *Consider the multivalued mapping* $F : \mathbb{R}^2 \to \mathcal{CC}(\mathbb{R}^2)$ *given by*

$$F(x) = -3x + A(x)Q, \tag{24}$$

where

$$A(x) = \begin{pmatrix} \cos(|x|) & -\sin(|x|) \\ \sin(|x|) & \cos(|x|) \end{pmatrix} \quad \text{and} \quad Q = \overline{co}\{(1,0),(0,-1),(-1,0),(0,1)\}$$

are a rotation matrix with angle depending on the norm of x and a square centered at the origin. It is easy to check that F is (-2)-ROSL and 3-Lipschitz, so that the statements of Proposition 7 hold. The solution set $S_F(0)$ and typical trajectories of the numerical method (12) applied to the problem $0 \in F(x)$ are depicted in Figure 3.

Appendix

The proof of the following proposition does not differ much from that of the classical intermediate value theorem and is therefore omitted.

Proposition 15. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $F : [a, b] \to \mathcal{CC}(\mathbb{R})$ be an usc mapping such that there exists some $f_a \in F(a)$ and $f_b \in F(b)$ with $f_a < 0$ and $f_b > 0$. Then there exists some $x^* \in (a, b)$ such that $0 \in F(x^*)$.*

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