SPATIAL DECAY OF ROTATING WAVES IN PARABOLIC SYSTEMS

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ABSTRACT. In this paper we study solutions of nonlinear systems

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

The linear operator is of Ornstein-Uhlenbeck type with an unbounded drift term defined by a skew-symmetric matrix $S \in \mathbb{R}^{d,d}$. Equations of this form determine the shape and angular speed of rotating waves in time-dependent reaction diffusion systems. We prove under certain conditions that every classical solution which falls below a certain threshold at infinity, must decay exponentially in space. For the proof we utilize the heat kernel matrix of a generalized Ornstein-Uhlenbeck operator, determine its maximal domain and analyze constant and variable coefficient perturbations.

CONTENTS

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1. Introduction and main result

1.1. **Assumptions and main result.** Consider a steady state problem of the form

$$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+f(v(x))=0,\ x\in\mathbb{R}^d,\ d\geqslant 2, \tag{1.1}$$

with diffusion matrix $A\in\mathbb{K}^{N,N}$ and a function $f:\mathbb{K}^N\to\mathbb{K}^N$ for $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, where $A\triangle v(x)+\langle Sx,\nabla v(x)\rangle$ is usually called the complex Ornstein-Uhlenbeck operator. The drift term is defined by a matrix $0\neq S\in\mathbb{R}^{d,d}$ as

$$\langle Sx,\nabla v(x)\rangle:=\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jD_iv(x), \tag{1.2}$$

where $D_i=\frac{\partial}{\partial x_i}$. Our interest is in skew-symmetric matrices $S=-S^T$, in which case (1.2) is a rotational term containing angular derivatives

$$\langle Sx,\nabla v(x)\rangle=\sum_{i=1}^{d-1}\sum_{j=i+1}^{d}S_{ij}\left(x_jD_i-x_iD_j\right)v(x). \tag{1.3}$$

We look for different types of solutions, which satisfy at least $v\in L^p(\mathbb{R}^d,\mathbb{K}^N)$ for some $1\leqslant p\leqslant\infty$ and $N\in\mathbb{N}$.

Under appropriate conditions our main result states that a solution $v_\star$ of (1.1) and its first order derivatives decay exponentially in space as the radius $|x|$ goes to infinity.

Investigating steady state problems of this type is motivated by the stability theory of rotating patterns in several spatial dimensions, [17]. There one considers reaction diffusion equations

$$\begin{aligned} u_t(x,t)=&A\triangle u(x,t)+f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0)=&u_0(x)\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d,\end{aligned} \tag{1.4}$$

where $A \in \mathbb{K}^{N,N}$ is a diffusion matrix, $f : \mathbb{K}^N \to \mathbb{K}^N$ a nonlinearity and u a solution that maps $\mathbb{R}^d \times [0,\infty[$ into $\mathbb{K}^N$.
Assume a rotating wave solution $u_\star : \mathbb{R}^d \times [0,\infty[\to \mathbb{K}^N$ of (1.4)

$$u_\star(x,t) = v_\star(e^{tS}x)$$

with profile (or pattern) $v_\star : \mathbb{R}^d \to \mathbb{K}^N$ and $0 \neq S \in \mathbb{R}^{d,d}$ skew-symmetric. In case $d = 2, 3$, S can be considered as the angular velocity tensor associated to the angular velocity vector $\omega \in \mathbb{R}^{\frac{d(d-1)}{2}}$ containing S_{ij}, $i = 1,\ldots,d-1$, $j = i+1,\ldots,d$. A transformation into a rotating frame shows that $u(x,t)$ solves (1.4) if and only if $v(x,t) = u(e^{tS}x,t)$ solves

$$\begin{aligned} v_t(x,t) =& A\triangle v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2, \\ v(x,0) =& u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\quad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned} \tag{1.5}$$

where the drift term is given by (1.3).
Note that $v_\star$ is a stationary solution of (1.5), meaning that $v_\star$ solves the nonlinear problem (1.1). In Section 2.2 we illustrate such rotating patterns by a series of examples.
In order to investigate exponential decay of the profile $v_\star$, we list a series of assumptions that will be important in the sequel. Throughout, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

Assumption 1.1. *Let $A \in \mathbb{K}^{N,N}$ be such that*

(A1) *A is diagonalizable (over $\mathbb{C}$) (system condition),*

(A2) $\sigma(A) \subset \mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$ *(ellipticity condition),*

(A3) $$\begin{aligned}\sigma(A) \subset \Sigma_p :=& \left\{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda|\,|p-2| \leqslant 2\sqrt{p-1}\operatorname{Re}\lambda\right\} \\ =& \left\{\lambda \in \mathbb{C} \mid |\arg\lambda| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right)\right\},\ 1<p<\infty,\end{aligned}$$
(L^p-dissipativity condition).

Assumption (A2) guarantees that the diffusion part $A\triangle$ is an elliptic operator and requires that all eigenvalues of A are contained in the right half-plane. Condition (A3) is more restrictive and postulates that all eigenvalues of A are even contained in a p-dependent sector in the right half-plane, see Figure 1.1. The opening angle $|\varphi|$ is close to 0 for small and large p, i.e. p close to 1 or ∞, and it is $\frac{\pi}{2}$ for $p = 2$. Assuming (A2), the condition (A3) is automatically satisfied for $p = 2$. If all eigenvalues of A are real and positive then assumption (A3) is satisfied for every $1 < p < \infty$. Condition (A1) ensures that all results for scalar equations can be extended to system cases.

Assumption 1.2. *Let $S \in \mathbb{R}^{d,d}$ be such that*

(A4) *S is skew-symmetric, i.e. $S = -S^T$, $S \in \operatorname{so}(d,\mathbb{R})$ (rotational condition).*

Assumption (A4) guarantees that the drift term (1.2) contains only angular derivatives, see (1.3). Our main result will be formulated for the real-valued case.

Assumption 1.3. *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be such that*

(A5) $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ *(smoothness condition).*

Later on we apply our results also to complex-valued nonlinearities of the form

$$f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u,$$

where $g : \mathbb{R} \to \mathbb{C}^{N,N}$ is a sufficiently smooth function. Such nonlinearities arise for example in Ginzburg-Landau equations, Schrödinger equations, $\lambda - \omega$ systems and many other equations from physical sciences, see Section 2.2. Note, that in

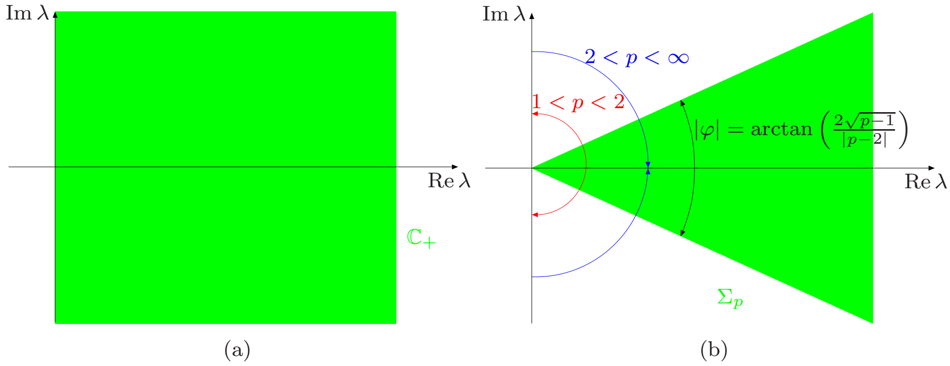


FIGURE 1.1. Sector for ellipticity assumption (A2) (left) and for dissipativity assumption (A3) (right)

this case, the function f is not holomorphic in $\mathbb{C}$, but its real-valued version in $\mathbb{R}^2$ satisfies (A5). For differentiable functions $f : \mathbb{R}^N \to \mathbb{R}^N$, Df denotes the Jacobian matrix in the real sense, see the following conditions (A7) and (A8).

Assumption 1.4. *Let $v_\infty \in \mathbb{R}^N$ be such that*

(A6) $f(v_\infty) = 0$ *(constant asymptotic state),*

(A7) $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ *are simultaneously diagonalizable (over $\mathbb{C}$) (system condition),*

(A8) $\sigma(Df(v_\infty)) \subset \mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$ *(spectral condition).*

Definition 1.5. *A function $v_\star : \mathbb{R}^d \to \mathbb{K}^N$ is called a classical solution of (1.1) if*

$$v_\star \in C^2(\mathbb{R}^d, \mathbb{K}^N) \cap C_b(\mathbb{R}^d, \mathbb{K}^N) \tag{1.6}$$

and $v_\star$ solves (1.1) pointwise.

For a matrix $C \in \mathbb{K}^{N,N}$ we denote by $\sigma(C)$ the spectrum of C, by $\rho(C) := \max_{\lambda\in\sigma(C)} |\lambda|$ the spectral radius of C and by $s(C) := \max_{\lambda\in\sigma(C)} \operatorname{Re}\lambda$ the spectral abscissa (or spectral bound) of C. Using this notation, we define the constants

$$\begin{aligned} a_{\min} &:= \left(\rho\left(A^{-1}\right)\right)^{-1}, & a_0 &:= -s(-A), \\ a_{\max} &:= \rho(A), & b_0 &:= -s(Df(v_\infty)). \end{aligned} \tag{1.7}$$

Our main tool for investigating exponential decay in space are exponentially weighted function spaces, which we introduce in Section 3 in detail. An essential ingredient for these function spaces is the choice of the weight function, which follows [60, Def. 3.1]:

Definition 1.6. (**1**) *A function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a weight function of exponential growth rate $\eta \geqslant 0$ provided that*

(W1) $\theta(x) > 0 \;\forall\, x \in \mathbb{R}^d$,

(W2) $\exists\, C_\theta > 0 : \; \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \;\forall\, x, y \in \mathbb{R}^d$.

(**2**) *A weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called radial provided that*

(W3) $\exists\, \phi : [0,\infty[\to \mathbb{R} : \; \theta(x) = \phi(|x|) \;\forall\, x \in \mathbb{R}^d$.

(**3**) *A radial weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ is called non-decreasing (or monotonically increasing) provided that*

$$\theta(x)\leqslant\theta(y)\;\forall\, x,y\in\mathbb{R}^d \text{ with } |x|\leqslant|y|. \tag{W4}$$

Note, that radial weight functions satisfy $\theta(x)=\theta(y)$ for every $x,y\in\mathbb{R}^d$ with $|x|=|y|$. Standard examples are

$$\theta_1(x)=\exp\left(-\mu|x|\right)\quad\text{and}\quad\theta_2(x)=\cosh\left(\mu|x|\right),$$

as well as their smooth analogs

$$\theta_3(x)=\exp\left(-\mu\sqrt{|x|^2+1}\right)\quad\text{and}\quad\theta_4(x)=\cosh\left(\mu\sqrt{|x|^2+1}\right),$$

for $\mu\in\mathbb{R}$ and $x\in\mathbb{R}^d$. Obviously, all these functions are radial weight functions of exponential growth rate $\eta=|\mu|$ with $C_\theta=1$ and they are non-decreasing if $\mu\leqslant 0$. Note, that for $\mu=0$ the examples include the weight function $\theta(x)=1$.
For weight functions of exponential growth we define the exponentially weighted Lebesgue and Sobolev spaces

$$\begin{aligned} L^p_\theta(\mathbb{R}^d,\mathbb{K}^N) :=& \{u\in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^N)\mid \|\theta u\|_{L^p}<\infty\},\\ W^{k,p}_\theta(\mathbb{R}^d,\mathbb{K}^N) :=& \{u\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N)\mid D^\beta u\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N)\;\forall\,|\beta|\leqslant k\}, \end{aligned}$$

for every $1\leqslant p\leqslant\infty$ and $k\in\mathbb{N}_0$. Our main result is the following:

Theorem 1.7 (Exponential decay of $v_\star$)**.** *Let the assumptions (A1)–(A8) be satisfied for $1<p<\infty$ and $\mathbb{K}=\mathbb{R}$. Then for every $0<\vartheta<1$ and for every radially nondecreasing weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ with*

$$0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2}$$

with $a_{\max}$, a_0, b_0 from (1.7), there exists a constant $K_1=K_1(A,f,v_\infty,d,p,\theta,\vartheta)>0$ with the following property:
Every classical solution $v_\star$ of

$$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+f(v(x))=0,\; x\in\mathbb{R}^d, \tag{1.8}$$

such that $v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and

$$\sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|\leqslant K_1 \text{ for some } R_0>0 \tag{1.9}$$

satisfies

$$v_\star-v_\infty\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{R}^N).$$

Remark. Roughly speaking, Theorem 1.7 states that every classical solution $v_\star$ which satisfies $v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and which is sufficiently close to the steady state v_∞ at infinity, see (1.9), must already decay exponentially in space. The exponential decay is expressed by the fact, that $v_\star-v_\infty$ belongs to an exponentially weighted Sobolev space. Moreover, the theorem gives an explicit bound for the exponential growth rate, that depends only on p, the spectral radius of A and the spectral abscissas of $-A$ and $Df(v_\infty)$. Note, that by (A2) assumption (A3) is automatically satisfied for $p=2$. The same holds for assumptions (A1) and (A7) if $N=1$.

Remark. Later on we apply Theorem 1.7 to complex systems with nonlinearities of the form

$$f:\mathbb{C}^N\to\mathbb{C}^N,\quad f(u)=g\left(|u|^2\right)u,$$

where $g:\mathbb{R}\to\mathbb{C}^{N,N}$ is a sufficiently smooth function. For this purpose, we transform the N-dimensional complex-valued system into a $2N$-dimensional real-valued system.

Remark. Theorem 1.7 replaces the theory of exponential dichotomies which is commonly used to show exponential decay of patterns in $\mathbb{R}^1$ and can also be considered as a type of generalization on $\mathbb{R}^d$. In the theory of exponential dichotomies, one considers ODEs of the form

$$u'(t)=f(u(t)),\ t\geqslant 0, \tag{1.10}$$

where $f\in C^1(\mathbb{R}^N,\mathbb{R}^N)$ and u maps $[0,\infty[$ into $\mathbb{R}^N$. Assuming that $u_\infty\in\mathbb{R}^N$ is a hyperbolic fixed point, i.e. $f(u_\infty)=0$ and $\sigma(Df(u_\infty))\cap i\mathbb{R}=\emptyset$, one finds a constant $K=K(f,u_\infty)>0$ such that every solution u of (1.10) with $u(t)\in B_K(u_\infty)$ for every $t\geqslant t_0$ satisfies $u(t)\to u_\infty$ exponentially fast as $t\to\infty$, cf. [53, Theorem III.7 (2)] for a time-discrete version.
To explicate the analogy, let us consider the Ornstein-Uhlenbeck operator instead of the time derivative in (1.10) and x instead of t. The smoothness assumption for f now corresponds to assumption (A5). If we consider v_∞ instead of u_∞, we see that $f(u_\infty)=0$ is expressed by assumption (A6). But the spectral condition $\sigma(Df(v_\infty))\subset\mathbb{C}_-$ from (A8) is more restrictive than the hyperbolicity condition $\sigma(Df(u_\infty))\subset\mathbb{C}_-\cup\mathbb{C}_+$.

Remark. Currently, we extend the theory to spaces of bounded continuous functions like $C_{rub}(\mathbb{R}^d,\mathbb{K}^N)$, which we introduce in Section 2.3. We believe the condition $v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$, contained in Theorem 1.7, can be omitted.

1.2. **Outline of proof.** In the following we explain the main steps of our approach that lead to the proof of Theorem 1.7.
Far-Field-Linearization: Consider the nonlinear problem

$$A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))=0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

Let $v_\infty\in\mathbb{R}^N$ be the constant asymptotic state satisfying (A6). Assume that $f\in C^1(\mathbb{R}^N,\mathbb{R}^N)$, compare (A5), then Taylor's theorem yields

$$f(v_\star(x))=\underbrace{f(v_\infty)}_{=0}+\underbrace{\int_0^1 Df(v_\infty+t(v_\star(x)-v_\infty))dt}_{=:a(x)}(v_\star(x)-v_\infty),\ x\in\mathbb{R}^d,$$

where $a\in L^\infty(\mathbb{R}^d,\mathbb{R}^{N,N})$ since $v_\star\in L^\infty(\mathbb{R}^d,\mathbb{R}^N)$ and $v_\star$ is a classical solution. Since $v_\infty\in\mathbb{R}^N$ is constant, it holds

$$A\triangle v_\infty+\langle Sx,\nabla v_\infty\rangle=0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

Hence, the difference $w(x):=v_\star(x)-v_\infty$ satisfies the linearized equation

$$A\triangle w(x)+\langle Sx,\nabla w(x)\rangle+a(x)w(x)=0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

In order to study the behavior of solutions as $|x|\to\infty$ we decompose the variable coefficient.
Decomposition of a: Let $a(x)=Df(v_\infty)+Q(x)$ with Q defined by

$$Q(x)=\int_0^1 Df\left(v_\infty+tw(x)\right)-Df\left(v_\infty\right)dt,\ x\in\mathbb{R}^d.$$

This yields $Q\in L^\infty(\mathbb{R}^d,\mathbb{R}^{N,N})$ and we obtain

$$A\triangle w(x)+\langle Sx,\nabla w(x)\rangle+\left(Df(v_\infty)+Q(x)\right)w(x)=0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

Decomposition of Q: Now, let $Q(x) = Q_\varepsilon(x)+Q_c(x)$, where $Q_\varepsilon, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$, Q_ε is small w.r.t. $\|\cdot\|_{L^\infty}$ and Q_c is compactly supported on $\mathbb{R}^d$, see Figure 1.2, then we arrive at

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + (Df(v_\infty) + Q_\varepsilon(x) + Q_c(x))\, w(x) = 0,\ x \in \mathbb{R}^d. \tag{1.11}$$

If we omit the term $Q_\varepsilon + Q_c$ we obtain the far-field-linearization.

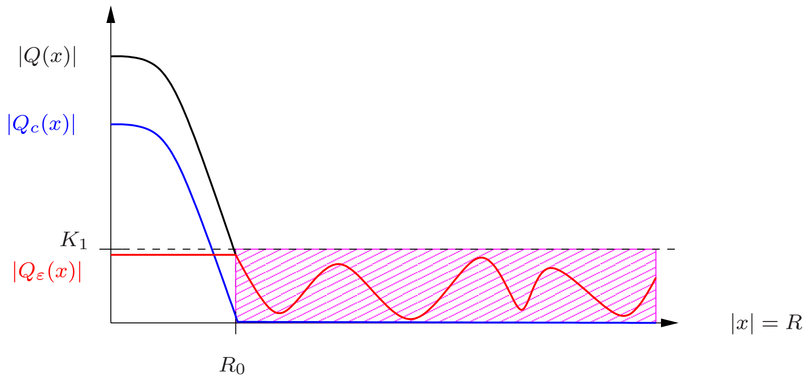


FIGURE 1.2. Decomposition of Q with data R_0 and K_1 from Theorem 1.7

Perturbations of Ornstein-Uhlenbeck operator: In order to show exponential decay for the solution $v_\star$ of the nonlinear steady state problem (1.1), it is sufficient to analyze the solutions of the linear system (1.11). Abbreviating $B := -Df(v_\infty)$, we will study the following linear differential operators:

$$\begin{aligned}
\left[\mathcal{L}_Q v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q_\varepsilon(x)v(x) + Q_c(x)v(x),\\
\left[\mathcal{L}_{Q_\varepsilon} v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x) + Q_\varepsilon(x)v(x),\\
\left[\mathcal{L}_\infty v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x),\\
\left[\mathcal{L}_0 v\right](x) &= A\triangle v(x) + \langle Sx, \nabla v(x)\rangle .
\end{aligned}$$

The operator $\mathcal{L}_0$, called the Ornstein-Uhlenbeck operator, is the sum of the diffusion term $\left[\mathcal{L}_0^{\text{diff}} v\right](x) := A\triangle v(x)$ and the drift term $\left[\mathcal{L}_0^{\text{drift}} v\right](x) := \langle Sx, \nabla v(x)\rangle$. The drift term has unbounded (in fact linearly increasing) coefficients. Afterwards, we will allow complex coefficients for the operators $\mathcal{L}_0$, $\mathcal{L}_\infty$, $\mathcal{L}_{Q_\varepsilon}$ and $\mathcal{L}_Q$. Therefore, we rewrite the assumptions (A7) and (A8) as follows:

Assumption 1.8. *Let $B \in \mathbb{K}^{N,N}$ be such that*

(A7$_B$) $A, B \in \mathbb{K}^{N,N}$ *are simultaneously diagonalizable (over $\mathbb{C}$), i.e.*

$$\exists\, Y \in \mathbb{C}^{N,N} \text{ invertible}: \ Y^{-1}AY = \Lambda_A \text{ and } Y^{-1}BY = \Lambda_B$$

where $\Lambda_A = \operatorname{diag}\left(\lambda_1^A, \ldots, \lambda_N^A\right), \Lambda_B = \operatorname{diag}\left(\lambda_1^B, \ldots, \lambda_N^B\right) \in \mathbb{C}^{N,N}$ *(system condition),*

(A8$_B$) $\sigma(B) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$ *(spectral condition).*

In this context b_0 is defined by $b_0 := -s(-B)$, cf. (1.7). Note that in case of $B = 0$ assumption (A7$_B$) coincides with (A1). We now give a short overview how this paper is organized.

In Section 2 we recall the derivation of the real scalar Ornstein-Uhlenbeck operator from an underlying stochastic ordinary differential equation (SODE). After that we motivate the complex Ornstein-Uhlenbeck operator in scalar and system cases. In the second part of Section 2 we give a series of examples from physical and

biological sciences, where the Ornstein-Uhlenbeck operator appears in the theory of rotating patterns.
In Section 3 we introduce in detail the exponentially weighted Lebesgue and Sobolev spaces as well as some general notation that will be used throughout this paper.
In Section 4 we extend, under the assumptions (A1), (A2), (A4) and ($A7_B$), the approach from [15], [4] and [20] to determine a heat kernel of the complex-valued operator $\mathcal{L}_\infty$ for the case, where A and B are complex simultaneously diagonalizable matrices. This leads to the following heat kernel matrix

$$H(x,\xi,t)=(4\pi tA)^{-\frac{d}{2}}\exp\left(-Bt-(4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right)$$

of $\mathcal{L}_\infty$, which we will denote later by H_∞. The choice $B=0$ provides us with a heat kernel, denoted by H_0, for the complex Ornstein-Uhlenbeck operator $\mathcal{L}_0$. Further, we show that H satisfies a Chapman-Kolmogorov formula, that is useful for the semigroup theory. In the remaining section we prove some integral properties for the modified kernel $K(\psi,t)=H(x,e^{tS}x-\psi,t)$, which will be needed in the sequel for the exponential decay.
Assuming (A1), (A2) and (A4) we will study in Section 5 the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ defined by the heat kernel of $\mathcal{L}_0$ as

$$\left[T_0(t)v\right](x):=\int_{\mathbb{R}^d}H_0(x,\xi,t)v_0(\xi)d\xi,\ t>0,\ x\in\mathbb{R}^d.$$

Here we show that the semigroup $(T_0(t))_{t\geqslant 0}$ (also known as the transition semigroup) is strongly continuous in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$. Hence, we can define the infinitesimal generator A_p of $(T_0(t))_{t\geqslant 0}$. Using abstract semigroup theory, [28], we are able to derive solvability and uniqueness results for the resolvent equation and resolvent estimates. Moreover, we show that the Schwartz space $\mathcal{S}$ is dense in the domain of A_p with respect to the graph norm of A_p for every $1\leqslant p<\infty$. This shows that A_p and $\mathcal{L}_0$ coincide on $\mathcal{S}$. To prove afterwards that A_p is indeed the maximal realization (extension) of $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$, we must restrict p to $1<p<\infty$ and require in addition the L^p-dissipativity assumption (A3) for $\mathcal{L}_0$, cf. [22]. Then, we derive some resolvent estimates for $\mathcal{L}_0$ in

$$\mathcal{D}^p(\mathcal{L}_0):=\left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N)\mid \mathcal{L}_0v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}$$

for $1<p<\infty$, [43]. This enables us to conclude that the maximal domain $\mathcal{D}^p_{max}$ of A_p is equal to $\mathcal{D}^p(\mathcal{L}_0)$ and that A_p and $\mathcal{L}_0$ coincide on $D^p(\mathcal{L}_0)$ for every $1<p<\infty$. Using exponentially weighted Sobolev spaces with radial weight functions of exponentially growth, we then obtain exponential decay of the solutions for the resolvent equation and its derivatives up to order 1, even if (A3) is not satisfied. In order to show that the maximal domain of the Ornstein-Uhlenbeck operator $\mathcal{L}_0=\mathcal{L}_0^{\mathrm{diff}}+\mathcal{L}_0^{\mathrm{drift}}$ coincides with the intersection of the domains of its diffusion and drift term, i.e.

$$\mathcal{D}\left(\mathcal{L}_0^{\mathrm{diff}}+\mathcal{L}_0^{\mathrm{drift}}\right)=\mathcal{D}\left(\mathcal{L}_0^{\mathrm{diff}}\right)\cap\mathcal{D}\left(\mathcal{L}_0^{\mathrm{drift}}\right),$$

we analyze the homogeneous and inhomogeneous Cauchy problem for $\mathcal{L}_0$, [43], and show for $1<p<\infty$ that the domain $\mathcal{D}^p(\mathcal{L}_0)$ coincides with

$$D^p:=\left\{v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\mid\langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.$$

In Section 6 we perturb the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ by adding the term $Bv(x)$ with constant coefficients, that leads us to the operator $\mathcal{L}_\infty$. To find a realization of $\mathcal{L}_\infty$, we assume (A1), (A2), (A4) and perturb the generator A_p by adding the operator $E_pv:=Bv$. Then the bounded perturbation $B_p:=A_p+E_p$, equipped with the same domain as A_p, generates a C^0-semigroup $(T_\infty(t))_{t\geqslant 0}$ on

$L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 \leqslant p < \infty$. If we require in addition assumption (A3), then the infinitesimal generator B_p is indeed the maximal realization of $\mathcal{L}_\infty$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 < p < \infty$ and the domain equals $\mathcal{D}^p(\mathcal{L}_0)$. Note, that in general we do not have an explicit formula for the semigroup $(T_\infty(t))_{t\geqslant 0}$ any more. But if A and B satisfy in addition to (A1), (A2), (A4) the assumption (A7$_B$), we are able to derive an explicit representation for the new semigroup $(T_\infty(t))_{t\geqslant 0}$, given by

$$[T_\infty(t)v](x) := \int_{\mathbb{R}^d} H_\infty(x,\xi,t)v_0(\xi)d\xi,\ t > 0,\ x \in \mathbb{R}^d.$$

Here, the function H_∞ coincide with the heat kernel for $\mathcal{L}_\infty$ computed in Section 4. Again, under the assumptions (A1), (A2), (A4) and (A7$_B$) we are able to derive solvability und uniqueness results for the resolvent equation and resolvent estimates. In particular, assuming (A8$_B$), we can derive an explicit representation of Green's function for B_p, as the time-integral over the heat kernel,

$$G(x,\xi) = -\int_0^\infty H_\infty(x,\xi,s)ds.$$

If in addition (A3) is satisfied, this turns out to be also a Green's function for $\mathcal{L}_\infty$. Again, we can prove exponential decay of solutions of the resolvent equation for B_p and its derivatives up to order 1, provided (A1), (A2), (A4) and (A7$_B$) are satisfied. Perturbing the operator $\mathcal{L}_\infty$ by adding the term $Q(x)v(x)$ with variable coefficients $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, leads us in Section 7 to the operator $\mathcal{L}_Q$. In order to find a realization of $\mathcal{L}_Q$, we assume (A1), (A2), (A4), (A7$_B$) and perturb this time the generator B_p by adding the bounded operator $F_p v := Qv$. Then the operator $C_p := B_p + F_p$, equipped with the domain of B_p, generates again a C^0-semigroup $(T_Q(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 \leqslant p < \infty$. Again, if we require in addition assumption (A3), then the infinitesimal generator C_p is the maximal realization of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 < p < \infty$ and the domain equals $\mathcal{D}^p(\mathcal{L}_0)$. We know again that the semigroup $(T_Q(t))_{t\geqslant 0}$ is uniquely determined but without further assumptions on Q we cannot derive an explicit representation in this case. However, under the assumptions (A1), (A2), (A4), (A7$_B$) and arbitrary $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, we are able to derive solvability und uniqueness results for the resolvent equation and resolvent estimates. Finally, assuming in addition (A3) and

$$\sup_{|x|\geqslant R} |Q(x)| \to 0 \text{ as } R \to \infty,$$

and following [17], we compute the essential spectrum of the operator $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1 < p < \infty$. This shows that neither $\mathcal{L}_Q$ nor C_p is sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $(T_Q(t))_{t\geqslant 0}$ does not generate an analytic semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1 < p < \infty$. This result can also be applied in case $Q = 0$ and $B = Q = 0$, which provides us the same statements for $\mathcal{L}_0$ and $(T_0(t))_{t\geqslant 0}$ as well as $\mathcal{L}_\infty$ and $(T_\infty(t))_{t\geqslant 0}$. In the rest of this section we apply this theory to perturbations $Q = Q_\varepsilon$, where Q_ε is assumed to be small with respect to $\|\cdot\|_{L^\infty}$, and to relatively compact perturbations $Q = Q_\varepsilon + Q_c$, where Q_c is compactly supported.

In Section 8 we analyze the steady state problem (1.1) and prove the main result from Theorem 1.7, stating that $v_\star - v_\infty$ and its derivatives up to order 1 decay exponentially in space at a certain rate, whenever $v_\star$ is a classical solution of (1.1). In the proof we use the above mentioned main steps of our approach. Afterwards we extend Theorem 1.7 to complex systems. Generalizing [17] from $d = 2$ to $d \geqslant 2$, we investigate the linearization of the nonlinear problem (1.1) of the Ornstein-Uhlenbeck operator on $\mathbb{R}^d$. We determine the eigenvalues on the imaginary axis and show that the associated eigenfunctions and their first order derivatives decay exponentially in space.

2. Derivation and applications of the Ornstein-Uhlenbeck operator

2.1. **The Ornstein-Uhlenbeck operator arising from stochastic ODEs.** In this section we recall the origin of the Ornstein-Uhlenbeck operator from stochastic differential equations. For this purpose we consider a stochastic ordinary differential equation and derive the Kolmogorov operator. The Ornstein-Uhlenbeck operator, which is an elliptic operator with unbounded linearly growing coefficients, is a special type of a Kolmogorov operator. Different types of Kolmogorov operators were treated in [34], [21], [3]. Applications of Kolmogorov operators in physics and finance can be found in [50]. For a motivation of the Ornstein-Uhlenbeck operator from SODE's we refer to [38, Chapter 9].

2.1.1. *From ODE to first-order PDE.* Let $d\in\mathbb{N}$ and let $\mu\in C^{\infty}(\mathbb{R}^d,\mathbb{R}^d)$ be a function, which is at most linearly growing, i.e.

$$\exists\, C>0:\ |\mu(x)|\leqslant C\left(1+|x|\right)\quad \forall\, x\in\mathbb{R}^d.$$

Then there exists a family

$$\Phi(\cdot;x):[0,\infty[\to\mathbb{R}^d,\ x\in\mathbb{R}^d,$$

of unique smooth functions, satisfying

$$\text{(ODE)}\qquad \begin{aligned}&\frac{\partial}{\partial t}\Phi(t;x)=\mu\left(\Phi(t;x)\right),\ t\in[0,\infty[,\ x\in\mathbb{R}^d,\\ &\Phi(0;x)=x.\end{aligned}$$

$\Phi(\cdot;x)$ is known as the solution flow of (ODE). These functions are smooth with respect to x for every fixed $t\in[0,\infty[$, i.e.

$$\Phi(t;\cdot):\mathbb{R}^d\to\mathbb{R}^d,\quad x\longmapsto\Phi(t;x)\text{ is smooth }\forall\, t\in[0,\infty[.$$

The family $T(t):C_b(\mathbb{R}^d,\mathbb{R})\to C_b(\mathbb{R}^d,\mathbb{R})$, $t\in[0,\infty[$, of linear operators defined by

$$\left[T(t)u_0\right](x):=u_0\left(\Phi(t;x)\right),\ x\in\mathbb{R}^d,\ t\in[0,\infty[,\ u_0\in C_b(\mathbb{R}^d,\mathbb{R}),$$

is called the transition semigroup of the (ODE). $(T(t))_{t\geqslant 0}$ satisfies the properties

$$T(0)=I\quad\text{and}\quad T(t_1)T(t_2)=T(t_1+t_2)\ \forall\, t_1,t_2\in[0,\infty[$$

and

$$T(t)C_b^k(\mathbb{R}^d,\mathbb{R})\subseteq C_b^k(\mathbb{R}^d,\mathbb{R})\ \forall\, t\in[0,\infty[\ \forall\, k\in\mathbb{N}_0\cup\{\infty\}.$$

Let us fix $u_0\in C_b^1(\mathbb{R}^d,\mathbb{R})$ and consider $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}$ given by

$$u(x,t):=\left[T(t)u_0\right](x)=u_0\left(\Phi(t;x)\right),\ t\in[0,\infty[,\ x\in\mathbb{R}^d,$$

then u is the classical solution of the first-order linear PDE

$$\text{(PDE}_{1st}\text{)}\qquad \begin{aligned}&\frac{\partial}{\partial t}u(x,t)=\sum_{i=1}^{d}\mu_i(x)\frac{\partial}{\partial x_i}u(x,t)=:\left\langle\mu(x),\nabla u(x,t)\right\rangle,\ x\in\mathbb{R}^d,\ t\in[0,\infty[,\\ &u(x,0)=u_0(x).\end{aligned}$$

As we will see in Section 2.1.2, (PDE$_{1st}$) is a special case of a Kolmogorov equation. In particular, the solution preserves the smoothness of the initial data, i.e. for every $k\in\mathbb{N}\cup\{\infty\}$

$$u(\cdot,0)=u_0(\cdot)\in C_b^k(\mathbb{R}^d,\mathbb{R})\quad\Rightarrow\quad u(\cdot,t)\in C_b^k(\mathbb{R}^d,\mathbb{R})\ \forall\, t\in[0,\infty[$$

Example 2.1 (Drift term of the Ornstein-Uhlenbeck operator)**.** Let $d\in\mathbb{N}$ and $\mu:\mathbb{R}^d\to\mathbb{R}^d$ with $\mu(x)=Sx$ for some $0\neq S\in\mathbb{R}^{d,d}$, then $\Phi(\cdot;x):[0,\infty[\to\mathbb{R}^d$ with $\Phi(t;x)=e^{tS}x$, $x\in\mathbb{R}^d$, $t\in[0,\infty[$, is the unique smooth function satisfying

$$\begin{aligned}\frac{\partial}{\partial t}\Phi(t;x)&=S\Phi(t;x),\,t\in[0,\infty[,\,x\in\mathbb{R}^d,\\ \Phi(0;x)&=x.\end{aligned}$$

The corresponding transition semigroup is given by $T(t):C_b(\mathbb{R}^d,\mathbb{R})\to C_b(\mathbb{R}^d,\mathbb{R})$, $t\in[0,\infty[$, with

$$[T(t)u_0](x):=u_0\left(e^{tS}x\right),\,x\in\mathbb{R}^d,\,t\in[0,\infty[,\,u_0\in C_b(\mathbb{R}^d,\mathbb{R}).$$

If we fix $u_0\in C_b^1(\mathbb{R}^d,\mathbb{R})$, then $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}$ given by

$$u(x,t):=[T(t)u_0](x)=u_0\left(e^{tS}x\right),\,t\in[0,\infty[,\,x\in\mathbb{R}^d,$$

is a classical solution of the first-order linear PDE

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t)&=\sum_{i=1}^d(Sx)_i\frac{\partial}{\partial x_i}u(x,t)=:\langle Sx,\nabla u(x,t)\rangle,\,x\in\mathbb{R}^d,\,t\in[0,\infty[,\\ u(x,0)&=u_0(x).\end{aligned}$$

2.1.2. *From SODE to second-order PDE.* Let $d,m\in\mathbb{N}$ and consider two functions $\mu\in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $\sigma\in C^\infty(\mathbb{R}^d,\mathbb{R}^{d,m})$, which are at most linearly growing, i.e.

$$\begin{aligned}&\exists\, C>0:\ |\mu(x)|\leqslant C\,(1+|x|)\quad\forall\, x\in\mathbb{R}^d,\\ &\exists\, C>0:\ |\sigma_j(x)|\leqslant C\,(1+|x|)\quad\forall\, x\in\mathbb{R}^d\,\forall\, j=1,\ldots,m.\end{aligned}$$

Furthermore, let $(\Omega,\mathcal{F},\mathbb{P})$ denote a probability space with a standard Brownian motion

$$W:[0,\infty[\times\Omega\to\mathbb{R}^m$$

Then there exists a family

$$\Phi(\cdot;x):[0,\infty[\times\Omega\to\mathbb{R}^d,\,x\in\mathbb{R}^d,$$

of solution processes of

$$\text{(SODE)}\qquad\begin{aligned}d\Phi(t;x)&=\mu\left(\Phi(t;x)\right)dt+\sigma\left(\Phi(t;x)\right)dW(t),\,t\in[0,\infty[,\,x\in\mathbb{R}^d,\\ \Phi(0;x)&=x,\end{aligned}$$

It is well known, that the solution processes are unique up to indistinguishability. More precisely, one has $\Phi(t;x)=\Phi(t,\omega;x)$, but we suppress the argument ω in the following. Note, that the (SODE) can describe the random motion of a particle in a fluid, [58]. The family $T(t):C_b(\mathbb{R}^d,\mathbb{R})\to C_b(\mathbb{R}^d,\mathbb{R})$, $t\in[0,\infty[$, of linear operators defined by

$$[T(t)u_0](x):=\mathbb{E}\left[u_0\left(\Phi(t;x)\right)\right],\,x\in\mathbb{R}^d,\,t\in[0,\infty[,\,u_0\in C_b(\mathbb{R}^d,\mathbb{R}),$$

is called the transition semigroup of the (SODE). $(T(t))_{t\geqslant 0}$ satisfies the properties

$$T(0)=I\quad\text{and}\quad T(t_1)T(t_2)=T(t_1+t_2)\,\forall\, t_1,t_2\in[0,\infty[$$

and

$$T(t)C_b(\mathbb{R}^d,\mathbb{R})\subseteq C_b^\infty(\mathbb{R}^d,\mathbb{R})\,\forall\, t\in[0,\infty[.$$

Such smoothing properties were established by Hörmander in 1967 under the Hörmander condition, [33]. This condition is for example satisfied, if

$$\operatorname{span}\{\sigma_1(x),\ldots,\sigma_m(x)\}=\mathbb{R}^d\,\forall\, x\in\mathbb{R}^d.$$

Let us fix $u_0 \in C_b^1(\mathbb{R}^d,\mathbb{R})$ and consider $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}$ given by

$$u(x,t) := [T(t)u_0](x) = \mathbb{E}\left[u_0\left(\Phi(t;x)\right)\right],\; t\in[0,\infty[,\; x\in\mathbb{R}^d.$$

If $u(\cdot,t)$ is smooth for all $t\in]0,\infty[$, then u is the classical solution of the second-order linear PDE

(PDE$_{2nd}$)

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \sum_{i=1}^d \mu_i(x)\frac{\partial}{\partial x_i}u(x,t) + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d \left(\sigma^T(x)\sigma(x)\right)_{ij}\frac{\partial^2}{\partial x_i\partial x_j}u(x,t),\\ &=: \langle \mu(x),\nabla u(x,t)\rangle + \frac{1}{2}\mathrm{Tr}\left(\sigma^T(x)\sigma(x)D^2u(x,t)\right)\\ u(x,0) &= u_0(x),\end{aligned}$$

for $x\in\mathbb{R}^d$ and $t\in]0,\infty[$. (PDE$_{2nd}$) is called the Kolmogorov equation. The second-order differential operator

$$\left[\mathcal{L}_{\mathrm{Kol}}u\right](x,t) := \frac{1}{2}\mathrm{Tr}\left(\sigma^T(x)\sigma(x)D^2u(x,t)\right) + \langle\mu(x),\nabla u(x,t)\rangle,\; x\in\mathbb{R}^d,\; t\in]0,\infty[$$

is called the Kolmogorov operator with diffusion term $\frac{1}{2}\mathrm{Tr}\left(\sigma^T(x)\sigma(x)D^2u(x,t)\right)$ and drift term $\langle\mu(x),\nabla u(x,t)\rangle$, $x\in\mathbb{R}^d$, $t\in]0,\infty[$. Note that the Kolmogorov operator $\mathcal{L}_{\mathrm{Kol}}$ can be considered as the infinitesimal generator of the transition semigroup of the (SODE).

Example 2.2 (Ornstein-Uhlenbeck operator)**.** Let $m=d\in\mathbb{N}$, $\mu:\mathbb{R}^d\to\mathbb{R}^d$ with $\mu(x)=Sx$ for some $0\neq S\in\mathbb{R}^{d,d}$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d,d}$ such that $\sigma^T(x)\sigma(x)=Q$ for every $x\in\mathbb{R}^d$ for some constant matrix $Q\in\mathbb{R}^{d,d}$. If we assume that Q is symmetric and positive definite, then there exists a unique symmetric and positive definite square root $\sqrt{Q}$ of Q, i.e. $\sigma(x)=\sqrt{Q}$ for every $x\in\mathbb{R}^d$. Furthermore, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a standard Brownian motion $W:[0,\infty[\times\Omega\to\mathbb{R}^d$. Then the family $\Phi(\cdot;x):[0,\infty[\times\Omega\to\mathbb{R}^d$ given by

$$\Phi(t;x) = e^{tS}x + \int_0^t e^{(t-\tau)S}dW(\tau),\; t\in[0,\infty[,\; x\in\mathbb{R}^d,$$

are the 'up to indistinguishability' unique solution processes of

$$\begin{aligned}d\Phi(t;x) &= S\Phi(t;x)dt + \sqrt{Q}dW(t),\; t\in[0,\infty[,\; x\in\mathbb{R}^d,\\ \Phi(0;x) &= x.\end{aligned}$$

The solution process $\Phi(\cdot,x)$ is called the Ornstein-Uhlenbeck process on $\mathbb{R}^d$ and the corresponing SODE is also known as the Langevin equation. A prototype of this equation, $u_t = u + xu_x + u_{xx}$, was considered by Ornstein and Uhlenbeck in 1930, [58]. The corresponding transition semigroup, or sometimes called the Ornstein-Uhlenbeck semigroup, is given by $T(t):C_b(\mathbb{R}^d,\mathbb{R})\to C_b(\mathbb{R}^d,\mathbb{R})$, $t\in[0,\infty[$, with

$$\begin{aligned}\left[T(t)u_0\right](x) :=& \mathbb{E}\left[u_0\left(\Phi(t;x)\right)\right]\\ =& \begin{cases}(4\pi)^{-\frac{d}{2}}\left(\det Q_t\right)^{-\frac{1}{2}}\int_{\mathbb{R}^d} e^{-\frac{1}{4}\left\langle Q_t^{-1}\psi,\psi\right\rangle} v_0(e^{tS}x-\psi)d\psi & ,\, t>0,\\ u_0(x) & ,\, t=0,\end{cases}\\ =& \begin{cases}\int_{\mathbb{R}^d} H(x,\xi,t)v_0(\xi)d\xi & ,\, t>0,\\ u_0(x) & ,\, t=0,\end{cases}\end{aligned}$$

for $x\in\mathbb{R}^d$, $t\in[0,\infty[$ and $u_0\in C_b(\mathbb{R}^d,\mathbb{R})$ where

$$H(x,\xi,t) = (4\pi)^{-\frac{d}{2}}\left(\det Q_t\right)^{-\frac{1}{2}} e^{-\frac{1}{4}\left\langle Q_t^{-1}(e^{tS}x-\xi),(e^{tS}x-\xi)\right\rangle},$$

for $x,\xi\in\mathbb{R}^d$, $t\in]0,\infty[$ and

$$Q_t=\int_0^t e^{\tau S}Q\left(e^{\tau S}\right)^T d\tau,$$

for $t\in]0,\infty[$. The explicit representation of $(T(t))_{t\geqslant 0}$ is due to Kolmogorov, [34]. The function $H:\mathbb{R}^d\times\mathbb{R}^d\times]0,\infty[\to\mathbb{R}$ denotes the heat kernel of the Ornstein-Uhlenbeck operator and is called the Kolmogorov kernel, or sometimes the Ornstein-Uhlenbeck kernel. Since $Q\in\mathbb{R}^{d,d}$ is symmetric and positive definite, it holds the following relation between the heat kernel and the d-dimensional Gaussian measure $\mathcal{N}_d$, see [38, Chapter 9.1] and [14, Satz 30.4],

$$\mathcal{N}_d\left(e^{tS}x,2Q_t\right)(d\xi)=H(x,\xi,t)d\xi,\ x\in\mathbb{R}^d,\ t>0,$$

i.e. $H(x,\cdot,t)$ is the density function of the normal distribution $\mathcal{N}_d\left(e^{tS}x,2Q_t\right)$ with respect to the Lebesgue measure. $2Q_t$ denotes the covariance matrix and $e^{tS}x$ the mean value vector. Let us fix $u_0\in C_b^1(\mathbb{R}^d,\mathbb{R})$ and let us define $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}$ by

$$u(x,t):=\left[T(t)u_0\right](x)=\mathbb{E}\left[u_0\left(\Phi(t;x)\right)\right],\ t\in[0,\infty[,\ x\in\mathbb{R}^d,$$

then, if $u(\cdot,t)$ is smooth for all $t\in]0,\infty[$, u is the classical solution of the Kolmogorov equation

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t)&=\sum_{i=1}^d (Sx)_i\frac{\partial}{\partial x_i}u(x,t)+\frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d Q_{ij}\frac{\partial^2}{\partial x_i\partial x_j}u(x,t)\\&=\langle Sx,\nabla u(x,t)\rangle+\frac{1}{2}\mathrm{Tr}\left(QD^2u(x,t)\right),\ x\in\mathbb{R}^d,\ t\in]0,\infty[,\\u(x,0)&=u_0(x).\end{aligned}$$

The second-order differential operator

$$\left[\mathcal{L}_{\mathrm{OU}}u\right](x,t):=\frac{1}{2}\mathrm{Tr}\left(QD^2u(x,t)\right)+\langle Sx,\nabla u(x,t)\rangle$$

is called the Ornstein-Uhlenbeck operator with diffusion term $\frac{1}{2}\mathrm{Tr}\left(QD^2u(x,t)\right)$ and drift term $\langle Sx,\nabla u(x,t)\rangle$. This operator can be considered as the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$. In addition, if Q is only assumed to be symmetric and positive semidefinit, $\mathcal{L}_{\mathrm{OU}}$ is called the degenerate Ornstein-Uhlenbeck operator. Several interpretations in physics and finance of this operator or its evolutionary counterpart - the Kolmogorov-Fokker-Planck operator $\mathcal{L}_{\mathrm{OU}}-\partial_t$ - are explained in the survey by Pascucci [50]. Finally, we observe that if $Q=2I_d$ then we have $\frac{1}{2}\mathrm{Tr}\left(QD^2u(x,t)\right)=\triangle u(x,t)$, where $\triangle$ denotes the Laplacian on $\mathbb{R}^d$.

2.2. **Rotating waves in reaction diffusion systems.** In Section 1.1 we have already motivated the nonlinear steady state problem (1.1) for the complex Ornstein-Uhlenbeck operator by the existence of rotating wave solutions. Such rotating waves arise in many applications from physical, chemical and biological sciences. In the following, we list a set of examples, where such rotating wave solutions exist. All the computations where done with the help of the package [1].

Example 2.3 (Ginzburg-Landau equation)**.** Consider the cubic-quintic complex Ginzburg-Landau equation (QCGL), [37],

$$u_t=\alpha\triangle u+u\left(\mu+\beta\left|u\right|^2+\gamma\left|u\right|^4\right)\tag{2.1}$$

with $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{C}$, $d\in\{2,3\}$ and $\alpha,\beta,\gamma,\mu\in\mathbb{C}$. The real-valued version of this equation reads as

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}_t=\begin{pmatrix}\alpha_1&-\alpha_2\\\alpha_2&\alpha_1\end{pmatrix}\triangle\begin{pmatrix}u_1\\u_2\end{pmatrix}+f\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

with

$$f\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}(u_1\mu_1-u_2\mu_2)+(u_1\beta_1-u_2\beta_2)\left(u_1^2+u_2^2\right)+(u_1\gamma_1-u_2\gamma_2)\left(u_1^2+u_2^2\right)^2\\(u_1\mu_2+u_2\mu_1)+(u_1\beta_2+u_2\beta_1)\left(u_1^2+u_2^2\right)+(u_1\gamma_2+u_2\gamma_1)\left(u_1^2+u_2^2\right)^2\end{pmatrix},$$

$u=u_1+iu_2$, $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$, $\gamma=\gamma_1+i\gamma_2$ and $u_i,\alpha_i,\beta_i,\gamma_i\in\mathbb{R}$ for $i=1,2$. This equation describes different aspects of signal propagation in heart tissue, superconductivity, superfluidity, nonlinear optical systems, see [47], photonics, plasmas, physics of lasers, Bose-Einstein condensation, liquid crystals, fluid dynamics, chemical waves, quantum field theory, granular media and is used in the study of hydrodynamic instabilities, see [45]. It shows a variety of coherent structures like stable and unstable pulses, fronts, sources and sinks in 1D, see [59], [55], [6] and [57], vortex solitons, see [23], spinning solitons, see [24], rotating spiral waves, propagating clusters, see [52], and exploding dissipative solitons, see [54] in 2D as well as scroll waves and spinning solitons in 3D, see [46].
Let us discuss the assumptions (A1)–(A8): Assumption (A1) is satisfied for every $\alpha\in\mathbb{C}$, assumption (A2) if $\operatorname{Re}\alpha=a_1>0$ and (A3) for some $1<p<\infty$ if

$$|\arg\alpha|\leqslant\arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

The condition (A4) is satisfied with

$$S=\begin{pmatrix}0&S_{12}\\-S_{12}&0\end{pmatrix}\quad\text{and}\quad S=\begin{pmatrix}0&S_{12}&S_{13}\\-S_{12}&0&S_{23}\\-S_{13}&-S_{23}&0\end{pmatrix}\tag{2.2}$$

for $d=2$ and $d=3$, respectively. In the examples below we determine S_{12}, S_{13} and S_{23} from a simulation. First we simulate the original system for some time then we switch to the freezing method, see [16], [19], [18], and [56], which then yields values for the velocities. The specific values of these variables we discuss in the examples below. Note that in case $d=2$ we have a clockwise rotation, if $S_{12}>0$, and a counter clockwise rotation, if $S_{12}<0$. Assumption (A5) is obviously satisfied. Using, for instance, $v_\infty=(0,0)^T$ then assumption (A6) is satisfied. Then, we have

$$Df(v_\infty)=\begin{pmatrix}\mu_1&-\mu_2\\\mu_2&\mu_1\end{pmatrix}$$

and assumption (A7) is also satisfied. Assumption (A8) is only satisfied if $\operatorname{Re}\mu<0$. The bound for the rate of the exponential decay from Theorem 1.7 equals

$$0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{\operatorname{Re}\alpha\,(-\operatorname{Re}\mu)}{|\alpha|^2p^2}.$$

for some $0<\vartheta<1$. Let us now consider some specific examples:
(**1**): For the parameters

$$\alpha=\frac{1}{2}+\frac{1}{2}i,\quad\beta=\frac{5}{2}+i,\quad\gamma=-1-\frac{1}{10}i,\quad\mu=-\frac{1}{2}\tag{2.3}$$

this equation exhibits so called spinning soliton solutions for space dimensions $d=2$ and $d=3$, see Figure 2.1.
Figure 2.1(a)–2.1(c) shows the spinning soliton in $\mathbb{R}^2$ as the solution of (2.1) on a circle of radius $R=20$ centered in the origin at time $t=150$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x=0.25$, the

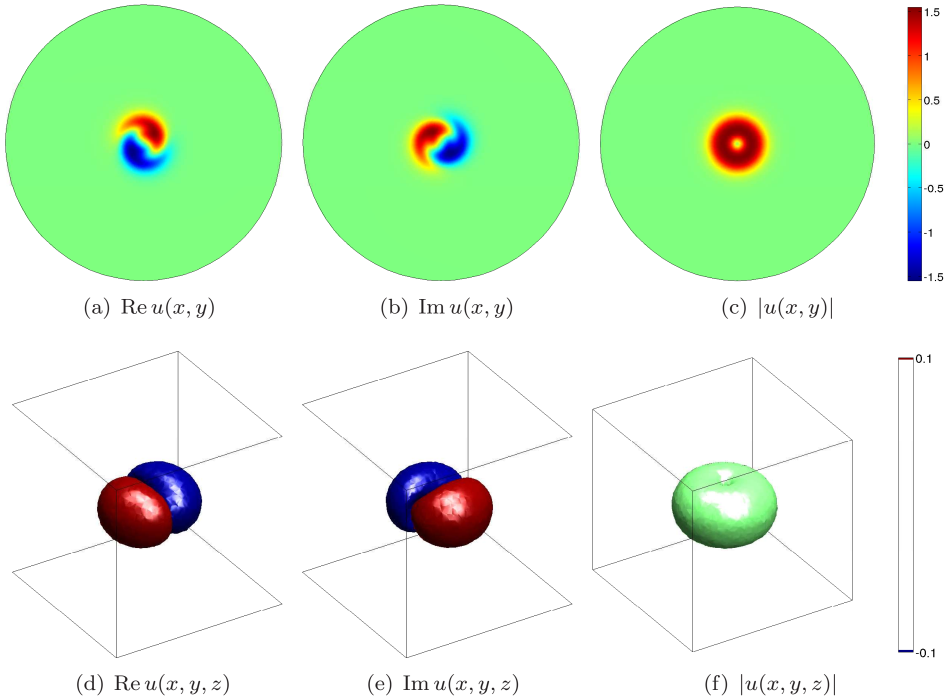


FIGURE 2.1. Spinning soliton of QCGL for $d=2$ (above) and $d=3$ (bottom)

BDF method of order 2 with stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0^{2D}(x,y) = \frac{1}{5}(x+iy)\exp\left(-\frac{x^2+y^2}{49}\right).$$

Figure 2.1(d)–2.1(f) shows the spinning soliton in $\mathbb{R}^3$ as the solution of (2.1) on a cube with edge length $L=20$ centered in the origin at time $t=100$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x = 0.8$, the BDF method of order 2 with stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0^{3D}(x,y,z) = u_0^{2D}(x,y)$$

for $|z|<9$ and otherwise 0.

The parameter values (2.3) satisfy our assumptions (A1)–(A8) for every $p \in]4-2\sqrt{2}, 4+2\sqrt{2}[$, i.e. $p=2,3,4,5,6$. At time $t=400$ we have the rotational velocities $S_{12}=1.027$ in case $d=2$ and $(S_{12},S_{13},S_{23}) = (0.6862,-0.01024,0.005671)$ at time $t=900$ in case $d=3$. The solitons are localized in the sense of Theorem 1.7 with the bound

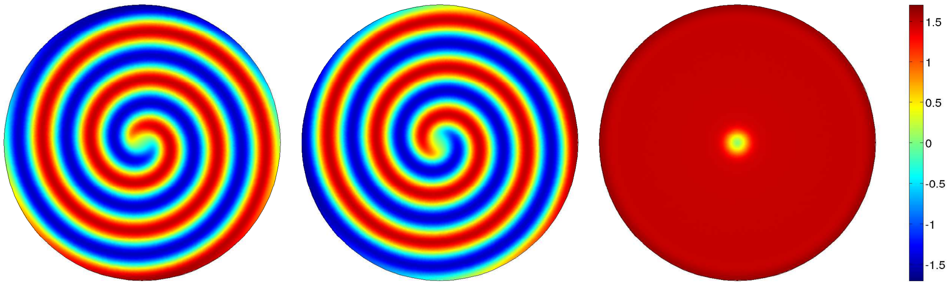
$$0 \leqslant \eta^2 \leqslant \vartheta\frac{1}{3p^2} < \frac{1}{3p^2} \quad \text{for} \quad p \in]4-2\sqrt{2}, 4+2\sqrt{2}[.$$

(**2**): For the parameters

$$\alpha = \frac{1}{2}+\frac{1}{2}i, \quad \beta = \frac{13}{5}+i, \quad \gamma = -1-\frac{1}{10}i, \quad \mu = -\frac{1}{2} \tag{2.4}$$

this equation exhibits so called rotating spiral soliton solutions, see Figure 2.2.

Figure 2.2(a)–2.2(c) shows the spiral soliton in $\mathbb{R}^2$ as the solution of (2.1) on a circle of radius $R=20$ centered in the origin at time $t=150$. For the computation



(a) $\mathrm{Re}\, u(x,y)$ (b) $\mathrm{Im}\, u(x,y)$ (c) $|u(x,y)|$

FIGURE 2.2. Rotating spiral soliton of QCGL for $d=2$

we used continuous piecewise linear finite elements with stepsize $\triangle x = 0.25$, the BDF method of order 2 with stepsize $\triangle t = 0.1$, homogeneous Neumann boundary conditions and initial data u_0^{2D} from above.

The only different in the choice of parameters in (2.4) to the previous example, compare (2.3), is the real part of β, which is now a little bit larger. The parameter values satisfy our assumptions (A1)–(A8) also for $p \in]4-2\sqrt{2}, 4+2\sqrt{2}[$. At time $t=300$ we have the rotational velocity $S_{12} = 1.323$. The solitons are also localized in the sense of Theorem 1.7 with the same bound as before. We observe that enlarging β from $\frac{5}{2}$ to $\frac{13}{5}$ generates a pattern with a larger support and a higher rotational velocity.

(**3**): For the parameters

$$\alpha = 1, \quad \beta = -(1+i), \quad \gamma = 0, \quad \mu = 1 \tag{2.5}$$

this equation exhibits so called twisted and untwisted scroll wave solutions, see Figure 2.4.

Figure 2.4(a)–2.4(c) shows the untwisted scroll wave in $\mathbb{R}^3$ as the solution of (2.1) and (2.6), respectively, on a cube with edge length $L=40$ centered in the origin at time $t=150$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x = 1.6$, the BDF method of order 2 with stepsize $\triangle t = 0.5$, homogeneous Neumann boundary conditions on the side surfaces, periodic boundary conditions on the faces for $z = \mp 10$ and initial data

$$u_0^{3D}(x,y,z) = \frac{1}{7}\left((x-20)+i(y-20)\right)\exp\left(-\frac{(x-20)^2+(y-20)^2}{49} + i\frac{z-20}{2\pi}\right).$$

The parameter values (2.5) satisfy only the assumptions (A1)–(A7) for every $1 < p < \infty$ but not condition (A8), since the real part of μ is not negative. In this case the pattern is not localized in the sense of Theorem 1.7. The rotational velocities at time $t=300$ are $(S_{12}, S_{13}, S_{23}) = (-0.1756, -0.00001, 0.002146)$.

Example 2.4 (λ-ω system)**.** Consider the λ-ω system, [36], [48],

$$u_t = \alpha \triangle u + u\left(\lambda\left(|u|^2\right) + i\omega\left(|u|^2\right)\right) \tag{2.6}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$, $\alpha \in \mathbb{C}$, $\lambda : [0,\infty[\to \mathbb{R}$ and $\omega : [0,\infty[\to \mathbb{R}$. The real-valued version of this equation reads as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix} \triangle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$f\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1\lambda\left(u_1^2+u_2^2\right) - u_2\omega\left(u_1^2+u_2^2\right) \\ u_1\omega\left(u_1^2+u_2^2\right) + u_2\lambda\left(u_1^2+u_2^2\right) \end{pmatrix},$$

$u = u_1 + iu_2$, $\alpha = \alpha_1 + i\alpha_2$ and $u_i, \alpha_i \in \mathbb{R}$ for $i = 1, 2$. This equation describes chemical reaction processes, see [36] and [35], physiological processes in the study of cardiac arrhythmias, time evolution of biological systems, see [48], and is often used to analyze the mechanism of pattern formation as well as to study the oneset of turbulent behavior. An example of an emerging technological application based on pattern forming systems is given by memory devices using magnetic domain patterns. This model exhibits rotating spirals as well as scroll wave and scroll ring solutions, see [27] and [29].

Let us again discuss the assumptions (A1)–(A8): Assumption (A1) is satisfied for every $\alpha \in \mathbb{C}$, assumption (A2) if $\mathrm{Re}\,\alpha = a_1 > 0$ and (A3) for some $1 < p < \infty$ if

$$|\arg \alpha| \leqslant \arctan\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

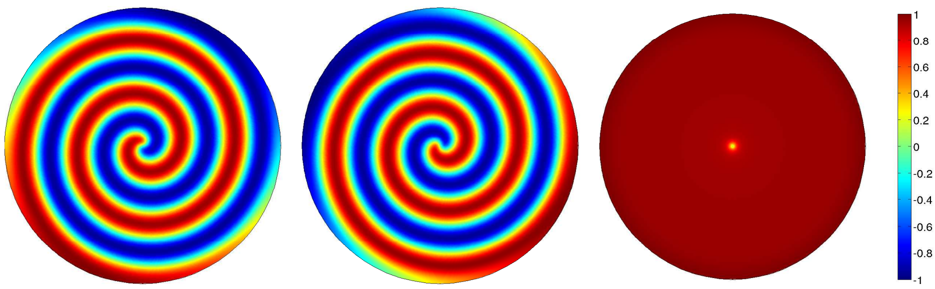
The condition (A4) is satisfied with S from (2.2). Assumption (A5) is satisfied if $\lambda, \omega \in C^2([0,\infty[, \mathbb{R})$. Since the assumptions (A6)–(A8) depends on the choice of λ and ω, we explain this conditions in the following example.

(1): For the parameters

$$\alpha = 1, \quad \lambda\left(|u|^2\right) = 1 - |u|^2, \quad \omega\left(|u|^2\right) = -|u|^2 \tag{2.7}$$

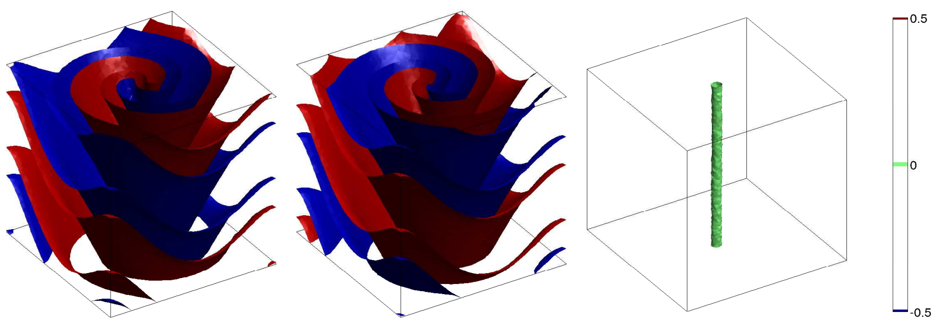
this equation exhibits so called rigidly rotating spiral wave solutions, see Figure 2.3, as well as twisted and untwisted scroll wave solutions, see Figure 2.4 for an untwisted scroll wave.

Figure 2.3(a)–2.3(c) shows the spiral wave in $\mathbb{R}^2$ as the solution of (2.6) on a



(a) $\mathrm{Re}\,u(x,y)$ (b) $\mathrm{Im}\,u(x,y)$ (c) $|u(x,y)|$

FIGURE 2.3. Rigidly rotating spiral wave of λ-ω system for $d = 2$



(a) $\mathrm{Re}\,u(x,y,z)$ (b) $\mathrm{Im}\,u(x,y,z)$ (c) $|u(x,y,z)|$

FIGURE 2.4. Untwisted scroll wave of CGL and of the λ-ω system for $d = 3$

circle of radius $R=50$ centered in the origin at time $t=150$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x=0.5$, the BDF method of order 2 with stepsize $\triangle t=0.1$, homogeneous Neumann boundary conditions and initial data

$$u_0(x,y)=\frac{1}{20}\,(x,y)^T\,.$$

For Figure 2.4(a)–2.4(c) see Example 2.3(3). The parameter values (2.7) satisfy only the assumptions (A1)–(A7) for every $1<p<\infty$ and with $v_\infty=(0,0)^T$ but not condition (A8), since $Df(0,0)$ has the eigenvalue 1 with algebraic multiplicity 2. In this case the pattern is not localized in the sense of Theorem 1.7. The rotational velocities of the rotating spiral is $S_{12}=-0.9091$ at time $t=300$. Since the λ-ω system (2.6) equipped with the parameter-values (2.7) is indeed a special case of the cubic-quintic complex Ginzburg-Landau equation (2.1), namely one has to choose $\beta=-(1+i)$, $\gamma=0$ and $\mu=1$, compare (2.5), we refer for a discussion about the assumptions also to Example 2.3(3).

Example 2.5 (Barkley model)**.** Consider the Barkley model, [12], [13], [11]

$$\begin{pmatrix} u_1\\ u_2\end{pmatrix}_t=\begin{pmatrix}1&0\\0&D\end{pmatrix}\triangle\begin{pmatrix} u_1\\ u_2\end{pmatrix}+\begin{pmatrix}\frac{1}{\varepsilon}u_1\,(1-u_1)\left(u_1-\frac{u_2+b}{a}\right)\\ g(u_1)-u_2\end{pmatrix} \tag{2.8}$$

with $u=(u_1,u_2)^T$, $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^2$, $d\in\{2,3\}$, $0\leqslant D<<1$, $0<\varepsilon<<1$, $0<a,b\in\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$. This equation describes excitable media, oscillatory media, see[12], catalytic surface reactions, see [10], the interaction of a fast activator u and a slow inhibitor v (in this case $g(u)$ describes a delayed production of the inhibitor) and is often used as a qualitative model in pattern forming systems (e.g. Belousov-Zhabotinsky reaction). This model exhibits rotating spiral wave and scroll wave solutions, see [13], [16] and [56].

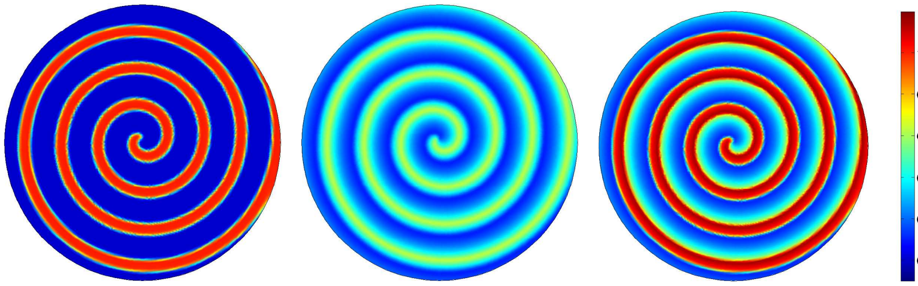
Let us discuss the assumptions (A1)–(A8): Assumption (A1) is satisfied for every $D\in\mathbb{R}$, assumption (A2) if $D>0$ and (A3) for every $1<p<\infty$ if $D>0$. The condition (A4) is satisfied with $S\in\mathbb{R}^{2,2}$ from (2.2). The specific values for S_{12} we discuss in the example below. Assumption (A5) is satisfied if $g\in C^2(\mathbb{R},\mathbb{R})$. The zeros of the nonlinearity are $(0,g(0))$, $(1,g(1))$ and one more. Using, for instance, $v_\infty=(0,g(0))^T$ then assumption (A6) is satisfied, (A7) is satisfied only for $D=1$ and (A8) only if $\frac{g(0)+b}{a^2}<0$, i.e. $g(0)<-b$, since the eigenvalues of $Df(v_\infty)$ are $\frac{g(0)+b}{a^2}$ and -1. Analogously, using $v_\infty=(1,g(1))^T$ then assumption (A6) is satisfied, (A7) is satisfied only for $D=1$ and (A8) only if $\frac{g(1)+b-a}{a^2}<0$, i.e. $g(1)<a-b$, since the eigenvalues of $Df(v_\infty)$ are $\frac{g(1)+b-a}{a^2}$ and -1. Let us now consider some specific examples:

(**1**): For the parameters

$$D=0,\quad \varepsilon=\frac{1}{50},\quad a=\frac{3}{4},\quad b=\frac{1}{100},\quad g(u_1)=u_1 \tag{2.9}$$

this equation exhibits so called rigidly rotating spiral wave solutions, see Figure 2.5. Figure 2.5(a)–2.5(c) shows the rotating spiral wave in $\mathbb{R}^2$ as the solution of (2.8) on a circle of radius $R=40$ centered in the origin at time $t=50$. For the computation we used continuous piecewise linear finite elements with stepsize $\triangle x=0.7$, the BDF method of order 2 with stepsize $\triangle t=0.2$, homogeneous Neumann boundary conditions and initial data

$$u_0^{(1)}(x,y)=\begin{cases}1 & ,\ x>0\\ 0 & ,\ x\leqslant 0\end{cases},\quad u_0^{(2)}(x,y)=\begin{cases}\frac{a}{2} & ,\ y>0\\ 0 & ,\ y\leqslant 0\end{cases}.$$



(a) $u_1(x,y)$ (b) $u_2(x,y)$ (c) $|u(x,y)|$

FIGURE 2.5. Rigidly rotating spiral wave of Barkley model for $d=2$

The parameter values (2.9) satisfy the assumptions (A1), (A5) since g is twice continuously differentiable, (A6) for $v_\infty=(0,0)^T$, $(1,1)^T$ and $\left(\frac{b}{a-1},\frac{b}{a-1}\right)^T$. At time $t=150$ we found the rotational velocity $S_{12}=2.108$ for the matrix S from (A4). All other assumptions are not satisfied. $D=0$ violates assumption (A2), (A3) and (A7). For $v_\infty=(0,0)^T$ condition (A8) needs $\frac{b}{a^2}<0$, which is not satisfied, and for $v_\infty=(1,1)^T$ assumption (A8) needs $\frac{1+b-a}{a^2}<0$, which is not true in this case. Assumption (A8) is also not satsfied for $v_\infty=\left(\frac{b}{a-1},\frac{b}{a-1}\right)^T$ with the parameters above.

2.3. **A review of the real-valued Ornstein-Uhlenbeck operator.** Before we start to investigate nonlinear Ornstein-Uhlenbeck problems in complex systems, let us present some well-known results about the scalar Ornstein-Uhlenbeck operator

$$[\mathcal{L}_{\mathrm{OU}}u](x):=\frac{1}{2}\mathrm{Tr}\left(QD^2u(x)\right)+\langle Sx,\nabla u(x)\rangle$$

considered in real-valued function spaces, where $Q\in\mathbb{R}^{d,d}$ with $Q=Q^T$, $Q>0$ and $0\neq S\in\mathbb{R}^{d,d}$. Note, that the properties of the matrix S play a fundamental role in the study of this operator.

The space $L^p(\mathbb{R}^d,\mathbb{R})$. The Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{R})$ related to the Lebesgue measure is indeed a semigroup for every $1\leqslant p\leqslant\infty$. A general problem is to show that $(T(t))_{t\geqslant 0}$ is strongly continuous. On $L^p(\mathbb{R}^d,\mathbb{R})$ one can verify that $(T(t))_{t\geqslant 0}$ is a C^0-semigroup for every $1\leqslant p<\infty$. A further problem that occurs, caused by the unbounded coefficients in the drift term, is to give an explicit representation for the domain of the infinitesimal generator A_p, which can be considered as the maximal realization of $\mathcal{L}_{\mathrm{OU}}$ in $L^p(\mathbb{R}^d,\mathbb{R})$ for $1<p<\infty$. In this context it was proved that the maximal domain is given by

$$\mathcal{D}^p(\mathcal{L}_{\mathrm{OU}})=\{v\in W^{2,p}(\mathbb{R}^d,\mathbb{R})\mid\langle Sx,\nabla v(x)\rangle\in L^p(\mathbb{R}^d,\mathbb{R})\}$$

for every $1<p<\infty$, which can be shown directly, [43], or with the aid of the Dore-Venni theorem, [51]. In case of $p=1$ no such representation is available, but it was proved that $\mathcal{D}^1(\mathcal{L}_{\mathrm{OU}})$ is the closure of $C_c^\infty(\mathbb{R}^d,\mathbb{R})$ with respect to the graph norm $\|\cdot\|_{\mathcal{L}_{\mathrm{OU}}}:=\|\cdot\|_{L^1}+\|\mathcal{L}_{\mathrm{OU}}\cdot\|_{L^1}$, i.e. $\mathcal{D}^1(\mathcal{L}_{\mathrm{OU}})=\overline{C_c^\infty}^{\|\cdot\|_{\mathcal{L}_{\mathrm{OU}}}}$. Moreover, it was established that the semigroup $(T(t))_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{R})$ for every $1\leqslant p<\infty$, if $S\neq 0$, which can be verified by analyzing the L^p-spectrum of $\mathcal{L}_{\mathrm{OU}}$, [42]. It was shown that the spectrum of the infinitesimal generator A_p of the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ considered on $L^p(\mathbb{R}^d,\mathbb{R})$ is given by

$$\sigma(A_p)=\left\{z\in\mathbb{C}\mid\mathrm{Re}\,z\leqslant-\frac{\mathrm{Tr}(S)}{p}\right\}$$

for every $1 < p < \infty$, if $\sigma(S) \subset \mathbb{C}_+$, $\sigma(S) \subset \mathbb{C}_-$ or S symmetric and Q and S commute, [42]. Thus, since $(T(t))_{t\geqslant 0}$ is not analytic for every $1 < p < \infty$, the parabolic equation $v_t = \mathcal{L}_{\mathrm{OU}}v$ does not satisfy the standard parabolic regularity properties on $L^p(\mathbb{R}^d,\mathbb{R})$.

The space $L^p(\mathbb{R}^d,\mathbb{R},\mu)$. Under the additional assumption that $\sigma(S) \subset \mathbb{C}_-$, which is very interesting from the point of view of diffusion processes, the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ considered on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ with uniquely determined invariant probability measure

$$\mu(x) = (4\pi)^{-\frac{d}{2}} (\det Q_\infty)^{-\frac{1}{2}} e^{-\frac{1}{4}\left\langle Q_\infty^{-1}x,x\right\rangle}$$

is a semigroup of positive contractions on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ for every $1 \leqslant p \leqslant \infty$ and a C^0-semigroup for every $1 \leqslant p < \infty$. The maximal domain is given by

$$\begin{aligned}\mathcal{D}^p_\mu(\mathcal{L}_{\mathrm{OU}}) =& W^{2,p}(\mathbb{R}^d,\mathbb{R},\mu)\\ =& \{v \in L^p(\mathbb{R}^d,\mathbb{R},\mu) \mid D_i v, D_j D_i v \in L^p(\mathbb{R}^d,\mathbb{R},\mu),\, i,j = 1,\ldots,d\}\end{aligned}$$

for every $1 < p < \infty$, [44], [40]. In case of $p = 1$ no such representation is available. A major difference to the usual L^p-cases is that $(T(t))_{t\geqslant 0}$ is compact and analytic on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ for every $1 < p < \infty$, [31]. In [41], it was shown for $1 < p < \infty$ that the spectrum of the infinitesimal generator A_p of the Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ considered on $L^p(\mathbb{R}^d,\mathbb{R},\mu)$ is a discrete set, independent of p and given by

$$\sigma(A_p) = \left\{\lambda = \sum_{i=1}^{r} n_i\lambda_i \mid n_i \in \mathbb{N}_0,\, i = 1,\ldots,r\right\},$$

where $\lambda_1,\ldots,\lambda_r$ denote the distinct eigenvalues of S. This is in strong contrast to the L^p-case. The eigenvalues are semisimple if and only if S is diagonalizable over $\mathbb{C}$. Moreover, the eigenfunctions of A_p are polynomials of degree at most $\frac{\mathrm{Re}\,\lambda}{s(S)}$. In case $p = 1$ the situation changes drastically and the spectrum is given by $\sigma(A_1) = \mathbb{C}_- \cup i\mathbb{R}$.

The space $C_b(\mathbb{R}^d,\mathbb{R})$. The Ornstein-Uhlenbeck semigroup $(T(t))_{t\geqslant 0}$ is a semigroup on $C_b(\mathbb{R}^d,\mathbb{R})$. To guarantee the strong continuity of $(T(t))_{t\geqslant 0}$ one usually considers the semigroup on the closed subspace $C_{ub}(\mathbb{R}^d,\mathbb{R})$ if the operator has constant or smooth bounded coefficients. But in case of the Ornstein-Uhlenbeck operator this space is not the right choice because the term $\langle Sx, \nabla v(x)\rangle$ has smooth but unbounded coefficients. One can show that $T(t)v_0$ tends to v_0 in $C_b(\mathbb{R}^d,\mathbb{R})$ as t tends to 0^+, if and only if $v_0 \in C_{ub}(\mathbb{R}^d,\mathbb{R})$ and $v_0(e^{tS}\cdot)$ tends to v_0 uniformly in $\mathbb{R}^d$ as t tends to 0^+. Hence, $(T(t))_{t\geqslant 0}$ is a C^0-semigroup on the much smaller subspace

$$C_{rub}(\mathbb{R}^d,\mathbb{R}) := \left\{f \in C_{ub}(\mathbb{R}^d,\mathbb{R}) \mid f(e^{tS}\cdot) \to f(\cdot) \text{ as } t \to 0^+ \text{ uniformly in } \mathbb{R}^d\right\},$$

[25], [26, see I.6]. The domain is completely characterized by

$$\mathcal{D}(\mathcal{L}_{\mathrm{OU}}) = \{v \in C_{rub}(\mathbb{R}^d,\mathbb{R}) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R})\;\forall p \geqslant 1 \mid \mathcal{L}_{\mathrm{OU}}v \in C_{rub}(\mathbb{R}^d,\mathbb{R})\},$$

[25]. Therein, it was also observed that $(T(t))_{t\geqslant 0}$ in not analytic on $C_{rub}(\mathbb{R}^d,\mathbb{R})$ and hence not analytic on $C_b(\mathbb{R}^d,\mathbb{R})$ and $C_{ub}(\mathbb{R}^d,\mathbb{R})$.

In table 1, we summarize these facts. For a detailed treatment of the Ornstein-Uhlenbeck operator we refer the reader e.g. to [38, Chapter 9].

3. Exponentially weighted function spaces

In this section we introduce the exponentially weighted Sobolev spaces, which we will use for all estimates in the sequal. For the weight functions of exponential growth rate and the exponentially weighted Sobolev spaces we follow [60, sec. 3].

Table 1. Properties of the Ornstein-Uhlenbeck operator

$T(t)$	**semigroup**	C^0**-semigroup**	**analytic semigroup**
$L^p(\mathbb{R}^d,\mathbb{R})$	$1\leqslant p\leqslant\infty$	$1\leqslant p<\infty$	no
$L^p(\mathbb{R}^d,\mathbb{R},\mu)$	$1\leqslant p\leqslant\infty$	$1\leqslant p<\infty$	$1<p<\infty$, if $\sigma(S)\subset\mathbb{C}_-$
$C_b(\mathbb{R}^d,\mathbb{R})$	yes	no	no
$C_{ub}(\mathbb{R}^d,\mathbb{R})$	yes	no	no
$C_{rub}(\mathbb{R}^d,\mathbb{R})$	yes	yes	no

The Euclidean Group $\mathrm{SE}(d)$**.** Let $d\in\mathbb{N}$ with $d\geqslant 2$ and let

$$\mathrm{SE}(d)=\mathbb{R}^d\ltimes\mathrm{SO}(d)$$

denote the special Euclidean group consisting of all pairs

$$\gamma=(\tau,R)\in\mathrm{SE}(d),\;\tau\in\mathbb{R}^d,\;R\in\mathrm{SO}(d)$$

with the group operation

$$\gamma_2\circ\gamma_1=(\tau_2,R_2)\circ(\tau_1,R_1)=(\tau_2+R_2\tau_1,R_2R_1),$$

the unit element $(0,I_d)$ and inverse element $(\tau,R)^{-1}=(-R^{-1}\tau,R^{-1})$. Here

$$\mathrm{SO}(d)=\{R\in\mathbb{R}^{d,d}\mid R^T=R^{-1}\text{ and }\det(R)=1\}$$

denotes the special orthogonal group. $\mathrm{SE}(d)$ is a Lie group of dimension $\frac{d(d+1)}{2}$. Its dimension is the sum of $\dim(\mathbb{R}^d)=d$ and $\dim(\mathrm{SO}(d))=\frac{d(d-1)}{2}$. The associated Lie algebra

$$\mathrm{se}(d)=\mathbb{R}^d\times\mathrm{so}(d)$$

is the product of $\mathbb{R}^d$ and the space

$$\mathrm{so}(d)=\{S\in\mathbb{R}^{d,d}\mid S^T=-S\}$$

of skew-symmetric matrices, which generate the rotations. Note, that the exponential mapping $\exp:\mathrm{so}(d)\to\mathrm{SO}(d)$ is onto and has the following properties

$$\begin{aligned}&\left(e^S\right)^{-1}=\left(e^S\right)^T=e^{S^T}=e^{-S},\;\det(e^S)=1,\;Se^{tS}=e^{tS}S\;\forall\,t\geqslant 0,\\&e^{S+S^T}=I_d\text{ and }\left|e^Sx\right|=|x|\;\;\forall\,x\in\mathbb{R}^d\end{aligned}\tag{3.1}$$

where $|\cdot|=\|\cdot\|_2$ denotes the Euclidean norm.

Sobolev Spaces. Let $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $N\in\mathbb{N}$, $p\in\mathbb{R}$ with $1\leqslant p\leqslant\infty$. We define the exponentially weighted L^p–spaces and their associated norms by

$$\begin{aligned}&L^p_\theta(\mathbb{R}^d,\mathbb{K}^N):=\{u\in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^N)\mid\|u\|_{L^p_\theta}<\infty\},\\&\|u\|_{L^p_\theta}:=\left(\int_{\mathbb{R}^d}\theta^p(x)\,|u(x)|^p\,dx\right)^{\frac{1}{p}},\;1\leqslant p<\infty,\\&\|u\|_{L^\infty_\theta}:=\operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\theta(x)\,|u(x)|\,,\;p=\infty.\end{aligned}$$

By definition $(L^p_\theta(\mathbb{R}^d,\mathbb{K}^N),\|\cdot\|_{L^p_\theta})$ is a Banach space.

Let $k\in\mathbb{N}_0$ and $1\leqslant p\leqslant\infty$ then we define the exponentially weighted Sobolev spaces of order k with exponent p and their associated norms by

$$\begin{aligned}&W^{k,p}_\theta(\mathbb{R}^d,\mathbb{K}^N):=\{u\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N)\mid D^\beta u\in L^p_\theta(\mathbb{R}^d,\mathbb{K}^N)\;\forall\;|\beta|\leqslant k\},\\&\|u\|_{W^{k,p}_\theta}:=\left(\sum_{|\beta|\leqslant k}\left\|D^\beta u\right\|^p_{L^p_\theta}\right)^{\frac{1}{p}},\;1\leqslant p<\infty,\end{aligned}$$

$$\|u\|_{W_\theta^{k,\infty}} := \max_{|\beta|\leqslant k} \left\|D^\beta u\right\|_{L_\theta^\infty}, \ p=\infty.$$

Let $k_1, k_2 \in \mathbb{N}_0$, $1 \leqslant p \leqslant \infty$, $T>0$ and $\Omega_T = \mathbb{R}^d \times]0,T[$ then we define the exponentially weighted anisotropic Sobolev spaces of order (k_1,k_2) with exponent p and their associated norms by, see [60, sec. 3]

$$\begin{aligned} W_\theta^{(k_1,k_2),p}(\Omega_T,\mathbb{K}^N) &:= \{u \in L^1_{\mathrm{loc}}(\Omega_T,\mathbb{K}^N) \mid D_x^\beta u \in L_\theta^p(\Omega_T,\mathbb{K}^N) \ \forall\, |\beta| \leqslant k_1 \\ &\qquad D_t^\gamma u \in L_\theta^p(\Omega_T,\mathbb{K}^N) \ \forall\, |\gamma| \leqslant k_2\}, \\ \|u\|_{W_\theta^{(k_1,k_2),p}(\Omega_T,\mathbb{K}^N)} &:= \Bigg(\|u\|^p_{L_\theta^p(\Omega_T,\mathbb{K}^N)} + \sum_{|\beta|=1}^{k_1} \left\|D_x^\beta u\right\|^p_{L_\theta^p(\Omega_T,\mathbb{K}^N)} \\ &\qquad + \sum_{|\gamma|=1}^{k_2} \|D_t^\gamma u\|^p_{L_\theta^p(\Omega_T,\mathbb{K}^N)} \Bigg)^{\frac{1}{p}}, \ 1 \leqslant p \leqslant \infty. \end{aligned}$$

Let $\Omega = \mathbb{R}^d$ or $\Omega = \Omega_T$ and $1 \leqslant p \leqslant \infty$ then we define the (anisotropic) local L^p–spaces by

$$L^p_{\mathrm{loc}}(\Omega,\mathbb{K}^N) := \{u \in L^1_{\mathrm{loc}}(\Omega,\mathbb{K}^N) \mid \|u\|_{L^p(A,\mathbb{K}^N)} < \infty \ \forall\, A \subset \Omega \text{ compact}\}.$$

The (anisotropic) local Sobolev spaces $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{K}^N)$ and $W^{(k_1,k_2),p}_{\mathrm{loc}}(\Omega_T,\mathbb{K}^N)$ can be defined in the same way.

Spaces of continuous functions. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $N \in \mathbb{N}$ and $k \in \mathbb{N}_0$.

$$\begin{aligned} &C_b(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C(\mathbb{R}^d,\mathbb{K}^N) \mid \|u\|_{C_b(\mathbb{R}^d,\mathbb{K}^N)} < \infty\}, \\ &\|u\|_{C_b(\mathbb{R}^d,\mathbb{K}^N)} := \|u\|_\infty := \sup_{x\in\mathbb{R}^d} |u(x)|, \\ &C_b^k(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_b(\mathbb{R}^d,\mathbb{K}^N) \mid D^\beta u \in C_b(\mathbb{R}^d,\mathbb{K}^N) \ \forall\, |\beta| \leqslant k\}, \\ &\|u\|_{C_b^k(\mathbb{R}^d,\mathbb{K}^N)} := \|u\|_{k,\infty} := \max_{|\beta|\leqslant k} \left\|D^\beta u\right\|_{C_b(\mathbb{R}^d,\mathbb{K}^N)}, \\ &C_{ub}(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_b(\mathbb{R}^d,\mathbb{K}^N) \mid u \text{ is uniformly continuous on } \mathbb{R}^d\}, \\ &C_{ub}^k(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_{ub}(\mathbb{R}^d,\mathbb{K}^N) \mid D^\beta u \in C_{ub}(\mathbb{R}^d,\mathbb{K}^N) \ \forall\, |\beta| \leqslant k\}, \\ &C_{rub}(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_{ub}(\mathbb{R}^d,\mathbb{K}^N) \mid \lim_{t\to 0} \left\|u(e^{tS}\cdot) - u(\cdot)\right\|_{C_b(\mathbb{R}^d,\mathbb{K}^N)} = 0\}, \\ &C_{rub}^k(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_{rub}(\mathbb{R}^d,\mathbb{K}^N) \mid D^\beta u \in C_{rub}(\mathbb{R}^d,\mathbb{K}^N) \ \forall\, |\beta| \leqslant k\}, \\ &C_{b,\theta}(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_b(\mathbb{R}^d,\mathbb{K}^N) \mid \|u\|_{C_{b,\theta}(\mathbb{R}^d,\mathbb{K}^N)} < \infty\}, \\ &\|u\|_{C_{b,\theta}(\mathbb{R}^d,\mathbb{K}^N)} := \|u\|_{\infty,\theta} := \|\theta u\|_{C_b(\mathbb{R}^d,\mathbb{K}^N)}, \\ &C_{b,\theta}^k(\mathbb{R}^d,\mathbb{K}^N) := \{u \in C_b^k(\mathbb{R}^d,\mathbb{K}^N) \mid \theta D^\beta u \in C_b(\mathbb{R}^d,\mathbb{K}^N) \ \forall\, |\beta| \leqslant k\}, \\ &\|u\|_{C_{b,\theta}^k(\mathbb{R}^d,\mathbb{K}^N)} := \|u\|_{k,\infty,\theta} := \max_{|\beta|\leqslant k} \left\|\theta D^\beta u\right\|_{C_b(\mathbb{R}^d,\mathbb{K}^N)}. \end{aligned}$$

Schwartz space. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and $N \in \mathbb{N}$. A function $\phi : \mathbb{R}^d \to \mathbb{K}^N$ is said to be rapidly decreasing if it is infinitely many times differentiable, i.e. $\phi \in C^\infty(\mathbb{R}^d,\mathbb{K}^N)$ and

$$\lim_{|x|\to\infty} x^\alpha D^\beta \phi(x) = 0 \in \mathbb{K}^N \quad \forall\, \alpha,\beta \in \mathbb{N}_0^d. \tag{3.2}$$

The space

$$\mathcal{S}(\mathbb{R}^d,\mathbb{K}^N) := \left\{\phi \in C^\infty(\mathbb{R}^d,\mathbb{K}^N) \mid \phi \text{ is rapidly decreasing}\right\}$$

is called the Schwartz space, [28, VI.5.1 Definition]. When endowed with the family of seminorms

$$|\phi|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \left|x^\alpha D^\beta \phi(x)\right|$$

the space $\mathcal{S}(\mathbb{R}^d,\mathbb{K}^N)$ becomes a Frechet space containing $C_c^{\infty}(\mathbb{R}^d,\mathbb{K}^N)$ as a dense subspace.

4. The heat kernel for operators of Ornstein-Uhlenbeck type in complex systems

In this section we derive a heat kernel for the complex-valued Ornstein-Uhlenbeck operator

$$[\mathcal{L}_\infty v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x),\; x\in\mathbb{R}^d,\; d\geqslant 2. \tag{4.1}$$

4.1. **Complex-valued Ornstein-Uhlenbeck kernel.** The aim of this subsection is to extend the appoach from [15], [4], [20, Chapter 13] to compute the heat kernel of the complex-valued operator $\mathcal{L}_\infty$ in the scalar case, in case of diagonal matrices and in case of simultaneously diagonalizable matrices. This will enable us in the next section to define the correspondig semigroup by an explicit respresentation. Therefore, we recall the definition of a heat kernel of $\mathcal{L}_\infty$, [20, Section 1.2]:

Definition 4.1. *A heat kernel (or a fundamental solution) of $\mathcal{L}_\infty$ given by (4.1) is a function*

$$H:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+^*\to\mathbb{C}^{N,N},\;(x,\xi,t)\mapsto H(x,\xi,t)$$

with $\mathbb{R}_+^*:=]0,\infty[$ such that

$$H\in C^{2,2,1}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+^*,\mathbb{C}^{N,N}), \tag{H1}$$

$$\frac{\partial}{\partial t}H(x,\xi,t)=\mathcal{L}_\infty H(x,\xi,t) \qquad \forall\,\xi\in\mathbb{R}^d,\; t>0, \tag{H2}$$

$$\lim_{t\downarrow 0}H(x,\xi,t)=\delta_x(\xi)I_N \qquad \forall\,\xi\in\mathbb{R}^d. \tag{H3}$$

where the convergence in (H3) is meant in the sense of distributions and $\delta_\xi(x)=\delta(x-\xi)$ denotes the Dirac delta function.

The following theorem provides an explicit representation for the heat kernel of $\mathcal{L}_\infty$ in the scalar case, i. e. with $N=1$. In addition, the proof contains a formal derivation of this heat kernel, which could be of interest for the computation of heat kernels for more general complex-valued heat operators. For the scalar real-valued but more general case a formal derivation of this kernel can be found in [15], [4] and [20, Section 13.2].

Theorem 4.2 (Scalar case)**.** *Let the assumptions (A2) and (A4) be satisfied for $\mathbb{K}=\mathbb{C}$ and $N=1$, then the function $H:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+^*\to\mathbb{C}$ defined by*

$$H(x,\xi,t)=(4\pi\alpha t)^{-\frac{d}{2}}\exp\left(-\delta t-(4\alpha t)^{-1}\left|e^{tS}x-\xi\right|^2\right) \tag{4.2}$$

is a heat kernel of $\mathcal{L}_\infty$ given by

$$[\mathcal{L}_\infty v](x) := \alpha\triangle v(x)+\langle Sx,\nabla v(x)\rangle-\delta v(x). \tag{4.3}$$

Remark. In the scalar case $N=1$ we write α and δ instead of A and B, respectively.

Proof. Before we verify that the heat kernel from (4.2) satisfies the properties (H1)–(H3) we discuss a formal derivation of this kernel. To compute the heat kernel (4.2) of (4.3) we generalize the approach from [15], [4] to the complex case and use the complexified ansatz

$$H(x,\xi,t)=\varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle M(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right) \tag{4.4}$$

where

$$\varphi:\mathbb{R}_+^*\to\mathbb{C},\, t\mapsto\varphi(t),$$
$$M:\mathbb{R}_+^*\to\mathbb{C}^{2d,2d},\, t\mapsto M(t)$$

have to be determined and $\langle\cdot,\cdot\rangle_{\mathbb{C}^{2d}}$ denotes the Euclidean inner product on $\mathbb{C}^{2d}$, i.e. $\langle x,y\rangle_{\mathbb{C}^{2d}}=\overline{x}^T y$. Note at this point that it is sufficient to determine the symmetric part of the complex-valued matrix M which we denote by N, i.e.

$$N:\mathbb{R}_+^*\to\mathbb{C}^{2d\times 2d},\, t\mapsto N(t):=\frac{1}{2}\left(M(t)+M^T(t)\right)=\begin{pmatrix}A(t) & B(t)\\ C(t) & D(t)\end{pmatrix},$$
$$A,B,C,D:\mathbb{R}_+^*\to\mathbb{C}^{d\times d},\, t\mapsto A(t),B(t),C(t),D(t).$$

Since $x,\xi\in\mathbb{R}^d$ we have

$$\begin{aligned}H(x,\xi,t)&=\varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle M(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right)\\ &=\varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle \frac{1}{2}\left(M(t)+M^T(t)\right)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right)\\ &=\varphi(t)\cdot\exp\left(-\frac{1}{2}\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right).\end{aligned}$$

Note that N is a symmetric but in general not a Hermitian matrix. In particular A and D are symmetric and $B^T=C$. Since the heat kernel must satisfy (H2) we obtain from the general Leibniz rule, the chain rule and the symmetry of N

$$H_t(x,\xi,t)=H(x,\xi,t)\left[\frac{\varphi_t(t)}{\varphi(t)}-\frac{1}{2}\left\langle N_t(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right],$$

$$\begin{aligned}\frac{\partial}{\partial x_i}H(x,\xi,t)&=H(x,\xi,t)\frac{\partial}{\partial x_i}\left[-\frac{1}{2}\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right]\\ &=H(x,\xi,t)\left[-\frac{1}{2}\left(\overline{\left\langle\overline{N}^T(t)\begin{pmatrix}x\\ \xi\end{pmatrix},e_i\right\rangle_{\mathbb{C}^{2d}}}+\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},e_i\right\rangle_{\mathbb{C}^{2d}}\right)\right]\\ &=-H(x,\xi,t)\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},e_i\right\rangle_{\mathbb{C}^{2d}},\end{aligned}$$

$$\frac{\partial^2}{\partial x_i^2}H(x,\xi,t)=H(x,\xi,t)\left[\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},e_i\right\rangle_{\mathbb{C}^{2d}}^2-\langle N(t)e_i,e_i\rangle_{\mathbb{C}^{2d}}\right]$$

$$\begin{aligned}\alpha\triangle H(x,\xi,t)&=\alpha H(x,\xi,t)\left[\sum_{i=1}^{d}\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},e_i\right\rangle_{\mathbb{C}^{2d}}^2-\sum_{i=1}^{d}\langle N(t)e_i,e_i\rangle_{\mathbb{C}^{2d}}\right]\\ &=\alpha H(x,\xi,t)\left[\sum_{i=1}^{d}\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},e_i\right\rangle_{\mathbb{C}^{2d}}\overline{\left\langle e_i,N(t)\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}}-\operatorname{Tr}\left(\overline{A}^T(t)\right)\right]\\ &=\alpha H(x,\xi,t)\left[\begin{pmatrix}x\\ \xi\end{pmatrix}^T\overline{N}^T(t)\left(\sum_{i=1}^{d}e_ie_i^T\right)\overline{N}(t)\begin{pmatrix}x\\ \xi\end{pmatrix}-\operatorname{Tr}\left(\overline{A}(t)\right)\right]\\ &=\alpha H(x,\xi,t)\left[\left\langle N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}I_d & 0\\ 0 & 0\end{pmatrix}\overline{N}(t)\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}-\operatorname{Tr}\left(\overline{A}(t)\right)\right]\\ &=H(x,\xi,t)\left[\left\langle\overline{\alpha}N(t)\tilde{P}N(t)\begin{pmatrix}x\\ \xi\end{pmatrix},\begin{pmatrix}x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}-\alpha\operatorname{Tr}\left(\overline{A}(t)\right)\right],\end{aligned}$$

$$\begin{aligned}
\langle Sx, \nabla H(x,\xi,t)\rangle &= \left(\frac{\partial}{\partial x_1}H(x,\xi,t),\ldots,\frac{\partial}{\partial x_d}H(x,\xi,t)\right)Sx\\
&= -H(x,\xi,t)\left(\left\langle N(t)\begin{pmatrix} x\\ \xi\end{pmatrix}, e_1\right\rangle_{\mathbb{C}^{2d}},\ldots,\left\langle N(t)\begin{pmatrix} x\\ \xi\end{pmatrix}, e_d\right\rangle_{\mathbb{C}^{2d}}\right)Sx\\
&= -H(x,\xi,t)\left\langle N(t)\begin{pmatrix} x\\ \xi\end{pmatrix}, \begin{pmatrix} S & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\\
&= -H(x,\xi,t)\left\langle \tilde{S}^T N(t)\begin{pmatrix} x\\ \xi\end{pmatrix}, \begin{pmatrix} x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\\
&= -H(x,\xi,t)\frac{1}{2}\left[\left\langle \tilde{S}^T N(t)\begin{pmatrix} x\\ \xi\end{pmatrix}, \begin{pmatrix} x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right.\\
&\quad\left. + \left\langle \begin{pmatrix} x\\ \xi\end{pmatrix}, \overline{N}^T(t)\tilde{S}\begin{pmatrix} x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right]\\
&= -H(x,\xi,t)\left\langle \frac{1}{2}\left(\tilde{S}^T N(t) + N(t)\tilde{S}\right)\begin{pmatrix} x\\ \xi\end{pmatrix}, \begin{pmatrix} x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}
\end{aligned}$$

where $i = 1,\ldots,d$. Introducing the extended matrices

$$\tilde{P} = \begin{pmatrix} I_d & 0\\ 0 & 0\end{pmatrix},\ \tilde{S} = \begin{pmatrix} S & 0\\ 0 & 0\end{pmatrix} \in \mathbb{R}^{2d,2d}$$

we end up with

$$\begin{aligned}
0 =& H(x,\xi,t)\left[\frac{\varphi_t(t)}{\varphi(t)} + \alpha\,\mathrm{tr}\left(\overline{A}(t)\right) + \delta\right.\\
&\left. + \left\langle\left(-\frac{1}{2}N_t(t) - \overline{\alpha}N(t)\tilde{P}N(t) + \frac{1}{2}\tilde{S}^T N(t) + \frac{1}{2}N(t)\tilde{S}\right)\begin{pmatrix} x\\ \xi\end{pmatrix}, \begin{pmatrix} x\\ \xi\end{pmatrix}\right\rangle_{\mathbb{C}^{2d}}\right].
\end{aligned}$$

Thus the kernel satisfies (H2) if the following differential equations hold

$$\varphi_t(t) = -\left(\alpha\,\mathrm{tr}\left(\overline{A}(t)\right) + \delta\right)\varphi(t) \qquad , t > 0, \tag{4.5}$$

$$N_t(t) = -2\overline{\alpha}N(t)\tilde{P}N(t) + \tilde{S}^T N(t) + N(t)\tilde{S} \qquad , t > 0. \tag{4.6}$$

Since (4.5) depends on the solution of (4.6), we will first solve the matrix-Riccati equation (4.6), see [5, sec. 3.1]. It is obvious that the solutions of (4.5) and (4.6) are not unique but one can select appropriate initial values, see [15] and [4].
Let us first eliminate linear terms in (4.6) by the following transformation

$$\begin{aligned}
\hat{N}(t) &= \exp\left(-t\tilde{S}^T\right)N(t)\exp\left(-t\tilde{S}\right)\\
&= \begin{pmatrix} \exp\left(-tS^T\right) & 0\\ 0 & I_d\end{pmatrix}\begin{pmatrix} A(t) & B(t)\\ C(t) & D(t)\end{pmatrix}\begin{pmatrix} \exp\left(-tS\right) & 0\\ 0 & I_d\end{pmatrix}\\
&= \begin{pmatrix} \exp(-tS^T)A(t)\exp(-tS) & \exp(-tS^T)B(t)\\ C(t)\exp(-tS) & D(t)\end{pmatrix}\\
&=: \begin{pmatrix} \hat{A}(t) & \hat{B}(t)\\ \hat{C}(t) & \hat{D}(t)\end{pmatrix}.
\end{aligned} \tag{4.7}$$

Differentiating $\hat{N}$ with respect to t and using (A4), $N = N^T$ and (4.6) we obtain

$$\begin{aligned}
\hat{N}_t(t) =& -\tilde{S}^T\exp\left(-t\tilde{S}^T\right)N(t)\exp\left(-t\tilde{S}\right)\\
&+ \exp\left(-t\tilde{S}^T\right)N_t(t)\exp\left(-t\tilde{S}\right)\\
&- \exp\left(-t\tilde{S}^T\right)N(t)\tilde{S}\exp\left(-t\tilde{S}\right)\\
=& -2\overline{\alpha}\hat{N}(t)\exp\left(t\tilde{S}\right)\tilde{P}\exp\left(t\tilde{S}^T\right)\hat{N}(t)
\end{aligned}$$

and hence

$$\hat{N}_t(t) = -2\overline{\alpha}\hat{N}(t)\exp\left(t\tilde{S}\right)\tilde{P}\exp\left(t\tilde{S}^T\right)\hat{N}(t) \qquad , t>0. \tag{4.8}$$

Writing this equation blockwise

$$\begin{aligned}
\hat{N}_t(t) &= -2\overline{\alpha}\hat{N}(t)\exp\left(t\tilde{S}\right)\tilde{P}\exp\left(t\tilde{S}^T\right)\hat{N}(t) \\
&= -2\overline{\alpha}\begin{pmatrix} \hat{A}(t)\exp\left(t\left(S+S^T\right)\right)\hat{A}(t) & \hat{A}(t)\exp\left(t\left(S+S^T\right)\right)\hat{B}(t) \\ \hat{C}(t)\exp\left(t\left(S+S^T\right)\right)\hat{A}(t) & \hat{C}(t)\exp\left(t\left(S+S^T\right)\right)\hat{B}(t) \end{pmatrix} \\
&= \begin{pmatrix} -2\overline{\alpha}\hat{A}^2(t) & -2\overline{\alpha}\hat{A}(t)\hat{B}(t) \\ -2\overline{\alpha}\hat{C}(t)\hat{A}(t) & -2\overline{\alpha}\hat{C}(t)\hat{B}(t) \end{pmatrix} =: \begin{pmatrix} \hat{A}_t(t) & \hat{B}_t(t) \\ \hat{C}_t(t) & \hat{D}_t(t) \end{pmatrix}
\end{aligned}$$

we arrive at the matrix ODE systems

$$\hat{A}_t(t) = -2\overline{\alpha}\hat{A}^2(t) \qquad , t>0, \tag{4.9}$$

$$\hat{B}_t(t) = -2\overline{\alpha}\hat{A}(t)\hat{B}(t) \qquad , t>0, \tag{4.10}$$

$$\hat{C}_t(t) = -2\overline{\alpha}\hat{C}(t)\hat{A}(t) \qquad , t>0, \tag{4.11}$$

$$\hat{D}_t(t) = -2\overline{\alpha}\hat{C}(t)\hat{B}(t) \qquad , t>0. \tag{4.12}$$

Note that $\hat{A} = \hat{A}^T$, $\hat{D} = \hat{D}^T$ and $\hat{B}^T = \hat{C}$ due to the corresponding properties of A, B, C and D. Therefore, solving (4.10) gives us automatically a solution of (4.11). Now we will successively solve the equations (4.9)–(4.12):

(4.9): Using the transformation $\tilde{A}(t) = \left(\hat{A}(t)\right)^{-1}$ we obtain

$$\begin{aligned}
\tilde{A}_t(t) &= \frac{d}{dt}\left(\hat{A}(t)\right)^{-1} = -\left(\hat{A}(t)\right)^{-1}\hat{A}_t(t)\left(\hat{A}(t)\right)^{-1} \\
&= 2\overline{\alpha}\left(\hat{A}(t)\right)^{-1}\left(\hat{A}(t)\right)^{2}\left(\hat{A}(t)\right)^{-1} = 2\overline{\alpha}I_d.
\end{aligned}$$

Componentwise integration of both sides from 0 to t w.r.t. t yields

$$\tilde{A}(t) - A_0 = \tilde{A}(t) - \tilde{A}(0) = \int_0^t \tilde{A}_s(s)ds = \int_0^t 2\overline{\alpha}I_d ds = 2\overline{\alpha}tI_d.$$

Using the transformation once more yields the solution of (4.9)

$$\hat{A}(t) = \left(2\overline{\alpha}tI_d + A_0\right)^{-1}, \, t>0.$$

Note that the initial data $A_0 \in \mathbb{C}^{d,d}$ must fullfill the relation $A_0 = A_0^T$ due to the symmetry of $\hat{A}(t)$ for $t>0$.

(4.10): Obviously, the general solution of (4.10) is of the form $\hat{B} = \hat{A}B_0$ for some constant matrix $B_0 \in \mathbb{C}^{d,d}$ and hence

$$\hat{B}(t) = \left(2\overline{\alpha}tI_d + A_0\right)^{-1}B_0, \, t>0.$$

(4.11): Thanks to the condition that $\hat{B}^T = \hat{C}$ we easily obtain the general solution of (4.11) by transposing $\hat{B}$ and using the symmetry of A_0

$$\hat{C}(t) = B_0^T\left(2\overline{\alpha}tI_d + A_0\right)^{-1}, \, t>0.$$

(4.12): Finally, the general solution of equation (4.12) has the form $B_0^T\hat{A}B_0 + D_0$ for some constant matrix $D_0 \in \mathbb{C}^{d,d}$ with $D_0 = D_0^T$ due to the symmetry of $\hat{D}$. This can be easily seen by rewriting the system as follows

$$\hat{D}_t(t) = -2\overline{\alpha}\hat{C}(t)\hat{B}(t) = -2\overline{\alpha}B_0^T\hat{A}^2(t)B_0.$$

Hence, we obtain

$$\hat{D}(t) = B_0^T\left(2\overline{\alpha}tI_d + A_0\right)^{-1}B_0 + D_0, \, t>0.$$

Now we choose $A_0 = 0$, $B_0 = -I_d$ and $D_0 = 0$. A resoning why the constants must have exactly these values can be found in [15] and [4]. Putting the solutions into (4.7) and this into (4.8) we have

$$\hat{N}_t(t) = \frac{1}{2\overline{\alpha}t}\begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}$$

Transforming $\hat{N}$ to N (see (4.7)) we obtain by (A4)

$$\begin{aligned} N(t) &= \exp\left(t\tilde{S}^T\right)\hat{N}(t)\exp\left(t\tilde{S}\right) \\ &= \begin{pmatrix} \exp(tS^T)\hat{A}(t)\exp(tS) & \exp(tS^T)\hat{B}(t) \\ \hat{C}(t)\exp(tS) & \hat{D}(t) \end{pmatrix} \\ &= \frac{1}{2\overline{\alpha}t}\begin{pmatrix} I_d & -\exp(tS^T) \\ -\exp(tS) & I_d \end{pmatrix}. \end{aligned} \tag{4.13}$$

Thus, $\operatorname{tr}\left(\overline{A}(t)\right) = \frac{d}{2\alpha t}$ and (4.5) can be written as

$$\varphi_t(t) = -\left(\alpha \operatorname{tr}\left(\overline{A}(t)\right) + \delta\right)\varphi(t) = -\left(\frac{d}{2t} + \delta\right)\varphi(t)$$

Hence, the general solution of (4.5) is given by

$$\varphi(t) = C\exp\left(-\int\left(\frac{d}{2t}+\delta\right)dt\right) = C\exp\left(-\frac{d}{2}\ln(t) - \delta t\right) = Ct^{-\frac{d}{2}}e^{-\delta t} \tag{4.14}$$

where $C \in \mathbb{C}$. Below we choose $C \in \mathbb{C}$ such that the normalization condition

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^d} H(x,\xi,t)d\xi = 1 \quad \forall x \in \mathbb{R}^d \tag{4.15}$$

holds. First note that from

$$\left\langle \frac{1}{2\overline{\alpha}t}\begin{pmatrix} I_d & -\exp(tS^T) \\ -\exp(tS) & I_d \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle_{\mathbb{C}^{2d}} = \frac{1}{2\alpha t}\left|e^{tS}x - \xi\right|^2$$

we obtain

$$H(x,\xi,t) = Ct^{-\frac{d}{2}}e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tS}x-\xi\right|^2}.$$

Now, integrating over $\mathbb{R}^d$ w.r.t. ξ, we obtain from the transformation theorem and assumption (A2)

$$\begin{aligned} \int_{\mathbb{R}^d} H(x,\xi,t)d\xi &= Ct^{-\frac{d}{2}}e^{-\delta t}\int_{\mathbb{R}^d} e^{-\frac{1}{4\alpha t}\left|e^{tS}x-\xi\right|^2}d\xi \\ &= Ct^{-\frac{d}{2}}e^{-\delta t}\int_{\mathbb{R}^d} e^{-\frac{1}{4\alpha t}|x-\psi|^2}d\psi \\ &= Ct^{-\frac{d}{2}}e^{-\delta t}\prod_{j=1}^{d}\int_{-\infty}^{\infty} e^{-\frac{1}{4\alpha t}x_j^2}dx_j \\ &= Ct^{-\frac{d}{2}}e^{-\delta t}\left(4\pi\alpha t\right)^{\frac{d}{2}} \\ &= C\left(4\pi\alpha\right)^{\frac{d}{2}}e^{-\delta t} \overset{t\to 0}{\to} C\left(4\pi\alpha\right)^{\frac{d}{2}} \overset{!}{=} 1. \end{aligned}$$

Hence, we choose $C = (4\pi\alpha)^{-\frac{d}{2}}$ such that (4.15) is satisfied. Here $\alpha^{-\frac{d}{2}}$ denotes the principal root (main branch) of α^{-d}. Finally we obtain the heat kernel (4.2) from (4.13) and (4.14). The properties (H1) and (H2) follow directly from the construction of the heat kernel. It remains to verify property (H3). Therefore, we need the integrate formula

$$\int_0^\infty r^{n-1}e^{-zr^2}dr = \frac{z^{-\frac{n}{2}}}{2\Gamma\left(\frac{n}{2}\right)}, \tag{4.16}$$

which holds for $n \in \mathbb{R}$ with $n > 0$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, [2]. Using the transformation theorem (with transformations for d-dimensional polar coordinates and $\Phi(\xi) = 2^{-1}t^{-\frac{1}{2}}\left(e^{tS}x - \xi\right)$) and formula (4.16) (with $n = d$ and $z = \alpha^{-1}$) we obtain, similarly to the proof of [20, Prop. 3.4.1], for every $\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$

$$
\begin{aligned}
&\lim_{t\downarrow 0} H(x,\xi,t)(\phi)\\
=&\lim_{t\downarrow 0}\int_{\mathbb{R}^d} H(x,\xi,t)\phi(\xi)d\xi\\
=&\lim_{t\downarrow 0}\int_{\mathbb{R}^d}(4\pi\alpha t)^{-\frac{d}{2}}\exp\left(-\delta t-(4\alpha t)^{-1}\left|e^{tS}x-\xi\right|^2\right)\phi(\xi)d\xi\\
=&\lim_{t\downarrow 0}(4\pi\alpha t)^{-\frac{d}{2}}(4t)^{\frac{d}{2}}\int_{\mathbb{R}^d}\exp\left(-\delta t-\alpha^{-1}|\psi|^2\right)\phi(e^{tS}x-2t^{\frac{1}{2}}\psi)d\psi\\
=&(\pi\alpha)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\exp\left(-\alpha^{-1}|\psi|^2\right)d\psi\phi(x)\\
=&(\pi\alpha)^{-\frac{d}{2}}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty r^{d-1}e^{-\alpha^{-1}r^2}dr\phi(x)\\
=&(\pi\alpha)^{-\frac{d}{2}}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\frac{\alpha^{\frac{d}{2}}}{2\Gamma\left(\frac{d}{2}\right)}\phi(x)\\
=&\phi(x)=\int_{\mathbb{R}^d}\delta_0(x-\xi)\phi(\xi)d\xi=\int_{\mathbb{R}^d}\delta_x(\xi)\phi(\xi)d\xi=\delta_x(\xi)(\phi).
\end{aligned}
$$

Note that $\operatorname{Re} z = \operatorname{Re}\left(\alpha^{-1}\right) = \frac{\operatorname{Re}\overline{\alpha}}{|\alpha|^2} = \frac{\operatorname{Re}\alpha}{|\alpha|^2} > 0$ is true by assumption (A2). ☐

The next statement is a direct consequence of Theorem 4.2.

Theorem 4.3 (Case of diagonal matrices)**.** *Let $\Lambda_A, \Lambda_B \in \mathbb{C}^{N,N}$ be two diagonal matrices and let the assumptions (A2) and (A4) be satisfied for $\mathbb{K} = \mathbb{C}$, then the function $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{C}^{N,N}$ defined by*

$$H(x,\xi,t) = (4\pi t\Lambda_A)^{-\frac{d}{2}}\exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x-\xi\right|^2\right) \tag{4.17}$$

is a heat kernel of $\mathcal{L}_\infty$ given by

$$\left[\mathcal{L}_\infty v\right](x) := \Lambda_A \triangle v(x) + \langle Sx, \nabla v(x)\rangle - \Lambda_B v(x). \tag{4.18}$$

Remark. In case of diagonal matrices we write Λ_A and Λ_B instead of A and B, respectively.

Proof. Using the notation $v = (v_1, \ldots, v_N)$, $\Lambda_A = \operatorname{diag}\left(\lambda_1^A, \ldots, \lambda_N^A\right)$ and $\Lambda_B = \operatorname{diag}\left(\lambda_1^B, \ldots, \lambda_N^B\right)$ the operator $\mathcal{L}_\infty$ from (4.18) is already decoupled

$$\left[\mathcal{L}_\infty v\right]_k(x) = \lambda_k^A \triangle v_k(x) + \langle Sx, \nabla v_k(x)\rangle - \lambda_k^B v_k(x),\ k = 1, \ldots, N.$$

Since (A2) and (A4) hold we infer from Theorem 4.2 that

$$H_k(x,\xi,t) := (4\pi t\lambda_k^A)^{-\frac{d}{2}}\exp\left(-\lambda_k^B t - (4t\lambda_k^A)^{-1}\left|e^{tS}x-\xi\right|^2\right)$$

is a heat kernel for the k-th component of $\mathcal{L}_\infty$. Indeed, an easy computation shows that $H(x,\xi,t) := \operatorname{diag}\left(H_1(x,\xi,t), \ldots, H_N(x,\xi,t)\right)$ is a heat kernel of $\mathcal{L}_\infty$ from (4.18) that coincides with H from (4.17). The properties (H1)–(H3) for the heat kernel H of $\mathcal{L}_\infty$ follow direcly from those of H_k for $k = 1, \ldots, N$. ☐

Theorem 4.4 (Case of simultaneously diagonalizable matrices)**.** *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $\mathbb{K} = \mathbb{C}$, then the function $H : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^* \to \mathbb{C}^{N,N}$ defined by*

$$H(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}}\exp\left(-Bt - (4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right) \tag{4.19}$$

is a heat kernel of $\mathcal{L}_\infty$ *given by*

$$[\mathcal{L}_\infty v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x). \tag{4.20}$$

Proof. Let us define the diagonalized operator $\tilde{\mathcal{L}}_\infty := Y^{-1}\mathcal{L}_\infty Y$ with Y from (A7$_B$). Multiplying (4.20) from left by Y^{-1} and using the transformations $A = Y\Lambda_A Y^{-1}$ and $B = Y\Lambda_B Y^{-1}$, the substitution $u(x) := Y^{-1}v(x)$, the property $Y^{-1}\langle Sx, \nabla v(x)\rangle = \langle Sx, \nabla Y^{-1}v(x)\rangle$ we obtain

$$\begin{aligned}\left[\tilde{\mathcal{L}}_\infty u\right](x) &= \left[Y^{-1}\mathcal{L}_\infty Y u\right](x) = Y^{-1}\left[\mathcal{L}_\infty v\right](x)\\ &= Y^{-1}\left(A\triangle v(x) + \langle Sx, \nabla v(x)\rangle - Bv(x)\right)\\ &= \Lambda_A Y^{-1}\triangle v(x) + Y^{-1}\langle Sx, \nabla v(x)\rangle - \Lambda_B Y^{-1}v(x)\\ &= \Lambda_A\triangle u(x) + \langle Sx, \nabla u(x)\rangle - \Lambda_B u(x)\end{aligned}$$

In this way we have decoupled the operator $\mathcal{L}_\infty$ from (4.20). Since $\Lambda_A, \Lambda_B \in \mathbb{C}^{N,N}$ are diagonal matrices, $\sigma(\Lambda_A) = \sigma(A) \subset \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\}$ by (A1) and (A2) hold, we deduce from Theorem 4.3 that

$$\tilde{H}(x,\xi,t) = (4\pi t\Lambda_A)^{-\frac{d}{2}} \exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x - \xi\right|^2\right)$$

is a heat kernel of $\tilde{\mathcal{L}}_\infty$. Again, an easy computation shows that $H(x,\xi,t) := Y\tilde{H}(x,\xi,t)Y^{-1}$ is a heat kernel of $\mathcal{L}_\infty$ from (4.20) that coincides with H from (4.19):

$$\begin{aligned}H(x,\xi,t) =& Y\tilde{H}(x,\xi,t)Y^{-1}\\ =& Y(4\pi t\Lambda_A)^{-\frac{d}{2}}Y^{-1}Y\exp\left(-\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x - \xi\right|^2\right)Y^{-1}\\ =& (4\pi t)^{-\frac{d}{2}}Y\Lambda_A^{-\frac{d}{2}}Y^{-1}\exp\left(-Y\left(\Lambda_B t - (4t\Lambda_A)^{-1}\left|e^{tS}x - \xi\right|^2\right)Y^{-1}\right)\\ =& (4\pi t)^{-\frac{d}{2}}Y\Lambda_A^{-\frac{d}{2}}Y^{-1}\exp\left(-Y\Lambda_B Y^{-1}t - (4t)^{-1}Y\Lambda_A^{-1}Y^{-1}\left|e^{tS}x - \xi\right|^2\right)\\ =& (4\pi tA)^{-\frac{d}{2}}\exp\left(-Bt - (4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right).\end{aligned}$$

The properties (H1)–(H3) for the heat kernel H of $\mathcal{L}_\infty$ follow again directly from those of $\tilde{H}$. □

Remark. Note that the condition (A7$_B$) in Theorem 4.4 is crucial. For arbitrary matrices $A, B \in \mathbb{C}^{N,N}$ satisfying only (A1) and (A2) the heat kernel of (4.20) is in general not given by (4.19), as we will see in Theorem 6.1 and Theorem 6.2. To generalize Theorem 4.4 to this more general case, one could try to use the Hadamard Lemma or the Baker-Campbell-Hausdorff formula.

Remark. For the computation of heat kernels of more general heat operators, Beals used in [15, (2)] - instead of (4.4) - the more general ansatz

$$H(x,\xi,t) = \varphi(t)\exp\left(-Q_t(x,\xi)\right),\ t > 0,\ x,\xi \in \mathbb{R}^d$$

where Q_t is a quadratic form of $2d$ variables. This formula is motivated by the Trotter product formula and the Feynman-Kac formula. Such a general ansatz was also used in [20, (13.2.14)] for the construction of heat kernels for degenerate elliptic operators.

Remark. Consider the more general N-dimensional complex-valued Ornstein-Uhlenbeck Operator on $\mathbb{R}^d$

$$[\mathcal{L}_{\mathrm{OU}}v](x) = A\mathrm{Tr}\left(QD^2v(x)\right) + \langle Sx, \nabla v(x)\rangle - Bv(x)$$

$$=A\sum_{i=1}^{d}\sum_{j=1}^{d}Q_{ij}D_iD_jv(x)+\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jD_iv(x)-Bv(x)$$

with $A,B\in\mathbb{C}^{N,N}$ satisfying (A1), (A2) and (A7$_B$), $Q\in\mathbb{R}^{d,d}$, $Q>0$, $Q=Q^T$ and $0\neq S\in\mathbb{R}^{d,d}$. We believe that

$$H(x,\xi,t)=(4\pi A)^{-\frac{d}{2}}\left(\det Q_t\right)^{-\frac{1}{2}}\exp\left(-Bt-(4A)^{-1}\left\langle Q_t^{-1}(e^{tS}x-\psi),(e^{tS}x-\psi)\right\rangle\right)$$

with

$$Q_t=\int_0^t\exp\left(\tau S\right)Q\exp\left(\tau S^T\right)d\tau$$

is a heat kernel of $\mathcal{L}_{\mathrm{OU}}$ even if (A4) is not satisfied, but this has not yet been proved.

4.2. **Some properties of the Ornstein-Uhlenbeck kernel.** The heat kernel satisfies the following Chapman-Kolmogorov formula, which plays a central role for the generation of semigroups, [38, Proposition C.3.2]. This formula can be understood as the semigroup property (5.6) on the basis of heat kernels.

Lemma 4.5 (Chapman-Kolmogorov formula). *Let the assumptions* (A1), (A2), (A4) *and* (A7$_B$) *be satisfied for* $\mathbb{K}=\mathbb{C}$. *Then*

$$\int_{\mathbb{R}^d}H(x,\tilde{\xi},t_1)H(\tilde{\xi},\xi,t_2)d\tilde{\xi}=H(x,\xi,t_1+t_2)\quad\forall\,x,\xi\in\mathbb{R}^d,\ \forall\,t_1,t_2>0.$$

Remark. For the proof we need the following integral

$$\begin{aligned}&\int_{-\infty}^{\infty}\exp\left(-c_1\left(a-\psi\right)^2-c_2\left(\psi-b\right)^2\right)d\psi\\&\qquad=\left(\frac{\pi}{c_1+c_2}\right)^{\frac{1}{2}}\exp\left(-\frac{c_1c_2}{c_1+c_2}\left(a-b\right)^2\right)\end{aligned}\tag{4.21}$$

for $a,b,c_1,c_2\in\mathbb{C}$ with $\operatorname{Re}c_1>0$, $\operatorname{Re}c_2>0$.

Proof. First let us prove the assertion for the diagonalized kernel

$$\tilde{H}(x,\xi,t)=(4\pi t\Lambda_A)^{-\frac{d}{2}}\exp\left(-\Lambda_Bt-(4t\Lambda_A)^{-1}\left|e^{tS}x-\xi\right|^2\right).$$

Because of (A4) we have $\left|e^{tS}x\right|=|x|$ and hence

$$\begin{aligned}&\int_{\mathbb{R}^d}\tilde{H}(x,\tilde{\xi},t_1)\tilde{H}(\tilde{\xi},\xi,t_2)d\tilde{\xi}\\&=(4\pi t_1\Lambda_A)^{-\frac{d}{2}}\left(4\pi t_2\Lambda_A\right)^{-\frac{d}{2}}\exp\left(-\Lambda_B(t_1+t_2)\right)\\&\quad\cdot\int_{\mathbb{R}^d}\exp\left(-\left(4t_1\Lambda_A\right)^{-1}\left|e^{t_1S}x-\tilde{\xi}\right|^2-\left(4t_2\Lambda_A\right)^{-1}\left|\tilde{\xi}-e^{-t_2S}\xi\right|^2\right)d\tilde{\xi}\end{aligned}\tag{4.22}$$

From (A2) we deduce that $\operatorname{Re}\lambda_j^A>0$ and hence $\operatorname{Re}\left(\lambda_j^A\right)^{-1}=\operatorname{Re}\frac{\overline{\lambda_j^A}}{|\lambda_j^A|^2}>0$ for every $j=1,\ldots,N$. Using formula (4.21) componentwise with $c_1=\left(4t_1\lambda_j^A\right)^{-1}$, $c_2=\left(4t_2\lambda_j^A\right)^{-1}$, $\psi=\tilde{\xi}_i$, $a=\left(e^{t_1S}x\right)_i$, $b=\left(e^{-t_2S}\xi\right)_i$, $i=1,\ldots,d$ we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty}\exp\left(-\left(4t_1\Lambda_A\right)^{-1}\left(\left(e^{t_1S}x\right)_i-\tilde{\xi}_i\right)^2-\left(4t_2\Lambda_A\right)^{-1}\left(\tilde{\xi}_i-\left(e^{-t_2S}\xi\right)_i\right)^2\right)d\tilde{\xi}_i\\&=\left(4\pi t_1\Lambda_A\right)^{\frac{1}{2}}\left(4\pi t_2\Lambda_A\right)^{\frac{1}{2}}\left(4\pi\left(t_1+t_2\right)\Lambda_A\right)^{-\frac{1}{2}}\\&\quad\cdot\exp\left(-\left(4\left(t_1+t_2\right)\Lambda_A\right)^{-1}\left(\left(e^{t_1S}x\right)_i-\left(e^{-t_2S}\xi\right)_i\right)^2\right)\end{aligned}$$

Using this integral and again $\left|e^{tS}x\right| = |x|$ we are able to compute the latter integral in (4.22)

$$\begin{aligned}
&\int_{\mathbb{R}^d} \exp\left(-(4t_1\Lambda_A)^{-1}\left|e^{t_1 S}x-\tilde{\xi}\right|^2 - (4t_2\Lambda_A)^{-1}\left|\tilde{\xi}-e^{-t_2 S}\xi\right|^2\right)d\tilde{\xi}\\
=&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\sum_{i=1}^{d}\Big[-(4t_1\Lambda_A)^{-1}\left((e^{t_1 S}x)_i-\tilde{\xi}_i\right)^2\right.\\
&\qquad\left.-(4t_2\Lambda_A)^{-1}\left(\tilde{\xi}_i-(e^{-t_2 S}\xi)_i\right)^2\Big]\right)d\tilde{\xi}_1\cdots d\tilde{\xi}_d\\
=&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^{d}\exp\Big(-(4t_1\Lambda_A)^{-1}\left((e^{t_1 S}x)_i-\tilde{\xi}_i\right)^2\\
&\qquad-(4t_2\Lambda_A)^{-1}\left(\tilde{\xi}_i-(e^{-t_2 S}\xi)_i\right)^2\Big)d\tilde{\xi}_1\cdots d\tilde{\xi}_d\\
=&\prod_{i=1}^{d}\int_{-\infty}^{\infty}\exp\left(-(4t_1\Lambda_A)^{-1}\left((e^{t_1 S}x)_i-\tilde{\xi}_i\right)^2-(4t_2\Lambda_A)^{-1}\left(\tilde{\xi}_i-(e^{-t_2 S}\xi)_i\right)^2\right)d\tilde{\xi}_i\\
=&(4\pi t_1\Lambda_A)^{\frac{d}{2}}(4\pi t_2\Lambda_A)^{\frac{d}{2}}(4\pi(t_1+t_2)\Lambda_A)^{-\frac{d}{2}}\\
&\qquad\cdot\exp\left(-(4(t_1+t_2)\Lambda_A)^{-1}\sum_{i=1}^{d}\left(\left(e^{t_1 S}x\right)_i-\left(e^{-t_2 S}\xi\right)_i\right)^2\right)\\
=&(4\pi t_1\Lambda_A)^{\frac{d}{2}}(4\pi t_2\Lambda_A)^{\frac{d}{2}}(4\pi(t_1+t_2)\Lambda_A)^{-\frac{d}{2}}\\
&\qquad\cdot\exp\left(-(4(t_1+t_2)\Lambda_A)^{-1}\left|e^{(t_1+t_2)S}x-\xi\right|^2\right)
\end{aligned}$$

Using this in (4.22) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\tilde{H}(x,\tilde{\xi},t_1)\tilde{H}(\tilde{\xi},\xi,t_2)d\tilde{\xi}\\
=&(4\pi(t_1+t_2)\Lambda_A)^{-\frac{d}{2}}\exp(-\Lambda_B(t_1+t_2))\exp\left(-(4(t_1+t_2)\Lambda_A)^{-1}\left|e^{(t_1+t_2)S}x-\xi\right|^2\right)\\
=&\tilde{H}(x,\xi,t_1+t_2)\quad\forall\,x,\xi\in\mathbb{R}^d,\ \forall\,t_1,t_2>0.
\end{aligned}$$

Let us now consider the general case: Since $H(x,\xi,t) = Y\tilde{H}(x,\xi,t)Y^{-1}$ with Y from (A7$_B$) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}H(x,\tilde{\xi},t_1)H(\tilde{\xi},\xi,t_2)d\tilde{\xi}\\
=&Y\int_{\mathbb{R}^d}\tilde{H}(x,\tilde{\xi},t_1)\tilde{H}(\tilde{\xi},\xi,t_2)d\tilde{\xi}Y^{-1}\\
=&Y\tilde{H}(x,\xi,t_1+t_2)Y^{-1} = H(x,\xi,t_1+t_2)\quad\forall\,x,\xi\in\mathbb{R}^d,\ \forall\,t_1,t_2>0.
\end{aligned}$$

□

The first two partial derivatives of H with respect to x are given by

$$\begin{aligned}
D_iH(x,\xi,t) =& -(2tA)^{-1}\left\langle e^{tS}x-\xi, e^{tS}e_i\right\rangle H(x,\xi,t),\\
D_jD_iH(x,\xi,t) =& \left(-(2tA)^{-1}\delta_{ij}+(2tA)^{-2}\left\langle e^{tS}x-\xi,e^{tS}e_i\right\rangle\left\langle e^{tS}x-\xi,e^{tS}e_j\right\rangle\right)\\
&\cdot H(x,\xi,t).
\end{aligned}$$

for $i,j=1,\ldots,d$ where we used (A7$_B$) once more. Let us define the kernels

$$\tilde{K}(\psi,t) := (4\pi t\Lambda_A)^{-\frac{d}{2}}\exp\left(-\Lambda_B t-(4t\Lambda_A)^{-1}|\psi|^2\right), \tag{4.23}$$

$$K(\psi,t) := H(x,e^{tS}x-\psi,t) = Y\tilde{K}(\psi,t)Y^{-1} = (4\pi tA)^{-\frac{d}{2}}\exp\left(-Bt-(4tA)^{-1}|\psi|^2\right), \tag{4.24}$$

$$\tilde{K}^i(\psi,t) := -(2t\Lambda_A)^{-1}\left\langle\psi,e^{tS}e_i\right\rangle\tilde{K}(\psi,t), \tag{4.25}$$

$$K^i(\psi,t) := [D_iH(x,\xi,t)]_{\xi=e^{tS}x-\psi} = Y\tilde{K}^i(\psi,t)Y^{-1} = -(2tA)^{-1}\left\langle\psi,e^{tS}e_i\right\rangle K(\psi,t), \tag{4.26}$$

$$\tilde{K}^{ji}(\psi,t) := \left((2t\Lambda_A)^{-2}\left\langle\psi,e^{tS}e_i\right\rangle\left\langle\psi,e^{tS}e_j\right\rangle-(2t\Lambda_A)^{-1}\delta_{ij}\right)\tilde{K}(\psi,t), \tag{4.27}$$

$$K^{ji}(\psi,t) := [D_jD_iH(x,\xi,t)]_{\xi=e^{tS}x-\psi} = Y\tilde{K}^{ji}(\psi,t)Y^{-1} = \left((2tA)^{-2}\left\langle\psi,e^{tS}e_i\right\rangle\left\langle\psi,e^{tS}e_j\right\rangle-(2tA)^{-1}\delta_{ij}\right)K(\psi,t). \tag{4.28}$$

In order to prove boundedness and exponentially decay of the associated semigroup, that we will perform in the next section, we need some upper bounds of the exponentially weighted integrals over the auxiliary kernels K, K^i and K^{ji}.

Lemma 4.6. *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $\mathbb{K}=\mathbb{C}$, $p,\eta\in\mathbb{R}$ and let K,K^i,K^{ji} be given by (4.24),(4.26),(4.28) for every $i,j=1,\ldots,d$, then*

(1) $\displaystyle\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K(\psi,t)\right|_2 d\psi \leqslant C_1(t)$, $t\geqslant 0$,

(2) $\displaystyle\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K^i(\psi,t)\right|_2 d\psi \leqslant C_2(t)$, $t>0$,

(3) $\displaystyle\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K^{ji}(\psi,t)\right|_2 d\psi \leqslant C_3(t)$, $t>0$,

where $|\cdot|_2$ denotes the spectral norm and the constants are given by

$$C_1(t) = M^{\frac{d}{2}}e^{-b_0t}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right],$$

$$C_2(t) = M^{\frac{d+1}{2}}e^{-b_0t}(ta_{\min})^{-\frac{1}{2}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\,{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\kappa t\right)\right],$$

$$C_3(t) = M^{\frac{d+2}{2}}e^{-b_0t}(ta_{\min})^{-1}\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\,{}_1F_1\left(\frac{d+2}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+3}{2};\frac{3}{2};\kappa t\right)+\frac{\delta_{ij}}{2}M^{-1}\,{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)+\delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right],$$

with $M:=\frac{a_{\max}^2}{a_{\min}a_0}\geqslant 1$ and $\kappa:=\frac{a_{\max}^2\eta^2p^2}{a_0}\geqslant 0$. Note that $C_{1+|\beta|}(t)\sim t^{\frac{d-1}{2}}e^{-(b_0-\kappa)t}$ as $t\to\infty$ and $C_{1+|\beta|}(t)\sim t^{-\frac{|\beta|}{2}}$ as $t\to 0$ for every $|\beta|=0,1,2$.

Remark. The function ${}_1F_1(a;b;z)$ denotes the Kummer confluent hypergeometric function $M(a,b,z)$ and satisfies the formula

$$\int_0^\infty s^n e^{-s^2+Bs}ds = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right){}_1F_1\left(\frac{n+1}{2};\frac{1}{2};\frac{B^2}{4}\right) + \frac{B}{2}\Gamma\left(\frac{n}{2}+1\right){}_1F_1\left(\frac{n}{2}+1;\frac{3}{2};\frac{B^2}{4}\right) \tag{4.29}$$

for $B\in\mathbb{R}$ with $B\geqslant 0$ and $n\in\mathbb{C}$ with $\mathrm{Re}\,n>-1$, see [2], that we need to prove Lemma 4.6. Moreover, in Lemma 4.8 we will need the connection formula

$${}_1F_1\left(a;b;x\right) = e^x {}_1F_1\left(b-a;b;-x\right) \tag{4.30}$$

for $a,b,x\in\mathbb{C}$ with $b\neq 0,-1,-2,\ldots$ (see [49] 13.2.39) and the integal

$$\int_0^\infty t^{\alpha-1}e^{-ct}{}_1F_1\left(a;b;-t\right)dt = c^{-\alpha}\Gamma\left(\alpha\right){}_2F_1\left(a,\alpha;b;-\frac{1}{c}\right) \tag{4.31}$$

for $a,b,c,\alpha\in\mathbb{C}$ with $b\neq 0,-1,-2,\ldots$, $\mathrm{Re}\,\alpha>0$ and $\mathrm{Re}\,c>0$ (see [49] 16.5.3) where ${}_2F_1\left(a_1,a_2;b_1;z\right)$ denotes the generalized hypergeometric function. To verify the asymptotic behavior of the function ${}_1F_1\left(a,b,z\right)$ at infinity we need the limiting form

$${}_1F_1\left(a,b,z\right)\sim\frac{\Gamma(b)}{\Gamma(a)}z^{a-b}e^z,\text{ as } z\to\infty,\ |\arg z|<\frac{\pi}{2} \tag{4.32}$$

for $z\in\mathbb{C}$ and $a,b\in\mathbb{C}\backslash\{0,-1,-2,\ldots\}$ (see [49] 13.2.4 and 13.2.23). Observe that ${}_1F_1\left(a;b;0\right)=1$ and ${}_2F_1\left(a_1,a_2;b_1;0\right)=1$ which induce a simplification of the constants in Lemma 4.6 in case of $\eta=0$.

Proof. First note that by (A7$_B$), (4.24),(4.26),(4.28) it hold

$$\left|K^\beta(\psi,t)\right|_2 = \left|Y\tilde{K}^\beta(\psi,t)Y^{-1}\right|_2 = \left|\tilde{K}^\beta(\psi,t)\right|_2 = \max_{k=1,\ldots,N}\left|\tilde{K}^\beta_{kk}(\psi,t)\right| \tag{4.33}$$

for every multi-index $\beta\in\mathbb{N}_0^d$ with $|\beta|\leqslant 2$. Note that $\tilde{K}^\beta(\psi,t)\in\mathbb{C}^{N,N}$ is diagonal. (**1**): Using (4.23) a simple computation shows that it holds for every $\psi\in\mathbb{R}^d$ and $t>0$

$$\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}(\psi,t)\right|\leqslant\left(4\pi t a_{\min}\right)^{-\frac{d}{2}}e^{-b_0t-\frac{a_0}{4ta_{\max}^2}|\psi|^2}. \tag{4.34}$$

From (4.33) with $|\beta|=0$, (4.34), the transformation theorem (with transformations for d-dimensional polar coordinates and $\Phi(r)=\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}}r)$ and formula (4.29) (since (A2) hold) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}e^{\eta p|\psi|}\left|K(\psi,t)\right|_2 d\psi\\
\leqslant&\int_{\mathbb{R}^d}e^{\eta p|\psi|}\left(4\pi t a_{\min}\right)^{-\frac{d}{2}}e^{-b_0t-\frac{a_0}{4ta_{\max}^2}|\psi|^2}d\psi\\
=&\left(4\pi t a_{\min}\right)^{-\frac{d}{2}}e^{-b_0t}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty r^{d-1}e^{-\frac{a_0}{4ta_{\max}^2}r^2+\eta pr}dr\\
=&\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}e^{-b_0t}\frac{2}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty s^{d-1}e^{-s^2+\left(\frac{4a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}}s}ds = C_1(t).
\end{aligned}$$

(**2**): Using (4.25) for every $i=1,\ldots,d$, $\psi\in\mathbb{R}^d$ and $t>0$ it holds

$$\max_{k=1,\ldots,N}\left|\tilde{K}^i_{kk}(\psi,t)\right|\leqslant\left(2ta_{\min}\right)^{-1}\left|\left\langle\psi,e^{tS}e_i\right\rangle\right|\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}(\psi,t)\right|. \tag{4.35}$$

From (4.33) with $|\beta|=1$, (4.35) with (4.34), Cauchy-Schwarz inequality with assumption (A4) ($\left|\left\langle\psi,e^{tS}e_i\right\rangle\right|\leqslant|\psi||e^{tS}e_i|=|\psi|$), the transformation theorem (with transformations from (1)) and formula (4.29) (since (A2) hold) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K^i(\psi,t)\right|_2 d\psi\\
\leqslant&\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left(2ta_{\min}\right)^{-1}\left|\left\langle\psi,e^{tS}e_i\right\rangle\right|\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}(\psi,t)\right|d\psi\\
\leqslant&\int_{\mathbb{R}^d} e^{\eta p|\psi|}(2ta_{\min})^{-1}\left|\psi\right|\left(4\pi ta_{\min}\right)^{-\frac{d}{2}}e^{-b_0t-\frac{a_0}{4ta_{\max}^2}|\psi|^2}d\psi\\
=&(2ta_{\min})^{-1}(4\pi ta_{\min})^{-\frac{d}{2}}e^{-b_0t}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty r^d e^{-\frac{a_0}{4ta_{\max}^2}r^2+\eta pr}dr\\
=&\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+1}{2}}e^{-b_0t}\frac{2}{\Gamma\left(\frac{d}{2}\right)}\left(ta_{\min}\right)^{-\frac{1}{2}}\int_0^\infty s^d e^{-s^2+\left(\frac{4a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}}s}ds=C_2(t).
\end{aligned}$$

(**3**): Using (4.27), the triangle inequality and Cauchy-Schwarz inequality with assumption (A4) (see (2)) yield for every $i,j=1,\ldots,d$, $\psi\in\mathbb{R}^d$ and $t>0$

(4.36)
$$\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}^{ji}(\psi,t)\right|\leqslant\left(\left(2ta_{\min}\right)^{-2}\left|\psi\right|^2+\left(2ta_{\min}\right)^{-1}\delta_{ij}\right)\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}(\psi,t)\right|.$$

From (4.33) with $|\beta|=2$, (4.36) with (4.34), the transformation theorem (with transformations from (1)) and formula (4.29) (since (A2) hold) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left|K^{ji}(\psi,t)\right|_2 d\psi\\
\leqslant&\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left(\left(2ta_{\min}\right)^{-2}\left|\psi\right|^2+\left(2ta_{\min}\right)^{-1}\delta_{ij}\right)\max_{k=1,\ldots,N}\left|\tilde{K}_{kk}(\psi,t)\right|d\psi\\
\leqslant&\int_{\mathbb{R}^d} e^{\eta p|\psi|}\left(\left(2ta_{\min}\right)^{-2}\left|\psi\right|^2+\left(2ta_{\min}\right)^{-1}\delta_{ij}\right)\left(4\pi ta_{\min}\right)^{-\frac{d}{2}}e^{-b_0t-\frac{a_0}{4ta_{\max}^2}|\psi|}d\psi\\
=&(4\pi ta_{\min})^{-\frac{d}{2}}e^{-b_0t}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty\left(\left(2ta_{\min}\right)^{-2}r^2+\left(2ta_{\min}\right)^{-1}\delta_{ij}\right)r^{d-1}e^{-\frac{a_0}{4ta_{\max}^2}r^2+\eta pr}dr\\
=&\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+2}{2}}e^{-b_0t}\frac{2}{\Gamma\left(\frac{d}{2}\right)}\left(ta_{\min}\right)^{-1}\int_0^\infty s^{d+1}e^{-s^2+\left(\frac{4a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}}s}ds\\
&+\delta_{ij}\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}e^{-b_0t}\frac{1}{\Gamma\left(\frac{d}{2}\right)}\left(ta_{\min}\right)^{-1}\int_0^\infty s^{d-1}e^{-s^2+\left(\frac{4a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}}s}ds=C_3(t).
\end{aligned}$$

□

In order to show that the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ coincide with the infinitesimal generator of the Ornstein-Uhlenbeck semigroup we need the following Lemma.

Lemma 4.7. *Let the assumptions (A1), (A2) and (A7$_B$) be satisfied for $\mathbb{K}=\mathbb{C}$ and let K be given by (4.24), then for every $i,j=1,\ldots,d$ and $t\geqslant 0$ we have*

$$(\mathbf{1})\quad\int_{\mathbb{R}^d}K(\psi,t)d\psi=e^{-Bt},$$

$$(\mathbf{2})\quad\int_{\mathbb{R}^d}K(\psi,t)\psi_i d\psi=0,$$

$$(\mathbf{3})\quad\int_{\mathbb{R}^d}K(\psi,t)\psi_i\psi_j=\begin{cases}2te^{-Bt}A & ,\ i=j\\ 0 & ,\ i\neq j\end{cases}.$$

Remark. Throughout this proof we will use d-dimensional polar coordinates: Let $x\in\mathbb{R}^d$, $\Omega:=]0,\infty[\times[0,2\pi[\times[0,\pi]^d$ and $(r,\phi,\theta_1,\ldots,\theta_{d-2})\in\Omega$, then we define

$$\begin{aligned} x_1 =&\Phi_1(r,\phi,\theta_1,\ldots,\theta_{d-2}) := r\cos\phi\prod_{k=1}^{d-2}\sin\theta_k,\\ x_2 =&\Phi_2(r,\phi,\theta_1,\ldots,\theta_{d-2}) := r\sin\phi\prod_{k=1}^{d-2}\sin\theta_k,\\ x_i =&\Phi_i(r,\phi,\theta_1,\ldots,\theta_{d-2}) := r\cos\theta_{i-2}\prod_{k=i-1}^{d-2}\sin\theta_k,\ 3\leqslant i\leqslant d. \end{aligned} \tag{4.37}$$

The transformation $\Phi:\Omega\to\mathbb{R}^d$ is a C^∞-diffeomorphism, [8, X.8.8 Lemma], satisfying $\Phi(\overline{\Omega})=\mathbb{R}^d$ and

$$\det D\Phi(r,\phi,\theta_1,\ldots,\theta_{d-2}) = (-1)^d r^{d-1}\prod_{k=1}^{d-2}(\sin\theta_k)^k.$$

Proof. First note that (A2), (A7$_B$) and componentwise integration yields for every $n>-1$

$$\begin{aligned} &\int_0^\infty r^n e^{-(4tA)^{-1}r^2}dr = \int_0^\infty r^n e^{-Y(4t\Lambda_A)^{-1}Y^{-1}r^2}dr\\ =&Y\int_0^\infty r^n e^{-(4t\Lambda_A)^{-1}r^2}drY^{-1} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2}Y(4t\Lambda_A)^{\frac{n+1}{2}}Y^{-1}\\ =&\frac{\Gamma\left(\frac{n+1}{2}\right)}{2}(4tA)^{\frac{n+1}{2}}. \end{aligned} \tag{4.38}$$

(**1**): From (4.24), (4.38) (with $n=d-1$), the transformation theorem (with d-dimensional polar coordinates) and (A7$_B$) we directly obtain for $t\geqslant 0$

$$\begin{aligned} \int_{\mathbb{R}^d}K(\psi,t)d\psi &= (4\pi tA)^{-\frac{d}{2}}e^{-Bt}\int_{\mathbb{R}^d}e^{-(4tA)^{-1}|\psi|^2}d\psi\\ &=(4\pi tA)^{-\frac{d}{2}}e^{-Bt}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty r^{d-1}e^{-(4tA)^{-1}r^2}dr\\ &=(4\pi tA)^{-\frac{d}{2}}e^{-Bt}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\frac{\Gamma\left(\frac{d}{2}\right)}{2}(4tA)^{\frac{d}{2}} = e^{-Bt}. \end{aligned}$$

(**2**): Now we must use d-dimensional polar coordinates. From the transformation theorem we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d}e^{-(4tA)^{-1}|\psi|^2}\psi_i d\psi\\ =&\int_\Omega e^{-(4tA)^{-1}r^2}\cdot\left\{\begin{array}{cl} r\cos\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\ i=1\\ r\sin\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\ i=2\\ r\cos\theta_{i-2}\prod_{k=i-1}^{d-2}\sin\theta_k & ,\ 3\leqslant i\leqslant d-2 \end{array}\right\}\\ &\cdot|\det D\Phi(r,\phi,\theta_1,\ldots,\theta_{d-2})|\,drd\phi d\theta_1\cdots d\theta_{d-2}\\ =&\int_\Omega e^{-(4tA)^{-1}r^2}\cdot\left\{\begin{array}{cl} r\cos\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\ i=1\\ r\sin\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\ i=2\\ r\cos\theta_{i-2}\prod_{k=i-1}^{d-2}\sin\theta_k & ,\ 3\leqslant i\leqslant d-2 \end{array}\right\}\\ &\cdot r^{d-1}\prod_{k=1}^{d-2}|\sin\theta_k|^k\,drd\phi d\theta_1\cdots d\theta_{d-2} \end{aligned}$$

$$= \left(\int_0^\infty r^d e^{-(4tA)^{-1}r^2} dr\right) \int_0^{2\pi}\int_0^\pi \cdots \int_0^\pi$$
$$\left\{\begin{array}{ll} \cos\phi \prod_{k=1}^{d-2}\sin\theta_k \prod_{k=1}^{d-2}|\sin\theta_k|^k & ,\, i=1 \\ \sin\phi \prod_{k=1}^{d-2}\sin\theta_k \prod_{k=1}^{d-2}|\sin\theta_k|^k & ,\, i=2 \\ \cos\theta_{i-2} \prod_{k=i-1}^{d-2}\sin\theta_k \prod_{k=1}^{d-2}|\sin\theta_k|^k & ,\, 3\leqslant i\leqslant d-2 \end{array}\right\} d\phi d\theta_1\cdots d\theta_{d-2}$$

In case of $i=1$ and $i=2$ the ϕ-integrals vanishes and in case of $3\leqslant i\leqslant d-2$ the θ_{i-2}-integral vanishes, since using for example

$$(\sin a)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos\left((n-2k)\left(a-\frac{\pi}{2}\right)\right),\ n\in\mathbb{N},$$

we obtain

$$\int_0^\pi \cos\theta_{i-2}\,|\sin\theta_{i-2}|^{i-2}\,d\theta_{i-2} = \int_0^\pi \cos\theta_{i-2}\,(\sin\theta_{i-2})^{i-2}\,d\theta_{i-2} = 0. \tag{4.39}$$

Hence, we have for every $i=1,\ldots,d$ and $t\geqslant 0$

$$\int_{\mathbb{R}^d} K(\psi,t)\psi_i d\psi = (4\pi tA)^{-\frac{d}{2}} e^{-Bt}\int_{\mathbb{R}^d} e^{-(4tA)^{-1}|\psi|^2}\psi_i d\psi = 0.$$

(**3**): Finally, let us use d-dimensional polar coordinates once more. Similar to (2) from the transformation theorem we obtain

$$\int_{\mathbb{R}^d} e^{-(4tA)^{-1}|\psi|^2}\psi_i\psi_j d\psi$$
$$= \left(\int_0^\infty r^{d+1} e^{-(4tA)^{-1}r^2} dr\right)\int_0^{2\pi}\int_0^\pi\cdots\int_0^\pi$$
$$\left\{\begin{array}{ll} \cos\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\, i=1\\ \sin\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\, i=2 \\ \cos\theta_{i-2}\prod_{k=i-1}^{d-2}\sin\theta_k & ,\, 3\leqslant i\leqslant d-2\end{array}\right\}\prod_{k=1}^{d-2}|\sin\theta_k|^k$$
$$\left\{\begin{array}{ll} \cos\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\, j=1\\ \sin\phi\prod_{k=1}^{d-2}\sin\theta_k & ,\, j=2 \\ \cos\theta_{j-2}\prod_{k=j-1}^{d-2}\sin\theta_k & ,\, 3\leqslant j\leqslant d-2\end{array}\right\} d\phi d\theta_1\cdots d\theta_{d-2}$$
$$= \begin{cases} \frac{\pi^{\frac{d}{2}}}{2}(4tA)^{\frac{d}{2}+1} & ,\, i=j\\ 0 & ,\, i\neq j\end{cases}.$$

Accept the last equality, we first deduce from (4.38) with $n=d+1$

$$\int_0^\infty r^{d+1}e^{-(4tA)^{-1}r^2}dr = \frac{\Gamma\left(\frac{d+2}{2}\right)}{2}(4tA)^{\frac{d}{2}+1}. \tag{4.40}$$

Moreover, for $\mathrm{Re}\, l>-1$, $a,b\in\mathbb{N}_0$ with $a\leqslant b$ it holds

$$\prod_{l=a}^{b}\int_0^\pi(\sin\theta)^l\,d\theta = \prod_{l=a}^{b}\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{l+1}{2}\right)}{\Gamma\left(\frac{l+2}{2}\right)} = \pi^{\frac{b-a+1}{2}}\frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{b+2}{2}\right)}. \tag{4.41}$$

Let is first consider the cases $i=j=1$ and $i=j=2$. Here we must use

$$\int_0^{2\pi}(\cos\phi)^2\,d\phi=\pi,\qquad \int_0^{2\pi}(\sin\phi)^2\,d\phi=\pi$$

and (4.41) with $a=3$ and $b=d$

$$\prod_{k=1}^{d-2}\int_0^\pi(\sin\theta_k)^2|\sin\theta_k|^k\,d\theta_k = \prod_{k=1}^{d-2}\int_0^\pi(\sin\theta)^{k+2}\,d\theta = \prod_{l=3}^{d}\int_0^\pi(\sin\theta)^l\,d\theta$$

$$=\pi^{\frac{d}{2}-1}\frac{\Gamma\left(\frac{4}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}=\frac{\pi^{\frac{d}{2}-1}}{\Gamma\left(\frac{d+2}{2}\right)}.$$

Now, let us consider the case $3\leqslant i=j\leqslant d$. Here we can deduce from (4.41) (with $a=1$ and $b=i-3$, $a=b=i-2$, $a=b=i$ as well as $a=i+1$ and $b=d$)

$$\int_0^{2\pi}1d\phi=2\pi,$$

$$\prod_{k=1}^{i-3}\int_0^{\pi}\left|\sin\theta_k\right|^k d\theta_k=\prod_{k=1}^{i-3}\int_0^{\pi}(\sin\theta)^k d\theta=\pi^{\frac{i-3}{2}}\frac{\Gamma(1)}{\Gamma\left(\frac{i-1}{2}\right)}=\frac{\pi^{\frac{i-3}{2}}}{\Gamma\left(\frac{i-1}{2}\right)},$$

$$\int_0^{\pi}(\cos\theta_{i-2})^2\left|\sin\theta_{i-2}\right|^{i-2}d\theta_{i-2}=\int_0^{\pi}\left(1-(\sin\theta_{i-2})^2\right)(\sin\theta_{i-2})^{i-2}d\theta_{i-2}$$

$$=\int_0^{\pi}(\sin\theta)^{i-2}d\theta-\int_0^{\pi}(\sin\theta)^{i}d\theta=\pi^{\frac{1}{2}}\left(\frac{\Gamma\left(\frac{i-1}{2}\right)}{\Gamma\left(\frac{i}{2}\right)}-\frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)}\right),$$

$$\prod_{k=i-1}^{d-2}\int_0^{\pi}(\sin\theta_k)^2\left|\sin\theta_k\right|^k d\theta_k=\prod_{k=i-1}^{d-2}\int_0^{\pi}(\sin\theta)^{k+2}d\theta$$

$$=\prod_{l=i+1}^{d}\int_0^{\pi}(\sin\theta)^l d\theta=\pi^{\frac{d-i}{2}}\frac{\Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}.$$

Multiplying these four terms with (4.40) and using $\Gamma(x+1)=x\Gamma(x)$ we obtain $\frac{\pi^{\frac{d}{2}}}{2}(4tA)^{\frac{d}{2}+1}$. Next, we consider the cases $3\leqslant i<j\leqslant d$ and $3\leqslant j<i\leqslant d$. Let w.l.o.g. $i<j$, then the term from (4.39) vanishes. For all the other cases exactly one term vanishes, namely

$$\int_0^{2\pi}\sin\phi\cos\phi d\phi=0,\qquad\text{if } (i=1,j=2)\text{ or }(i=2,j=1),$$

$$\int_0^{2\pi}\cos\phi d\phi=0,\qquad\text{if } (i=1,3\leqslant j\leqslant d)\text{ or }(3\leqslant i\leqslant d,j=1),$$

$$\int_0^{2\pi}\sin\phi d\phi=0,\qquad\text{if } (i=2,3\leqslant j\leqslant d)\text{ or }(3\leqslant i\leqslant d,j=2).$$

□

4.3. **Some useful integrals.** Under the assumption (A2) and with the notation from Section 1.1 we define

$$C_4(t)=C_\theta M^{\frac{d}{2}}e^{-b_0t}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},$$

$$C_5(t)=C_\theta M^{\frac{d+1}{2}}e^{-b_0t}(ta_{\min})^{-\frac{1}{2}}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\kappa t\right)\right.$$
$$\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},$$

$$C_6(t)=C_\theta M^{\frac{d+2}{2}}e^{-b_0t}(ta_{\min})^{-1}\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+2}{2};\frac{1}{2};\kappa t\right)\right.$$
$$+2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+3}{2};\frac{3}{2};\kappa t\right)+\frac{\delta_{ij}}{2}M^{-1}{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)$$
$$\left.+\delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},$$

with $M:=\frac{a_{\max}^2}{a_{\min}a_0}\geqslant 1$, $\kappa:=\frac{a_{\max}^2\eta^2p^2}{a_0}\geqslant 0$, $1\leqslant p\leqslant\infty$ and $\eta\geqslant 0$. In case of $p=\infty$ the constants are given by $C_{4+|\beta|}(t)$ with $p=1$ for every $|\beta|=0,1,2$. Moreover, in case of $p=1$ it holds $C_{4+|\beta|}(t)=C_\theta C_{1+|\beta|}(t)$. In order to show that the solutions of the steady state problems for the Ornstein-Uhlenbeck operator decay exponentially, see Theorem 5.14, we need the following Lemma. The upper bound for η^2 can be considered as the maximal decay rate.

Lemma 4.8. *Let the assumption* (A2) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Moreover, let* $0<\vartheta<1$, $\tilde{\omega}\in\mathbb{R}$, $\omega:=\tilde{\omega}-b_0$, $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda>\omega$ *and* $0\leqslant\eta^2\leqslant\vartheta\frac{a_0(\operatorname{Re}\lambda-\omega)}{a_{\max}^2p^2}$, *then we have*

$$\textbf{(1)}\quad \int_0^\infty e^{-\operatorname{Re}\lambda t}C_4(t)dt\leqslant\frac{C_7}{\operatorname{Re}\lambda-\omega},$$
$$\textbf{(2)}\quad \int_0^\infty e^{-\operatorname{Re}\lambda t}C_5(t)dt\leqslant\frac{C_8}{(\operatorname{Re}\lambda-\omega)^{\frac12}},$$

with $M:=\frac{a_{\max}^2}{a_{\min}a_0}\geqslant 1$ *and*

$$\begin{aligned}
C_7 =&C_\theta M^{\frac d2}\left(\frac{1}{1-\vartheta}\right)^{\frac1p}\left({}_2F_1\left(-\frac{d-1}{2},1;\frac12;-\frac{\vartheta}{1-\vartheta}\right)\right.\\
&\left.+\pi^{\frac12}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac d2\right)}\left(\frac{\vartheta}{1-\vartheta}\right)^{\frac12}{}_2F_1\left(-\frac{d-2}{2},\frac32;\frac32;-\frac{\vartheta}{1-\vartheta}\right)\right)^{\frac1p},\\
C_8 =&C_\theta M^{\frac{d+1}{2}}\frac{\Gamma\left(\frac12\right)}{a_{\min}^{\frac12}}\left(\frac{1}{1-\vartheta}\right)^{\frac{1}{2p}}\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac d2\right)}{}_2F_1\left(-\frac d2,\frac12;\frac12;-\frac{\vartheta}{1-\vartheta}\right)\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac12\right)\Gamma\left(\frac d2\right)}\left(\frac{\vartheta}{1-\vartheta}\right)^{\frac12}{}_2F_1\left(-\frac{d-1}{2},1;\frac32;-\frac{\vartheta}{1-\vartheta}\right)\right)^{\frac1p}.
\end{aligned}$$

Proof. **(1)**: From $c_0:=\operatorname{Re}\lambda-\omega$, Hölder's inequality (with $\frac1p+\frac1q=1$ and $1\leqslant p<\infty$), the transformation theorem (with transformation $\Phi(t)=\frac{a_{\max}^2\eta^2p^2t}{a_0}$), formula (4.30) (with $a=\frac d2$, $b=\frac12$, $x=s$ and $a=\frac{d+1}{2}$, $b=\frac32$, $x=s$) and formula (4.31) (with $\alpha=1$, $c=\frac{a_0c_0-a_{\max}^2\eta^2p^2}{a_{\max}^2\eta^2p^2}$, $a=-\frac{d-1}{2}$, $b=\frac12$ and $\alpha=\frac32$, $c=\frac{a_0c_0-a_{\max}^2\eta^2p^2}{a_{\max}^2\eta^2p^2}$, $a=-\frac{d-2}{2}$, $b=\frac32$ – note that because of (A2), $c_0>0$ and $\eta^2<\frac{a_0c_0}{a_{\max}^2p^2}$ we have $\operatorname{Re}c>0$) we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\operatorname{Re}\lambda t}C_4(t)dt\\
=&\int_0^\infty C_\theta\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac d2}e^{-c_0t}\left[{}_1F_1\left(\frac d2;\frac12;\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac d2\right)}\left(\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac12}{}_1F_1\left(\frac{d+1}{2};\frac32;\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)\right]^{\frac1p}dt\\
\leqslant&C_\theta\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac d2}\left(\int_0^\infty e^{-c_0t}dt\right)^{\frac1q}\left(\int_0^\infty e^{-c_0t}{}_1F_1\left(\frac d2;\frac12;\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)dt\right.\\
&\left.+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac d2\right)}\int_0^\infty\left(\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac12}e^{-c_0t}{}_1F_1\left(\frac{d+1}{2};\frac32;\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)dt\right)^{\frac1p}\\
=&C_\theta M^{\frac d2}\left(\frac{1}{c_0}\right)^{\frac1q}\left(\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{-1}\int_0^\infty e^{-\frac{a_0c_0}{a_{\max}^2\eta^2p^2}s}{}_1F_1\left(\frac d2;\frac12;s\right)ds\right.
\end{aligned}$$

$$
\begin{aligned}
&+ 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{-1}\int_0^\infty s^{\frac{1}{2}}e^{-\frac{a_0c_0}{a_{\max}^2\eta^2p^2}s}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};s\right)ds\Bigg)^{\frac{1}{p}}\\
=&C_\theta M^{\frac{d}{2}}\left(\frac{1}{c_0}\right)\Bigg(\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0}\right)^{-1}\int_0^\infty e^{-\left(\frac{a_0c_0}{a_{\max}^2\eta^2p^2}-1\right)s}\,{}_1F_1\left(-\frac{d-1}{2};\frac{1}{2};-s\right)ds\\
&+ 2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0}\right)^{-1}\int_0^\infty s^{\frac{1}{2}}e^{-\left(\frac{a_0c_0}{a_{\max}^2\eta^2p^2}-1\right)s}\,{}_1F_1\left(-\frac{d-2}{2};\frac{3}{2};-s\right)ds\Bigg)^{\frac{1}{p}}\\
=&C_\theta M^{\frac{d}{2}}\left(\frac{1}{c_0}\right)\left(\frac{a_0c_0}{a_0c_0-a_{\max}^2\eta^2p^2}\right)^{\frac{1}{p}}\Bigg({}_2F_1\left(-\frac{d-1}{2},1;\frac{1}{2};-\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)\\
&+\pi^{\frac{1}{2}}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)^{\frac{1}{2}}{}_2F_1\left(-\frac{d-2}{2},\frac{3}{2};\frac{3}{2};-\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)\Bigg)^{\frac{1}{p}}.
\end{aligned}
$$

Finally, to obtain C_7 we must use that ${}_2F_1$ is strictly monotone decreasing in $]-\infty,0]$ as well as the inequalities

$$
\frac{a_0c_0}{a_0c_0-a_{\max}^2\eta^2p^2}\leqslant\frac{1}{1-\vartheta}\quad\text{and}\quad\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\leqslant\frac{\vartheta}{1-\vartheta}.
$$

(**2**): From $c_0:=\operatorname{Re}\lambda-\omega$, Hölder's inequality (with $\frac{1}{p}+\frac{1}{q}=1$ and $1\leqslant p<\infty$), the transformation theorem (with transformation $\Phi(t)=\frac{a_{\max}^2\eta^2p^2t}{a_0}$), formula (4.30) (with $a=\frac{d+1}{2}$, $b=\frac{1}{2}$, $x=s$ and $a=\frac{d+2}{2}$, $b=\frac{3}{2}$, $x=s$) and formula (4.31) (with $\alpha=\frac{1}{2}$, $c=\frac{a_0c_0-a_{\max}^2\eta^2p^2}{a_{\max}^2\eta^2p^2}$, $a=-\frac{d}{2}$, $b=\frac{1}{2}$ and $\alpha=1$, $c=\frac{a_0c_0-a_{\max}^2\eta^2p^2}{a_{\max}^2\eta^2p^2}$, $a=-\frac{d-1}{2}$, $b=\frac{3}{2}$ – note that because of (A2), $c_0>0$ and $\eta^2<\frac{a_0c_0}{a_{\max}^2p^2}$ we have $\operatorname{Re}c>0$) we obtain

$$
\begin{aligned}
&\int_0^\infty e^{-\operatorname{Re}\lambda t}C_5(t)dt\\
=&\int_0^\infty C_\theta\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+1}{2}}e^{-c_0t}\left(ta_{\min}\right)^{-\frac{1}{2}}\Bigg[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)\\
&+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)^{\frac{1}{2}}{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)\Bigg]^{\frac{1}{p}}dt\\
\leqslant&C_\theta\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d+1}{2}}a_{\min}^{-\frac{1}{2}}\left(\int_0^\infty t^{-\frac{1}{2}}e^{-c_0t}dt\right)^{\frac{1}{q}}\\
&\cdot\Bigg(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty t^{-\frac{1}{2}}e^{-c_0t}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)dt\\
&+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{\frac{1}{2}}\int_0^\infty e^{-c_0t}{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\frac{a_{\max}^2\eta^2p^2t}{a_0}\right)dt\Bigg)^{\frac{1}{p}}\\
=&C_\theta M^{\frac{d+1}{2}}\left(\left(\frac{1}{c_0}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\right)^{\frac{1}{q}}a_{\min}^{-\frac{1}{2}}\\
&\cdot\Bigg(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{-\frac{1}{2}}\int_0^\infty s^{-\frac{1}{2}}e^{-\frac{a_0c_0}{a_{\max}^2\eta^2p^2}s}\,{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};s\right)ds\\
&+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0}\right)^{-\frac{1}{2}}\int_0^\infty e^{-\frac{a_0c_0}{a_{\max}^2\eta^2p^2}s}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};s\right)ds\Bigg)^{\frac{1}{p}}
\end{aligned}
$$

$$=C_\theta M^{\frac{d+1}{2}}\left(\frac{1}{c_0}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}\right)}{a_{\min}^{\frac{1}{2}}}$$

$$\cdot\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0}\right)^{-\frac{1}{2}}\int_0^\infty s^{-\frac{1}{2}}e^{-\left(\frac{a_0c_0}{a_{\max}^2\eta^2p^2}-1\right)s}\,{}_1F_1\left(-\frac{d}{2};\frac{1}{2};-s\right)ds\right.$$

$$\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0}\right)^{-\frac{1}{2}}\int_0^\infty e^{-\left(\frac{a_0c_0}{a_{\max}^2\eta^2p^2}-1\right)s}\,{}_1F_1\left(-\frac{d-1}{2};\frac{3}{2};-s\right)ds\right)^{\frac{1}{p}}$$

$$=C_\theta M^{\frac{d+1}{2}}\left(\frac{1}{c_0}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}\right)}{a_{\min}^{\frac{1}{2}}}\left(\frac{a_0c_0}{a_0c_0-a_{\max}^2\eta^2p^2}\right)^{\frac{1}{2p}}$$

$$\cdot\left(\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_2F_1\left(-\frac{d}{2},\frac{1}{2};\frac{1}{2};-\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)\right.$$

$$\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{d}{2}\right)}\left(\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)^{\frac{1}{2}}{}_2F_1\left(-\frac{d-1}{2},1;\frac{3}{2};-\frac{a_{\max}^2\eta^2p^2}{a_0c_0-a_{\max}^2\eta^2p^2}\right)\right)^{\frac{1}{p}}.$$

Finally, to obtain C_8 we use the same tools as in (1). ☐

5. The complex Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d,\mathbb{C}^N)$

In this section we apply semigroup theory to the Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0v](x) := A\triangle v(x)+\langle Sx,\nabla v(x)\rangle\,,\; x\in\mathbb{R}^d,\; d\geqslant 2,$$

and characterize its maximal domain in $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

5.1. **Application of semigroup theory.** Let us consider the Ornstein-Uhlenbeck kernel of $\mathcal{L}_0$ from Theorem 4.4 (with $B=0$)

$$H_0(x,\xi,t)=(4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|e^{tS}x-\xi\right|^2\right)$$

and the family of mappings $(T_0(t))_{t\geqslant 0}$ given by

$$[T_0(t)v](x) := \begin{cases}\int_{\mathbb{R}^d}H_0(x,\xi,t)v(\xi)d\xi & ,\, t>0\\ v(x) & ,\, t=0\end{cases}\quad ,x\in\mathbb{R}^d \tag{5.1}$$

on the (complex-valued) Banach space $\left(L^p(\mathbb{R}^d,\mathbb{C}^N),\|\cdot\|_{L^p}\right)$, $1\leqslant p\leqslant\infty$. In the scalar real-valued case, formula (5.1) is due to Kolmogorov, [34]. The next three theorems show that the semigroup defined in (5.1) is strongly continuous on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, $1\leqslant p<\infty$. In order to show exponential decay of the solutions of the resolvent equation via a-priori estimates, we have to prove the boundedness of T_0 and its derivatives up to order 2 in exponentially weighted norms.

Theorem 5.1 (Boundedness on $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A4) *be satisfied for* $1\leqslant p\leqslant\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then for every radial weight function* $\theta\in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta\geqslant 0$ *and for every* $v\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$

$$\|T_0(t)v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant C_4(t)\,\|v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad ,\, t\geqslant 0, \tag{5.2}$$

$$\|D_iT_0(t)v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant C_5(t)\,\|v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad ,\, t>0,\, i=1,\ldots,d, \tag{5.3}$$

$$\|D_jD_iT_0(t)v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant C_6(t)\,\|v\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad ,\, t>0,\, i,j=1,\ldots,d, \tag{5.4}$$

where the constants $C_{4+|\beta|}(t)=C_{4+|\beta|}(t;b_0=0)$ *are from Section 4.3 for every* $|\beta|=0,1,2$, *i.e.*

$$C_4(t;b_0=0)=C_\theta M^{\frac{d}{2}}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac{1}{2}}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac{1}{p}},$$

$$\begin{aligned}
C_5(t;b_0=0) =& C_\theta M^{\frac{d+1}{2}}\left(ta_{\min}\right)^{-\frac12}\Bigg[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac12;\kappa t\right)\\
&+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\kappa t\right)^{\frac12}{}_1F_1\left(\frac{d+2}{2};\frac32;\kappa t\right)\Bigg]^{\frac1p},\\
C_6(t;b_0=0) =& C_\theta M^{\frac{d+2}{2}}\left(ta_{\min}\right)^{-1}\Bigg[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+2}{2};\frac12;\kappa t\right)\\
&+2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\kappa t\right)^{\frac12}{}_1F_1\left(\frac{d+3}{2};\frac32;\kappa t\right)+\frac{\delta_{ij}}{2}M^{-1}{}_1F_1\left(\frac{d}{2};\frac12;\kappa t\right)\\
&+\delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\left(\kappa t\right)^{\frac12}{}_1F_1\left(\frac{d+1}{2};\frac32;\kappa t\right)\Bigg]^{\frac1p}.
\end{aligned}$$

In case $p=\infty$ *they are given by* $C_{4+|\beta|}(t;b_0=0)$ *with* $p=1$*, where* $M:=\frac{a_{\max}^2}{a_{\min}a_0}\geqslant 1$ *and* $\kappa:=\frac{a_{\max}^2\eta^2p^2}{a_0}\geqslant 0$*. Note that* $C_{4+|\beta|}(t)\sim t^{\frac{-p|\beta|+d+|\beta|-1}{2p}}e^{\frac{\kappa}{p}t}$ *as* $t\to\infty$ *and* $C_{4+|\beta|}(t)\sim t^{-\frac{|\beta|}{2}}$ *as* $t\to 0$ *for every* $|\beta|=0,1,2$*.*

Proof. Let $v\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. In the following $\beta\in\mathbb{N}_0^d$ denotes a d-dimensional multi-index with $|\beta|\leqslant 2$ and we will use the notation

$$D^\beta v=\begin{cases}v & ,|\beta|=0\\ D_iv & ,|\beta|=1\\ D_jD_iv & ,|\beta|=2\end{cases},\quad D^\beta H_0=\begin{cases}H_0 & ,|\beta|=0\\ D_iH_0 & ,|\beta|=1\\ D_jD_iH_0 & ,|\beta|=2\end{cases},\quad K_\beta=\begin{cases}K & ,|\beta|=0\\ K^i & ,|\beta|=1\\ K^{ji} & ,|\beta|=2\end{cases}$$

where $i,j=1,\ldots,d$. Note that $H_0(x,\xi,t)=H(x,\xi,t)$ since we have $B=0$. Moreover, in this proof K, K_i and K_{ji} are given by (4.24), (4.26) and (4.28) with $B=0$. To show (5.2), (5.3) and (5.4) for $1\leqslant p<\infty$ we use (5.1), the transformation theorem (with transformation $\Phi(\xi)=e^{tS}x-\xi$ in ξ and $\Phi(x)=e^{tS}x-\psi$ in x), (4.24), (4.26), (4.28), the triangle inequality, Hölder's inequality (with q such that $\frac1p+\frac1q=1$), Fubini's theorem, (3.1), (W1)–(W2), Lemma 4.6 (1),(2),(3)

$$\begin{aligned}
&\left\|D^\beta T_0(t)v\right\|_{L^p_\theta}=\left(\int_{\mathbb{R}^d}\theta^p(x)\left|D^\beta\left[T_0(t)v\right](x)\right|^p dx\right)^{\frac1p}\\
=&\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_{\mathbb{R}^d}\left[D^\beta H_0(x,\xi,t)\right]v(\xi)d\xi\right|^p dx\right)^{\frac1p}\\
=&\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_{\mathbb{R}^d}K^\beta(\psi,t)v(e^{tS}x-\psi)d\psi\right|^p dx\right)^{\frac1p}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\theta(x)\left|K^\beta(\psi,t)\right|_2\left|v(e^{tS}x-\psi)\right|d\psi\right)^p dx\right)^{\frac1p}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2 d\psi\right)^{\frac pq}\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2\left(\theta(x)\left|v(e^{tS}x-\psi)\right|\right)^p d\psi dx\right)^{\frac1p}\\
=&\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2 d\psi\right)^{\frac1q}\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(x)\left|v(e^{tS}x-\psi)\right|\right)^p dxd\psi\right)^{\frac1p}\\
=&\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2 d\psi\right)^{\frac1q}\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(e^{-tS}(y+\psi))\left|v(y)\right|\right)^p dyd\psi\right)^{\frac1p}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2 d\psi\right)^{\frac1q}\left(\int_{\mathbb{R}^d}C_\theta^p e^{\eta p|\psi|}\left|K^\beta(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(y)\left|v(y)\right|\right)^p dyd\psi\right)^{\frac1p}
\end{aligned}$$

$$\leqslant C_\theta \left(\int_{\mathbb{R}^d} \left|K^\beta(\psi,t)\right|_2 d\psi\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} e^{\eta p|\psi|} \left|K^\beta(\psi,t)\right|_2 d\psi\right)^{\frac{1}{p}} \|v\|_{L^p_\theta}$$
$$\leqslant C_{4+|\beta|}(t;b_0=0)\|v\|_{L^p_\theta}$$

for $t \geqslant 0$, if $|\beta|=0$ and for $t>0$, if $|\beta|=1$ or $|\beta|=2$. Similarly, to show (5.2), (5.3) and (5.4) for $p=\infty$ we use (5.1), the transformation theorem (with transformation $\Phi(\xi)=e^{tS}x-\xi$ in ξ and $\Phi(x)=e^{tS}x-\psi$ in x), (4.24), (4.26), (4.28), the triangle inequality, (3.1), (W1)–(W2), Lemma 4.6 (1),(2),(3) and obtain

$$\begin{aligned}
\left\|D^\beta T_0(t)v\right\|_{L^\infty_\theta} =& \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \theta(x)\left|D^\beta\left[T_0(t)v\right](x)\right| \\
=& \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \theta(x)\left|\int_{\mathbb{R}^d}\left[D^\beta H_0(x,\xi,t)\right]v(\xi)d\xi\right| \\
=& \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \theta(x)\left|\int_{\mathbb{R}^d} K^\beta(\psi,t)v(e^{tS}x-\psi)d\psi\right| \\
\leqslant& \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \theta(x)\left|K^\beta(\psi,t)\right|_2 \left|v(e^{tS}x-\psi)\right| d\psi \\
\leqslant& \int_{\mathbb{R}^d} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \theta(x)\left|K^\beta(\psi,t)\right|_2 \left|v(e^{tS}x-\psi)\right| d\psi \\
=& \int_{\mathbb{R}^d} \left|K^\beta(\psi,t)\right|_2 \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} \theta(e^{-tS}(y+\psi))\left|v(y)\right| d\psi \\
\leqslant& C_\theta\left(\int_{\mathbb{R}^d} e^{\eta|\psi|}\left|K^\beta(\psi,t)\right|_2 d\psi\right)\|v\|_{L^\infty_\theta} \leqslant C_{4+|\beta|}(t;b_0=0)\|v\|_{L^\infty_\theta}.
\end{aligned}$$

□

Theorem 5.2 (Semigroup on $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2), (A4) be satisfied for $1\leqslant p\leqslant\infty$ and $\mathbb{K}=\mathbb{C}$. Then the operators $(T_0(t))_{t\geqslant 0}$ given by (5.1) generate a semigroup on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, i.e. $T_0(t): L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is linear and bounded for every $t\geqslant 0$ and satisfies the semigroup properties*

$$T_0(0)=I, \tag{5.5}$$
$$T_0(t)T_0(s)=T_0(t+s),\ \forall\, s,t\geqslant 0. \tag{5.6}$$

Proof. The boundedness of $T_0(t)$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $t\geqslant 0$ can be deduced from (5.2) (with $\theta\equiv 1$, $\eta=0$, $C_\theta=1$). The linearity of $T_0(t)$ and property (5.5) follow from the definition of $T_0(t)$ in (5.1). Property (5.6) can easily be verified by using (5.1), Lemma 4.5 (with $B=0$, i.e. with H_0 instead of H) and Fubini's theorem

$$\begin{aligned}
\left[T_0(t)\left(T_0(s)v\right)\right](x) =& \int_{\mathbb{R}^d} H_0(x,\tilde\xi,t)\left[T_0(s)v\right](\tilde\xi)d\tilde\xi \\
=& \int_{\mathbb{R}^d} H_0(x,\tilde\xi,t)\int_{\mathbb{R}^d} H_0(\tilde\xi,\xi,s)v(\xi)d\xi d\tilde\xi \\
=& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} H_0(x,\tilde\xi,t)H_0(\tilde\xi,\xi,s)d\tilde\xi v(\xi)d\xi \\
=& \int_{\mathbb{R}^d} H_0(x,\xi,t+s)v(\xi)d\xi = \left[T_0(t+s)v\right](x),\ x\in\mathbb{R}^d.
\end{aligned}$$

□

The next theorem states that the semigroup $(T_0(t))_{t\geqslant 0}$ is strongly continuous on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$, which justifies to define its infinitesimal generator.

Theorem 5.3 (Strong continuity on $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A4) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then* $(T_0(t))_{t\geqslant 0}$ *is a* C^0*-semigroup (or strongly continuous semigroup) on* $L^p(\mathbb{R}^d,\mathbb{C}^N)$*, i.e.*

$$\lim_{t\downarrow 0}\left\|T_0(t)v-v\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}=0\;\forall\, v\in L^p(\mathbb{R}^d,\mathbb{C}^N). \tag{5.7}$$

Proof. 1. Let us define the (d-dimensional) diffusion semigroup (Gaussian semigroup, heat semigroup)

$$\begin{aligned}\left[G(t)v\right](y) &:= \int_{\mathbb{R}^d} H_0(e^{-tS}y,\xi,t)v(\xi)d\xi \\ &= \int_{\mathbb{R}^d} (4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|y-\xi\right|^2\right)v(\xi)d\xi\end{aligned} \tag{5.8}$$

then we have $\left[T_0(t)v\right](x)=\left[G(t)v\right](e^{tS}x)$. Let $1\leqslant p<\infty$. Motivated by [25], we consider the decomposition

$$\begin{aligned}\left\|T_0(t)v-v\right\|_{L^p} &\leqslant \left\|\left[G(t)v\right](e^{tS}\cdot)-v(e^{tS}\cdot)\right\|_{L^p}+\left\|v(e^{tS}\cdot)-v(\cdot)\right\|_{L^p} \\ &=: \left\|v_1(\cdot,t)\right\|_{L^p}+\left\|v_2(\cdot,t)\right\|_{L^p}\end{aligned}$$

Here and in the sequel of the proof we abbreviate $\|\cdot\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}$ by $\|\cdot\|_{L^p}$.
2. First we compute the v_1-term. Therefore, we use the transformation theorem with $\Phi(x)=e^{tS}x$ and consider the decomposition

$$\begin{aligned}&\left\|v_1(\cdot,t)\right\|_{L^p} = \left\|\left[G(t)v\right](e^{tS}\cdot)-v(e^{tS}\cdot)\right\|_{L^p} = \left\|\left[G(t)v\right](\cdot)-v(\cdot)\right\|_{L^p} \\ \leqslant &\left\|\int_{\mathbb{R}^d} H_0(e^{-tS}\cdot,\xi,t)\left(v(\xi)-v(\cdot)\right)d\xi\right\|_{L^p}+\left\|\left(\int_{\mathbb{R}^d} H_0(e^{-tS}\cdot,\xi,t)d\xi-I_N\right)v(\cdot)\right\|_{L^p} \\ =: &\left\|v_3(\cdot,t)\right\|_{L^p}+\left\|v_4(\cdot,t)\right\|_{L^p}\end{aligned}$$

3. Let us consider the v_4-term. Using the transformation theorem (with transformation $\Phi(\xi)=y-\xi$) and Lemma 4.7 (1) (with $B=0$), we obtain

$$\begin{aligned}&\left\|v_4(\cdot,t)\right\|_{L^p} \\ =&\left(\int_{\mathbb{R}^d}\left|\left(\int_{\mathbb{R}^d} H_0(e^{-tS}y,\xi,t)d\xi-I_N\right)v(y)\right|^p dy\right)^{\frac{1}{p}} \\ =&\left(\int_{\mathbb{R}^d}\left|\left(\int_{\mathbb{R}^d}(4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|y-\xi\right|^2\right)d\xi-I_N\right)v(y)\right|^p dy\right)^{\frac{1}{p}} \\ =&\left(\int_{\mathbb{R}^d}\left|\left(\int_{\mathbb{R}^d} K(\psi,t)d\psi-I_N\right)v(y)\right|^p dy\right)^{\frac{1}{p}} \\ \leqslant&\left|\int_{\mathbb{R}^d} K(\psi,t)d\psi-I_N\right|_2\left\|v\right\|_{L^p} \\ =&\left|I_N-I_N\right|_2\left\|v\right\|_{L^p}=0\quad\text{for}\quad t\geqslant 0.\end{aligned}$$

4. The v_3-term is much more delicate: First we need the following integral for $b_0=0$ and some constant $\delta_0>0$, compare proof of Lemma 4.6,

$$\begin{aligned}&\int_{|\psi|\geqslant\delta_0}\left|K(\psi,t)\right|_2 d\psi \\ \leqslant&\int_{|\psi|\geqslant\delta_0}(4\pi ta_{\min})^{-\frac{d}{2}}e^{-b_0t-\frac{a_0}{4ta_{\max}^2}|\psi|^2}d\psi \\ =&(4\pi ta_{\min})^{-\frac{d}{2}}e^{-b_0t}\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_{\delta_0}^{\infty}r^{d-1}e^{-\frac{a_0}{4ta_{\max}^2}r^2}dr\end{aligned}$$

$$= \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}} e^{-b_0 t}\frac{2}{\Gamma\left(\frac{d}{2}\right)}\int_{\left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}}\delta_0}^{\infty} s^{d-1}e^{-s^2}ds =: C(t,\delta_0)$$

where we used the transformation theorem (with transformations for d-dimensional polar coordinates and $\Phi(r) = \left(\frac{a_0}{4ta_{\max}^2}\right)^{\frac{1}{2}} r$). Note, that $C(t,\delta_0) \to 0$ as $t \to 0$ for every fixed $\delta_0 > 0$. Using the transformation theorem (with transformations $\Phi(\xi) = y - \xi$ and $\Phi(y) = y - \psi$), the triangle inequality, Hölder's inequality (with q such that $\frac{1}{p} + \frac{1}{q} = 1$), Fubini's theorem, the L^p-continuity from [7, Satz 2.14(1)], (4.24) and Lemma 4.6(1) (with $\eta = 0$ and $b_0 = 0$) we obtain

$$\begin{aligned}
&\left\|v_3(\cdot,t)\right\|_{L^p}\\
=&\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} H_0(e^{-tS}y,\xi,t)\left(v(\xi)-v(y)\right)d\xi\right|^p dy\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} (4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|y-\xi\right|^2\right)\left(v(\xi)-v(y)\right)d\xi\right|^p dy\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} K(\psi,t)\left(v(y-\psi)-v(y)\right)d\psi\right|^p dy\right)^{\frac{1}{p}}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \left|K(\psi,t)\right|_2\left|v(y-\psi)-v(y)\right|d\psi\right)^p dy\right)^{\frac{1}{p}}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \left|K(\psi,t)\right|_2 d\psi\right)^{\frac{p}{q}}\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2\left|v(y-\psi)-v(y)\right|^p d\psi dy\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2\int_{\mathbb{R}^d}\left|v(y-\psi)-v(y)\right|^p dy d\psi\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2\left\|v(\cdot-\psi)-v(\cdot)\right\|_{L^p}^p d\psi\right)^{\frac{1}{p}}\\
=&\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\int_{|\psi|\leqslant\delta_0}\left|K(\psi,t)\right|_2\left\|v(\cdot-\psi)-v(\cdot)\right\|_{L^p}^p d\psi\right.\\
&\left.+\int_{|\psi|\geqslant\delta_0}\left|K(\psi,t)\right|_2\left\|v(\cdot-\psi)-v(\cdot)\right\|_{L^p}^p d\psi\right)^{\frac{1}{p}}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\varepsilon_0^p\int_{|\psi|\leqslant\delta_0}\left|K(\psi,t)\right|_2 d\psi\right.\\
&\left.+2^{p-1}\int_{|\psi|\geqslant\delta_0}\left|K(\psi,t)\right|_2\left(\left\|v(\cdot-\psi)\right\|_{L^p}^p+\left\|v\right\|_{L^p}^p\right)d\psi\right)^{\frac{1}{p}}\\
\leqslant&\left(\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2 d\psi\right)^{\frac{1}{q}}\left(\varepsilon_0^p\int_{\mathbb{R}^d}\left|K(\psi,t)\right|_2 d\psi+2^p\int_{|\psi|\geqslant\delta_0}\left|K(\psi,t)\right|_2 d\psi\left\|v\right\|_{L^p}^p\right)^{\frac{1}{p}}\\
\leqslant& C_1^{\frac{1}{q}}(t)\left(\varepsilon_0^p C_1(t)+2^p C(t,\delta_0)\left\|v\right\|_{L^p}^p\right)^{\frac{1}{p}}
\end{aligned}$$

Hence, $\lim_{t\to 0}\left\|v_3(\cdot,t)\right\|_{L^p} \leqslant \varepsilon_0 C_1(0) = \varepsilon_0 M^{\frac{d}{2}}$. Now, choose $\varepsilon_0 > 0$ arbitrary small.

5. Finally, let us consider the v_2-term. Let $\varepsilon > 0$. Since $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d,\mathbb{C}^N)$ is dense in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$ for every $1 \leqslant p < \infty$, see [7, Satz 2.14(3)], we can choose $\varphi_\varepsilon \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d,\mathbb{C}^N)$ such that $\|v-\varphi_\varepsilon\|_{L^p} \leqslant \frac{\varepsilon}{3}$. Since $\varphi_\varepsilon \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d,\mathbb{C}^N)$, φ_ε is uniformly continuous on $\mathrm{supp}(\varphi_\varepsilon)$, i.e.

$$\forall\,\varepsilon_0 > 0\;\exists\,\delta_0 = \delta_0(\varepsilon_0) > 0\;\forall\, x, x_0 \in \mathrm{supp}(\varphi_\varepsilon)$$

$$\text{with } |x-x_0|\leqslant\delta_0: \; |\varphi_\varepsilon(x)-\varphi_\varepsilon(x_0)|\leqslant\varepsilon_0$$

Choosing $x_0 := e^{tS}x$ we have

$$\exists\, t_0=t_0(\delta_0)>0\;\forall\, 0\leqslant t\leqslant t_0:\; \left|e^{tS}x-x\right|\leqslant\delta_0$$

Thus, choosing $\varepsilon_0 := \varepsilon\left(3\left|\mathrm{supp}(\varphi_\varepsilon)\right|^{\frac{1}{p}}\right)^{-1}$ and combining this facts yields

$$\left\|\varphi_\varepsilon(e^{tS}\cdot)-\varphi_\varepsilon(\cdot)\right\|_{L^p}=\left(\int_{\mathrm{supp}(\varphi_\varepsilon)}\left|\varphi_\varepsilon(e^{tS}x)-\varphi_\varepsilon(x)\right|^p\right)^{\frac{1}{p}}\leqslant\varepsilon\quad\forall\,0\leqslant t\leqslant t_0(\varepsilon).$$

This implies

$$\begin{aligned}&\|v_2(\cdot,t)\|_{L^p}=\left\|v(e^{tS}\cdot)-v(\cdot)\right\|_{L^p}\\ \leqslant&\left\|v(e^{tS}\cdot)-\varphi_\varepsilon(e^{tS}\cdot)\right\|_{L^p}+\left\|\varphi_\varepsilon(e^{tS}\cdot)-\varphi_\varepsilon(\cdot)\right\|_{L^p}+\|\varphi_\varepsilon(\cdot)-v(\cdot)\|_{L^p}\\ \leqslant&\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\quad\forall\,0\leqslant t\leqslant t_0(\varepsilon).\end{aligned}$$

Hence, $\lim_{t\to0}\|v_2(\cdot,t)\|_{L^p}\leqslant\varepsilon$. Now, choose $\varepsilon>0$ arbitrary small. □

Now, the infinitesimal generator $A_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ of $(T_0(t))_{t\geqslant0}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, short $(A_p,\mathcal{D}(A_p))$, can be defined by, [28, II.1.2 Definition],

$$A_pv:=\lim_{t\downarrow0}\frac{T_0(t)v-v}{t},\; 1\leqslant p<\infty$$

for every $v\in\mathcal{D}(A_p)$, where the domain of A_p is given by

$$\begin{aligned}\mathcal{D}(A_p):=&\left\{v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\mid\lim_{t\downarrow0}\frac{T_0(t)v-v}{t}\text{ exists in }L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}\\ =&\left\{v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\mid A_pv\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.\end{aligned}$$

Note that $\mathcal{D}(A_p)$ is a linear subspace of $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, from [28, II.1.3 Lemma, II.1.4 Theorem], we obtain the following result:

Lemma 5.4. *Let the assumptions (A1), (A2), (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$.*
(1) *$A_p:\mathcal{D}(A_p)\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is a linear, closed and densely defined operator that determines the semigroup $(T_0(t))_{t\geqslant0}$ uniquely.*
(2) *For every $v\in\mathcal{D}(A_p)$ and $t\geqslant0$ we have*

$$\begin{aligned}&T_0(t)v\in\mathcal{D}(A_p)\\ &\frac{d}{dt}T_0(t)v=T_0(t)A_pv=A_pT_0(t)v\end{aligned}$$

(3) *For every $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and every $t\geqslant0$ we have*

$$\int_0^t T_0(s)vds\in\mathcal{D}(A_p)$$

(4) *For every $t\geqslant0$ we have*

$$\begin{aligned}T_0(t)v-v=&A_p\int_0^t T_0(s)vds &&,\text{ for } v\in L^p(\mathbb{R}^d,\mathbb{C}^N)\\ =&\int_0^t T_0(s)A_pvds &&,\text{ for } v\in\mathcal{D}(A_p)\end{aligned}$$

Since $(A_p,\mathcal{D}(A_p))$ is a closed operator on the Banach space $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, we can define

$$\begin{aligned}
\sigma(A_p) &:= \{\lambda\in\mathbb{C}\mid \lambda I-A_p \text{ is not bijective}\} && \text{spectrum of } A_p,\\
\rho(A_p) &:= \mathbb{C}\backslash\sigma(A_p) && \text{resolvent set of } A_p,\\
R(\lambda,A_p) &:= (\lambda I-A_p)^{-1}\,,\ \text{for } \lambda\in\rho(A_p) && \text{resolvent of } A_p.
\end{aligned}$$

The next identities follow from [28, II.1.9 Lemma].

Lemma 5.5. *Let the assumptions (A1), (A2), (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Then for every $\lambda\in\mathbb{C}$ and $t>0$,*

$$\begin{aligned}
e^{-\lambda t}T_0(t)v-v &= (A_p-\lambda I)\int_0^t e^{-\lambda s}T_0(s)vds && \text{, for } v\in L^p(\mathbb{R}^d,\mathbb{C}^N),\\
&= \int_0^t e^{-\lambda s}T_0(s)\,(A_p-\lambda I)\,vds && \text{, for } v\in\mathcal{D}(A_p).
\end{aligned}$$

By (5.2) from Theorem 5.1 (with $\theta\equiv 1$, $\eta=0$ and $C_\theta=1$) we have

$$\exists\,\omega_0\in\mathbb{R}\ \wedge\ \exists\,M_0\geqslant 1:\ \|T_0(t)\|_{\mathcal{L}(L^p,L^p)}\leqslant M_0e^{\omega_0 t}\quad\forall\,t\geqslant 0, \tag{5.9}$$

where $M_0:=\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}$ and $\omega_0:=0$. For the next statement we refer to [28, II.1.10 Theorem].

Theorem 5.6. *Let the assumptions (A1), (A2), (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$.*
(1) *For every $\lambda\in\mathbb{C}$ such that $R(\lambda)v:=\int_0^\infty e^{-\lambda s}T_0(s)vds$ exists for every $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ we have*

$$\lambda\in\rho(A_p)\quad\text{and}\quad R(\lambda,A_p)=R(\lambda).$$

(2) *For every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_0$ we have*

$$\lambda\in\rho(A_p),\quad R(\lambda,A_p)=R(\lambda)$$

and

$$\|R(\lambda,A_p)\|_{\mathcal{L}(L^p,L^p)}\leqslant\frac{M_0}{\operatorname{Re}\lambda-\omega_0}.$$

A direct consequence of Theorem 5.6 is the following:

Corollary 5.7 (Solvability and uniqueness in $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2), (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_0$. Then for every $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ the resolvent equation*

$$(\lambda I-A_p)\,v=g$$

admits a unique solution $v_\star\in\mathcal{D}(A_p)$, which is given by the integral expression

$$\begin{aligned}
v_\star=R(\lambda)g &= \int_0^\infty e^{-\lambda s}T_0(s)gds\\
&= \int_0^\infty e^{-\lambda s}\int_{\mathbb{R}^d}H_0(\cdot,\xi,s)g(\xi)d\xi ds.
\end{aligned}$$

Moreover, the following resolvent estimate holds

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{M_0}{\operatorname{Re}\lambda-\omega_0}\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\,.$$

For the next statement we refer to [28, II.1.11 Corollary].

Corollary 5.8. *Let the assumptions (A1), (A2), (A4) be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Moreover, let* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$*. Then, for every* $n \in \mathbb{N}$ *and every* $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *it hold*

$$\begin{aligned} R(\lambda, A_p)^n v =& \frac{(-1)^n}{(n-1)!} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A_p) v \\ =& \frac{1}{(n-1)!} \int_0^\infty s^{n-1} e^{-\lambda s} T_0(s) v ds \end{aligned}$$

and the estimate

$$\|R(\lambda, A_p)^n\|_{\mathcal{L}(L^p, L^p)} \leqslant \frac{M_0}{(\operatorname{Re}\lambda - \omega_0)^n}.$$

Let us now define the spectral bound $s(A_p)$ of A_p, [28, II.1.12 Definition]:

$$-\infty \leqslant s(A_p) := \sup_{\lambda \in \sigma(A_p)} \operatorname{Re}\lambda \leqslant \omega_0 = 0 < +\infty$$

5.2. **A core for the infinitesimal generator.** Let $1 \leqslant p < \infty$. A subspace $D \subset \mathcal{D}(A_p)$ of the domain $\mathcal{D}(A_p)$ of the linear operator $A_p : \mathcal{D}(A_p) \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ is called a core for $(A_p, \mathcal{D}(A_p))$ if D is dense in $\mathcal{D}(A_p)$ with respect to the graph norm

$$\|v\|_{A_p} := \|A_p v\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)} + \|v\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)}, \quad v \in \mathcal{D}(A_p),$$

see [28, II.1.6 Definition]. The next theorem states that the formally defined Ornstein-Uhlenbeck operator $\mathcal{L}_0$, which was defined only for smooth functions, and the infinitesimal generator A_p of the Ornstein-Uhlenbeck semigroup coincide on the Schwartz space $\mathcal{S}$. Moreover, the Schwartz space is a core for $(A_p, \mathcal{D}(A_p))$.

Theorem 5.9 (Core for the infinitesimal generator)**.** *Let the assumptions (A1), (A2) and (A4) be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Then:*
(1) $\mathcal{S} \subset L^p(\mathbb{R}^d, \mathbb{C}^N)$ *is dense for the* L^p*–norm* $\|\cdot\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)}$*.*
(2) $\mathcal{S}$ *is a subspace of* $\mathcal{D}(A_p)$*, i.e.* $\mathcal{S} \subset \mathcal{D}(A_p)$ *and* $A_p\phi = \mathcal{L}_0\phi$ *for every* $\phi \in \mathcal{S}$*.*
(3) $\mathcal{S}$ *is invariant under the semigroup* $(T_0(t))_{t \geqslant 0}$*, i.e.*

$$T_0(t)\mathcal{S} \subseteq \mathcal{S} \text{ for every } t \geqslant 0.$$

(4) $\mathcal{S} \subset \mathcal{D}(A_p)$ *is a core for* $(A_p, \mathcal{D}(A_p))$*, i.e.*

$$\mathcal{D}(A_p) = \overline{\mathcal{S}}^{\|\cdot\|_{A_p}} =: \mathcal{D}^p_{max}.$$

This is an extension of the real-valued scalar result in [42, Proposition 2.2 and 3.2] to complex valued systems. The details have still to be filled in and will be discussed in the appendix.

5.3. **Characterization of the maximal domain (Part 1).** In this subsection we characterize the maximal domain $\mathcal{D}(A_p) = \mathcal{D}^p_{max}$ of the infinitesimal generator $A_p : \mathcal{D}(A_p) \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ of the semigroup $(T_0(t))_{t \geqslant 0}$ and the infinitesimal generator A_p itself. Assuming in addition $1 < p < \infty$ and the L^p-dissipativity condition (A3) for $\mathcal{L}_0$, we show that the domain $\mathcal{D}^p_{max}$ coincides with

$$\begin{aligned} \mathcal{D}^p(\mathcal{L}_0) :=& \left\{ v \in W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p(\mathbb{R}^d, \mathbb{C}^N) \mid A\triangle v + \langle S\cdot, \nabla v \rangle \in L^p(\mathbb{R}^d, \mathbb{C}^N) \right\} \\ =& \left\{ v \in W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p(\mathbb{R}^d, \mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N) \right\}, \end{aligned}$$

where $\mathcal{L}_0 v$ is meant in the distributional sense, i.e. $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ if and only if there exists $u \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ such that

$$\int_{\mathbb{R}^d} \overline{[\mathcal{L}_0^* \varphi](x)}^T v(x) dx = \int_{\mathbb{R}^d} \overline{\varphi(x)}^T u(x) dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^N).$$

$\mathcal{L}_0^*$ denotes the formal adjoint operator of $\mathcal{L}_0$ and is given by

$$[\mathcal{L}_0^* v](x) = A^H \triangle v(x) - \langle Sx, \nabla v(x)\rangle - \mathrm{Tr}(S)v(x),$$

with $A^H = \overline{A}^T$. We then conclude that the infinitesimal generator A_p and the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ coincide on $\mathcal{D}^p(\mathcal{L}_0)$. For this purpose, it will be sufficient to verify the closedness of $\mathcal{L}_0 : \mathcal{D}^p(\mathcal{L}_0) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ and the uniqueness of the resolvent equation for $\mathcal{L}_0$ in $\mathcal{D}^p(\mathcal{L}_0)$. The following Lemma shows the closedness of $\mathcal{L}_0$. A proof for the real-valued case can be found in [42].

Lemma 5.10. *Let the assumptions (A1), (A2) and (A4) be satisfied for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$, then the operator $\mathcal{L}_0 : \mathcal{D}^p(\mathcal{L}_0) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ is closed in $L^p(\mathbb{R}^d, \mathbb{C}^N)$.*

Proof. Let $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}^p(\mathcal{L}_0)$ be such that v_n converges to $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$ and $\mathcal{L}_0 v_n$ converges to $u \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ w.r.t. $\|\cdot\|_{L^p}$, then we obtain

$$\int_{\mathbb{R}^d} \overline{[\mathcal{L}_0^*\varphi](x)}^T v(x)dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \overline{[\mathcal{L}_0^*\varphi](x)}^T v_n(x)dx$$
$$= \lim_{n\to\infty} \int_{\mathbb{R}^d} \overline{\varphi(x)}^T [\mathcal{L}_0 v_n](x)dx = \int_{\mathbb{R}^d} \overline{\varphi(x)}^T u(x)dx \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^N).$$

Hence $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ and $\mathcal{L}_0 v = u$. $v \in W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}^N)$ follows by local elliptic regularity, see [30]. We conclude $v \in \mathcal{D}^p(\mathcal{L}_0)$, thus $\mathcal{L}_0$ is closed. □

In order to prove uniqueness of the resolvent equation for $\mathcal{L}_0$ we need the following Lemma. This is the scalar complex-valued version of [43, Lemma 2.1].

Lemma 5.11. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a C^2-boundary or $\Omega = \mathbb{R}^d$ and $v \in W^{2,p}(\Omega, \mathbb{C}) \cap W_0^{1,p}(\Omega, \mathbb{C})$. Moreover, let $\eta \in C_b^1(\Omega, \mathbb{R})$ be nonnegative and $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$. Then for $1 < p < \infty$ it holds*

$$\begin{aligned} -\mathrm{Re} \int_\Omega \eta \overline{v} |v|^{p-2} \alpha \triangle v \geqslant & \mathrm{Re} \int_\Omega |v|^{p-2} \sum_{i=1}^d D_i\eta \alpha \overline{v} D_i v \\ & + \mathrm{Re}\,(p-1)\alpha \int_\Omega \eta |v|^{p-2} \sum_{i=1}^d |D_i v|^2 \chi_{\{v\neq 0\}} \\ & - \mathrm{Re}\,(p-2)\alpha i \int_\Omega \eta |v|^{p-4} \sum_{i=1}^d \mathrm{Im}\,(v D_i \overline{v})\, \overline{v} D_i v \chi_{\{v\neq 0\}}. \end{aligned}$$

Proof. We only provide the proof for $\Omega \subset \mathbb{R}^d$ bounded and $2 \leqslant \infty$. In case $\Omega = \mathbb{R}^d$ integration by parts yields no boundary terms due to decay at infinity and the case $1 < p < \infty$ can be treated in an analogous way.
Let $\Omega \subset \mathbb{R}^d$ be bounded and $2 \leqslant \infty$. First note that we obtain from $z + \overline{z} = 2\mathrm{Re}\,z$ for every $v \in \mathbb{C}$ and $p \geqslant 2$

$$\begin{aligned} D_i(|v|^p) =& D_i\left(\left(|v|^2\right)^{\frac{p}{2}}\right) = \frac{p}{2}\left(|v|^2\right)^{\frac{p}{2}-1} D_i\left(|v|^2\right) \\ =& \frac{p}{2}|v|^{p-2}\left[\left(\overline{D_i v}\right)v + \overline{v}\left(D_i v\right)\right] = \frac{p}{2}|v|^{p-2}\left[\overline{\overline{v}\left(D_i v\right)} + \overline{v}\left(D_i v\right)\right] \\ =& p|v|^{p-2}\,\mathrm{Re}\,(\overline{v} D_i v) \end{aligned} \tag{5.10}$$

Note that this formula holds also for every $v \neq 0$ and $p \geqslant 0$. Using $\mathrm{Re}\,z = \mathrm{Re}\,\overline{z}$ and $\mathrm{Re}\,z = z - i\mathrm{Im}\,z$ this implies for every $v \neq 0$ and $p \geqslant 2$

$$\begin{aligned} D_i\left(\overline{v}|v|^{p-2}\right) &= (D_i\overline{v})|v|^{p-2} + \overline{v} D_i\left(|v|^{p-2}\right) \\ &= (D_i\overline{v})|v|^{p-2} + (p-2)\overline{v}|v|^{p-4}\,\mathrm{Re}\,(\overline{v} D_i v) \end{aligned}$$

$$
\begin{aligned}
&= (D_i\overline{v})\left|v\right|^{p-2} + (p-2)\overline{v}\left|v\right|^{p-4}\operatorname{Re}\left(v\overline{D_i v}\right)\\
&= (D_i\overline{v})\left|v\right|^{p-2} + (p-2)\overline{v}\left|v\right|^{p-4}\left[v\overline{D_i v} - i\operatorname{Im}\left(v\overline{D_i v}\right)\right]\\
&= (D_i\overline{v})\left|v\right|^{p-2} + (p-2)\left|v\right|^{p-2}\overline{D_i v} - (p-2)i\overline{v}\left|v\right|^{p-4}\operatorname{Im}\left(v\overline{D_i v}\right)\\
&=(p-1)\left|v\right|^{p-2}(D_i\overline{v}) - (p-2)\left|v\right|^{p-4} i\overline{v}\operatorname{Im}\left(v\overline{D_i v}\right)
\end{aligned}
$$

Multiplying $\alpha\triangle v$ from left by $\eta\overline{v}\left|v\right|^{p-2}$, integrating over Ω, taking real parts and using integration by parts formula we obtain

$$
\begin{aligned}
&-\operatorname{Re}\int_\Omega \eta\overline{v}\left|v\right|^{p-2}\alpha\triangle v = -\operatorname{Re}\int_\Omega \eta\overline{v}\left|v\right|^{p-2}\alpha\sum_{i=1}^d D_i^2 v\\
=&-\operatorname{Re}\sum_{i=1}^d \alpha\int_\Omega \eta\overline{v}\left|v\right|^{p-2} D_i^2 v = \operatorname{Re}\sum_{i=1}^d\alpha\int_\Omega D_i(\eta\overline{v}\left|v\right|^{p-2})D_i v\\
=&\operatorname{Re}\sum_{i=1}^d\alpha\int_\Omega (D_i\eta)\overline{v}\left|v\right|^{p-2}D_i v + \operatorname{Re}\sum_{i=1}^d\alpha\int_\Omega \eta D_i(\overline{v}\left|v\right|^{p-2})D_i v\\
=&\operatorname{Re}\sum_{i=1}^d\alpha\int_\Omega (D_i\eta)\overline{v}\left|v\right|^{p-2}D_i v + \operatorname{Re}(p-1)\sum_{i=1}^d\alpha\int_\Omega \eta\left|v\right|^{p-2}\overline{D_i v}D_i v\chi_{\{v\neq 0\}}\\
&-\operatorname{Re}(p-2)\sum_{i=1}^d\alpha i\int_\Omega \eta\overline{v}\left|v\right|^{p-4}\operatorname{Im}\left(v\overline{D_i v}\right)D_i v\chi_{\{v\neq 0\}}\\
=&\operatorname{Re}\int_\Omega\left|v\right|^{p-2}\sum_{i=1}^d(D_i\eta)\alpha\overline{v}D_i v + \operatorname{Re}(p-1)\alpha\int_\Omega\eta\left|v\right|^{p-2}\sum_{i=1}^d\left|D_i v\right|^2\chi_{\{v\neq 0\}}\\
&-\operatorname{Re}(p-2)\alpha i\int_\Omega\eta\left|v\right|^{p-4}\sum_{i=1}^d\operatorname{Im}\left(v\overline{D_i v}\right)\overline{v}D_i v\chi_{\{v\neq 0\}}
\end{aligned}
$$

□

We are now able to prove the uniqueness of the resolvent equation for the Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $\mathcal{D}^p(\mathcal{L}_0)$. The main idea of the following proof comes from [43, Theorem 2.2] for the scalar real-valued case. For the the maximal domain of the scalar real-valued Ornstein-Uhlenbeck operator we refer to [43] and [51] for the L^p-spaces and to [44] for the L^p-spaces with invariant measures.

Theorem 5.12 (Uniqueness in $\mathcal{D}^p(\mathcal{L}_0)$ – Resolvent Estimates)**.** *Let the assumptions (A1), (A2), (A3) and (A4) be satisfied for $1<p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_0$ and let $v_\star\in\mathcal{D}^p(\mathcal{L}_0)$ denote a solution of*

$$(\lambda I-\mathcal{L}_0)\,v=g$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for some $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Then $v_\star$ is the unique solution in $\mathcal{D}^p(\mathcal{L}_0)$ and satisfies the resolvent estimate

$$\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{1}{\operatorname{Re}\lambda-\omega_0}\left\|g\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

Remark. (**1**): The L^p-dissipativity condition from (A3) for the complex Ornstein-Uhlenbeck operator $\mathcal{L}_0$ comes originally for the scalar case with $N=1$ from [22]. There it was shown, that the diffusion part $\mathcal{L}_0^{\text{diff}}$ of $\mathcal{L}_0$ is dissipative in $L^p(\mathbb{R}^d,\mathbb{C})$ if and only if (A3) with $N=1$ is satisfied.
(**2**): In the scalar real-valued case, i.e. $\alpha,\lambda\in\mathbb{R}$ with $\alpha>0$ and $\lambda>\omega_0$, the solution $v_\star$ is real valued and hence the dissipativity condition (A3) can be dropped, since $\operatorname{Im}\left(v_\star D_i\overline{v_\star}\right)=0$, see 4th term in step 3 of the proof.

(**3**): For $p=2$ the L^p-dissipativity condition is automatically satisfied for both the real-valued and complex-valued case.

Proof. It is sufficient to consider only the scalar case with $N=1$. Assume $v_\star \in \mathcal{D}^p(\mathcal{L}_0)$ satisfies

$$(\lambda I - \mathcal{L}_0)\, v_\star = g \tag{5.11}$$

in $L^p(\mathbb{R}^d,\mathbb{C})$ for some $g\in L^p(\mathbb{R}^d,\mathbb{C})$ with $1<p<\infty$. Let us define

$$\eta_n(x)=\eta\left(\frac{x}{n}\right),\quad \eta\in C_0^\infty(\mathbb{R}^d,\mathbb{R}),\quad \eta(x)=\begin{cases}1 & ,\ |x|\leqslant 1\\ \in[0,1],\ \text{smooth} & ,\ 1<|x|<2\\ 0 & ,\ |x|\geqslant 2\end{cases}.$$

1. Multiply (5.11) from left by $\eta_n^2\overline{v_\star}\,|v_\star|^{p-2}$ and integrate over $\mathbb{R}^d$, $1<p<\infty$

$$\begin{aligned}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\,\overline{v_\star}g =&\lambda\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^p-\alpha\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d D_i^2 v_\star\right)\overline{v_\star}\,|v_\star|^{p-2}\\ &-\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d (Sx)_i D_i v_\star\right)\overline{v_\star}\,|v_\star|^{p-2}.\end{aligned}$$

2. Taking real parts yields

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\,\overline{v_\star}g =&\operatorname{Re}\lambda\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^p-\operatorname{Re}\left(\alpha\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d D_i^2 v_\star\right)\overline{v_\star}\,|v_\star|^{p-2}\right)\\ &-\operatorname{Re}\left(\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d (Sx)_i D_i v_\star\right)\overline{v_\star}\,|v_\star|^{p-2}\right).\end{aligned}$$

3. Note that $S\in\mathbb{R}^{d,d}$, $-S=S^T$, the integration by parts formula and (5.10) imply

$$\begin{aligned}0=&\frac{1}{p}\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d S_{ii}\right)|v_\star|^p=\frac{1}{p}\int_{\mathbb{R}^d}\eta_n^2\operatorname{div}(Sx)\,|v_\star|^p\\ =&\frac{1}{p}\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d D_i\left((Sx)_i\right)\right)|v_\star|^p=\frac{1}{p}\sum_{i=1}^d\int_{\mathbb{R}^d}\eta_n^2 D_i\left((Sx)_i\right)|v_\star|^p\\ =&-\frac{1}{p}\sum_{i=1}^d\int_{\mathbb{R}^d}D_i\left(\eta_n^2\right)(Sx)_i\,|v_\star|^p-\frac{1}{p}\sum_{i=1}^d\int_{\mathbb{R}^d}\eta_n^2(Sx)_i D_i\left(|v_\star|^p\right)\\ =&-\frac{2}{p}\sum_{i=1}^d\int_{\mathbb{R}^d}\eta_n(D_i\eta_n)(Sx)_i\,|v_\star|^p-\sum_{i=1}^d\int_{\mathbb{R}^d}\eta_n^2(Sx)_i\operatorname{Re}\left(\overline{v_\star}D_i v_\star\right)|v_\star|^{p-2}\\ =&-\frac{2}{p}\int_{\mathbb{R}^d}\eta_n\left(\sum_{i=1}^d(Sx)_i(D_i\eta_n)\right)|v_\star|^p-\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d(Sx)_iD_iv_\star\right)\overline{v_\star}\,|v_\star|^{p-2}.\end{aligned}$$

Applying Lemma 5.11 (with $\Omega=\mathbb{R}^d$ and $\eta=\eta_n^2$) we obtain

$$\begin{aligned}&\operatorname{Re}\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^{p-2}\,\overline{v_\star}g\\ \geqslant&\operatorname{Re}\lambda\int_{\mathbb{R}^d}\eta_n^2\,|v_\star|^p+\operatorname{Re}\int_{\mathbb{R}^d}\alpha\left(\sum_{i=1}^d D_i\left(\eta_n^2\right)D_iv_\star\right)\overline{v_\star}\,|v_\star|^{p-2}\\ &+\operatorname{Re}(p-1)\alpha\int_{\mathbb{R}^d}\eta_n^2\left(\sum_{i=1}^d|D_iv_\star|^2\right)|v_\star|^{p-2}\end{aligned}$$

$$
\begin{aligned}
&- \operatorname{Re}(p-2)i\alpha \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^d \operatorname{Im}\left(v_\star D_i\overline{v_\star}\right) D_i v_\star\right) \overline{v_\star}\, |v_\star|^{p-4} \\
&- \operatorname{Re} \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^d (Sx)_i D_i v_\star\right) \overline{v_\star}\, |v_\star|^{p-2} \\
=&\operatorname{Re}\lambda \int_{\mathbb{R}^d} \eta_n^2 |v_\star|^p + \operatorname{Re}\int_{\mathbb{R}^d} 2\alpha\eta_n \left(\sum_{i=1}^d (D_i\eta_n)(D_i v_\star)\right) \overline{v_\star}\, |v_\star|^{p-2} \\
&+ (\operatorname{Re}\alpha)(p-1) \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^d |D_i v_\star|^2\right) |v_\star|^{p-2} \\
&- \operatorname{Re}(p-2)i\alpha \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^d \operatorname{Im}\left(v_\star D_i\overline{v_\star}\right) D_i v_\star\right) \overline{v_\star}\, |v_\star|^{p-4} \\
&+ \frac{2}{p} \int_{\mathbb{R}^d} \eta_n \left(\sum_{i=1}^d (Sx)_i (D_i\eta_n)\right) |v_\star|^p
\end{aligned}
$$

4. Putting the 2nd, 4th and 5th term on the left hand side yields

$$
\begin{aligned}
&\operatorname{Re}\lambda \int_{\mathbb{R}^d} \eta_n^2 |v_\star|^p + (\operatorname{Re}\alpha)(p-1) \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^d |D_i v_\star|^2\right) |v_\star|^{p-2} \\
\leqslant&\operatorname{Re}\int_{\mathbb{R}^d} \eta_n^2 |v_\star|^{p-2}\, \overline{v_\star} g - \operatorname{Re}\int_{\mathbb{R}^d} 2\alpha\eta_n \left(\sum_{i=1}^d (D_i\eta_n)(D_i v_\star)\right) \overline{v_\star}\, |v_\star|^{p-2} \\
&+ \operatorname{Re}(p-2)i\alpha \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^d \operatorname{Im}\left(v_\star D_i\overline{v_\star}\right) D_i v_\star\right) \overline{v_\star}\, |v_\star|^{p-4} \\
&- \frac{2}{p} \int_{\mathbb{R}^d} \eta_n \left(\sum_{i=1}^d (Sx)_i (D_i\eta_n)\right) |v_\star|^p
\end{aligned}
$$

For the 1st term we use $\operatorname{Re} z \leqslant |z|$ and Hölder's inequality (with q such that $\frac{1}{p} + \frac{1}{q} = 1$)

$$
\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}^d} \eta_n^2 |v_\star|^{p-2}\, \overline{v_\star} g = \int_{\mathbb{R}^d} \eta_n^2 |v_\star|^{p-2} \operatorname{Re}\left(\overline{v_\star} g\right) \\
\leqslant& \int_{\mathbb{R}^d} \eta_n^2 |v_\star|^{p-1} |g| \leqslant \left(\int_{\mathbb{R}^d} \left(\eta_n^{\frac{2(p-1)}{p}} |v_\star|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \left(\eta_n^{\frac{2}{p}} |g|\right)^p\right)^{\frac{1}{p}} \\
=& \left(\int_{\mathbb{R}^d} \eta_n^2 |v_\star|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \eta_n^2 |g|^p\right)^{\frac{1}{p}}
\end{aligned}
$$

For the 2st term we use $\operatorname{Re} z \leqslant |z|$, Hölder's inequality (with $p = q = 2$) and Cauchy's inequality (with $\varepsilon > 0$)

$$
\begin{aligned}
&- \operatorname{Re}\int_{\mathbb{R}^d} 2\alpha\eta_n \left(\sum_{i=1}^d (D_i\eta_n)(D_i v_\star)\right) \overline{v_\star}\, |v_\star|^{p-2} \\
\leqslant& 2|\alpha| \int_{\mathbb{R}^d} \eta_n \left(\sum_{i=1}^d |D_i\eta_n|\, |D_i v_\star|\right) |v_\star|^{p-1} \\
\leqslant& \frac{2|\alpha|\, \|\eta\|_{1,\infty}}{n} \sum_{i=1}^d \int_{\mathbb{R}^d} \eta_n |D_i v_\star|\, |v_\star|^{p-1}
\end{aligned}
$$

$$\leqslant \frac{2|\alpha| \, \|\eta\|_{1,\infty}}{n} \sum_{i=1}^{d} \left(\int_{\mathbb{R}^d} \eta_n^2 \, |D_i v_\star|^2 \, |v_\star|^{p-2} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |v_\star|^p \right)^{\frac{1}{2}}$$
$$\leqslant \frac{2|\alpha| \, \|\eta\|_{1,\infty} \, \varepsilon}{n} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \eta_n^2 \, |D_i v_\star|^2 \, |v_\star|^{p-2} + \frac{2d|\alpha| \, \|\eta\|_{1,\infty}}{4n\varepsilon} \int_{\mathbb{R}^d} |v_\star|^p$$
$$\leqslant \frac{2|\alpha| \, \|\eta\|_{1,\infty} \, \varepsilon}{n} \sum_{i=1}^{d} \int_{\mathbb{R}^d} |D_i v_\star|^2 \, |v_\star|^{p-2} + \frac{2d|\alpha| \, \|\eta\|_{1,\infty}}{4n\varepsilon} \int_{\mathbb{R}^d} |v_\star|^p$$

Here we used that for every $x \in \mathbb{R}^d$ and $i = 1, \ldots, d$

$$|D_i \eta_n(x)| = \left| D_i \left(\eta \left(\frac{x}{n} \right) \right) \right| = \frac{1}{n} \left| (D_i \eta) \left(\frac{x}{n} \right) \right| \leqslant \frac{1}{n} \max_{i=1,\ldots,d} \max_{y \in \mathbb{R}^d} |D_i \eta(y)| = \frac{\|\eta\|_{1,\infty}}{n}$$

For the 3st term we use $\mathrm{Re}\,(i\alpha z) = -\mathrm{Im}\,\alpha \mathrm{Re}\,z - \mathrm{Re}\,\alpha \mathrm{Im}\,z$, $\mathrm{Im}\,\overline{z} = -\mathrm{Im}\,z$ for $z = \overline{v_\star} D_i v_\star$ and $x \leqslant |x|$

$$- \mathrm{Re}\,(p-2) i\alpha \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^{d} \mathrm{Im}\,(v_\star D_i \overline{v_\star}) \, D_i v_\star \right) \overline{v_\star} \, |v_\star|^{p-4}$$
$$= - (p-2) \int_{\mathbb{R}^d} \eta_n^2 \, |v_\star|^{p-4} \sum_{i=1}^{d} \mathrm{Im}\,(v_\star D_i \overline{v_\star}) \, \mathrm{Re}\,(i\alpha \overline{v_\star} D_i v_\star)$$
$$= (\mathrm{Re}\,\alpha)(p-2) \int_{\mathbb{R}^d} \eta_n^2 \, |v_\star|^{p-4} \sum_{i=1}^{d} (\mathrm{Im}\,(v_\star D_i \overline{v_\star}))^2$$
$$+ (\mathrm{Im}\,\alpha)(p-2) \int_{\mathbb{R}^d} \eta_n^2 \, |v_\star|^{p-4} \sum_{i=1}^{d} \mathrm{Im}\,(v_\star D_i \overline{v_\star}) \, \mathrm{Re}\,(\overline{v_\star} D_i v_\star)$$
$$\leqslant (\mathrm{Re}\,\alpha)(p-2) \int_{\mathbb{R}^d} \eta_n^2 \, |v_\star|^{p-4} \sum_{i=1}^{d} |\mathrm{Im}\,(v_\star D_i \overline{v_\star})|^2$$
$$+ |\mathrm{Im}\,\alpha| \, |p-2| \int_{\mathbb{R}^d} \eta_n^2 \, |v_\star|^{p-4} \sum_{i=1}^{d} |\mathrm{Im}\,(\overline{v_\star} D_i v_\star)| \, |\mathrm{Re}\,(\overline{v_\star} D_i v_\star)|$$

For the 4st term we use that $\eta_n(x) = 0$ for $|x| \geqslant 2n$ and $\eta_n(x) = 1$ for $|x| \leqslant n$. Hence $D_i \eta_n(x) = 0$ for $|x| \leqslant n$ and we obtain

$$- \frac{2}{p} \int_{\mathbb{R}^d} \eta_n \left(\sum_{i=1}^{d} (Sx)_i (D_i \eta_n) \right) |v_\star|^p \leqslant \frac{2}{p} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \eta_n \, |(Sx)_i| \, |D_i \eta_n| \, |v_\star|^p$$
$$= \frac{2}{p} \sum_{i=1}^{d} \int_{n \leqslant |x| \leqslant 2n} \eta_n \, |(Sx)_i| \, |D_i \eta_n| \, |v_\star|^p \leqslant 2d \, |S| \, \|\eta\|_{1,\infty} \int_{n \leqslant |x| \leqslant 2n} |v_\star|^p \, .$$

The last inequality is justified by $\eta_n(x) \leqslant 1$ and

$$|(Sx)_i| \, |D_i \eta_n(x)| = \frac{1}{n} \, |(Sx)_i| \left| (D_i \eta) \left(\frac{x}{n} \right) \right| \leqslant \frac{1}{n} \, |Sx| \left| (D_i \eta) \left(\frac{x}{n} \right) \right|$$
$$\leqslant \frac{1}{n} \left(\sup_{n \leqslant |x| \leqslant 2n} |Sx| \right) \max_{i=1,\ldots,d} \max_{y \in \mathbb{R}^d} |D_i \eta(y)| \leqslant \frac{|S|}{n} \left(\sup_{n \leqslant |x| \leqslant 2n} |x| \right) \|\eta\|_{1,\infty}$$
$$= 2 \, |S| \, \|\eta\|_{1,\infty} \, .$$

Altogether, we obtain

$$\mathrm{Re}\,\lambda \int_{\mathbb{R}^d} \eta_n^2 \, |v_\star|^p + (\mathrm{Re}\,\alpha)(p-1) \int_{\mathbb{R}^d} \eta_n^2 \left(\sum_{i=1}^{d} |D_i v_\star|^2 \right) |v_\star|^{p-2}$$

$$\leqslant \left(\int_{\mathbb{R}^d} \eta_n^2 |v_\star|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} \eta_n^2 |g|^p\right)^{\frac{1}{p}} + \frac{2|\alpha|\, \|\eta\|_{1,\infty}\, \varepsilon}{n} \sum_{i=1}^{d} \int_{\mathbb{R}^d} |D_i v_\star|^2 |v_\star|^{p-2}$$
$$+ \frac{2d|\alpha|\, \|\eta\|_{1,\infty}}{4n\varepsilon} \int_{\mathbb{R}^d} |v_\star|^p + (\operatorname{Re}\alpha)(p-2) \int_{\mathbb{R}^d} \eta_n^2 |v_\star|^{p-4} \sum_{i=1}^{d} \left|\operatorname{Im}\,(v_\star D_i \overline{v_\star})\right|^2$$
$$+ |\operatorname{Im}\alpha|\, |p-2| \int_{\mathbb{R}^d} \eta_n^2 |v_\star|^{p-4} \sum_{i=1}^{d} |\operatorname{Im}\,(\overline{v_\star} D_i v_\star)|\, |\operatorname{Re}\,(\overline{v_\star} D_i v_\star)|$$
$$+ 2d\,|S|\, \|\eta\|_{1,\infty} \int_{n\leqslant|x|\leqslant 2n} |v_\star|^p .$$

5. Let $n \to \infty$, using $\operatorname{Re}\lambda = \operatorname{Re}\lambda - \omega_0 > 0$ and $\operatorname{Re}\alpha > 0$, use that if $0 \leqslant f_n \leqslant g_n$ then $\lim_{n\to\infty} f_n \leqslant \lim_{n\to\infty} g_n$ and that the last integral from step 4 tends to 0 (since $v_\star \in L^p(\mathbb{R}^d, \mathbb{C})$) we obtain

$$(\operatorname{Re}\lambda - \omega_0) \int_{\mathbb{R}^d} |v_\star|^p + (\operatorname{Re}\alpha)(p-1) \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} |D_i v_\star|^2\right) |v_\star|^{p-2}$$
$$\leqslant \left(\int_{\mathbb{R}^d} |v_\star|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} |g|^p\right)^{\frac{1}{p}} + (\operatorname{Re}\alpha)(p-2) \int_{\mathbb{R}^d} |v_\star|^{p-4} \sum_{i=1}^{d} \left|\operatorname{Im}\,(v_\star D_i \overline{v_\star})\right|^2$$
$$+ |\operatorname{Im}\alpha|\, |p-2| \int_{\mathbb{R}^d} |v_\star|^{p-4} \sum_{i=1}^{d} |\operatorname{Im}\,(\overline{v_\star} D_i v_\star)|\, |\operatorname{Re}\,(\overline{v_\star} D_i v_\star)|$$

6. Putting the last two terms on the left hand side and using (A3) yields

$$(\operatorname{Re}\lambda - \omega_0)\, \|v_\star\|^p_{L^p(\mathbb{R}^d,\mathbb{C})} = (\operatorname{Re}\lambda - \omega_0) \int_{\mathbb{R}^d} |v_\star|^p$$
$$\leqslant (\operatorname{Re}\lambda - \omega_0) \int_{\mathbb{R}^d} |v_\star|^p + \int_{\mathbb{R}^d} |v_\star|^{p-4} \sum_{i=1}^{d} \Big[(\operatorname{Re}\alpha)(p-1)\, |\overline{v_\star} D_i v_\star|^2$$
$$- (\operatorname{Re}\alpha)(p-2)\, \left|\operatorname{Im}\,(v_\star D_i \overline{v_\star})\right|^2 - |\operatorname{Im}\alpha|\, |p-2|\, |\operatorname{Re}\,(\overline{v_\star} D_i v_\star)|\, |\operatorname{Im}\,(\overline{v_\star} D_i v_\star)| \Big]$$
$$= (\operatorname{Re}\lambda - \omega_0) \int_{\mathbb{R}^d} |v_\star|^p + (\operatorname{Re}\alpha)(p-1) \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} |D_i v_\star|^2\right) |v_\star|^{p-2}$$
$$- (\operatorname{Re}\alpha)(p-2) \int_{\mathbb{R}^d} |v_\star|^{p-4} \sum_{i=1}^{d} \left|\operatorname{Im}\,(v_\star D_i \overline{v_\star})\right|^2$$
$$- |\operatorname{Im}\alpha|\, |p-2| \int_{\mathbb{R}^d} |v_\star|^{p-4} \sum_{i=1}^{d} |\operatorname{Im}\,(\overline{v_\star} D_i v_\star)|\, |\operatorname{Re}\,(\overline{v_\star} D_i v_\star)|$$
$$\leqslant \left(\int_{\mathbb{R}^d} |v_\star|^p\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^d} |g|^p\right)^{\frac{1}{p}} = \|v_\star\|^{p-1}_{L^p(\mathbb{R}^d,\mathbb{C})}\, \|g\|_{L^p(\mathbb{R}^d,\mathbb{C})}$$

To accept the first inequality, let us abbreviate $z = \overline{v_\star} D_i v_\star$. Using (A3) we can show for every $z \in \mathbb{C}$

$$(\operatorname{Re}\alpha)(p-1)\, |z|^2 - (\operatorname{Re}\alpha)(p-2)\, |\operatorname{Im} z|^2 - |\operatorname{Im}\alpha|\, |p-2|\, |\operatorname{Re} z|\, |\operatorname{Im} z| \geqslant 0.$$

Therefore, let us define the positive constants $a := ((\operatorname{Re}\alpha)(p-1))^{\frac{1}{2}}$ and $b := (\operatorname{Re}\alpha)^{\frac{1}{2}}$, then we have for every $z \in \mathbb{C}$

$$(\operatorname{Re}\alpha)(p-1)\, |z|^2 - (\operatorname{Re}\alpha)(p-2)\, |\operatorname{Im} z|^2 - |\operatorname{Im}\alpha|\, |p-2|\, |\operatorname{Re} z|\, |\operatorname{Im} z|$$
$$= (\operatorname{Re}\alpha)(p-1)\, |\operatorname{Re} z|^2 + (\operatorname{Re}\alpha)\, [(p-1)-(p-2)]\, |\operatorname{Im} z|^2 - |\operatorname{Im}\alpha|\, |p-2|\, |\operatorname{Re} z|\, |\operatorname{Im} z|$$

$$=(\operatorname{Re}\alpha)(p-1)\left|\operatorname{Re}z\right|^2+(\operatorname{Re}\alpha)\left|\operatorname{Im}z\right|^2-\left|\operatorname{Im}\alpha\right|\left|p-2\right|\left|\operatorname{Re}z\right|\left|\operatorname{Im}z\right|$$
$$=(a\left|\operatorname{Re}z\right|-b\left|\operatorname{Im}z\right|)^2+(2ab-\left|\operatorname{Im}\alpha\right|\left|p-2\right|)\left|\operatorname{Re}z\right|\left|\operatorname{Im}z\right|$$

The first term is nonnegative without any restrictions. In the second term the coefficient $(2ab-|\operatorname{Im}\alpha|\,|p-2|)$ coincides exactly with the L^p-dissipativity condition for $\mathcal{L}_0$ from (A3). Hence, also this term cannot be negative.
7. Using $\operatorname{Re}\lambda-\omega_0>0$ once more and dividing both hand sides by $\operatorname{Re}\lambda-\omega_0$ and by $\|v_\star\|^{p-1}_{L^p(\mathbb{R}^d,\mathbb{C})}$ we end up with

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C})}\leqslant\frac{1}{\operatorname{Re}\lambda-\omega_0}\|g\|_{L^p(\mathbb{R}^d,\mathbb{C})}.$$

8. To show uniqueness in $\mathcal{D}^p(\mathcal{L}_0)$, let both $u_\star,v_\star\in\mathcal{D}^p(\mathcal{L}_0)$ be a solution of

$$(\lambda I-\mathcal{L}_0)\,u_\star=g\quad\text{and}\quad(\lambda I-\mathcal{L}_0)\,v_\star=g$$

in $L^p(\mathbb{R}^d,\mathbb{C})$. Then $w_\star:=v_\star-u_\star\in\mathcal{D}^p_{max}(\mathcal{L}_0)$ is a solution of $(\lambda I-\mathcal{L}_0)\,w_\star=0$ in $L^p(\mathbb{R}^d,\mathbb{C})$. From the resolvent estimate we obtain $\|w_\star\|_{L^p}\leqslant 0$, hence $u_\star$ and $v_\star$ coincide in $L^p(\mathbb{R}^d,\mathbb{C})$. Since $u_\star,v_\star\in\mathcal{D}^p(\mathcal{L}_0)$ and $\mathcal{D}^p(\mathcal{L}_0)\subset L^p(\mathbb{R}^d,\mathbb{C}^N)$ we deduce that $v_\star=u_\star$ in $\mathcal{D}^p(\mathcal{L}_0)$. □

The next theorem gives us a complete characterization of the infinitesimal generator A_p and its maximal domain $\mathcal{D}(A_p)$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$. It shows that A_p is the maximal realization (or maximal extension) of the complex-valued Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$. The main idea for the first part of the proof comes from [42, Proposition 2.2 and 3.2].

Theorem 5.13 (Maximal domain, Part 1)**.** *Let the assumptions* (A1)*,* (A2)*,* (A3) *and* (A4) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$*, then it holds* $\mathcal{D}(A_p)=\mathcal{D}^p(\mathcal{L}_0)$*, where* $\mathcal{D}^p(\mathcal{L}_0)$ *is given by*

$$\mathcal{D}^p(\mathcal{L}_0):=\left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N)\cap L^p(\mathbb{R}^d,\mathbb{C}^N)\mid A\triangle v+\langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.$$

Proof. $\subseteq$: Let $v\in\mathcal{D}(A_p)$. Since $\mathcal{S}$ is dense in $\mathcal{D}(A_p)$ with respect to the graph norm $\|\cdot\|_{A_p}$ by Theorem 5.9(4), we have

$$\exists\ (v_n)_{n\in\mathbb{N}}\subset\mathcal{S}:\ \|v_n-v\|_{A_p}\to 0\text{ as }n\to\infty.$$

This yields

$$\|v_n-v\|_{L^p}\to 0\text{ as }n\to\infty.$$

and using Theorem 5.9(2)

$$\|\mathcal{L}_0v_n-A_pv\|_{L^p}=\|A_pv_n-A_pv\|_{L^p}\to 0\text{ as }n\to\infty,$$

where $A_pv\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ because $v\in\mathcal{D}(A_p)$. Since obviously $\mathcal{S}\subset\mathcal{D}^p(\mathcal{L}_0)$, we have $(v_n)_{n\in\mathbb{N}}\subset\mathcal{D}^p(\mathcal{L}_0)$ and we deduce by the closedness of $\mathcal{L}_0:\mathcal{D}^p(\mathcal{L}_0)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ from Lemma 5.10 that $v\in\mathcal{D}(\mathcal{L}_0)$ and $\mathcal{L}_0v=A_pv$.
$\supseteq$: Let $v\in\mathcal{D}^p(\mathcal{L}_0)$. Choose $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_0$ and define $g:=(\lambda I-\mathcal{L}_0)\,u\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Then Corollary 5.7 yields a unique solution $v_\star\in\mathcal{D}(A_p)$ of $(\lambda I-A_p)\,v_\star=g$. Since $\mathcal{D}(A_p)\subseteq\mathcal{D}^p(\mathcal{L}_0)$ we conclude $v_\star\in\mathcal{D}^p(\mathcal{L}_0)$ and since $A_pu=\mathcal{L}_0u$ for every $u\in\mathcal{D}^p(\mathcal{L}_0)$ we have

$$(\lambda I-\mathcal{L}_0)\,v_\star=g\quad\text{and}\quad(\lambda I-\mathcal{L}_0)\,v=g.$$

From the uniqueness of the resolvent equation for $\mathcal{L}_0$ from Theorem 5.12 we deduce $v=v_\star$, thus $v\in\mathcal{D}(A_p)$ and $\mathcal{L}_0v=A_pv$. □

5.4. **Exponential decay.**

Theorem 5.14 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2) and (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $0<\vartheta<1$ and $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_0$. Then for every radially nondecreasing weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ with $0\leqslant\eta^2\leqslant\vartheta\frac{a_0(\operatorname{Re}\lambda-\omega_0)}{a_{\max}^2p^2}$ and for every $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ we have $v_\star\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ with*

$$\left\|v_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_7}{\operatorname{Re}\lambda-\omega_0}\left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)},\tag{5.12}$$

$$\left\|D_iv_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_8}{(\operatorname{Re}\lambda-\omega_0)^{\frac12}}\left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)},\ i=1,\ldots,d,\tag{5.13}$$

where $v_\star\in\mathcal{D}^p_{max}$ denotes the unique solution of $(\lambda I-A_p)v=g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and the λ-independent constants C_7, C_8 are given by Lemma 4.8 (with $b_0=0$ and $\omega=\omega_0$).

Proof. By Corollary 5.7 we have the representation

$$v_\star(x)=\int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d}H_0(x,\xi,t)g(\xi)d\xi dt,\tag{5.14}$$

where $H_0(x,\xi,t)=H(x,\xi,t)$ since we have $B=0$. In the following we make use of the notation from Theorem 5.1 once more. To show (5.12) and (5.13) for $1\leqslant p<\infty$ we use (5.14), the transformation theorem (with transformation $\Phi(\xi)=e^{tS}x-\xi$ in ξ and $\Phi(x)=e^{tS}x-\psi$ in x), (4.24) and (4.26) (with $B=0$), the triangle inequality, Hölder's inequality (with q such that $\frac1p+\frac1q=1$), Fubini's theorem, (3.1), (W1)–(W2), Lemma 4.8 (with $b_0=0$ and $\omega=\omega_0$) and obtain for every $\beta\in\mathbb{N}_0^d$ with $|\beta|\in\{0,1\}$

$$
\begin{aligned}
&\left\|D^\beta v_\star\right\|_{L^p_\theta}=\left(\int_{\mathbb{R}^d}\theta^p(x)\left|D^\beta v_\star(x)\right|^p dx\right)^{\frac1p}\\
&=\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d}\left[D^\beta H_0(x,\xi,t)\right]g(\xi)d\xi dt\right|^p dx\right)^{\frac1p}\\
&=\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d}K^\beta(\psi,t)g(e^{tS}x-\psi)d\psi dt\right|^p dx\right)^{\frac1p}\\
&\leqslant\int_0^\infty e^{-\operatorname{Re}\lambda t}\left(\int_{\mathbb{R}^d}\theta^p(x)\left|\int_{\mathbb{R}^d}K^\beta(\psi,t)g(e^{tS}x-\psi)d\psi\right|^p dx\right)^{\frac1p}dt\\
&\leqslant\int_0^\infty e^{-\operatorname{Re}\lambda t}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\theta(x)\left|K^\beta(\psi,t)\right|_2\left|g(e^{tS}x-\psi)\right|d\psi\right)^p dx\right)^{\frac1p}dt\\
&\leqslant\int_0^\infty e^{-\operatorname{Re}\lambda t}\left(\int_{\mathbb{R}^d}Z^{\frac pq}(t)\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2\left(\theta(x)\left|g(e^{tS}x-\psi)\right|\right)^p d\psi dx\right)^{\frac1p}dt\\
&=\int_0^\infty e^{-\operatorname{Re}\lambda t}Z^{\frac1q}(t)\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(x)\left|g(e^{tS}x-\psi)\right|\right)^p dxd\psi\right)^{\frac1p}dt\\
&=\int_0^\infty e^{-\operatorname{Re}\lambda t}Z^{\frac1q}(t)\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2\int_{\mathbb{R}^d}\left(\theta(e^{-tS}(y+\psi))\left|g(y)\right|\right)^p dyd\psi\right)^{\frac1p}dt\\
&\leqslant\int_0^\infty e^{-\operatorname{Re}\lambda t}Z^{\frac1q}(t)\left(\int_{\mathbb{R}^d}C_\theta^pe^{\eta p|\psi|}\left|K^\beta(\psi,t)\right|_2\int_{\mathbb{R}^d}\theta^p(y)\left|g(y)\right|^p dyd\psi\right)^{\frac1p}dt\\
&=\int_0^\infty e^{-\operatorname{Re}\lambda t}C_\theta\left(\int_{\mathbb{R}^d}\left|K^\beta(\psi,t)\right|_2d\psi\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^d}e^{\eta p|\psi|}\left|K^\beta(\psi,t)\right|_2d\psi\right)^{\frac1p}dt\left\|g\right\|_{L^p_\theta}
\end{aligned}
$$

$$\leqslant \int_0^\infty e^{-\operatorname{Re}\lambda t} C_{4+|\beta|}(t; b_0 = 0) dt \, \|g\|_{L^p_\theta} \leqslant \frac{C_{7+|\beta|}}{(\operatorname{Re}\lambda - \omega_0)^{1-\frac{|\beta|}{2}}} \|g\|_{L^p_\theta},$$

where we used the abbreviation

$$Z(t) := \int_{\mathbb{R}^d} \left|K^\beta(\psi, t)\right|_2 d\psi.$$

□

Remark. Theorem 5.14 states for $\theta \equiv 1$ (with $\eta = 0$ and $C_\theta = 1$) that

$$\mathcal{D}^p_{max} \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N), \text{ for every } 1 \leqslant p < \infty.$$

Remark. In the proof of Theorem 5.14 it is in general not possible to specify also an estimate for $\|D_j D_i v_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}$ since $C_{4+|\beta|}(t) \sim t^{-\frac{|\beta|}{2}}$ as $t \to 0$, see Theorem 5.1, and consequently we have the singularity t^{-1} at $t = 0$ for $|\beta| = 2$.

5.5. **Cauchy problems and exponential decay.** In this subsection we study abstract (i.e. Banach-space-valued) linear initial value problems of the form

$$\begin{aligned} v_t(t) =& A_p v(t) + f(t), \, t \geqslant 0, \\ v(0) =& v_0, \end{aligned}$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$, where $v : \mathbb{R}_+ \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ with $\mathbb{R}_+ = [0, \infty[$, $A_p : \mathcal{D}^p_{max} \subset L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ denotes the infinitesimal generator of the strongly continuous semigroup $(T_0(t))_{t\geqslant 0}$, $v_0 \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ the initial data and $f : \mathbb{R}_+ \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ the inhomogenity. During this subsection we require that the assumptions (A1), (A2), (A4) and $1 \leqslant p < \infty$ are satisfied. In the first part of this subsection we investigate the homogeneous, and in the second part the inhomogeneous abstract Cauchy problem in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. For the homogeneous and inhomogeneous abstract Cauchy problem we refer the reader to [28, Chapter II.6] and [28, Chapter VI.7], respectively, where this theory was done in a more general framework.

5.5.1. *Homogeneous Cauchy problems.* In the following we consider the (homogeneous) abstract linear initial value problem, [28, Chapter II.6],

$$\begin{aligned} v_t(t) =& A_p v(t), \, t \in [0, T], \\ v(0) =& v_0, \end{aligned} \tag{5.15}$$

in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$. (5.15) is called the abstract Cauchy problem in $[0, T]$ associated to $(A_p, \mathcal{D}^p_{max})$ and the initial value $v_0 \in L^p(\mathbb{R}^d, \mathbb{C}^N)$.

Definition 5.15. *Let $T > 0$ and let the assumptions (A1), (A2) and (A4) be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A function $v : [0, T] \to L^p(\mathbb{R}^d, \mathbb{K}^N)$ is called a (classical) solution of (5.15) in $[0, T]$ if*

$$v \in C^1([0, T], L^p(\mathbb{R}^d, \mathbb{K}^N)), \quad v(t) \in \mathcal{D}^p_{max} \, \forall \, t \in [0, T] \quad \textit{and} \quad (5.15) \textit{ holds.}$$

A function $v : [0, \infty[\to L^p(\mathbb{R}^d, \mathbb{K}^N)$ is called a (classical) solution of (5.15) in $[0, \infty[$, if $v\,|_{[0,T]}$ is a classical solution of (5.15) in $[0, T]$ for every $T > 0$.

We already know from the privious subsections that A_p is the generator of the strongly continuous Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. Hence, we can deduce from Lemma 5.4(2) that the semigroup $(T_0(t))_{t\geqslant 0}$ provides us solutions of the associated abstract Cauchy problem (5.15), [28, Proposition II.6.2].

Proposition 5.16 (Existence and uniqueness of classical solution in $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** Let $T>0$ and let the assumptions (A1), (A2) and (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Then for every $v_0\in\mathcal{D}^p_{max}$ the function

$$v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad v(t):=T_0(t)v_0$$

is the unique classical solution of (5.15) in $[0,\infty[$.

Definition 5.17. *Let $T>0$ and let the assumptions (A1), (A2) and (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. A function $v:[0,T]\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ is called a mild solution of (5.15) in $[0,T]$ if*

$$\int_0^t v(s)ds\in\mathcal{D}^p_{max}\;\forall\,t\in[0,T]\quad\text{and}\quad v(t)=A_p\int_0^t v(s)ds+v_0.$$

A function $v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ is called a mild solution of (5.15) in $[0,\infty[$ if $v\,|_{[0,T]}$ is a mild solution of (5.15) in $[0,T]$ for every $T>0$.

Proposition 5.18 (Existence and uniqueness of mild solution in $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** Let the assumptions (A1), (A2) and (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Then for every $v_0\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ the function

$$v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad v(t):=T_0(t)v_0$$

is the unique mild solution of (5.15) in $[0,\infty[$.

In the following we investigate the regularity of the mild solution of (5.15). The result and its proof is motivated by [43, Theorem 3.3].

Theorem 5.19 (Regularity for mild solution)**.** *Let the assumptions (A1), (A2), (A3) and (A4) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $T>0$, $v_0\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and let $v(t)=T_0(t)v_0$ denote the unique mild solution of (5.15) in $[0,T]$. Then*

$$v\in C\left([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N)\right)\cap C\left(]0,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\right)\cap C^1\left(]0,T],L^p_{\text{loc}}(\mathbb{R}^d,\mathbb{C}^N)\right).$$

Proof. Let $u(x,t)$ be a solution of

$$\begin{aligned}u_t(x,t)&=A\triangle u(x,t),\\ u(x,0)&=v_0(x),\end{aligned}$$

then $v(x,t)=u(e^{tS}x,t)$ solves

$$\begin{aligned}v_t(x,t)&=A\triangle v(x,t)+\langle Sx,\nabla v(x)\rangle\,,\\ v(x,0)&=v_0(x).\end{aligned}$$

Applying [39, Corollary 6.1.6(i)] with $X=L^p(\mathbb{R}^d,\mathbb{C}^N)$, $A(t)=A\triangle$, $\mathcal{D}(A(t))=W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$, $f(t)=0$, $x=v_0$, $s=0$ with arbitrary θ we obtain

$$u\in C\left([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N)\right)\cap C\left(]0,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\right)\cap C^1\left(]0,T],L^p(\mathbb{R}^d,\mathbb{C}^N)\right).$$

Note that, in order to apply [39, Corollary 6.1.6(i)], the diffusion operator $A\triangle:W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\supset L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1<p<\infty$ by assumption (A2). Now, using $v(x,t)=u(e^{tS}x,t)$ we deduce from the regularity for u that

$$v\in C\left([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N)\right)\cap C\left(]0,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\right)\cap C^1\left(]0,T],L^p_{\text{loc}}(\mathbb{R}^d,\mathbb{C}^N)\right).$$

□

The proof of the following regularity result for the mild solution is a direct consequence of Proposition 5.18 and Theorem 5.1.

Theorem 5.20 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2) and (A4) be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K}=\mathbb{C}$. Then for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ and for every initial data $v_0 \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ we have $v(t) \in W^{2,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ for every $t>0$ with*

$$\|v(t)\|_{L^p_\theta} \leqslant C_4(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t \geqslant 0, \tag{5.16}$$

$$\|D_i v(t)\|_{L^p_\theta} \leqslant C_5(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t > 0,\ i=1,\ldots,d, \tag{5.17}$$

$$\|D_j D_i v(t)\|_{L^p_\theta} \leqslant C_6(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t > 0,\ i,j=1,\ldots,d, \tag{5.18}$$

where $v : [0,\infty[\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ denotes the unique mild solution of (5.15) in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and the constants $C_{4+|\beta|}(t)$ are given by Theorem 5.1 for every $|\beta|=0,1,2$.

5.5.2. *Inhomogeneous Cauchy problems.* In the following we consider the (inhomogeneous) abstract linear initial value problem, [28, Chapter VI.7],

$$\begin{aligned} v_t(t) =& A_p v(t) + f(t),\ t \in [0,T], \\ v(0) =& v_0, \end{aligned} \tag{5.19}$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1 \leqslant p < \infty$. (5.19) is called the inhomogeneous abstract Cauchy problem associated to $(A_p,\mathcal{D}^p_{max})$, the initial value $v_0 \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and the inhomogenity $f : [0,T] \to L^p(\mathbb{R}^d,\mathbb{C}^N)$.

Definition 5.21. *Let the assumptions (A1), (A2) and (A4) be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Moreover, let $T>0$, $v_0 \in L^p(\mathbb{R}^d,\mathbb{K}^N)$ and $f \in L^p([0,T],L^p(\mathbb{R}^d,\mathbb{K}^N))$. Then the function $v : [0,T] \to L^p(\mathbb{R}^d,\mathbb{K}^N)$ given by*

$$v(t) := T_0(t)v_0 + \int_0^t T_0(t-s)f(s)ds,\ t \in [0,T], \tag{5.20}$$

is called the mild solution of (5.19) in $[0,T]$. A function $v : [0,\infty[\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ is called the mild solution of (5.19) in $[0,\infty[$ if $v\,|_{[0,T]}$ is the mild solution of (5.19) in $[0,T]$ for every $T>0$.

Note, that by definition the mild solution is unique. In the following we investigate the regularity of the mild solution of (5.19). This result and its proof is again motivated by [43, Theorem 3.4]. Note, that one can identify $L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ with $L^p(\mathbb{R}^d\times[0,T],\mathbb{C}^N)$. The details for the proof will be filled in later.

Theorem 5.22 (Regularity for mild solution)**.** *Let the assumptions (A1), (A2), (A3) and (A4) be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $T>0$, $v_0=0$, $f \in L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ and let v given by (5.20) denote the unique mild solution of (5.19) in $[0,T]$. Then*

$$v \in W^{(2,1),p}(\mathbb{R}^d \times]0,T[,\mathbb{C}^N)$$

and satisfies

$$v, v_t - \langle S\cdot,\nabla v\rangle, D_i v, D_j D_i v \in L^p(\mathbb{R}^d \times [0,T],\mathbb{C}^N).$$

The proof of the following regularity result for the mild solution (5.20) is a direct consequence of Theorem 5.1.

Theorem 5.23 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2) and (A4) be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K}=\mathbb{C}$. Then for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta \geqslant 0$, for every $v_0 \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ and $f \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ we have $v(t) \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ for every $t>0$ with*

$$\|v(t)\|_{L^p_\theta} \leqslant C_4(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} + C_9(t)\,\|f\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad , t \geqslant 0,$$

$$\|D_i v(t)\|_{L^p_\theta} \leqslant C_5(t)\,\|v_0\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} + C_{10}(t)\,\|f\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \quad, t>0,\, i=1,\ldots,d,$$

where $v:\mathbb{R}_+\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ *given by* (5.20) *denotes the unique mild solution of* (5.19) *in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and the constants* $C_{4+|\beta|}(t)$ *and* $C_{9+|\beta|}(t)$ *are given by Theorem 5.1 and*

$$C_{9+|\beta|}(t) := \int_0^t C_{4+|\beta|}(s)ds,$$

respectively, for every $|\beta|=0,1$.

Definition 5.24. *Let the assumptions* (A1), (A2) *and* (A4) *be satisfied for* $1\leqslant p<\infty$ *and* $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$. *Moreover, let* $T>0$, $v_0\in\mathcal{D}^p_{max}$ *and* $f\in L^p([0,T],L^p(\mathbb{R}^d,\mathbb{K}^N))$. *A function* $v:[0,T]\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ *is called a (classical) solution of* (5.19) *in* $[0,T]$ *if*

$$v\in C^1([0,T[,L^p(\mathbb{R}^d,\mathbb{K}^N)),\quad v(t)\in\mathcal{D}^p_{max}\;\forall\, t\in[0,T]\quad \textit{and}\quad (5.19)\ \textit{holds.}$$

A function $v:[0,\infty[\to L^p(\mathbb{R}^d,\mathbb{K}^N)$ *is called a (classical) solution of* (5.19) *in* $[0,\infty[$ *if* $v\mid_{[0,T]}$ *is a classical solution of* (5.19) *in* $[0,T]$ *for every* $T>0$.

5.6. **Characterization of the maximal domain (Part 2).** The next theorem states that the domain of the Ornstein-Uhlenbeck operator coincides with the intersection of the domains of its diffusion and drift part, i.e.

$$\mathcal{D}\left(\mathcal{L}_0\right)=\mathcal{D}\left(\mathcal{L}_0^{\text{diff}}+\mathcal{L}_0^{\text{drift}}\right)=\mathcal{D}\left(\mathcal{L}_0^{\text{diff}}\right)\cap\mathcal{D}\left(\mathcal{L}_0^{\text{drift}}\right).$$

This result was proved in [43, Theorem 1] for the scalar real case, where Metafune, Pallara and Vespri used $\nabla(P\nabla\cdot)$ as the diffusion part of the Ornstein-Uhlenbeck operator.

Theorem 5.25 (Maximal domain, Part 2)**.** *Let the assumptions* (A1), (A2), (A3) *and* (A4) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then it holds* $\mathcal{D}^p(\mathcal{L}_0)=\mathcal{D}^p$, *where* $\mathcal{D}^p$ *is given by*

$$\mathcal{D}^p := \left\{v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)\mid \langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)\right\}.$$

Proof. • $\mathcal{D}^p\subset\mathcal{D}^p(\mathcal{L}_0)$: Let $v\in\mathcal{D}^p$, then we have $v\in W^{2,p}_{\text{loc}}(\mathbb{R}^d,\mathbb{C}^N)$ and $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ since $v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$. Moreover, $v\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ implies $A\triangle v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Thus, using $\langle S\cdot,\nabla v\rangle\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ we conclude $\mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$.
• $\mathcal{D}^p\supset\mathcal{D}^p(\mathcal{L}_0)$: Let $v\in\mathcal{D}^p(\mathcal{L}_0)$, then $g:=\mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$, i.e. $0=\mathcal{L}_0 v-g$. Then $w(t)=v$ is a stationary solution of

$$\begin{aligned} w(t)&=\mathcal{L}_0 w(t)-g,\, t\in[0,T]\\ w(0)&=v. \end{aligned}$$

Since $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $g\in L^p([0,T],L^p(\mathbb{R}^d,\mathbb{C}^N))$ for every fixed $T>0$, the unique mild solution is given by

$$v=w(t)=T_0(t)v-\int_0^t T_0(t-s)g ds =: w_1(t)+w_2(t),\, t\in[0,T].$$

w_1 is the mild solution of (5.15) in $[0,T]$ with initial data $v_0=v$. Theorem 5.19 states now that $w(t)\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ for every $t\in]0,T]$. w_2 is the mild solution of (5.19) in $[0,T]$ with initial data $v_0=0$ and inhomogenity $f=-g$. Theorem 5.22 states now that $w_2\in L^p([0,T],W^{2,p}(\mathbb{R}^d,\mathbb{C}^N))$, i.e. $w_2(t)\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$ for almost every $t\in[0,T]$. If we consider such a $\bar{t}\in]0,T]$, we can deduce that

$$v=w(\bar{t})=T_0(\bar{t})v+\int_0^{\bar{t}} T_0(\bar{t}-s)f ds = w_1(\bar{t})+w_2(\bar{t})\in W^{2,p}(\mathbb{R}^d,\mathbb{C}^N)$$

and thus we have $A\triangle v \in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Consequently, using $\mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^N)$, we conclude

$$\langle S\cdot,\nabla v\rangle = \mathcal{L}_0 v - A\triangle v \in L^p(\mathbb{R}^d,\mathbb{C}^N),$$

that means $v \in \mathcal{D}^p$. □

5.7. **The essential spectrum.** The study of the essential spectrum shows that the semigroup $(T_0(t))_{t\geqslant 0}$ is not analytic. For a detailed study see Theorem 7.8 and Corollary 7.9. Here we just make a general remark about the structure of the essential spectrum. Consider the operator $A\triangle v(x) + \langle Sx,\nabla v(x)\rangle$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$ with A satisfying (A1), (A2), (A3) and S satisfying (A4). It is well-known that the operators $A\triangle$ and $\langle S\cdot,\nabla\rangle$ commute, see e.g. proof of Theorem 8.3. Since $A\triangle$ generates a holomorphic semigroup, we can apply [9, Theorem 7.3] to deduce that the spectrum of $A\triangle+\langle S\cdot,\nabla\rangle$ is contained in the algebraic sum $\sigma(A\triangle)+\sigma(\langle S\cdot,\nabla\rangle)$. Moreover, we can deduce from [42, Theorem 2.6] that $\sigma(\langle S\cdot,\nabla\rangle)=G$, where G is a discrete subgroup of $i\mathbb{R}$ (independent of p). Altogether, we have

$$\sigma(A\triangle+\langle S\cdot,\nabla\rangle)\subset\sigma(A\triangle)+\sigma(\langle S\cdot,\nabla\rangle)=\sigma(A\triangle)+G.$$

Note that this fact means that the spectrum of the Ornstein-Uhlenbeck operator is contained in the algebraic sum of the spectra of its diffusion and drift terms. In case of $A=I_N$ we have $\sigma(\triangle)=]-\infty,0]$ and hence, $\sigma(\triangle+\langle S\cdot,\nabla\rangle)$ is contained in a countable union of half-lines. For the case $d=p=2$ the essential spectrum of $A\triangle v(x)+\langle Sx,\nabla v(x)\rangle - Bv(x)$ in $L^2(\mathbb{R}^2,\mathbb{R}^N)$ was computed explicitly in [17, Theorem 8.1] for matrices $A,B\in\mathbb{R}^{N,N}$ with $A>0$ and also in case of variable coefficients by using polar coordinates and a Fourier ansatz.

6. Constant coefficient perturbations of the complex Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d,\mathbb{C}^N)$

Consider the operator

$$[\mathcal{L}_\infty\phi](x) := [\mathcal{L}_0\phi](x) - B\phi(x),\ \phi\in\mathcal{S}$$

6.1. **Application of semigroup theory.** During this section we claim that the assumptions (A1), (A2) and (A4) are satisfied. In the following $(A_p,\mathcal{D}^p_{max})$ denotes the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $(T_0(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, that is given by (5.1). We already know that $(T_0(t))_{t\geqslant 0}$ is a strongly continuous semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$ satisfying

$$\|T_0(t)\|_{\mathcal{L}(L^p,L^p)}\leqslant M_0 e^{\omega_0 t}\quad\forall\, t\geqslant 0,$$

where $M_0 := \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}\geqslant 1$ and $\omega_0:=0\in\mathbb{R}$, compare (5.9). Moreover, under the additional assumptions (A3), we know that A_p is the maximal realization of the complex Ornstein-Uhlenbeck operator $\mathcal{L}_0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$ and we have a complete characterization of its maximal domain. In this section we investigate constant coefficient perturbations of A_p in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Therefore, let $E_p\in\mathcal{L}(L^p,L^p)$ be a perturbation defined by

$$E_p : L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad [E_p v](x) := -Bv(x)$$

where $B\in\mathbb{C}^{N,N}$. If we define $[\mathcal{L}_{bdd}\phi](x) := -B\phi(x)$ then it is obvious, that E_p is the maximal realization of $\mathcal{L}_{bdd}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Applying [28, III.1.3 Bounded Perturbation Theorem, III.1.7 Corollary and III.1.10 Theorem] we obtain the following theorem.

Theorem 6.1 (Bounded Perturbation Theorem)**.** *Let the assumptions* (A1)*,* (A2)*,* (A4) *and $B \in \mathbb{K}^{N,N}$ be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} = \mathbb{C}$. Then the operator*

$$B_p := A_p + E_p \quad \text{with} \quad \mathcal{D}(B_p) := \mathcal{D}^p_{max}$$

generates a strongly continuous semigroup $\left(T_\infty(t)\right)_{t\geqslant 0}$ on $L^p(\mathbb{R}^d, \mathbb{C}^N)$ satisfying

$$\|T_\infty(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_0 e^{\left(\omega_0 + M_0\|E_p\|_{\mathcal{L}(L^p,L^p)}\right)t} \quad \forall\, t \geqslant 0. \tag{6.1}$$

Moreover, for every $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ and $t \geqslant 0$ the semigroup $\left(T_\infty(t)\right)_{t\geqslant 0}$ satisfies the integral equation (variation of parameters formula)

$$T_\infty(t)v = T_0(t)v + \int_0^t T_0(t-s)E_pT_\infty(s)vds,$$

is unique and can be obtained by

$$T_\infty(t) = \sum_{n=0}^{\infty} T_\infty^{(n)}(t) \tag{6.2}$$

where

$$T_\infty^{(0)}(t) := T_0(t), \quad T_\infty^{(n+1)}(t) := \int_0^t T_0(t-s)E_pT_\infty^{(n)}(s)ds. \tag{6.3}$$

Theorem 6.1 states that $(B_p, \mathcal{D}^p_{max})$ is the infinitesimal generator of $\left(T_\infty(t)\right)_{t\geqslant 0}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, $1 \leqslant p < \infty$. Assuming in addition (A3), $(B_p, \mathcal{D}^p_{max})$ is the maximal realization of $\mathcal{L}_\infty$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$ and its maximal domain $\mathcal{D}^p_{max}$ is given by $\mathcal{D}^p$ from Theorem 5.25. Under the assumption that the matrices $A, B \in \mathbb{C}^{N,N}$ are simultaneously diagonalizable (over $\mathbb{C}$), see (A7$_B$), we are able to derive an explicit representation for the semigroup $\left(T_\infty(t)\right)_{t\geqslant 0}$. We require this additonal assumption for the rest of this section.

Theorem 6.2 (Semigroup representation)**.** *Let the assumptions* (A1)*,* (A2)*,* (A4) *and* (A7$_B$) *be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} = \mathbb{C}$. Then the semigroup $\left(T_\infty(t)\right)_{t\geqslant 0}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ is given by*

$$\left[T_\infty(t)v\right](x) := \begin{cases} \int_{\mathbb{R}^d} H_\infty(x,\xi,t)v(\xi)d\xi & ,\ t > 0 \\ v(x) & ,\ t = 0 \end{cases} \quad , x \in \mathbb{R}^d, \tag{6.4}$$

where $H_\infty(x,\xi,t) = H(x,\xi,t)$ is the heat kernel from Theorem 4.4.

Remark. As we have already mentioned before (see remarks after Theorem 4.4), the situation changes dramatically if the assumption (A7$_B$) is not satisfied. In this case one could use for example the Hadamard Lemma, since the kernel H_0 and B in general do not commute.

Proof. From Theorem 6.1 we know that the semigroup $\left(T_\infty(t)\right)_{t\geqslant 0}$ is given by (6.2) where (6.3) is satisfied. By induction over $n \in \mathbb{N}_0$ we show that

$$T_\infty^{(n)}(t) = T_0(t)\frac{(-tB)^n}{n!},\ n \in \mathbb{N}_0,\ t \geqslant 0. \tag{6.5}$$

The case $n = 0$ is satisfied by (6.3). Let us consider the case $n \to n+1$:

$$\begin{aligned} T_\infty^{(n+1)}(t) &= \int_0^t T_0(t-s)E_pT_\infty^{(n)}(s)ds \\ &= -\int_0^t T_0(t-s)BT_0(s)\frac{(-sB)^n}{n!}ds \\ &= \int_0^t T_0(t-s)T_0(s)s^n ds\frac{(-B)^{n+1}}{n!} \end{aligned}$$

$$=T_0(t)\int_0^t s^n ds\frac{(-B)^{n+1}}{n!}$$
$$=T_0(t)\frac{(-tB)^{n+1}}{(n+1)!}.$$

This proves (6.5). Thus, using (6.2) it yields

$$T_\infty(t)=T_0(t)\sum_{n=0}^{\infty}\frac{(-tB)^n}{n!}=T_0(t)e^{-Bt}.$$

□

Remark. The representation from Theorem 6.2 is true if and only if the operators $T_0(t)$ and E_p commute. Since for both operators we have explicit expressions, we can interchange them, whenever the matrix B commutes with the Ornstein-Uhlenbeck kernel H_0. If A and B are simultaneously diagonalizable, it is obvious that $BH_0 = H_0B$ hold. In order to investigate the nonlinear problem for the Ornstein-Uhlenbeck operator, it is obligatory to have a representation for the semigroup of the bounded perturbation, since the estimate (6.1) has no good behavior as $t\to\infty$.

Theorem 6.3 (Boundedness on $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A4) *and* ($A7_B$) *be satisfied for $1\leqslant p\leqslant\infty$ and $\mathbb{K}=\mathbb{C}$. Then for every radial weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ and for every $v\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$*

(6.6) $$\left\|T_\infty(t)v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant C_4(t)\left\|v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad,\ t\geqslant 0,$$

(6.7) $$\left\|D_iT_\infty(t)v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant C_5(t)\left\|v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad,\ t>0,\ i=1,\ldots,d,$$

(6.8) $$\left\|D_jD_iT_\infty(t)v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant C_6(t)\left\|v\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\quad,\ t>0,\ i,j=1,\ldots,d,$$

where the constants $C_{4+|\beta|}(t)$ are from Section 4.3 for every $|\beta|=0,1,2$, i.e.

$$C_4(t)=C_\theta M^{\frac{d}{2}}e^{-b_0t}\left[{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)+2\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac12}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac1p},$$

$$C_5(t)=C_\theta M^{\frac{d+1}{2}}e^{-b_0t}\left(ta_{\min}\right)^{-\frac12}\left[\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+1}{2};\frac{1}{2};\kappa t\right)\right.$$
$$\left.+2\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac12}\,{}_1F_1\left(\frac{d+2}{2};\frac{3}{2};\kappa t\right)\right]^{\frac1p},$$

$$C_6(t)=C_\theta M^{\frac{d+2}{2}}e^{-b_0t}\left(ta_{\min}\right)^{-1}\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}{}_1F_1\left(\frac{d+2}{2};\frac{1}{2};\kappa t\right)\right.$$
$$+2\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac12}\,{}_1F_1\left(\frac{d+3}{2};\frac{3}{2};\kappa t\right)+\frac{\delta_{ij}}{2}M^{-1}{}_1F_1\left(\frac{d}{2};\frac{1}{2};\kappa t\right)$$
$$\left.+\delta_{ij}M^{-1}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(\kappa t)^{\frac12}\,{}_1F_1\left(\frac{d+1}{2};\frac{3}{2};\kappa t\right)\right]^{\frac1p},$$

and in case of $p=\infty$ they are given by $C_{4+|\beta|}(t)$ with $p=1$, where $M:=\frac{a_{\max}^2}{a_{\min}a_0}\geqslant 1$ and $\kappa:=\frac{a_{\max}^2\eta^2p^2}{a_0}\geqslant 0$. Note that $C_{4+|\beta|}(t)\sim t^{\frac{-p|\beta|+d+|\beta|-1}{2p}}e^{-(b_0-\frac{\kappa}{p})t}$ as $t\to\infty$ and $C_{4+|\beta|}(t)\sim t^{-\frac{|\beta|}{2}}$ as $t\to 0$ for every $|\beta|=0,1,2$.

Proof. Using the semigroup representation from Theorem 6.2, the proof can be adopted from Theorem 5.1, where we have to replace T_0 and H_0 by T_∞ and $H_\infty = H$, respectively. □

By (6.6) from Theorem 6.3 (with $\theta\equiv 1$, $\eta=0$ and $C_\theta=1$) we have

$$\exists\,\omega_\infty\in\mathbb{R}\;\wedge\;\exists\,M_\infty\geqslant 1:\;\|T_\infty(t)\|_{\mathcal{L}(L^p,L^p)}\leqslant M_\infty e^{\omega_\infty t}\quad\forall\,t\geqslant 0, \tag{6.9}$$

where $M_\infty := M_0 = \left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}\geqslant 1$ and $\omega_\infty := -b_0$, which gives a better estimate as in (6.1). For the next statement we refer to [28, II.1.3 Lemma, II.1.4 Theorem]:

Lemma 6.4. *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$.*
(1) *$B_p:\mathcal{D}^p_{max}\subseteq L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N)$ is a linear, closed and densely defined operator.*
(2) *For every $v\in\mathcal{D}^p_{max}$ and $t\geqslant 0$ we have*

$$T_\infty(t)v\in\mathcal{D}^p_{max}$$
$$\frac{d}{dt}T_\infty(t)v = T_\infty(t)B_pv = B_pT_\infty(t)v$$

(3) *For every $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and every $t\geqslant 0$ we have*

$$\int_0^t T_\infty(s)vds\in\mathcal{D}^p_{max}$$

(4) *For every $t\geqslant 0$ we have*

$$\begin{aligned}T_\infty(t)v-v =&B_p\int_0^t T_\infty(s)vds && \text{, for } v\in L^p(\mathbb{R}^d,\mathbb{C}^N),\\ =&\int_0^t T_\infty(s)B_pvds && \text{, for } v\in\mathcal{D}^p_{max}.\end{aligned}$$

Since $(B_p,\mathcal{D}^p_{max})$ is a closed operator on the Banach space $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, we can define

$$\begin{aligned}\sigma(B_p) :=&\{\lambda\in\mathbb{C}\mid\lambda I-B_p\text{ is not bijective}\} && \text{spectrum of } B_p,\\ \rho(B_p) :=&\mathbb{C}\backslash\sigma(B_p) && \text{resolvent set of } B_p,\\ R(\lambda,B_p) :=&(\lambda I-B_p)^{-1}\text{, for }\lambda\in\rho(B_p) && \text{resolvent of } B_p.\end{aligned}$$

The next identities follow from [28, II.1.9 Lemma].

Lemma 6.5. *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Then for every $\lambda\in\mathbb{C}$ and $t>0$,*

$$\begin{aligned}e^{-\lambda t}T_\infty(t)v-v =&(B_p-\lambda I)\int_0^t e^{-\lambda s}T_\infty(s)vds && \text{, for } v\in L^p(\mathbb{R}^d,\mathbb{C}^N),\\ =&\int_0^t e^{-\lambda s}T_\infty(s)\left(B_p-\lambda I\right)vds && \text{, for } v\in\mathcal{D}^p_{max}.\end{aligned}$$

Theorem 6.6. *Let the assumptions (A1), (A2), (A4), (A7$_B$) and $1\leqslant p<\infty$ be satisfied for $\mathbb{K}=\mathbb{C}$.*
(1) *For every $\lambda\in\mathbb{C}$ such that $R(\lambda)v:=\int_0^\infty e^{-\lambda s}T_\infty(s)vds$ exists for every $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ we have*

$$\lambda\in\rho(B_p)\quad\text{and}\quad R(\lambda,B_p)=R(\lambda).$$

(2) *For every $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>\omega_\infty$ we have*

$$\lambda\in\rho(B_p),\quad R(\lambda,B_p)=R(\lambda)$$

and

$$\|R(\lambda,B_p)\|_{\mathcal{L}(L^p,L^p)}\leqslant\frac{M_\infty}{\mathrm{Re}\,\lambda-\omega_\infty}.$$

A direct consequence of Theorem 6.6 is the following:

Corollary 6.7 (Solvability and uniqueness in $L^p(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_\infty$. Then for every $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ the resolvent equation*

$$(\lambda I-B_p)\,v=g$$

admits a unique solution $v_\star\in\mathcal{D}^p_{max}$, which is given by the integral expression

$$\begin{aligned} v_\star=R(\lambda)g&=\int_0^\infty e^{-\lambda s}T_\infty(s)gds\\ &=\int_0^\infty e^{-\lambda s}\int_{\mathbb{R}^d}H_\infty(\cdot,\xi,s)g(\xi)d\xi ds. \end{aligned} \tag{6.10}$$

Moreover, it holds the resolvent estimate

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{M_\infty}{\operatorname{Re}\lambda-\omega_\infty}\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

Remark. Let the assumptions (A1), (A2), (A4), (A7$_B$) and (A8$_B$) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. By Corollary 6.7 (with $\lambda=0$) we obtain for every $g\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ a unique solution $v_\star\in\mathcal{D}^p_{max}$ of the resolvent equation

$$B_pv=g$$

which is given by (6.10). We believe one can apply Fubini's theorem in equation (6.10) to obtain

$$v_\star(x)=-\left[R(0)g\right](x)=\int_{\mathbb{R}^d}G(x,\xi)g(\xi)d\xi,$$

where

$$G(x,\xi):=-\int_0^\infty H_\infty(x,\xi,s)ds$$

denotes the Green's function of B_p. In particular, by Corollary 6.7 the following resolvent estimate holds:

$$\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{M_\infty}{-\omega_\infty}\|g\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}.$$

For the next statement we refer to [28, II.1.11 Corollary].

Corollary 6.8. *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega_\infty$. Then, for every $n\in\mathbb{N}$ and every $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ it hold*

$$\begin{aligned} R(\lambda,B_p)^nv=&\frac{(-1)^n}{(n-1)!}\cdot\frac{d^{n-1}}{d\lambda^{n-1}}R(\lambda,B_p)v\\ =&\frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-\lambda s}T_\infty(s)vds. \end{aligned}$$

and the estimate

$$\|R(\lambda,B_p)^n\|_{\mathcal{L}(L^p,L^p)}\leqslant\frac{M_\infty}{(\operatorname{Re}\lambda-\omega_\infty)^n}.$$

Let us now define the spectral bound $s(B_p)$ of B_p, [28, II.1.12 Definition]:

$$-\infty\leqslant s(B_p):=\sup_{\lambda\in\sigma(B_p)}\operatorname{Re}\lambda\leqslant\omega_\infty=-b_0<+\infty.$$

6.2. **Exponential decay.**

Theorem 6.9 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2), (A4) and (A7$_B$) be satisfied for $1\leqslant p<\infty$ and $\mathbb{K}=\mathbb{C}$. Moreover, let $0<\vartheta<1$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>\omega_\infty$. Then for every radially nondecreasing weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ with $0\leqslant\eta^2\leqslant\vartheta\frac{a_0(\mathrm{Re}\,\lambda-\omega_\infty)}{a_{\max}^2 p^2}$ and for every $g\in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ we have $v_\star\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ with*

$$\left\|v_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_7}{\mathrm{Re}\,\lambda-\omega_\infty}\left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}, \tag{6.11}$$

$$\left\|D_iv_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}\leqslant\frac{C_8}{(\mathrm{Re}\,\lambda-\omega_\infty)^{\frac{1}{2}}}\left\|g\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)},\ i=1,\ldots,d, \tag{6.12}$$

where $v_\star\in\mathcal{D}^p_{max}$ denotes the unique solution of $(\lambda I-B_p)v=g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and the λ-independent constants C_7, C_8 are given by Lemma 4.8 (with $\omega=\omega_\infty$).

Proof. By Corollary 6.7 we have the representation

$$v_\star(x)=\int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d}H_\infty(x,\xi,t)g(\xi)d\xi dt, \tag{6.13}$$

where $H_\infty(x,\xi,t)=H(x,\xi,t)$. Using this representation, the proof can be adopted from Theorem 5.14. In the last inequality, we must apply Lemma 4.8 with $\omega=\omega_\infty$. ☐

Remark. Theorem 6.9 states for $\theta\equiv 1$ (with $\eta=0$ and $C_\theta=1$) that

$$\mathcal{D}^p_{max}\subseteq W^{1,p}(\mathbb{R}^d,\mathbb{C}^N),\ \text{for every } 1\leqslant p<\infty.$$

Remark. In the proof of Theorem 6.9 it is in general not possible to specify also an estimate for $\left\|D_jD_iv_\star\right\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}$ since $C_{4+|\beta|}(t)\sim t^{-\frac{|\beta|}{2}}$ as $t\to 0$, see Theorem 6.3, and consequently we have the singularity t^{-1} at $t=0$ for $|\beta|=2$.

7. Variable coefficient perturbations of the complex Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d,\mathbb{C}^N)$

Consider the operator

$$\left[\mathcal{L}_Q\phi\right](x):=\left[\mathcal{L}_\infty\phi\right](x)+Q(x)\phi(x),\ \phi\in\mathcal{S}$$

7.1. **Application of semigroup theory.** During this section we claim that the assumptions (A1), (A2), (A4) and (A7$_B$) are satisfied. In the following $(B_p,\mathcal{D}^p_{max})$ denotes the infinitesimal generator of the semigroup $(T_\infty(t))_{t\geqslant 0}$ on $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for $1\leqslant p<\infty$, that is given by (5.1). We already know that $(T_\infty(t))_{t\geqslant 0}$ is a strongly continuous semigroup in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1\leqslant p<\infty$ satisfying

$$\left\|T_\infty(t)\right\|_{\mathcal{L}(L^p,L^p)}\leqslant M_\infty e^{\omega_\infty t}\quad\forall\,t\geqslant 0,$$

where $M_\infty:=\left(\frac{a_{\max}^2}{a_{\min}a_0}\right)^{\frac{d}{2}}\geqslant 1$ and $\omega_\infty:=-b_0\in\mathbb{R}$, compare (6.9). Moreover, under the additional assumptions (A3), we know that B_p is the maximal realization of the perturbed complex Ornstein-Uhlenbeck operator $\mathcal{L}_\infty$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ for every $1<p<\infty$ and we have a complete characterization of its maximal domain. In this section we investigate variable coefficient perturbations of B_p in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. for this purpose, let $F_p\in\mathcal{L}(L^p,L^p)$ be a perturbation defined by

$$F_p:L^p(\mathbb{R}^d,\mathbb{C}^N)\to L^p(\mathbb{R}^d,\mathbb{C}^N),\quad\left[F_pv\right](x):=Q(x)v(x)$$

where $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. If we define $\left[\mathcal{L}_{var}\phi\right](x):=Q(x)\phi(x)$ then it is obvious, that F_p is the maximal realization of $\mathcal{L}_{var}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Applying [28, III.1.3

Bounded Perturbation Theorem, III.1.7 Corollary and III.1.10 Theorem] we obtain the following result.

Theorem 7.1 (Bounded Perturbation Theorem)**.** *Let the assumptions* (A1)*,* (A2)*,* (A4)*,* ($A7_B$) *and* $Q \in L^{\infty}(\mathbb{R}^d, \mathbb{K}^{N,N})$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*. Then the operator*

$$C_p := B_p + F_p \quad \text{with} \quad \mathcal{D}(C_p) := \mathcal{D}^p_{max}$$

generates a strongly continuous semigroup $(T_Q(t))_{t\geqslant 0}$ *on* $L^p(\mathbb{R}^d, \mathbb{C}^N)$ *satisfying*

$$\|T_Q(t)\|_{\mathcal{L}(L^p,L^p)} \leqslant M_{\infty} e^{\left(\omega_{\infty}+M_{\infty}\|F_p\|_{\mathcal{L}(L^p,L^p)}\right)t} \quad \forall\, t \geqslant 0.$$

Moreover, for every $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *and* $t \geqslant 0$ *the semigroup satisfies the integral equation (variation of parameters formula)*

$$T_Q(t)v = T_{\infty}(t)v + \int_0^t T_{\infty}(t-s)F_p T_Q(s)v ds,$$

is unique and can be obtained by

$$T_Q(t) = \sum_{n=0}^{\infty} T_Q^{(n)}(t)$$

where

$$T_Q^{(0)}(t) := T_{\infty}(t), \quad T_Q^{(n+1)}(t) := \int_0^t T_{\infty}(t-s)F_p T_Q^{(n)}(s)ds.$$

Theorem 7.1 states that $(C_p, \mathcal{D}^p_{max})$ is the infinitesimal generator of $(T_Q(t))_{t\geqslant 0}$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, $1 \leqslant p < \infty$. Assuming in addition (A3), $(C_p, \mathcal{D}^p_{max})$ is the maximal realization of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$ and its maximal domain $\mathcal{D}^p_{max}$ is given by $\mathcal{D}^p$ from Theorem 5.25. Contrary to the case of constant coefficients, we cannot assume here, that $Q(x)$ commutes with both A and B for every $x \in \mathbb{R}^d$, since this is in general not satisfied in order to investigate the nonlinear problem of the Ornstein-Uhlenbeck operator. Thus, we are not able to derive a closed form for the representation of the semigroup $(T_Q(t))_{t\geqslant 0}$. In particular, it is not possible in this case to optimize the boundedness property of $\|T_Q(t)\|_{\mathcal{L}(L^p,L^p)}$ from Theorem 7.1. For the next statement we refer to [28, II.1.3 Lemma, II.1.4 Theorem]:

Lemma 7.2. *Let the assumptions* (A1)*,* (A2)*,* (A4)*,* ($A7_B$) *and* $Q \in L^{\infty}(\mathbb{R}^d, \mathbb{K}^{N,N})$ *be satisfied for* $1 \leqslant p < \infty$ *and* $\mathbb{K} = \mathbb{C}$*.*

(**1**) $C_p : \mathcal{D}^p_{max} \subseteq L^p(\mathbb{R}^d, \mathbb{C}^N) \to L^p(\mathbb{R}^d, \mathbb{C}^N)$ *is a linear, closed and densely defined operator.*

(**2**) *For every* $v \in \mathcal{D}^p_{max}$ *and* $t \geqslant 0$ *we have*

$$T_Q(t)v \in \mathcal{D}^p_{max}$$
$$\frac{d}{dt}T_Q(t)v = T_Q(t)C_p v = C_p T_Q(t)v$$

(**3**) *For every* $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ *and every* $t \geqslant 0$ *we have*

$$\int_0^t T_Q(s)v ds \in \mathcal{D}^p_{max}$$

(**4**) *For every* $t \geqslant 0$ *we have*

$$\begin{aligned} T_Q(t)v - v &= C_p \int_0^t T_Q(s)v ds && , \text{ for } v \in L^p(\mathbb{R}^d, \mathbb{C}^N), \\ &= \int_0^t T_Q(s)C_p v ds && , \text{ for } v \in \mathcal{D}^p_{max}. \end{aligned}$$

Since $(C_p, \mathcal{D}^p_{max})$ is a closed operator on the Banach space $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for $1 \leqslant p < \infty$, we can define

$$\begin{aligned}
\sigma(C_p) &:= \{\lambda \in \mathbb{C} \mid \lambda I - C_p \text{ is not bijective}\} && \text{spectrum of } C_p,\\
\rho(C_p) &:= \mathbb{C}\backslash\sigma(C_p) && \text{resolvent set of } C_p,\\
R(\lambda, C_p) &:= (\lambda I - C_p)^{-1}, \text{ for } \lambda \in \rho(C_p) && \text{resolvent of } C_p.
\end{aligned}$$

The next identities follow from [28, II.1.9 Lemma].

Lemma 7.3. *Let the assumptions (A1), (A2), (A4), (A7$_B$), $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$ and $1 \leqslant p < \infty$ be satisfied for $\mathbb{K} = \mathbb{C}$. Then for every $\lambda \in \mathbb{C}$ and $t > 0$ it hold*

$$\begin{aligned}
e^{-\lambda t} T_Q(t)v - v &= (C_p - \lambda I) \int_0^t e^{-\lambda s} T_Q(s) v ds && , \text{for } v \in L^p(\mathbb{R}^d, \mathbb{C}^N),\\
&= \int_0^t e^{-\lambda s} T_Q(s) (C_p - \lambda I) v ds && , \text{for } v \in \mathcal{D}^p_{max}.
\end{aligned}$$

The following statement comes from [28, II.1.10 Theorem].

Theorem 7.4. *Let the assumptions (A1), (A2), (A4), (A7$_B$) and $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} = \mathbb{C}$.*
(1) *For every $\lambda \in \mathbb{C}$ such that $R(\lambda)v := \int_0^\infty e^{-\lambda s} T_Q(s) v ds$ exists and for every $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ it holds*

$$\lambda \in \rho(C_p) \quad \text{and} \quad R(\lambda, C_p) = R(\lambda).$$

(2) *For every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_\infty + M_\infty \|Q\|_{L^\infty}$ it holds*

$$\lambda \in \rho(C_p), \quad R(\lambda, C_p) = R(\lambda)$$

and

$$\|R(\lambda, C_p)\|_{\mathcal{L}(L^p, L^p)} \leqslant \frac{M_\infty}{\operatorname{Re}\lambda - \left(\omega_\infty + M_\infty \|Q\|_{C_b}\right)}.$$

Corollary 7.5 (Solvability and uniqueness in $L^p(\mathbb{R}^d, \mathbb{C}^N)$)**.** *Let the assumptions (A1), (A2), (A4), (A7$_B$) and $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} = \mathbb{C}$. Moreover, let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_\infty + M_\infty \|Q\|_{L^\infty}$. Then for every $g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ the resolvent equation*

$$(\lambda I - C_p)\, v = g$$

admits a unique solution $v_\star \in \mathcal{D}^p_{max}$ which satisfies the integral expression

$$\begin{aligned}
v_\star = R(\lambda) g &= \int_0^\infty e^{-\lambda s} T_Q(s) g ds\\
&= \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda s} H_\infty(\cdot, \xi, s) \left(g(\xi) + Q(\xi) v_\star(\xi)\right) d\xi ds.
\end{aligned}$$

Moreover, it holds the resolvent estimate

$$\|v_\star\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)} \leqslant \frac{M_\infty}{\operatorname{Re}\lambda - (\omega_\infty + M_\infty \|Q\|_{L^\infty})} \|g\|_{L^p(\mathbb{R}^d, \mathbb{C}^N)}.$$

The following statement comes from [28, II.1.11 Corollary].

Corollary 7.6. *Let the assumptions (A1), (A2), (A4), (A7$_B$) and $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ be satisfied for $1 \leqslant p < \infty$ and $\mathbb{K} = \mathbb{C}$. Moreover, let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_\infty + M_\infty \|Q\|_{L^\infty}$. Then, for every $n \in \mathbb{N}$ and every $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ it hold*

$$R(\lambda, C_p)^n v = \frac{(-1)^n}{(n-1)!} \cdot \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, C_p) v$$

$$=\frac{1}{(n-1)!}\int_0^\infty s^{n-1}e^{-\lambda s}T_Q(s)vds.$$

and the estimate

$$\left\|R(\lambda,C_p)^n\right\|_{\mathcal{L}(L^p,L^p)}\leqslant\frac{M_\infty}{\left(\operatorname{Re}\lambda-(\omega_\infty+M_\infty\left\|Q\right\|_{L^\infty})\right)^n}.$$

Let us now define the spectral bound $s(C_p)$ of C_p, [28, II.1.12 Definition]:

$$-\infty\leqslant s(C_p)=\sup_{\lambda\in\sigma(C_p)}\operatorname{Re}\lambda\leqslant\omega_\infty+M_\infty\left\|Q\right\|_{L^\infty}<+\infty$$

7.2. **The essential spectrum.** In this section we extend the approach from [17, Theorem 8.1] to compute the essential spectrum of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and $1<p<\infty$. The paper [17] analyzes the case $p=d=2$. For the spectrum of the Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d,\mathbb{R})$, $1<p<\infty$, without perturbation terms, we refer to [42]. First, let us introduce the definition of the essential spectrum, [32]:

Definition 7.7. *Let X be a (complex-valued) Banach space and let $A:\mathcal{D}(A)\subset X\to X$ be a closed, densely defined, linear operator. Moreover, let $\lambda\in\mathbb{C}$.*

(**1**) *$\lambda\in\rho(A)$ if and only if the following properties hold*

- *$(\lambda I-A):\mathcal{D}(A)\to X$ is $1-1$ (injective) and onto (surjective),*
- *$(\lambda I-A)^{-1}$ is bounded on X.*

(**2**) *$\lambda_0\in\rho(A)$ is called isolated if and only if*

$$\exists\,\varepsilon>0\,\forall\,\lambda\in\mathbb{C}\text{ with }0<|\lambda-\lambda_0|<\varepsilon:\lambda\in\rho(A).$$

(**3**) *The multiplicity of $\lambda_0\in\sigma(A)$ is defined as the dimension of the algebraic eigenspace*

$$\left\{v\in X\mid(\lambda_0 I-A)^k=0\text{ for some }k\in\mathbb{N}\right\}.$$

(**4**) *$\lambda\in\mathbb{C}$ is called a normal point of A if and only if one of the following properties hold*

- *$\lambda\in\rho(A)$,*
- *$\lambda\in\sigma_{\text{point}}(A):=\{\lambda\in\sigma(A)\mid\lambda\text{ is isolated with finite multiplicity}\}$.*

(**5**) *The set*

$$\sigma_{\text{ess}}(A)=\{\lambda\in\mathbb{C}\mid\lambda\text{ is not a normal point of }A\}$$

is called the essential spectrum of A.

By definition it holds

$$\mathbb{C}=\rho(A)\,\dot\cup\,\sigma(A)=\rho(A)\,\dot\cup\left(\sigma_{\text{ess}}(A)\,\dot\cup\,\sigma_{\text{point}}(A)\right).$$

We first give a short motivation, how we can determine the essential spectrum of $\mathcal{L}_Q$, see [17, Section 8.2] for the casee $d=p=2$ and see [42, Theorem 2.6] for the essential spectrum of the drift term in $L^p(\mathbb{R}^d,\mathbb{R})$, $1<p<\infty$:

1. (Orthogonal transformation). Let us consider the operator

$$\left[\mathcal{L}_Q v\right](x)=A\triangle v(x)+\langle Sx,\nabla v(x)\rangle-Bv(x)+Q(x)v(x),\,x\in\mathbb{R}^d.$$

Note that for space dimensions $d\geqslant3$ the axis of rotation is in general not orthogonal to some plane (x_l,x_k), $1\leqslant l,k\leqslant d$. Furthermore, in space dimensions $d\geqslant4$ the pattern can also rotate rigidly around several axes of rotation simultaneously. In order to investigate the essential spectrum of $\mathcal{L}_Q$ the main idea of the first step is to seperate the axes of rotation such that they are orthogonal to (completely) different planes. For this purpose we perform an orthogonal transformation: Since

$S \in \mathbb{R}^{d,d}$ with $S^T = -S$ we have $\sigma(S) \subset i\mathbb{R}$. Let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S, $1 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$, then

$$\exists\, P \in \mathbb{R}^{d,d} \text{ orthogonal matrix}: \; S = P\Lambda^S_{\text{block}}P^T,$$

where

$$\Lambda^S_{\text{block}} = \begin{pmatrix} \Lambda^S_1 & & & & & 0 \\ & \ddots & & & & \\ & & \Lambda^S_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix} \in \mathbb{R}^{d,d}, \quad \Lambda^S_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \in \mathbb{R}^{2,2},$$

for every $j = 1, \ldots, k$. The orthogonal transformation of coordinates

$$T_1(x) = Px, \quad T_1^{-1}(x) = P^T x$$

yields the transformed operator $\mathcal{L}_{Q,T_1}v = \mathcal{L}_Q\left(v \circ T_1^{-1}\right) \circ T_1$ given by

$$\left[\mathcal{L}_{Q,T_1}v\right](x) = A\triangle v(x) + \left\langle \Lambda^S_{\text{block}}x, \nabla v(x)\right\rangle - Bv(x) + Q(Px)v(x),\; x \in \mathbb{R}^d,$$

with

$$\left\langle \Lambda^S_{\text{block}}x, \nabla v(x)\right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l}D_{2l-1} - x_{2l-1}D_{2l}\right) v(x).$$

2. (Transformation into several planar polar coordinates). Since now we have k angular derivatives in k different planes it is helpful to use the following transformations into planar polar coordinates

$$\begin{pmatrix} x_{2l-1} \\ x_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l \cos\phi_l \\ r_l \sin\phi_l \end{pmatrix},\; l = 1, \ldots, k,\; \phi_l \in]-\pi, \pi],\; r_l > 0.$$

All further coordinates, i.e. $x_{2k+1}, \ldots, x_d$, remain fixed. Denoting the total transformation by $T_2(r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d)$ we obtain the transformed operator $\mathcal{L}_{Q,T_2}v = \mathcal{L}_{Q,T_1}\left(v \circ T_2^{-1}\right) \circ T_2$ given by

$$\begin{aligned} \left[\mathcal{L}_{Q,T_2}v\right](x) =& A\left[\sum_{l=1}^{k}\left(\partial^2_{r_l} + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial^2_{\phi_l}\right) + \sum_{l=2k+1}^{d}\partial^2_{x_l}\right] v(\xi) \\ & - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) - Bv(\xi) + Q(\xi)v(\xi), \end{aligned}$$

where $\xi := (r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d)$ and $Q(T_1(T_2(\xi)))$ is abbreviated by $Q(\xi)$.

3. (Simplified operator and finite-dimensional eigenvalue problem). Neglecting the terms of order $\mathcal{O}\left(\frac{1}{r}\right)$ we obtain the simplified operator

$$\left[\mathcal{L}^{\text{sim}}_{Q,T_2}v\right](x) = A\left[\sum_{l=1}^{k}\partial^2_{r_l} + \sum_{l=2k+1}^{d}\partial^2_{x_l}\right] v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) - Bv(\xi).$$

If we choose

$$v(\xi) = \exp\left(i\kappa\sum_{l=1}^{k} r_l\right)\exp\left(i\sum_{l=1}^{k} n_l\phi_l\right)\hat{v},\, n_l \in \mathbb{Z},\, \kappa \in \mathbb{R},\, \hat{v} \in \mathbb{C}^N,\, |\hat{v}| = 1$$
$$\phi_l \in]-\pi, \pi],\, r_l > 0,\, l = 1, \ldots, k,$$

then

$$\left[\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right) v\right](\xi) = \left(\lambda I_N + \kappa^2 A + i\sum_{l=1}^{k} n_l\sigma_l I_N + B\right) v(\xi).$$

Therefore, $\left[\left(\lambda I - \mathcal{L}_{Q,T_2}^{\mathrm{sim}}\right) v\right](\xi) = 0$ for every ξ if and only if $\lambda \in \mathbb{C}$ satisfies

$$\left(\kappa^2 A + B\right)\hat{v} = -\left(\lambda + i\sum_{l=1}^{k} n_l\sigma_l\right)\hat{v}.$$

Theorem 7.8. *Let the assumptions (A1), (A2), (A3), (A4), ($A7_B$) and $Q \in L^\infty(\mathbb{R}^d, \mathbb{K}^{N,N})$ with*

$$\eta_R := \operatorname*{ess\,sup}_{|x|\geqslant R} |Q(x)|_2 \to 0 \text{ as } R \to \infty$$

be satisfied for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$. Moreover, let $\lambda_j(\kappa)$ denote the eigenvalues of $\kappa^2 A + B$ for $j = 1, \ldots, N$. Then every number $\lambda \in \mathbb{C}$ with

$$\lambda = -\lambda_j(\kappa) - i\sum_{l=1}^{k} n_l\sigma_l,\ n_l \in \mathbb{Z},\ \kappa \in \mathbb{R},\ j = 1, \ldots, N,$$

belongs to the essential spectrum of $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, i.e. $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L}_Q)$.

Remark. (**1**): Theorem 7.8 shows that $\sigma_{\mathrm{ess}}(\mathcal{L}_Q) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant b_0\}$, where $b_0 = -s(-B)$. If there exists σ_n, σ_m such that $\sigma_n\sigma_m^{-1} \notin \mathbb{Q}$ then $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ is dense in $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant b_0\}$, i.e. $\sigma_{\mathrm{ess}}(\mathcal{L}_Q) = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant b_0\}$. Otherwise $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ is a discrete subgroup of $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant b_0\}$ (independently of p). The reason for this conclusion is given by Metafune in [42, Theorem 2.6]: It is proved that the essential spectrum of the drift term is dense in $i\mathbb{R}$, i.e. $\sigma_{\mathrm{ess}}(\langle Sx, \nabla v(x)\rangle) = i\mathbb{R}$, if and only if there exists σ_m, σ_m such that $\sigma_n\sigma_m \notin \mathbb{Q}$. Otherwise, $\sigma_{\mathrm{ess}}(\langle Sx, \nabla v(x)\rangle)$ is a discrete subgroup of $i\mathbb{R}$ (independently of p).
(**2**): If we require in addition assumption ($A8_B$) then $\operatorname{Re}\lambda_j(\kappa) > 0$ and thus $\sigma_{\mathrm{ess}}(\mathcal{L}_Q) \subset \mathbb{C}_-$, where $\mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$. If in addition there exists σ_n, σ_m such that $\sigma_n\sigma_m^{-1} \notin \mathbb{Q}$ then $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ is dense in $\mathbb{C}_-$, i.e. $\sigma_{\mathrm{ess}}(\mathcal{L}_Q) = \mathbb{C}_-$. Otherwise $\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$ is a discrete subgroup of $\mathbb{C}_-$ (independently of p).
(**3**): Note that in Theorem 7.8 we can choose $Q = 0$ and $B = Q = 0$, which yields information about the essential spectrum of $\mathcal{L}_\infty$ and $\mathcal{L}_0$, respectively.

Remark. The condition

$$\det\left(\lambda I_N + \kappa^2 A + B + i\sum_{l=1}^{k} n_l\sigma_l I_N\right) = 0 \qquad (\Leftrightarrow \lambda \in \sigma_{\mathrm{ess}}(\mathcal{L}_Q))$$

is called the dispersion relation for $\mathcal{L}_Q$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, $1 < p < \infty$. This condition is necessary for the localization (and the existence) of the essential spectrum. Note, that the essential spectrum provides information about the stability of the solution.

Proof. 1. Let $R \geqslant 2$ be large and let $\chi_R : [0, \infty[\to [0, 1]$ be a cut-off function such that $\chi_R \in C^2([0, \infty[, [0, 1])$ with bounded derivatives independently of R and

$$\chi_R(r) = \begin{cases} 0 & , r \in [0, R-1], \\ \in [0, 1] & , r \in [R-1, R], \\ 1 & , r \in [R, 2R], \\ \in [0, 1] & , r \in [2R, 2R+1], \\ 0 & , r \in [2R+1, \infty[. \end{cases}$$

2. Define

$$v_R(\xi) := \left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)v(\xi)$$
$$= \left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)\exp\left(i\kappa\sum_{l=1}^{k}r_l\right)\exp\left(i\sum_{l=1}^{k}n_l\phi_l\right)\hat{v},$$

where $\tilde{x} := (x_{2k+1},\ldots,x_d)$, $\xi := (r_1,\phi_1,\ldots,r_k,\phi_k,\tilde{x})$, $n_l\in\mathbb{Z}$, $\kappa\in\mathbb{R}$, $\hat{v}\in\mathbb{C}^N$ with $|\hat{v}|=1$, $\phi_l\in]-\pi,\pi]$, $r_l>0$ and $l=1,\ldots,k$. By definition of χ_R we have

$$\left(\lambda I-\mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right)v_R(\xi)=0, \tag{7.1}$$

whenever $|\tilde{x}|\in[0,R-1]\cup[2R+1,\infty[$ or $r_l\in[0,R-1]\cup[2R+1,\infty[$ for some $1\leqslant l\leqslant k$. Moreover, by the choice of λ and by definition of χ_R we have

$$\left(\lambda I-\mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right)v_R(\xi)=0, \tag{7.2}$$

if $|\tilde{x}|, r_l\in[R,2R]$ for every $l=1,\ldots,k$.

3. By the choice of λ,

$$\partial_{r_l}^2\left(\chi_R(r_l)e^{i\kappa r_l}\right)=\chi_R''(r_l)e^{i\kappa r_l}+2i\kappa\chi_R'(r_l)e^{i\kappa r_l}+\chi_R(r_l)\partial_{r_l}^2e^{i\kappa r_l},\ l=1,\ldots,k,$$
$$\partial_{x_l}^2\left(\chi_R(|\tilde{x}|)\right)=\frac{|\tilde{x}|^2-x_l^2}{|\tilde{x}|^3}\chi_R'(|\tilde{x}|)+\frac{x_l^2}{|\tilde{x}|^2}\chi_R''(|\tilde{x}|),\ l=2k+1,\ldots,d,$$

the triangle inequality, $|\chi_R(r)|\leqslant 1$, $\chi_R'(r)\leqslant\|\chi_R\|_{C_b^2}$, $\chi_R''(r)\leqslant\|\chi_R\|_{C_b^2}$, $|v(\xi)|=1$ and $\frac{1}{|\tilde{x}|}\leqslant\frac{1}{R-1}\leqslant 1$, since $R\geqslant 2$, we have

$$\begin{aligned}
&\left|\left(\lambda I-\mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right)v_R(\xi)\right|\\
=&\left|\left(\lambda I-A\left[\sum_{l=1}^{k}\partial_{r_l}^2+\sum_{l=2k+1}^{d}\partial_{x_l}^2\right]+\sum_{l=1}^{k}\sigma_l\partial_{\phi_l}+B\right)\left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)v(\xi)\right|\\
=&\left|\left(\prod_{l=1}^{k}\chi_R(r_l)\right)\chi_R(|\tilde{x}|)\underbrace{\left(\lambda I-A\left[\sum_{l=1}^{k}\partial_{r_l}^2+\sum_{l=2k+1}^{d}\partial_{x_l}^2\right]+\sum_{l=1}^{k}n_l\partial_{\phi_l}+B\right)v(\xi)}_{=0\text{ (by the choice of }\lambda)}\right.\\
&-A\sum_{l=1}^{k}\left(\chi_R''(r_l)+2i\kappa\chi_R'(r_l)\right)\left(\prod_{\substack{j=1\\j\neq l}}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)\\
&\left.-A\sum_{l=2k+1}^{d}\left(\frac{|\tilde{x}|^2-x_l^2}{|\tilde{x}|^3}\chi_R'(|\tilde{x}|)+\frac{x_l^2}{|\tilde{x}|^2}\chi_R''(|\tilde{x}|)\right)\left(\prod_{j=1}^{k}\chi_R(r_j)\right)v(\xi)\right|\\
\leqslant&|A|_2\sum_{l=1}^{k}\left(|\chi_R''(r_l)|+2|\kappa|\,|\chi_R'(r_l)|\right)\left(\prod_{\substack{j=1\\j\neq l}}^{k}|\chi_R(r_j)|\right)|\chi_R(|\tilde{x}|)|\,|v(\xi)|\\
&+|A|_2\left(\frac{|d-2k-1|}{|\tilde{x}|}|\chi_R'(|\tilde{x}|)|+|\chi_R''(|\tilde{x}|)|\right)\left(\prod_{j=1}^{k}|\chi_R(r_j)|\right)|v(\xi)|\\
\leqslant&|A|_2\left(k(1+2|\kappa|)+|d-2k-1|+1\right)\|\chi_R\|_{C_b^2}=:C,
\end{aligned}$$

for every $|\tilde{x}|, r_l\in[R-1,R]\cup[R,2R]\cup[2R,2R+1]$ and $1\leqslant l\leqslant k$.

4. Furthermore, we have by the definition of χ_R, $|v(\xi)|^p=1$ and by the transformation theorem

$$\|v_R\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}=\int_{\mathbb{R}^d}|v_R(x)|^p\,dx$$

$$
\begin{aligned}
&= \int_0^\infty \int_{-\pi}^{\pi} \cdots \int_0^\infty \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-2k}} \left(\prod_{l=1}^k r_l\right) |v_R(\xi)|^p \, d\tilde{x} d\phi_k dr_k \cdots d\phi_1 dr_1 \\
&= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1} \left(\prod_{l=1}^k r_l\right) |v_R(\xi)|^p \, d\xi \\
&= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1} \left(\prod_{l=1}^k r_l\right) \left(\prod_{l=1}^k \chi_R^p(r_l)\right) \chi_R^p(|\tilde{x}|) d\xi \\
&= \int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1} \chi_R^p(|\tilde{x}|) d\tilde{x} \prod_{l=1}^k \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} r_l \chi_R^p(r_l) d\phi_l dr_l \\
&= \Bigg(\int_{R-1\leqslant|\tilde{x}|\leqslant R} \underbrace{\chi_R^p(|\tilde{x}|)}_{\geqslant 0} d\tilde{x} + \int_{R\leqslant|\tilde{x}|\leqslant 2R} \underbrace{\chi_R^p(|\tilde{x}|)}_{=1} d\tilde{x} + \int_{2R\leqslant|\tilde{x}|\leqslant 2R+1} \underbrace{\chi_R^p(|\tilde{x}|)}_{\geqslant 0} d\tilde{x} \Bigg) \\
&\quad \cdot \prod_{l=1}^k 2\pi \Bigg(\int_{R-1}^{R} r_l \underbrace{\chi_R^p(r_l)}_{\geqslant 0} dr_l + \int_R^{2R} r_l \underbrace{\chi_R^p(r_l)}_{=1} dr_l + \int_{2R}^{2R+1} r_l \underbrace{\chi_R^p(r_l)}_{\geqslant 0} dr_l \Bigg) \\
&\geqslant \int_{R\leqslant|\tilde{x}|\leqslant 2R} 1 d\tilde{x} \cdot \prod_{l=1}^k 2\pi \int_R^{2R} r_l dr_l = CR^{\tilde{d}} \prod_{l=1}^k 3\pi R^2 = (3\pi)^k CR^{2k+\tilde{d}} = CR^d,
\end{aligned}
$$

where $d\xi := d\tilde{x} d\phi_k dr_k \cdots d\phi_1 dr_1$ and $\tilde{d} := d - 2k$ denotes the dimension of the $\tilde{x}$-integral. Moreover, we used the following formula with $a = R$ and $b = 2R$

$$
\int_{a\leqslant|\tilde{x}|\leqslant b} 1 d\tilde{x} = \begin{cases} 1 & , \tilde{d} = 0, \\ b - a & , \tilde{d} = 1, \\ 2\pi \frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)} \frac{\left(b^{\tilde{d}} - a^{\tilde{d}}\right)}{\tilde{d}} & , \tilde{d} \geqslant 2. \end{cases} \tag{7.3}
$$

5. Furthermore, we have by (7.1)

$$
\begin{aligned}
&\left\| \left(\lambda I - \mathcal{L}_{Q,T_2}^{\text{sim}}\right) v_R \right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}^p = \int_{\mathbb{R}^d} \left| \left(\lambda I - \mathcal{L}_{Q,T_2}^{\text{sim}}\right) v_R(x) \right|^p dx \\
&= \int_0^\infty \int_{-\pi}^{\pi} \cdots \int_0^\infty \int_{-\pi}^{\pi} \int_{\mathbb{R}^{d-2k}} \left(\prod_{l=1}^k r_l\right) \left| \left(\lambda I - \mathcal{L}_{Q,T_2}^{\text{sim}}\right) v_R(\xi) \right|^p d\tilde{x} d\phi_k dr_k \cdots d\phi_1 dr_1 \\
&= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1} \left(\prod_{l=1}^k r_l\right) \left| \left(\lambda I - \mathcal{L}_{Q,T_2}^{\text{sim}}\right) v_R(\xi) \right|^p d\xi
\end{aligned}
$$

Defining $\tilde{d} := d - 2k$ we distinguish between the following cases:
Case 1: ($\tilde{d} = 0$). From step 3, (7.2), the multinomial theorem,

$$
\int_{R-1}^{R} r_l dr_l = \frac{1}{2}(2R-1), \quad \int_R^{2R} r_l dr_l = \frac{1}{2} 3R^2, \quad \int_{2R}^{2R+1} r_l dr_l = \frac{1}{2}(4R+1), \tag{7.4}
$$

$k = \frac{d}{2}$ and

$$
(2R-1)^{j_1} (3R^2)^{j_2} (4R+1)^{j_3} \leqslant CR^{j_1+2j_2+j_3} = CR^{k+j_2} \leqslant CR^{k+k-1} = CR^{d-1}
$$

we further obtain

$$
= \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \cdots \int_{R-1}^{2R+1} \int_{-\pi}^{\pi} \left(\prod_{l=1}^k r_l\right) \left| \left(\lambda I - \mathcal{L}_{Q,T_2}^{\text{sim}}\right) v_R(\xi) \right|^p d\tilde{x} d\phi_k dr_k \cdots d\phi_1 dr_1
$$

$$\leqslant \sum_{\substack{j_1+j_2+j_3=k\\ j_2\neq k}} \binom{k}{j_1,j_2,j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3} C^p \left(\prod_{l=1}^{k} r_l\right) (2\pi)^k dr_1\cdots dr_k$$
$$= \sum_{\substack{j_1+j_2+j_3=k\\ j_2\neq k}} \binom{k}{j_1,j_2,j_3} C^p(2\pi)^k \frac{1}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3} \leqslant CR^{d-1}.$$

Case 2: ($\tilde{d}\geqslant 1$). Again from step 3, (7.2), the multinomial theorem, (7.3), $(2R-1)^{j_1}\leqslant CR^{j_1}$, $(3R^2)^{j_2}\leqslant CR^{2j_2}$ and $(4R+1)^{j_3}\leqslant CR^{j_3}$ we further obtain

$$\leqslant \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3}$$
$$\int_{R-1\leqslant|\tilde{x}|\leqslant R} C^p \left(\prod_{l=1}^{k} r_l\right)(2\pi)^k d\tilde{x}dr_1\cdots dr_k$$
$$+ \sum_{\substack{j_1+j_2+j_3=k\\ j_2\neq k}} \binom{k}{j_1,j_2,j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3}$$
$$\int_{R\leqslant|\tilde{x}|\leqslant 2R} C^p \left(\prod_{l=1}^{k} r_l\right)(2\pi)^k d\tilde{x}dr_1\cdots dr_k$$
$$+ \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} \left(\int_{R-1}^{R}\right)^{j_1} \left(\int_{R}^{2R}\right)^{j_2} \left(\int_{2R}^{2R+1}\right)^{j_3}$$
$$\int_{2R\leqslant|\tilde{x}|\leqslant 2R+1} C^p \left(\prod_{l=1}^{k} r_l\right)(2\pi)^k d\tilde{x}dr_1\cdots dr_k$$
$$= \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} \frac{1}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3}$$
$$\left\{\begin{array}{ll} 1 & ,\ \tilde{d}=1 \\ 2\pi\frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)}\frac{\left(R^{\tilde{d}}-(R-1)^{\tilde{d}}\right)}{\tilde{d}} & ,\ \tilde{d}\geqslant 2 \end{array}\right\} C^p(2\pi)^k$$
$$+ \sum_{\substack{j_1+j_2+j_3=k\\ j_2\neq k}} \binom{k}{j_1,j_2,j_3} \frac{1}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3}$$
$$\left\{\begin{array}{ll} R & ,\ \tilde{d}=1 \\ 2\pi\frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)}\frac{\left((2R)^{\tilde{d}}-R^{\tilde{d}}\right)}{\tilde{d}} & ,\ \tilde{d}\geqslant 2 \end{array}\right\} C^p(2\pi)^k$$
$$+ \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} \frac{1}{2^k}(2R-1)^{j_1}(3R^2)^{j_2}(4R+1)^{j_3}$$
$$\left\{\begin{array}{ll} 1 & ,\ \tilde{d}=1 \\ 2\pi\frac{\pi^{\frac{\tilde{d}-2}{2}}}{\Gamma\left(\frac{\tilde{d}}{2}\right)}\frac{\left((2R)^{\tilde{d}}-R^{\tilde{d}}\right)}{\tilde{d}} & ,\ \tilde{d}\geqslant 2 \end{array}\right\} C^p(2\pi)^k$$
$$\leqslant \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} CR^{j_1+2j_2+j_3+\tilde{d}-1} + \sum_{\substack{j_1+j_2+j_3=k\\ j_2\neq k}} \binom{k}{j_1,j_2,j_3} CR^{j_1+2j_2+j_3+\tilde{d}}$$
$$+ \sum_{j_1+j_2+j_3=k} \binom{k}{j_1,j_2,j_3} CR^{j_1+2j_2+j_3+\tilde{d}-1} \leqslant CR^{d-1}.$$

6. Now, let us consider the operator $\mathcal{L}_{Q,T_2}$ instead of $\mathcal{L}^{\mathrm{sim}}_{Q,T_2}$. By definition of χ_R we have

$$\left[\left(\lambda I - \mathcal{L}_{Q,T_2}\right) v_R\right](\xi) = 0,$$

whenever $|\tilde{x}| \in [0,R-1] \cup [2R+1,\infty[$ or $r_l \in [0,R-1] \cup [2R+1,\infty[$ for some $1 \leqslant l \leqslant k$. Moreover, we have by the choice of λ, by definition of χ_R and since $R \geqslant 1$

$$\begin{aligned}
&\left|\left(\lambda I - \mathcal{L}_{Q,T_2}\right) v_R(\xi)\right| \\
=&\left|\left(\lambda I - \mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right) v_R(\xi) - A\sum_{l=1}^{k}\left(\frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}\right) v_R(\xi) - Q(\xi)v_R(\xi)\right| \\
=&\left|A\sum_{l=1}^{k}\left(\frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}\right) v_R(\xi) + Q(\xi)v_R(\xi)\right| \\
\leqslant&\, |A|_2 \sum_{j=1}^{k}\left(\frac{|i\kappa|}{r_l} + \frac{|in_l|^2}{r_l^2}\right) + |Q(\xi)| \\
\leqslant&\, |A|_2 \sum_{j=1}^{k}\left(|\kappa| + |n_l|^2\right)\frac{1}{r_l} + \eta_R \\
\leqslant&\left(|A|_2 \sum_{j=1}^{k}\left(|\kappa| + |n_l|^2\right)\frac{1}{r_l} + \eta_R\right)^{\frac{1}{p}}, \; 1 < p < \infty
\end{aligned}$$

if $|\tilde{x}|, r_l \in [R,2R]$ for every $l = 1,\ldots,k$.

7. From the choice of λ, step 3, $\frac{1}{r_l} \leqslant \frac{1}{R-1} \leqslant 1$ (since $R \geqslant 2$), $\frac{1}{r_l^2} \leqslant 1$, $|\chi_R(y)| \leqslant 1$, $|\chi'_R(y)| \leqslant \|\chi_R\|_{C_b^2}$ and $|v(\xi)| = 1$ we obtain

$$\begin{aligned}
&\left|\left(\lambda I - \mathcal{L}_{Q,T_2}\right) v_R(\xi)\right| \\
=&\left|\left(\lambda I - \mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right) v_R(\xi) - A\sum_{l=1}^{k}\left(\frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}\right) v_R(\xi) - Q(\xi)v_R(\xi)\right| \\
=&\Bigg|\left(\lambda I - \mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right) v_R(\xi) - A\sum_{l=1}^{k}\frac{1}{r_l}\chi'_R(r_l)\left(\prod_{\substack{j=1\\j\neq l}}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi) \\
&- A\sum_{l=1}^{k}\frac{1}{r_l}i\kappa\left(\prod_{j=1}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi) - A\sum_{l=1}^{k}\frac{1}{r_l^2}in_l\left(\prod_{j=1}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi) \\
&- Q(\xi)\left(\prod_{j=1}^{k}\chi_R(r_j)\right)\chi_R(|\tilde{x}|)v(\xi)\Bigg| \\
\leqslant&\left|\left(\lambda I - \mathcal{L}^{\mathrm{sim}}_{Q,T_2}\right) v_R(\xi)\right| + |A|_2\sum_{l=1}^{k}\frac{1}{r_l}\left|\chi'_R(r_l)\right|\left(\prod_{\substack{j=1\\j\neq l}}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right| \\
&+ |A|_2\sum_{l=1}^{k}\frac{1}{r_l}|\kappa|\left(\prod_{j=1}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right| \\
&+ |A|_2\sum_{l=1}^{k}\frac{1}{r_l^2}|n_l|\left(\prod_{j=1}^{k}\left|\chi_R(r_j)\right|\right)\left|\chi_R(|\tilde{x}|)\right|\left|v(\xi)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\|Q\|_{L^\infty}\left(\prod_{j=1}^{k}|\chi_R(r_j)|\right)|\chi_R(|\tilde{x}|)|\,|v(\xi)| \\
\leqslant & C+|A|_2\left(k\,\|\chi_R\|_{C_b^2}+k\,|\kappa|+\sum_{l=1}^{k}|n_l|+\|Q\|_{L^\infty}\right)=C,
\end{aligned}
$$

for every $|\tilde{x}|, r_l \in [R-1,R]\cup[R,2R]\cup[2R,2R+1]$ and $1\leqslant l\leqslant k$.
8. Hence, we obtain from the transformation theorem and step 6

$$
\begin{aligned}
&\|(\lambda I-\mathcal{L}_Q)\,v_R\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)} \\
&=\int_{\mathbb{R}^d}|(\lambda I-\mathcal{L}_Q)\,v_R(x)|^p\,dx=\int_{\mathbb{R}^d}|(\lambda I-\mathcal{L}_{Q,T_1})\,v_R(x)|^p\,dx \\
&=\int_0^\infty\int_{-\pi}^{\pi}\cdots\int_0^\infty\int_{-\pi}^{\pi}\int_{\mathbb{R}^{d-2k}}\left(\prod_{l=1}^{k}r_l\right)|(\lambda I-\mathcal{L}_{Q,T_2})\,v_R(\xi)|^p\,d\xi \\
&=\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\cdots\int_{R-1}^{2R+1}\int_{-\pi}^{\pi}\int_{R-1\leqslant|\tilde{x}|\leqslant 2R+1}\left(\prod_{l=1}^{k}r_l\right)|(\lambda I-\mathcal{L}_{Q,T_2})\,v_R(\xi)|^p\,d\xi
\end{aligned}
$$

Using the abbreviation $\tilde{d}:=d-2k$ we distinguish again between the following cases:
Case 1: ($\tilde{d}=0$). From step 6, step 7 and (7.4) we deduce

$$
\begin{aligned}
=&\int_R^{2R}\int_{-\pi}^{\pi}\cdots\int_R^{2R}\int_{-\pi}^{\pi}\left(\prod_{l=1}^{k}r_l\right)\left[\sum_{l=1}^{k}\frac{|A|_2\left(|\kappa|+|n_l|^2\right)}{r_l}+\eta_R\right]d\phi_k dr_k\cdots d\phi_1 dr_1 \\
&+\sum_{\substack{j_1+j_2+j_3\\ j_2\neq k}}\binom{k}{j_1,j_2,j_3}\left(\int_{R-1}^{R}\right)^{j_1}\left(\int_{R}^{2R}\right)^{j_2}\left(\int_{2R}^{2R+1}\right)^{j_3}C^p\left(\prod_{l=1}^{k}r_l\right)(2\pi)^k dr_1\cdots dr_k \\
\leqslant&\int_R^{2R}\cdots\int_R^{2R}(2\pi)^k\left[\sum_{l=1}^{k}\left(\prod_{\substack{j=1\\ j\neq l}}^{k}r_j\right)|A|_2\left(|\kappa|+|n_l|^2\right)\right] \\
&+(2\pi)^k\left(\prod_{l=1}^{k}r_l\right)\eta_R dr_k\cdots dr_1+CR^{d-1} \\
=&\sum_{l=1}^{k}|A|_2\left(|\kappa|+|n_l|^2\right)\int_R^{2R}\cdots\int_R^{2R}\prod_{\substack{j=1\\ j\neq l}}^{k}r_j dr_1\cdots dr_k \\
&+(2\pi)^k\eta_R\int_R^{2R}\cdots\int_R^{2R}\left(\prod_{l=1}^{k}r_l\right)dr_1\cdots dr_k+CR^{d-1} \\
=&\sum_{l=1}^{k}(2\pi)^k|A|_2\left(|\kappa|+|n_l|^2\right)\left(\prod_{\substack{j=1\\ j\neq l}}^{k}\int_R^{2R}r_j dr_j\right)\int_R^{2R}dr_l \\
&+(2\pi)^k\eta_R\prod_{j=1}^{k}\int_R^{2R}r_j dr_j+CR^{d-1} \\
=&\left(\sum_{l=1}^{k}(2\pi)^k|A|_2\left(|\kappa|+|n_l|^2\right)\left(\frac{3}{2}\right)^{k-1}R^{2k-1}\right)+(2\pi)^k\eta_R\left(\frac{3}{2}\right)^k R^{2k}+CR^{d-1} \\
\leqslant& CR^{d-1}+CR^d\eta_R.
\end{aligned}
$$

Here we refer to case 1 from step 5 for an estimate of the sum.
Case 2: ($\tilde{d}\geqslant 1$). From the procedure used in case 2 from step 5 and in case 1 and (7.3) we obtain

$$\begin{aligned}&\leqslant \int_R^{2R}\int_{-\pi}^{\pi}\cdots\int_R^{2R}\int_{-\pi}^{\pi}\int_{R\leqslant|\tilde{x}|\leqslant 2R}\left(\prod_{l=1}^{k} r_l\right)\left[\sum_{l=1}^{k}\frac{|A|_2\left(|\kappa|+|n_l|^2\right)}{r_l}+\eta_R\right]d\xi\\&\quad+CR^{d-1}\\&\leqslant\left(CR^{2k-1}+CR^{2k}\eta_R\right)\int_{R\leqslant|\tilde{x}|\leqslant 2R}d\tilde{x}+CR^{d-1}\\&\leqslant CR^{2k-1+\tilde{d}}+CR^{d-1}+CR^{2k+\tilde{d}}\eta_R=CR^{d-1}+CR^{d}\eta_R.\end{aligned}$$

The constant CR^{d-1} in the first inequality comes from an estimate of three sums, compare case 2 from step 5. For the second inequality compare case 1.
9. Define

$$w_R:=\frac{v_R}{\|v_R\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}}\in L^p(\mathbb{R}^d,\mathbb{C}^N),$$

which belongs to $L^p(\mathbb{R}^d,\mathbb{C}^N)$ by step 4, then we obtain from step 4 and step 8

$$\begin{aligned}\left\|\left(\lambda I-\mathcal{L}_Q\right)w_R\right\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}=&\frac{\left\|\left(\lambda I-\mathcal{L}_Q\right)v_R\right\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}}{\|v_R\|^p_{L^p(\mathbb{R}^d,\mathbb{C}^N)}}\\\leqslant&\frac{CR^{d-1}+CR^{d}\eta_R}{CR^d}=\frac{C}{R}+\eta_R\to 0\text{ as }R\to\infty.\end{aligned}$$

10. Hence, we must have

$$\lambda\in\sigma(\mathcal{L}_Q)\text{ or }\left(\lambda I-\mathcal{L}_Q\right)^{-1}\text{ is unbounded on }L^p(\mathbb{R}^d,\mathbb{C}^N).$$

If $\lambda=-\lambda_j(\kappa)-i\sum_{l=1}^{k}n_l\sigma_l\in\sigma(\mathcal{L}_Q)$, i.e. λ is an eigenvalue of $\mathcal{L}_Q$, then varying $\kappa\in\mathbb{R}$ shows that λ cannot be isolated, i.e. λ is not a normal point of $\mathcal{L}_Q$. Therefore, all such numbers λ belongs to the essential spectrum of $\mathcal{L}_Q$, i.e. $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L}_Q)$. □

For the next Corollary recall the definition of an analytic semigroup, [28, II.4.5], and a sectorial operator, [28, II.4.1].

Corollary 7.9. *Let the assumptions* (A1), (A2), (A3), (A4), ($A7_B$), *and* $Q\in L^\infty(\mathbb{R}^d,\mathbb{K}^{N,N})$ *with*

$$\eta_R:=\operatorname*{ess\,sup}_{|x|\geqslant R}|Q(x)|_2\to 0\text{ as }R\to\infty$$

be satisfied for $1<p<\infty$ *and* $\mathbb{K}=\mathbb{C}$. *Then the operator* $\mathcal{L}_Q$ *is not sectorial in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and, consequently, the corresponding semigroup* $\left(T_Q(t)\right)_{t\geqslant 0}$ *is not analytic on* $L^p(\mathbb{R}^d,\mathbb{C}^N)$.

Remark. Note that in Theorem 7.8 and in Corollary 7.9 it is allowed to choose $Q=0$. Thus, the operator $\mathcal{L}_\infty$ is not sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and, consequently, the corresponding semigroup $\left(T_\infty(t)\right)_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, $1<p<\infty$. Analogously, if we choose $B=Q=0$, we can deduce that the operator $\mathcal{L}_0$ is not sectorial in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and, consequently, the corresponding semigroup $\left(T_0(t)\right)_{t\geqslant 0}$ is not analytic on $L^p(\mathbb{R}^d,\mathbb{C}^N)$, $1<p<\infty$.

7.3. **Exponential decay for small perturbations.** Let us consider the operator

$$\left[\mathcal{L}_{Q_\varepsilon}\phi\right](x) := \left[\mathcal{L}_\infty\phi\right](x) + Q_\varepsilon(x)\phi(x),\ \phi\in\mathcal{S}$$

for some small perturbation $Q_\varepsilon \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. In this subsection we want to apply the theory from subsection 7.1 with $Q = Q_\varepsilon$. Thus, in this subsection $C_p^\varepsilon := C_p$ denotes the maximal realization of $\mathcal{L}_{Q_\varepsilon}$.

Theorem 7.10 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2), (A3), (A4), (A7$_B$) *and* (A8$_B$) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Then for every* $0 < \vartheta < 1$ *and for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with* $0 \leqslant \eta^2 \leqslant \vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2}$, *for every* $Q_\varepsilon \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ *with* $\|Q_\varepsilon\|_{L^\infty} \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}$, *for every* $\lambda\in\mathbb{C}$ *with* $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ *and for every* $g \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *we have* $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *with*

$$\|v_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{C_7}{\operatorname{Re}\lambda+\frac{2b_0}{3}}\|g\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)}, \tag{7.5}$$

$$\|D_iv_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{\sqrt{2}C_8}{\left(\operatorname{Re}\lambda+\frac{2b_0}{3}\right)^{\frac{1}{2}}}\|g\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)},\ i=1,\ldots,d, \tag{7.6}$$

where $v_\star \in \mathcal{D}^p_{max}$ *denotes the unique solution of* $(\lambda I - C_p^\varepsilon)v = g$ *in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$ *and the* λ*-independent constants* C_7, C_8 *are given by Lemma* 4.8 *(with* $\omega = \omega_\infty$*).*

Proof. 1. First, let us show that there exists a unique solution $v_\star$ of $(\lambda I - C_p^\varepsilon)v = g$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ with $v_\star \in \mathcal{D}^p_{max}$. Therefore, note that since θ is nondecreasing we have $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N) \subset L^p(\mathbb{R}^d,\mathbb{C}^N)$ and hence $g \in L^p(\mathbb{R}^d,\mathbb{C}^N)$. Using

$$\operatorname{Re}\lambda \geqslant -\frac{b_0}{3} = -\frac{2}{3}b_0 + \frac{b_0}{3} \geqslant -\frac{2}{3}b_0 + M_\infty\|Q_\varepsilon\|_{L^\infty} > \omega_\infty + M_\infty\|Q_\varepsilon\|_{L^\infty}$$

the statement follows directly from Corollary 7.5 (with $Q = Q_\varepsilon$ and $C_\infty = C_\infty^\varepsilon$). Moreover, $v_\star$ satisfies the integral expression

$$v_\star = \int_{\mathbb{R}^d}\int_0^\infty e^{-\lambda s}H_\infty(\cdot,\xi,s)ds\,(g(\xi)+Q_\varepsilon(\xi)v_\star(\xi))\,d\xi. \tag{7.7}$$

2. In order to 'apply' Theorem 6.9 (with $g = g - Q_\varepsilon v_\star$) we have to check that the assumptions are satisfied. Indeed, since $\omega_\infty = -b_0 < 0$ by (A8$_B$) we have

$$\operatorname{Re}\lambda \geqslant -\frac{b_0}{3} = \frac{\omega_\infty}{3} > \omega_\infty$$

and from $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ it follows

$$\begin{aligned}0 \leqslant \eta^2 \leqslant \vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2} &= \vartheta\frac{a_0b_0}{a_{\max}^2p^2} + \vartheta\frac{a_0}{a_{\max}^2p^2}\left(-\frac{b_0}{3}\right)\\ &\leqslant \vartheta\frac{a_0(\operatorname{Re}\lambda+b_0)}{a_{\max}^2p^2} = \vartheta\frac{a_0(\operatorname{Re}\lambda-\omega_\infty)}{a_{\max}^2p^2}.\end{aligned}$$

Moreover, by step 1, $v_\star \in D^p_{max}$ is the unique solution of

$$(\lambda I - \mathcal{L}_\infty)\,v_\star = (\lambda I - B_p)\,v_\star = g - Q_\varepsilon v_\star$$

in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Note that for the moment we don't know if $g - Q_\varepsilon v_\star \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$ and consequently, we cannot apply Theorem 6.9 directly. Nevertheless, the inequalities (6.11) and (6.12) hold with $g - Q_\varepsilon v_\star$ instead of g. To accept this, one can replicate the proof of the inequalities using the representation (7.7) instead of (6.13). Thus (6.11) implies

$$\|v_\star\|_{L^p_\theta} \leqslant \frac{C_7}{\operatorname{Re}\lambda-\omega_\infty}\|g - Q_\varepsilon v_\star\|_{L^p_\theta}$$

$$\leqslant \frac{C_7}{\operatorname{Re}\lambda + b_0}\left(\|g\|_{L^p_\theta} + \|Q_\varepsilon\|_{L^\infty}\|v_\star\|_{L^p_\theta}\right)$$
$$\leqslant \frac{C_7}{\operatorname{Re}\lambda + b_0}\|g\|_{L^p_\theta} + \frac{\frac{b_0}{3}}{\operatorname{Re}\lambda + b_0}\|v_\star\|_{L^p_\theta}$$

Subtracting the second term from both hand sides

$$\left(\frac{\operatorname{Re}\lambda + \frac{2b_0}{3}}{\operatorname{Re}\lambda + b_0}\right)\|v_\star\|_{L^p_\theta} \leqslant \frac{C_7}{\operatorname{Re}\lambda + b_0}\|g\|_{L^p_\theta}$$

and dividing by the coefficient of $\|v_\star\|_{L^p_\theta}$ yields (7.5). Furthermore, (7.6) can be deduced from the inequalities (6.12) and (7.5) for every $i = 1,\ldots,d$

$$\begin{aligned}
\|D_i v_\star\|_{L^p_\theta} \leqslant & \frac{C_8}{(\operatorname{Re}\lambda - \omega_\infty)^{\frac{1}{2}}}\|g - Q_\varepsilon v_\star\|_{L^p_\theta} \\
\leqslant & \frac{C_8}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}}\left(\|g\|_{L^p_\theta} + \|Q_\varepsilon\|_{L^\infty}\|v_\star\|_{L^p_\theta}\right) \\
\leqslant & \frac{C_8}{(\operatorname{Re}\lambda + b_0)^{\frac{1}{2}}}\left(\|g\|_{L^p_\theta} + \frac{b_0}{3C_7}\frac{C_7}{\operatorname{Re}\lambda + \frac{2b_0}{3}}\|g\|_{L^p_\theta}\right) \\
= & \frac{C_8}{(\operatorname{Re}\lambda + \frac{2b_0}{3})^{\frac{1}{2}}}\left(\frac{\operatorname{Re}\lambda + b_0}{\operatorname{Re}\lambda + \frac{2b_0}{3}}\right)^{\frac{1}{2}}\|g\|_{L^p_\theta} \\
\leqslant & \frac{\sqrt{2}C_8}{(\operatorname{Re}\lambda + \frac{2b_0}{3})^{\frac{1}{2}}}\|g\|_{L^p_\theta}.
\end{aligned}$$

For the last inequality we used $\operatorname{Re}\lambda \leqslant -\frac{b_0}{3}$ and

$$\frac{\operatorname{Re}\lambda + b_0}{\operatorname{Re}\lambda + \frac{2b_0}{3}} = 1 + \frac{\frac{b_0}{3}}{\operatorname{Re}\lambda + \frac{2b_0}{3}} \leqslant 2.$$

This shows that $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$. □

7.4. **Exponential decay for relatively compact perturbations.** Let us consider the operator

$$\begin{aligned}
\left[\mathcal{L}_Q\phi\right](x) :=& \left[\mathcal{L}_{Q_\varepsilon}\phi\right](x) + Q_c(x)\phi(x) \\
=& \left[\mathcal{L}_\infty\phi\right](x) + Q(x)\phi(x),\ \phi\in\mathcal{S}
\end{aligned}$$

with $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ such that $Q = Q_\varepsilon + Q_c$ with a small perturbation $Q_\varepsilon \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ and a compact perturbation $Q_c \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$. $\mathcal{L}_Q$ can be understood as a compact perturbation of $\mathcal{L}_{Q_\varepsilon}$ or - with $Q = Q_\varepsilon + Q_c$ - as a relatively compact perturbation of $\mathcal{L}_\infty$. In this subsection we want to apply the theory from subsection 7.1 with $Q = Q_\varepsilon + Q_c$. Thus, in this subsection C_p denotes the maximal realization of $\mathcal{L}_Q$.

Theorem 7.11 (A-priori estimates in $L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$)**.** *Let the assumptions* (A1), (A2)*,* (A3)*,* (A4)*,* ($A7_B$) *and* ($A8_B$) *be satisfied for* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{C}$. *Moreover, let* $0 < \vartheta < 1$, $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$ *satisfy*

$$\operatorname*{ess\,sup}_{|x|\geqslant R_0} |Q(x)|_2 \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\} \tag{7.8}$$

for some $R_0 > 0$, $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ *and let* $v_\star \in \mathcal{D}^p_{max}$ *denote a solution of* $(\lambda I - \mathcal{L}_Q)v = 0$ *in* $L^p(\mathbb{R}^d,\mathbb{C}^N)$. *Then we have* $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ *for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with* $0 \leqslant \eta^2 \leqslant \vartheta\frac{2}{3}\frac{a_0 b_0}{a^2_{\max}p^2}$.

Proof. 1. For positive real R choose a C^{∞} cut-off function $\chi_R : [0,\infty[\to[0,1]$ with

$$\chi_R(r) = \begin{cases} 0 & , r \leqslant R \\ \text{smooth} & , R \leqslant r \leqslant 2R \\ 1 & , r \geqslant 2R \end{cases}.$$

We decompose Q as follows

$$Q(x) = \chi_{R_0}(|x|)Q(x) + (1-\chi_{R_0}(|x|))\,Q(x) =: Q_{\varepsilon}(x) + Q_c(x),$$

where $R_0 > 0$ comes from (7.8).
2. Since $Q \in L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})$ and $\chi_R(|\cdot|) \in C_b(\mathbb{R}^d,[0,1])$ we conclude that $Q_{\varepsilon}, Q_c \in L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})$. Moreover, Q_c is compactly supported because $Q_c(x) = 0$ for every $|x| \geqslant 2R_0$ and Q_{ε} is bounded by

$$\begin{aligned}\|Q_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})} &= \|\chi_{R_0}(|\cdot|)Q(\cdot)\|_{L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})} \\ &= \|\chi_{R_0}(|\cdot|)Q(\cdot)\|_{L^{\infty}(\mathbb{R}^d\backslash B_{R_0},\mathbb{C}^{N,N})} \\ &\leqslant \|\chi_{R_0}(|\cdot|)\|_{C_b(\mathbb{R}^d\backslash B_{R_0},[0,1])}\,\|Q(\cdot)\|_{L^{\infty}(\mathbb{R}^d\backslash B_{R_0},\mathbb{C}^{N,N})} \\ &\leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_{\infty}}\right\}.\end{aligned}$$

3. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$ and let $v_{\star} \in \mathcal{D}^p_{max}$ denote a solution of $(\lambda I - \mathcal{L}_Q)\,v = 0$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$, i.e.

$$\left(\lambda I - C_p^{\varepsilon}\right) v_{\star} = (\lambda I - \mathcal{L}_{Q_{\varepsilon}})\,v_{\star} = Q_c v_{\star},\ \text{in } L^p(\mathbb{R}^d,\mathbb{C}^N). \tag{7.9}$$

Now, let $\theta \in C(\mathbb{R}^d,\mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta \geqslant 0$ with $0 \leqslant \eta^2 \leqslant \vartheta\frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2}$ and let us consider the problem

$$\left(\lambda I - C_p^{\varepsilon}\right) u_{\star} = Q_c v_{\star},\ \text{in } L^p_{\theta}(\mathbb{R}^d,\mathbb{C}^N). \tag{7.10}$$

Our aim is to show by Corollary 7.5 that $u_{\star} = v_{\star}$ (in $L^p(\mathbb{R}^d,\mathbb{C}^N)$) is the unique solution of (7.10) in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ and by Theorem 7.10 that $u_{\star} \in W^{1,p}_{\theta}(\mathbb{R}^d,\mathbb{C}^N)$.
4. First we consider the case $\theta \equiv 1$ (with $\eta = 0$). Defining $w_{\star} := u_{\star} - v_{\star}$ we obtain by substracting (7.9) from (7.10)

$$\left(\lambda I - C_p^{\varepsilon}\right) w_{\star} = 0,\ \text{in } L^p(\mathbb{R}^d,\mathbb{C}^N). \tag{7.11}$$

Since $Q_{\varepsilon} \in L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})$ and $\lambda \in \mathbb{C}$ satisfies

$$\operatorname{Re}\lambda \geqslant -\frac{b_0}{3} = -\frac{2}{3}b_0 + \frac{b_0}{3} \geqslant -\frac{2}{3}b_0 + M_{\infty}\|Q_{\varepsilon}\|_{L^{\infty}} > \omega_{\infty} + M_{\infty}\|Q_{\varepsilon}\|_{L^{\infty}},$$

where we used the bound from step 2, we can apply Corollary 7.5 (with $Q = Q_{\varepsilon}$ and $g = 0$) that yields a unique solution $w_{\star} \in \mathcal{D}^p_{max}$ of (7.11), which satisfies $\|w_{\star}\|_{L^p} = 0$, i.e. $u_{\star} = v_{\star}$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$. Since $v_{\star}, w_{\star} \in \mathcal{D}^p_{max}$ and since $\mathcal{D}^p_{max}$ is a closed subspace of $L^p(\mathbb{R}^d,\mathbb{C}^N)$, we deduce $u_{\star} = w_{\star} + v_{\star} \in \mathcal{D}^p_{max}$. Moreover, since $w_{\star}$ is the unique solution of (7.11), we conclude that $u_{\star} = v_{\star}$ is the unique solution of (7.10).
5. Finally, we consider equation (7.10) in $L^p_{\theta}(\mathbb{R}^d,\mathbb{C}^N)$. In order to apply Theorem 7.10, we have to check that the assumptions are satisfied. Indeed, we have $0 \leqslant \eta^2 \leqslant \vartheta\frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2}$, $Q_{\varepsilon} \in L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})$ with $\|Q_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})} \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_{\infty}}\right\}$ (compare stap 2) and $\operatorname{Re}\lambda \geqslant -\frac{b_0}{3}$. Furthermore, since $v_{\star} \in \mathcal{D}^p_{max} \subset L^p(\mathbb{R}^d,\mathbb{C}^N)$ and Q_c is compactly supported in $B_{2R_0}(0)$ it holds

$$\begin{aligned}\|Q_c v_{\star}\|_{L^p_{\theta}(\mathbb{R}^d,\mathbb{C}^N)} &= \|\theta Q_c v_{\star}\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)} = \|\theta Q_c v_{\star}\|_{L^p(B_{2R_0},\mathbb{C}^N)} \\ &\leqslant \|\theta\|_{C_b(B_{2R_0},\mathbb{R})}\,\|Q_c\|_{L^{\infty}(B_{2R_0},\mathbb{C}^{N,N})}\,\|v_{\star}\|_{L^p(B_{2R_0},\mathbb{C}^N)} \\ &\leqslant 2\,\|\theta\|_{C_b(B_{2R_0},\mathbb{R})}\,\|Q\|_{L^{\infty}(\mathbb{R}^d,\mathbb{C}^{N,N})}\,\|v_{\star}\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}\end{aligned}$$

$$=C_{\theta,Q,R_0}\left\|v_\star\right\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)}$$

i.e. $Q_c v_\star \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Since $u_\star \in \mathcal{D}^p_{max}$ is the unique solution of (7.10), we can apply Theorem 7.10 that implies $u_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$. Since $u_\star = v_\star$ in $L^p(\mathbb{R}^d,\mathbb{C}^N)$ ($\|w_\star\|_{L^p}=0$ by step 3) and $u_\star \in L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)\subset L^p(\mathbb{R}^d,\mathbb{C}^N)$, we conclude that $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$ as well. ☐

Remark. From (7.5), (7.6) and the estimate from step 4 we can deduce

$$\|v_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{C_7 C_{\theta,Q,R_0}}{\operatorname{Re}\lambda+\frac{2b_0}{3}}\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)},$$

$$\|D_i v_\star\|_{L^p_\theta(\mathbb{R}^d,\mathbb{C}^N)} \leqslant \frac{\sqrt{2}C_8 C_{\theta,Q,R_0}}{\left(\operatorname{Re}\lambda+\frac{2b_0}{3}\right)^{\frac{1}{2}}}\|v_\star\|_{L^p(\mathbb{R}^d,\mathbb{C}^N)},\ i=1,\ldots,d,$$

where the constants C_7, C_8 are from Lemma 4.8 (with $\omega=\omega_\infty$).

8. The nonlinear problem and complex Ornstein-Uhlenbeck operators

8.1. **Proof of main theorem.** In this section we investigate the nonlinear problem

$$[\mathcal{L}_0 v](x)+f(v(x)) := A\triangle v(x)+\langle Sx,\nabla v(x)\rangle + f(v(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

We are now able to prove our main result from Theorem 1.7:

Proof. Let $0<\vartheta<1$ be fixed, $1<p<\infty$ and $\theta\in C(\mathbb{R}^d,\mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta\geqslant 0$ with $0\leqslant\eta^2\leqslant \vartheta\frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2}$, where $a_{\max}$, a_0 and b_0 are from (1.7).
1. Let $v_\star$ denote a classical solution of (1.9) satisfying $v_\star - v_\infty \in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and (1.8). From Taylor's theorem, (A5) and (A6) we obtain

$$\begin{aligned}
f(v_\star(x)) =& \underbrace{f(v_\infty)}_{=0}+\underbrace{Df(v_\infty)}_{=:-B}(v_\star(x)-v_\infty)\\
&+\underbrace{\int_0^1\left(Df(v_\infty+t(v_\star(x)-v_\infty))-Df(v_\infty)\right)dt}_{=:Q(x)}(v_\star(x)-v_\infty)\\
=&-B\left(v_\star(x)-v_\infty\right)+Q(x)\left(v_\star(x)-v_\infty\right).
\end{aligned}$$

2. Defining $w_\star := v_\star - v_\infty$ then $w_\star\in C^2(\mathbb{R}^d,\mathbb{R}^N)\cap C_b(\mathbb{R}^d,\mathbb{R}^N)\cap L^p(\mathbb{R}^d,\mathbb{R}^N)$ since $v_\star$ is a classical solution of (1.1) and $v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$, and we obtain

$$\begin{aligned}
0 =& A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))\\
=& A\triangle\left(v_\star(x)-v_\infty\right)+\langle Sx,\nabla\left(v_\star(x)-v_\infty\right)\rangle\\
&-B\left(v_\star(x)-v_\infty\right)+Q(x)\left(v_\star(x)-v_\infty\right)\\
=& A\triangle w_\star(x)+\langle Sx,\nabla w_\star(x)\rangle - Bw_\star(x)+Q(x)w_\star(x) = [\mathcal{L}_Q w_\star](x).
\end{aligned}$$

3. In order to apply Theorem 7.11 (with $\lambda=0$) we have to check, that the assumptions are satisfied. The assumptions (A1), (A2), (A3) and (A4) are directly satisfied, (A7$_B$) follows from (A7) and (A8$_B$) follows from (A8), by definition of B. To verify (7.8), let us choose $K_1=K_1(A,f,v_\infty,d,p,\theta,\vartheta)>0$ such that

$$K_1\left(\sup_{z\in B_{K_1}(v_\infty)}\left|D^2 f(z)\right|_2\right)\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}$$

is satisfied, where $C_7 = C_7(A,d,p,\theta,\vartheta)$ is from Lemma 4.8, $M_\infty = M_\infty(A,d)$ from (6.9), $b_0 = b_0(f,v_\infty)$ from (1.7) and

$$\left|D^2 f(z)\right|_2 := \left\|D^2 f(z)\right\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^{N,N})} := \sup_{\substack{v\in\mathbb{R}^N\\ |v|=1}} \left|D^2 f(z)v\right|_2 .$$

Using the fundamental theorem of calculus, (A5), (1.9) and the choice of K_1 we obtain for every $|x| \geqslant R_0$

$$\begin{aligned}
&|Q(x)|_2\\
=&\left|\int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt\right|_2\\
=&\left|\int_0^1\int_0^1 D^2 f(v_\infty + s(v_\infty + tw_\star(x) - v_\infty))ds(v_\infty + tw_\star(x) - v_\infty)dt\right|_2\\
=&\left|\int_0^1\int_0^1 D^2 f(v_\infty + stw_\star(x))ds\cdot tw_\star(x)dt\right|_2\\
\leqslant&\int_0^1\int_0^1 \sup_{|x|\geqslant R_0}\left|D^2 f(v_\infty + st(v_\star(x) - v_\infty))\right|_2 ds\cdot t\,|v_\star(x) - v_\infty|\,dt\\
\leqslant&K_1\left(\sup_{z\in B_{K_1}(v_\infty)}\left|D^2 f(z)\right|_2\right) \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}.
\end{aligned}$$

Taking the suprema over $|x| \geqslant R_0$ yields

$$\sup_{|x|\geqslant R_0}|Q(x)|_2 \leqslant \frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}.$$

4. To verify $Q \in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, note that $w_\star \in C_b(\mathbb{R}^d,\mathbb{R}^N)$. Hence, we have for every $x \in \mathbb{R}^d$ and $0 \leqslant t \leqslant 1$

$$|v_\infty + tw_\star(x)| \leqslant |v_\infty| + t\,|w_\star(x)| \leqslant |v_\infty| + \|w_\star\|_\infty =: R_1 = R_1(v_\infty, v_\star).$$

This implies by (A5) for every $x \in \mathbb{R}^d$

$$\begin{aligned}
|Q(x)|_2 &\leqslant \int_0^1 |Df(v_\infty + tw_\star(x))|_2 + |Df(v_\infty)|_2\,dt\\
&\leqslant \sup_{z\in B_{R_1}(0)} |Df(z)|_2 + |Df(v_\infty)|_2\,,
\end{aligned}$$

which is of course finite by the continuity of Df on compact sets. Taking the suprema over $x \in \mathbb{R}^d$ we obtain $Q \in C_b(\mathbb{R}^d,\mathbb{R}^{N,N})$, thus Q belongs to $L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$
5. Next we verify that $w_\star \in \mathcal{D}^p_{max}$. By assumption we know that $w_\star = v_\star - v_\infty \in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and $w_\star \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^N)$ since $w_\star \in C^2(\mathbb{R}^d,\mathbb{R}^N)$. It remains to prove that $\mathcal{L}_0 w_\star \in L^p(\mathbb{R}^d,\mathbb{R}^N)$: Since $v_\star \in C_b(\mathbb{R}^d,\mathbb{R}^N)$ there exists a constant $R_1 > 0$ such that $|v_\star(x) - v_\infty| \leqslant R_1$ for every $x \in \mathbb{R}^d$. From (A5) we deduce that f is locally Lipschitz continuous, i.e. there exists $L = L(R_1) \geqslant 0$ such that

$$|f(v_\star(x)) - f(v_\infty)| \leqslant L\,|v_\star(x) - v_\infty|\ \ \forall\, x \in \mathbb{R}^d$$

Now, we obtain from (A6) and (1.1)

$$\begin{aligned}
&\|\mathcal{L}_0 w_\star\|^p_{L^p} = \int_{\mathbb{R}^d} |[\mathcal{L}_0 w_\star](x)|^p\,dx = \int_{\mathbb{R}^d} |[\mathcal{L}_0 v_\star](x)|^p\,dx\\
=&\int_{\mathbb{R}^d} |f(v_\star(x))|^p\,dx = \int_{\mathbb{R}^d} |f(v_\star(x)) - f(v_\infty)|^p\,dx \leqslant L^p\int_{\mathbb{R}^d} |v_\star(x) - v_\infty|^p\,dx\\
=&L^p\,\|v_\star - v_\infty\|^p_{L^p}
\end{aligned}$$

Thus $\mathcal{L}_0 w_\star \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ since $w_\star = v_\star - v_\infty \in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and hence $w_\star \in \mathcal{D}^p_{max}$. Moreover, we have the equality $\mathcal{L}_Q w_\star = 0$ in $L^p(\mathbb{R}^d,\mathbb{R}^N)$: Since $w_\star \in \mathcal{D}^p_{max}$ we have $\mathcal{L}_Q w_\star \in L^p(\mathbb{R}^d,\mathbb{R}^N)$ and since $w_\star \in C^2(\mathbb{R}^d,\mathbb{R}^N)$ satisfies $[\mathcal{L}_Q w_\star](x) = 0$ for every $x \in \mathbb{R}^d$ we obtain $\mathcal{L}_Q w_\star = 0$ in $L^p(\mathbb{R}^d,\mathbb{R}^N)$. Now, we can deduce from Theorem 7.11 that $v_\star - v_\infty = w_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{R}^N)$. □

8.2. **Application to complex-valued systems.** Next we apply Theorem 1.7 to complex systems

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,$$

whose nonlinearities are of the form

$$f:\mathbb{C}^N \to \mathbb{C}^N,\quad f(u) = g\left(|u|^2\right)u,$$

where $g:\mathbb{R}\to\mathbb{C}^{N,N}$ is a sufficiently smooth function with some additional properties. More precisely, we prove the following:

Corollary 8.1. *Let the assumptions (A1)–(A4) be satisfied for $1 < p < \infty$ and $\mathbb{K} = \mathbb{C}$. Moreover, let $g \in C^2(\mathbb{R},\mathbb{C}^{N,N})$ such that, $g(0)$ is diagonalizable (over $\mathbb{C}$), $g(0)A = Ag(0)$, $\sigma(g(0)) \subset \mathbb{C}_-$ and define*

$$f:\mathbb{C}^N \to \mathbb{C}^N,\quad f(u) = g\left(|u|^2\right)u.$$

Then for every $0 < \vartheta < 1$ and for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ with

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3}\frac{a_0 b_0}{a_{\max}^2 p^2}$$

with $a_{\max}$, a_0 from (1.7), $b_0 = -s(g(0))$, there exists a constant $K_1 = K_1(A,g,d,p,\theta,\vartheta) > 0$ with the following property:
Every classical solution $v_\star$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x\in\mathbb{R}^d, \tag{8.1}$$

such that $v_\star \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and

$$\sup_{|x|\geqslant R_0} |v_\star(x)| \leqslant K_1 \text{ for some } R_0 > 0 \tag{8.2}$$

satisfies

$$v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N).$$

Proof. 1. We transform the N-dimensional complex-valued system (8.1) into the coupled $2N$-dimensional real-valued system

$$A_\mathbb{R}\triangle v_\mathbb{R}(x) + \langle Sx, \nabla v_\mathbb{R}(x)\rangle + f_\mathbb{R}(v_\mathbb{R}(x)) = 0,\ x\in\mathbb{R}^d, \tag{8.3}$$

where $A = A_1 + iA_2$ with $A_1, A_2 \in \mathbb{R}^{N,N}$, $v = v_1 + iv_2$ with $v_1, v_2 : \mathbb{R}^d \to \mathbb{R}^N$, $f_1, f_2 : \mathbb{R}^{2N} \to \mathbb{R}^N$ with $f_1(u_1,u_2) = \operatorname{Re} f(u_1+iu_2)$ and $f_2(u_1,u_2) = \operatorname{Im} f(u_1+iu_2)$, $g = g_1 + ig_2$ with $g_1, g_2 : \mathbb{R}\to\mathbb{R}$,

$$A_\mathbb{R} := \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix},\quad v_\mathbb{R} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad\text{and}\quad f_\mathbb{R} := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 v_1 - g_2 v_2 \\ g_1 v_2 + g_2 v_1 \end{pmatrix}.$$

2. In order to apply Theorem 1.7 to the $2N$-dimensional problem (8.3), we have to check, that the assumptions (A1)–(A8) are satisfied for $\mathbb{K} = \mathbb{R}$: First note, that

$$\lambda \in \sigma(A) \quad\Longleftrightarrow\quad \lambda, \overline{\lambda} \in \sigma(A_\mathbb{R}) \tag{8.4}$$

$$Y^{-1}AY = \Lambda_A \quad\Longleftrightarrow\quad \begin{pmatrix} iY & \overline{Y} \\ Y & -i\overline{Y} \end{pmatrix} A_\mathbb{R} \begin{pmatrix} iY & \overline{Y} \\ Y & -i\overline{Y} \end{pmatrix}^{-1} = \begin{pmatrix} \Lambda_A & 0 \\ 0 & \overline{\Lambda_A} \end{pmatrix} \tag{8.5}$$

Since A satisfies (A1) we deduce that $A_{\mathbb{R}}$ satisfies (A1) by (8.5). Analogously, since A satisfies (A2) and (A3) we deduce that $A_{\mathbb{R}}$ satisfies (A2) and (A3) by (8.4) and by the symmetry of the sector σ_p with respect to the real-axis. The condition (A4) is directly satisfied. Since $g \in C^2(\mathbb{R},\mathbb{C})$ the assumption (A5) is clearly satisfied. Choosing $v_\infty = 0 \in \mathbb{R}^{2N}$, condition (A6) is satisfied, i.e. $f_{\mathbb{R}}(v_\infty) = 0$, by definition of $f_{\mathbb{R}}$. Since

$$Df_{\mathbb{R}}(0) = \begin{pmatrix} g_1(0) & -g_2(0) \\ g_2(0) & g_1(0) \end{pmatrix}$$

and since $g(0)$ is diagonalizable (over $\mathbb{C}$) we conclude (8.5) that $Df_{\mathbb{R}}(0)$ is diagonalizable (over $\mathbb{C}$). Since A and $g(0)$ commute, we deduce that also $A_{\mathbb{R}}$ and $Df_{\mathbb{R}}(0)$ commute. Thus, $A_{\mathbb{R}}$ and $Df_{\mathbb{R}}(0)$ are simultaneously diagonalizable (over $\mathbb{C}$) and assumption (A7) is satisfied. Finally, since $\sigma(g(0)) \subset \mathbb{C}_-$ we deduce by (8.4) that $\sigma(Df_{\mathbb{R}}(0)) \subset \mathbb{C}_-$, i.e. assumption (A8) is also satisfied.
3. Let $0 < \vartheta < 1$ be fixed, $1 < p < \infty$ and $\theta \in C(\mathbb{R}^d,\mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta \geqslant 0$ with $0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$, where $a_{\max}$, a_0 and b_0 are from (1.7) with $A_{\mathbb{R}}$ and $Df_{\mathbb{R}}(0)$ instead of A and $Df(v_\infty)$. Moreover, let $v_\star$ be a classical solution of (8.1) satisfying $v_\star \in L^p(\mathbb{R}^d,\mathbb{C}^N)$ and (8.2). Then the function

$$v_{\mathbb{R},\star} := \begin{pmatrix} \operatorname{Re} v_\star \\ \operatorname{Im} v_\star \end{pmatrix}$$

is a classical solution of (8.3), which also satisfies $v_{\mathbb{R},\star} \in L^p(\mathbb{R}^d,\mathbb{R}^{2N})$ and (8.2) since $|v_\star(x)| = |v_{\mathbb{R},\star}(x)|$. Applying Theorem 1.7 we obtain $v_{\mathbb{R},\star} \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{R}^{2N})$ and thus $v_\star \in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^N)$. ☐

8.3. **Eigenfunctions of the linearized problem.** In this section we consider the linearized operator

$$[\mathcal{L}v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) \tag{8.6}$$

under the conditions of Theorem 1.7 and we will analyze the corresponding eigenvalue problem

$$[\mathcal{L}v](x) = \lambda v(x),\ x \in \mathbb{R}^d, \tag{8.7}$$

with $\lambda \in \mathbb{C}$.

Definition 8.2. *A function $v : \mathbb{R}^d \to \mathbb{C}^N$ is called a classical solution of (8.7) if*

$$v \in C^2(\mathbb{R}^d,\mathbb{C}^N) \tag{8.8}$$

and v solves (8.7) pointwise.

Theorem 8.3 (Eigenfunctions of $\mathcal{L}$ with eigenvalues on the imaginary axis)**.** *Let the assumptions (A1), (A2), (A3), (A4) and (A5) be satisfied for $1 < p < \infty$ and $\mathbb{K} = \mathbb{R}$. Moreover, let $v_\star \in C^3(\mathbb{R}^d,\mathbb{C}^N)$ be a classical solution of (1.8). Then the following assertions hold:*
(1) *The function*

$$v(x) = \langle Sx, \nabla v_\star(x)\rangle$$

solves $\mathcal{L}v = 0$ in the classical sense, i.e. v is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda = 0$.
(2) *Let $(w,\mu) \in \mathbb{R}^{2d} \times \mathbb{R}$ denote a solution of the $2d$-dimensional eigenvalue problem*

$$\begin{pmatrix} 0 & S^T \\ S & 0 \end{pmatrix} w = \mu w. \tag{8.9}$$

The functions

$$v(x)=\sum_{j=1}^{d}w_jD_jv_\star(x)+i\sum_{j=1}^{d}w_{j+d}D_jv_\star(x) \tag{8.10}$$

solve $\mathcal{L}v=i\mu v$ in the classical sense, i.e. v is an eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda=i\mu$.

Proof. (**1**): Using (A4) and $v_\star\in C^3(\mathbb{R}^d,\mathbb{R}^N)$ we deduce that the operators $A\triangle v_\star(x)$ and $\langle Sx,\nabla v(x)\rangle$ commute, i.e.

$$\begin{aligned}
&A\triangle\left(\langle Sx,\nabla v_\star(x)\rangle\right)\\
=&A\sum_{k=1}^{d}D_k^2\left(\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jD_iv_\star(x)\right)\\
=&A\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}D_k^2\left(S_{ij}x_jD_iv_\star(x)\right)\\
=&A\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}D_k\left(S_{ij}\left(D_kx_j\right)D_iv_\star(x)+S_{ij}x_jD_kD_iv_\star(x)\right)\\
=&A\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}\left(S_{ij}\underbrace{\left(D_k^2x_j\right)}_{=0}D_iv_\star(x)+2S_{ij}\underbrace{\left(D_kx_j\right)}_{=\delta_{jk}}D_kD_iv_\star(x)+S_{ij}x_jD_k^2D_iv_\star(x)\right)\\
=&A\sum_{j=1}^{d}\sum_{i=1}^{d}S_{ji}D_jD_iv_\star(x)+A\sum_{j=1}^{d}\sum_{i=1}^{d}S_{ji}D_jD_iv_\star(x)+A\sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{d}S_{ij}x_jD_iD_k^2v_\star(x)\\
=&A\sum_{i=1}^{d}\sum_{j=1}^{d}\underbrace{\left(S_{ij}+S_{ji}\right)}_{=0}D_jD_iv_\star(x)+\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jD_i\left(A\sum_{k=1}^{d}D_k^2v_\star(x)\right)\\
=&\left\langle Sx,\nabla\left(A\triangle v_\star(x)\right)\right\rangle.
\end{aligned}$$

This fact was already mentioned in [42, Remark 5.2] for the scalar real-valued case and in [17, Proof of Lemma 2.3] for the complex-valued case in 2 space dimensions with polar coordinates. In particular, we conclude from (A5) and $v_\star\in C^1(\mathbb{R}^d,\mathbb{R}^N)$

$$\begin{aligned}
\left\langle Sx,\nabla\left(f(v_\star(x))\right)\right\rangle=&\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jD_i\left(f(v_\star(x))\right)\\
=&\sum_{i=1}^{d}\sum_{j=1}^{d}S_{ij}x_jDf(v_\star(x))D_iv_\star(x)\\
=&Df(v_\star(x))\left\langle Sx,\nabla v_\star(x)\right\rangle.
\end{aligned}$$

To verify that $v(x)=\langle Sx,\nabla v_\star(x)\rangle$ is a eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda=0$ we multiply (1.1) from left by $S_{ij}x_jD_i$ and sum this equation for both i and j from 1 to d

$$\begin{aligned}
0&=\left\langle Sx,\nabla\left(A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))\right)\right\rangle\\
&=\left\langle Sx,\nabla\left(A\triangle v_\star(x)\right)\right\rangle+\left\langle Sx,\nabla\left(\langle Sx,\nabla v_\star(x)\rangle\right)\right\rangle+\left\langle Sx,\nabla\left(f(v_\star(x))\right)\right\rangle\\
&=A\triangle\left(\langle Sx,\nabla v_\star(x)\rangle\right)+\left\langle Sx,\nabla\left(\langle Sx,\nabla v_\star(x)\rangle\right)\right\rangle+Df(v_\star(x))\left\langle Sx,\nabla v_\star(x)\right\rangle\\
&=\mathcal{L}\left\langle Sx,\nabla v_\star(x)\right\rangle.
\end{aligned}$$

(**2**): Applying D_k to (1.1) for every $k=1,\ldots,d$ yields

$$0=D_k\left(A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))\right)$$

$$=A\triangle\left(D_k v_\star(x)\right)+\sum_{i=1}^{d}\sum_{j=1}^{d}\left[S_{ij}\underbrace{\left(D_k x_j\right)}_{\delta_{jk}}D_i v_\star(x)+S_{ij}x_j D_i D_k v_\star(x)\right]$$
$$+Df(v_\star(x))D_k v_\star(x)$$
$$=A\triangle\left(D_k v_\star(x)\right)+\langle Sx,\nabla D_k v_\star(x)\rangle+Df(v_\star(x))D_k v_\star(x)+\sum_{i=1}^{d}S_{ik}D_i v_\star(x)$$
$$=\mathcal{L}D_k v_\star(x)+\sum_{i=1}^{d}S_{ik}D_i v_\star(x).$$

To find the eigenfunctions v of the eigenvalue problem (8.7) with corresponding eigenvalues $\lambda=i\mu$ on the imaginary axis, i.e. $\mu\in\mathbb{R}$, we put the ansatz (8.10) into (8.7)

$$i\left(\sum_{j=1}^{d}\mu w_j D_j v_\star(x)+i\sum_{j=1}^{d}\mu w_{j+d}D_j v_\star(x)\right)$$
$$=i\mu v(x)=\mathcal{L}v(x)$$
$$=\sum_{j=1}^{d}w_j\mathcal{L}D_j v_\star(x)+i\sum_{j=1}^{d}w_{j+d}\mathcal{L}D_j v_\star(x)$$
$$=\sum_{j=1}^{d}w_j\left(-\sum_{i=1}^{d}S_{ij}D_i v_\star(x)\right)+i\sum_{j=1}^{d}w_{j+d}\left(-\sum_{i=1}^{d}S_{ij}D_i v_\star(x)\right)$$
$$=i\left(\sum_{i=1}^{d}\left(-\sum_{j=1}^{d}w_{j+d}S_{ij}\right)D_i v_\star(x)+i\sum_{i=1}^{d}\left(\sum_{j=1}^{d}w_j S_{ij}\right)D_i v_\star(x)\right)$$
$$=i\left(\sum_{j=1}^{d}\left(-\sum_{k=1}^{d}w_{k+d}S_{jk}\right)D_j v_\star(x)+i\sum_{j=1}^{d}\left(\sum_{k=1}^{d}w_k S_{jk}\right)D_j v_\star(x)\right)$$

that leads us to the equations

$$\mu w_j=-\sum_{k=1}^{d}w_{k+d}S_{jk}\quad ,\ j=1,\ldots,d,$$
$$\mu w_{j+d}=\sum_{k=1}^{d}w_k S_{jk}\qquad ,\ j=1,\ldots,d,$$

which can be formulated as the $2d$-dimensional eigenvalue problem (8.9). Finding real eigenvalues μ and eigenvectors $w\in\mathbb{R}^{2d}$ of this finite dimensional eigenvalue problem will give us directly the eigenvalues $\lambda=i\mu$ on the imaginary axis and the corresponding eigenfunctions v from (8.10) for the eigenvalue problem (8.7). □

Remark. (**1**) By assumption (A4) the matrix $\begin{pmatrix}0 & S^T\\ S & 0\end{pmatrix}$ is real and symmetric. Hence the correspondig eigenvalues μ for the eigenvalue problem (8.9) are real. Moreover, every eigenvalue μ of (8.9) has algebraic and geometric multiplicity 2. Since every eigenvalue μ has geometric multiplicity 2, they admit exactly two linearly independent eigenvectors $w_1,w_2\in\mathbb{R}^{2d}$. This two eigenvectors generate two linearly dependent eigenfunctions v_1,v_2 of (8.7). For odd space dimensions d the eigenvalue problem (8.9) admits the eigenvalue $\mu=0$ and the resulting eigenfunction v of (8.7) differs from the eigenfunction from (1) in Theorem 8.3, since the eigenfunction from (1) has unbounded coefficients. For even space dimensions the eigenvalue $\mu = 0$ does never appears for the eigenvalue problem (8.9). Thus, assertion (2) in Theorem 8.3 provides us d pairwise different eigenvalues $\lambda = i\mu \in i\mathbb{R}$ (see Figure 8.3) and their associated eigenfunctions v.

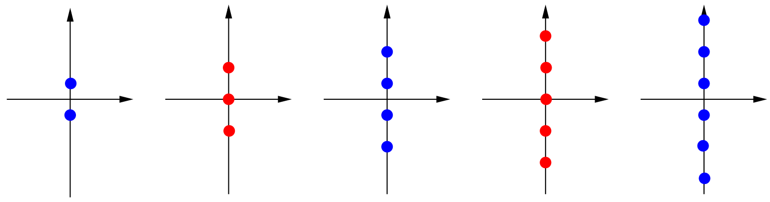


FIGURE 8.1. Eigenvalues $\lambda = i\mu \in i\mathbb{R}$ of $\mathcal{L}$ on the imaginary axis given by Theorem 8.3 (2) for $d = 2$ (left) to $d = 6$ (right)

(**2**) Consider the coupled eigenvalue problem (8.9) with $w = (w_1, W_2)^T \in \mathbb{R}^{2d}$ and $\mu \in \mathbb{R}$, i.e.

$$S^T w_2 = \mu w_1$$
$$S w_1 = \mu w_2$$

To decouple (8.9) we must distinguish between the case $\mu = 0$ (which appears if the space dimension d is odd) and $\mu \neq 0$.
Case 1: ($\mu = 0$). In this case we have

$$-Sw_2 = S^T w_2 = S w_1 \;\Rightarrow\; Sz := S(w_1 + w_2) = 0$$

and hence, we can choose w_1, w_2 such that $z = w_1 + w_2$, where $z \in \mathbb{R}^d$ solves $Sz = 0$.
Case 2: ($\mu \neq 0$). In this case we have

$$\mu w_1 = S^T w_2 = S^T \left(\frac{1}{\mu} S w_1\right) = \frac{1}{\mu} S^T S w_1$$

Let (μ^2, w_1) be a solution of $S^T S w_1 = \mu^2 w_1$ and let $w_2 := \frac{1}{\mu} S w_1$, then (μ, w_1, w_2) solves (8.9). Since $S^T S \in \mathbb{R}^{d,d}$ is a real and symmetric, the corresponding eigenvalues μ^2 are real.

Example 8.4. In the two dimensional case, i.e. $d = 2$, the eigenvalue problem (8.9) admits the solutions

$$\mu_1 = +S_{12}, \quad e_1 = \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}^T, \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^T,$$
$$\mu_2 = -S_{12}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}^T, \begin{pmatrix} 0 & -1 & 1 & 0 \end{pmatrix}^T.$$

By (8.10) this yields the following eigenfunctions and eigenvalues of the operator $\mathcal{L}$

$$\lambda_1 = +iS_{12}, \quad v_1(x) = -D_1 v_\star(x) + iD_2 v_\star(x),$$
$$\lambda_1 = +iS_{12}, \quad v_2(x) = D_2 v_\star(x) + iD_1 v_\star(x),$$
$$\lambda_2 = -iS_{12}, \quad v_3(x) = D_1 v_\star(x) + iD_2 v_\star(x),$$
$$\lambda_2 = -iS_{12}, \quad v_4(x) = -D_2 v_\star(x) + iD_1 v_\star(x).$$

Note that $v_1(x) = iv_2(x)$ and $v_3(x) = iv_4(x)$. Hence, v_1, v_2 and v_3, v_4 are linearly dependent, respectively. Thus we have (see [17, Lemma 2.3] for the case $d = 2$)

$$\lambda = 0, \quad v(x) = S_{12} \left(x_2 D_1 v_\star(x) - x_1 D_2 v_\star(x) \right),$$
$$\lambda = \pm iS_{12}, \quad v(x) = D_1 v_\star(x) \mp iD_2 v_\star(x).$$

Example 8.5. In the three dimensional case, i.e. $d=3$, the eigenvalue problem (8.9) admits the solutions

$$\begin{aligned}
\mu_1 = 0,\quad & e_1 = \begin{pmatrix} 0 & 0 & 0 & S_{23} & -S_{13} & S_{12} \end{pmatrix}^T,\\
& \begin{pmatrix} S_{23} & -S_{13} & S_{12} & 0 & 0 & 0 \end{pmatrix}^T,\\
\mu_2 = +E,\quad & e_2 = \begin{pmatrix} -S_{13}S_{23} & -(S_{12}^2+S_{23}^2) & -S_{12}S_{13} & -ES_{12} & 0 & ES_{23} \end{pmatrix}^T,\\
& \begin{pmatrix} -S_{12}S_{23} & S_{12}S_{13} & S_{13}^2+S_{23}^2 & ES_{13} & ES_{23} & 0 \end{pmatrix}^T,\\
\mu_3 = -E,\quad & e_3 = \begin{pmatrix} S_{13}S_{23} & S_{12}^2+S_{23}^2 & S_{12}S_{13} & -ES_{12} & 0 & ES_{23} \end{pmatrix}^T,\\
& \begin{pmatrix} S_{12}S_{23} & -S_{12}S_{13} & -(S_{13}^2+S_{23}^2) & ES_{13} & ES_{23} & 0 \end{pmatrix}^T.
\end{aligned}$$

where $E:=\sqrt{-S^TS}=\sqrt{S_{12}^2+S_{13}^2+S_{23}^2}$ denotes the length of the angular velocity vector. By (8.10) this yields the following eigenfunctions and eigenvalues of the operator $\mathcal{L}$

$$\begin{aligned}
\lambda_1 = 0,\quad v_1(x) =& i\left(S_{23}D_1v_\star(x) - S_{13}D_2v_\star(x) + S_{12}D_3v_\star(x)\right),\\
\lambda_1 = 0,\quad v_2(x) =& S_{23}D_1v_\star(x) - S_{13}D_2v_\star(x) + S_{12}D_3v_\star(x),\\
\lambda_2 = +iE,\quad v_3(x) =& -S_{13}S_{23}D_1v_\star(x) - (S_{12}^2+S_{23}^2)D_2v_\star(x) - S_{12}S_{13}D_3v_\star(x)\\
&+ i\left(-ES_{12}D_1v_\star(x) + ES_{23}D_3v_\star(x)\right),\\
\lambda_2 = +iE,\quad v_4(x) =& -S_{12}S_{23}D_1v_\star(x) + S_{12}S_{13}D_2v_\star(x) + (S_{13}^2+S_{23}^2)D_3v_\star(x)\\
&+ i\left(ES_{13}D_1v_\star(x) + ES_{23}D_2v_\star(x)\right),\\
\lambda_2 = -iE,\quad v_5(x) =& S_{13}S_{23}D_1v_\star(x) + (S_{12}^2+S_{23}^2)D_2v_\star(x) + S_{12}S_{13}D_3v_\star(x)\\
&+ i\left(-ES_{12}D_1v_\star(x) + ES_{23}D_3v_\star(x)\right),\\
\lambda_2 = -iE,\quad v_6(x) =& S_{12}S_{23}D_1v_\star(x) - S_{12}S_{13}D_2v_\star(x) - (S_{13}^2+S_{23}^2)D_3v_\star(x)\\
&+ i\left(ES_{13}D_1v_\star(x) + ES_{23}D_2v_\star(x)\right).
\end{aligned}$$

Note that v_1, v_2 as well as v_3, v_4 and v_5, v_6 are linearly dependent. Thus we have

$$\begin{aligned}
\lambda = 0,\quad v(x) =& S_{12}\left(x_2D_1v_\star(x) - x_1D_2v_\star(x)\right)\\
&+ S_{13}\left(x_3D_1v_\star(x) - x_1D_3v_\star(x)\right)\\
&+ S_{23}\left(x_3D_2v_\star(x) - x_2D_3v_\star(x)\right),\\
\lambda = 0,\quad v(x) =& S_{23}D_1v_\star(x) - S_{13}D_2v_\star(x) + S_{12}D_3v_\star(x),\\
\lambda = \pm iE,\quad v(x) =& \pm\left(-S_{12}S_{23}D_1v_\star(x) + S_{12}S_{13}D_2v_\star(x) + (S_{13}^2+S_{23}^2)D_3v_\star(x)\right)\\
&+ i\left(ES_{13}D_1v_\star(x) + ES_{23}D_2v_\star(x)\right).
\end{aligned}$$

Theorem 8.6 (Exponential decay of eigenfunctions)**.** *Let the assumptions* (A1)–(A8) *be satisfied for* $1<p<\infty$ *and* $\mathbb{K}=\mathbb{R}$. *Then for every* $0<\vartheta<1$ *and for every radially nondecreasing weight function* $\theta\in C(\mathbb{R}^d,\mathbb{R})$ *of exponential growth rate* $\eta\geqslant 0$ *with*

$$0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2}$$

with $a_{\max}$, a_0, b_0 *from* (1.7), *there exists a constant* $K_1=K_1(A,f,v_\infty,d,p,\theta,\vartheta)>0$ *with the following property:*
Given a classical solution $v_\star$ *of* (1.8) *such that* $v_\star-v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^N)$ *and* (1.9) *hold. Then every classical solution* $v\in L^p(\mathbb{R}^d,\mathbb{C}^N)$ *of the eigenvalue problem*

$$A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+Df(v_\star(x))v(x)=\lambda v(x),\ x\in\mathbb{R}^d, \tag{8.11}$$

with $\lambda\in\mathbb{C}$ *and* $\operatorname{Re}\lambda\geqslant\frac{b_0}{3}$ *satisfies*

$$v\in W^{1,p}_{\theta}(\mathbb{R}^d,\mathbb{C}^N).$$

Proof. Let $0<\vartheta<1$ be fixed, $1<p<\infty$ and $\theta\in C(\mathbb{R}^d,\mathbb{R})$ be a radially nondecreasing weight function of exponential growth rate $\eta\geqslant 0$ with $0\leqslant\eta^2\leqslant\vartheta\frac{2}{3}\frac{a_0b_0}{a_{\max}^2p^2}$, where $a_{\max}$, a_0 and b_0 are from (1.7).
1. Let v be a classical solution of (8.7) with $v\in L^p(\mathbb{R}^d,\mathbb{R}^N)$, then we can write (8.7) as

$$\begin{aligned}0&=\lambda v(x)-[\mathcal{L}v](x)\\&=\lambda v(x)-\left(A\triangle v(x)+\langle Sx,\nabla v(x)\rangle+Df(v_\infty)v(x)+\left(Df(v_\star(x))-Df(v_\infty)\right)v(x)\right)\\&=\lambda v(x)-\left(A\triangle v(x)+\langle Sx,\nabla v(x)\rangle-Bv(x)+Q(x)v(x)\right)\\&=\left(\lambda I-\mathcal{L}_Q\right)v(x)\end{aligned}$$

where $B:=-Df(v_\infty)$ and $Q(x):=Df(v_\star(x))-Df(v_\infty)$.
2. In order to apply Theorem 7.11 we have to check, that the assumptions are satisfied. Assumptions (A1), (A2), (A3) and (A4) are directly satisfied, ($A7_B$) follows from (A7) and ($A8_B$) follows from (A8), by definition of B. To verify (7.8), let us choose $K_1=K_1(A,f,v_\infty,d,p,\theta,\vartheta)>0$ (as in Theorem 1.7) such that

$$K_1\left(\sup_{z\in B_{K_1}(v_\infty)}\left|D^2f(z)\right|_2\right)\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}$$

is satisfied, where $C_7=C_7(A,d,p,\theta,\vartheta)$ is from Lemma 4.8, $M_\infty=M_\infty(A,d)$ from (6.9), $b_0=b_0(f,v_\infty)$ from (1.7) and

$$\left|D^2f(z)\right|_2:=\left\|D^2f(z)\right\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^{N,N})}:=\sup_{\substack{v\in\mathbb{R}^N\\|v|=1}}\left|D^2f(z)v\right|_2.$$

Using the fundamental theorem of calculus, (A5), (1.9) and the choice of K_1 we obtain for every $|x|\geqslant R_0$

$$\begin{aligned}&|Q(x)|_2\\=&|Df(v_\star(x))-Df(v_\infty)|_2\\=&\left|\int_0^1 D^2f(v_\infty+s(v_\star(x)-v_\infty))ds(v_\star(x)-v_\infty)\right|_2\\\leqslant&\int_0^1\left|D^2f(v_\infty+s(v_\star(x)-v_\infty))\right|_2ds\,|v_\star(x)-v_\infty|\\\leqslant&\int_0^1\left(\sup_{|x|\geqslant R_0}\left|D^2f(v_\infty+s(v_\star(x)-v_\infty))\right|_2\right)ds\left(\sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|\right)\\\leqslant&K_1\left(\sup_{z\in B_{K_1}(v_\infty)}\left|D^2f(z)\right|_2\right)\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}.\end{aligned}$$

Taking the suprema over $|x|\geqslant R_0$ yields

$$\sup_{|x|\geqslant R_0}|Q(x)|_2\leqslant\frac{b_0}{3}\min\left\{\frac{1}{C_7},\frac{1}{M_\infty}\right\}.$$

3. To verify $Q\in L^\infty(\mathbb{R}^d,\mathbb{C}^{N,N})$, note that $v_\star-v_\infty\in C_b(\mathbb{R}^d,\mathbb{R}^N)$ since $v_\star$ is a classical solution of (1.8). Thus

$$\exists\,R_1>0:\ |v_\star(x)-v_\infty|\leqslant R_1\quad\forall\,x\in\mathbb{R}^d.$$

This implies by (A5) for every $x\in\mathbb{R}^d$

$$|Q(x)|_2=|Df(v_\star(x))-Df(v_\infty)|_2$$

$$\leqslant \sup_{x\in\mathbb{R}^d} |Df(v_\star(x))|_2 + |Df(v_\infty)|_2$$
$$\leqslant \sup_{z\in B_{R_1}(0)} |Df(z)|_2 + |Df(v_\infty)|_2 ,$$

which is of course finite by the continuity of Df on compact sets. Taking the suprema over $x \in \mathbb{R}^d$ we obtain $Q \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N})$, thus Q belongs also to $Q \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N,N})$.

4. Next we verify that $v \in \mathcal{D}^p_{max}$. By assumption we know that $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ and $v \in W^{2,p}_{loc}(\mathbb{R}^d, \mathbb{C}^N)$ since $v \in C^2(\mathbb{R}^d, \mathbb{C}^N)$. It remains to prove that $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$: From (A5) we deduce that Df is locally Lipschitz continuous, i.e. there exists $L = L(R_1) \geqslant 0$ such that

$$|Df(v_\star(x)) - Df(v_\infty)| \leqslant L\,|v_\star(x) - v_\infty| \;\forall\, x \in \mathbb{R}^d.$$

Now, we obtain from (8.11) and Hölder's inequality

$$\|\mathcal{L}_0 v\|_{L^p} \leqslant |\lambda|\,\|v\|_{L^p} + |Df(v_\infty)|\,\|v\|_{L^p} + \|Df(v_\star(x)) - Df(v_\infty)\|_{L^\infty}\,\|v\|_{L^p}$$
$$\leqslant (|\lambda| + |Df(v_\infty)| + LR_1)\,\|v\|_{L^p} .$$

Thus $\mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ since $v \in L^p(\mathbb{R}^d, \mathbb{C}^N)$ and hence $v \in \mathcal{D}^p_{max}$. Moreover, we have the equality $\lambda v - \mathcal{L}_Q v = 0$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$: Since $v \in \mathcal{D}^p_{max}$ we have that $\mathcal{L}_Q v$ and thus $\lambda v - \mathcal{L}_Q v$ belongs to $L^p(\mathbb{R}^d, \mathbb{C}^N)$ and since $[\mathcal{L}_Q v](x) = \lambda v(x)$ for every $x \in \mathbb{R}^d$ we obtain $\mathcal{L}v = \lambda v$ in $L^p(\mathbb{R}^d, \mathbb{C}^N)$. Finally, applying Theorem 7.11 we conclude that $v \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{C}^N)$. ☐

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