QUALITATIVE ANALYSIS OF A NONAUTONOMOUS BEVERTON-HOLT RICKER MODEL

THORSTEN HÜLS* AND CHRISTIAN PÖTZSCHE†

Abstract. We explore a planar discrete-time model from population dynamics subject to a general aperiodic time-varying environment in order to illustrate the recent theory of nonautonomous dynamical systems. Given such a setting, the mathematical standard tools from classical dynamical systems and bifurcation theory cannot be employed, since for instance equilibria typically do not exist or eigenvalues yield no stability information. For this reason, we apply a combination of recent analytical and numerical techniques adequate to tackle such situations.

Key words. Nonautonomous dynamics, numerical dynamics, pullback attractor, nonautonomous bifurcation, exponential dichotomy, Bohl exponent, population models

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1. Nonautonomous models. Mathematical models to describe evolutions with discrete time-steps are formulated in terms of recursions (or difference equations)

$$x_{n+1} = f_\lambda(x_n), \tag{1.1}$$

where x_n represents a vector containing, for instance in the life sciences, sizes or densities of populations or biochemical substances involved at time $n \in \mathbb{Z}$. Naturally, equations (1.1) contain parameters λ describing the influence of the environment on the models. In population dynamics, such factors might be seasonal and governed by the weather, describe possible catastrophes (floods, storms), unspecific predators, parasites, inner or intra-specific competition, social stress or infective diseases. They might, as well, capture dosing and control strategies including harvesting, hunting or the use of pesticides.

The complexity of these influences suggests that constant parameters λ in (1.1) are hardly realistic. As an improvement one should better work with time-varying parameter sequences λ_n and thus equations of the form

$$x_{n+1} = f_{\lambda_n}(x_n). \tag{1.2}$$

However, the drawback and challenge of such a *nonautonomous* framework (1.2) is that the classical mathematical theory of dynamical systems turns out to be hardly applicable. For instance, periodic solutions of (1.1) obtained from the fixed point relation $x = f_\lambda^p(x)$, $p \in \mathbb{N}$, typically fail to be periodic solutions to (1.2) (for general parameter sequences), eigenvalues are of no use for indicating stability properties, and forward limit sets or attractors are not invariant. Hence, in absence of equilibria, what should classical bifurcation theory or numerical continuation software be applied to?

For the above reasons, an extension of the conventional dynamical systems theory is required. It is based on the leitmotiv that equilibria or periodic solutions to (1.1) persist as bounded entire solutions to (1.2) under moderately sized time-varying parameter sequences. These entire solutions also reflect the particular temporal forcing, i.e. for example almost periodic (almost automorphic, asymptotically constant, etc.)

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parameter sequences $(\lambda_n)_{n\in\mathbb{Z}}$ in (1.2); thus one also speaks of *nonautonomous equilibria*. In this sense, for instance a bifurcation is understood as a change in the structure or attraction properties of such entire solutions. We point out that an analogous theory as presented here also exists in the setting of continuous time nonautonomous dynamical systems (cf. [3, 18, 19, 22]). Yet, for the sake of conceptual clarity we have restricted to a discrete-time set-up.

Our considerations are based on an autonomous model

$$\begin{cases} x_{n+1} = \dfrac{\alpha x_n}{1 + x_n + \beta y_n}, \\ y_{n+1} = y_n e^{\gamma - \delta x_n - y_n} \end{cases} \tag{Δ'}$$

from [6] (see also [20, 16]) describing the interaction between two populations by coupling a Beverton-Holt with a Ricker equation. This system is interesting from an ecological point of view, since according to [6] it yields a counterexample to the so-called *exclusion principle* indicating that one species goes extinct, provided there exists no fixed point in the interior of the first quadrant. The latter holds for general d-dimensional systems, when exclusively Beverton-Holt- resp. Ricker-type equations are present, but [6, Sect. 4] demonstrates that for certain parameter constellations coexistence is possible in form of a period-2-solution in the open first quadrant, although there is no fixed-point in $(0, \infty)^2$.

Nevertheless, our goal is to a minor extend biologically motivated and should rather be seen as a contribution to nonautonomous and numerical dynamics. From this mathematical perspective alone, (Δ') is a rewarding prototype example featuring dynamical behavior of different complexity, since it links the fully understood and dynamically well-behaved Beverton-Holt model $x_{n+1} = \frac{\alpha x_n}{1+x_n}$ with the possibly chaotic Ricker difference equation $y_{n+1} = y_n e^{\gamma - y_n}$ (cf. [24]).

In this paper, we extend the above autonomous set-up and study (Δ') in an environment with bounded but otherwise arbitrary temporal fluctuations in the parameters $\alpha, \beta, \gamma, \delta$. This requires various recent techniques from nonautonomous dynamics (cf. [19, 25]) in order to obtain information on the forward, as well as the pullback dynamics and the corresponding attractors. Changes in them are caused by multiple nonautonomous bifurcation scenarios (see [22, 30, 11, 26]), among which we particularly address counterparts to transcritical and flip (period doubling) bifurcations. The associate stability transitions are described using Bohl exponents which are boundary points of the dichotomy (also known as Sacker-Sell) spectrum (see [32] and [2, 1, 13]). Gaps in the dichotomy spectrum in turn give rise to invariant manifolds and we particularly perform a nonautonomous center manifold reduction (cf. [29]). In this endeavor it turns out at an early stage that up-to-date numerical techniques are indispensable, when quantitative information on the dichotomy spectrum [12, 13] or the continuation of bounded entire solutions [10, 14] is required. Also explicit perturbation bounds for the persistence of hyperbolicity under nonautonomous forcing are given in a representative special case. In conclusion, only a combination of analytical and numerical methods yields a necessary insight into the long-term behavior of our in fact merely 2-dimensional, but nonautonomous dynamical system.

For the reader's convenience, we summarized some basic, as well as required new results on nonautonomous dynamics, in the appendix.

2. Preliminaries and the model. Throughout the paper, let us suppose that $\mathbb{I}$ denotes a discrete interval unbounded above, i.e. the intersection of a real interval with the integers $\mathbb{Z}$. For a sequence $(a_n)_{n\in\mathbb{I}}$ we briefly write $a_{\mathbb{I}}$. The symbol ℓ^∞ denotes the space of bounded, and ℓ^1 (resp. ℓ_0) of absolutely summable sequences (resp. those with limit 0). Moreover, we abbreviate the nonnegative half-axis $\mathbb{R}_+ := [0, \infty)$.

Initial point is a general planar Beverton-Holt Ricker model

$$\begin{cases} v_{n+1} = \dfrac{\alpha_n^1 v_n}{\alpha_n + \beta_n^1 v_n + \gamma_n^1 w_n}, \\ w_{n+1} = w_n e^{\alpha_n^2 - \beta_n^2 v_n - \gamma_n^2 w_n} \end{cases}$$

in discrete time $n \in \mathbb{I}$ and with parameter sequences $\alpha_{\mathbb{I}}, \alpha_{\mathbb{I}}^i$, $\beta_{\mathbb{I}}^i$, $\gamma_{\mathbb{I}}^i$, $i = 1, 2$. Related autonomous models, where both components are of Beverton-Holt- or of Ricker-type, have been suggested in [4]. In combining these two kinds of nonlinearities, we are able to recover both possible behaviors in a single model. However, this type of problem has also been investigated in [6, 16] with a focus on biological and ecological implications. Consequently our approach has more than just a didactical motivation.

First, we get rid of several parameters and simplify this model to its canonical form. By means of the abbreviations $x_n := \frac{\beta_n^1}{\alpha_n} v_n$, $y_n := \gamma_n^2 w_n$ the reduced model is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F_n(x_n, y_n), \qquad F_n(x, y) := \begin{pmatrix} \frac{a_n x}{1+x+b_n y}, \\ y e^{c_n - d_n x - y} \end{pmatrix} \tag{Δ}$$

with the right-hand side $F_n : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, $n \in \mathbb{I}$, and parameter sequences

$$a_n := \frac{\alpha_n^1 \beta_{n+1}^1}{\alpha_{n+1} \beta_n^1}, \qquad b_n := \frac{\gamma_n^1}{\alpha_n \gamma_n^2},$$
$$c_n := \alpha_n^2 + \ln\left(\frac{\gamma_{n+1}^2}{\gamma_n^2}\right), \qquad d_n := \frac{\alpha_n \beta_n^2}{\beta_n^1}.$$

Referring to this simplification, we can exclusively consider (Δ) with the right-hand side F_n from now on. In addition, let us impose the global assumption throughout the paper that the real parameter sequences $a_{\mathbb{I}}$, $b_{\mathbb{I}}$, $c_{\mathbb{I}}$, $d_{\mathbb{I}}$ in (Δ) are bounded; furthermore $a_{\mathbb{I}}, b_{\mathbb{I}}, d_{\mathbb{I}}$ are supposed to have positive values, while $c_{\mathbb{I}}$ can be nonnegative.

A solution $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ of (Δ) is a sequence satisfying $(\xi_{n+1}, \eta_{n+1}) = F_n(\xi_n, \eta_n)$. Given an initial time $n_0 \in \mathbb{I}$ and initial states $\bar{x}, \bar{y} \geq 0$, we denote the forward solution to (Δ) satisfying the initial condition $x_{n_0} = \bar{x}$, $y_{n_0} = \bar{y}$ by $\varphi(\cdot; n_0, \bar{x}, \bar{y})$ and its components by φ_1, φ_2. Further basics from nonautonomous dynamics are summarized in Appendix A.

We start with general remarks on the stability of solutions to nonautonomous difference equations. For a solution $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ to (Δ), the *variational equation* reads as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F_n'(\xi_n, \eta_n) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{V}$$

with the coefficient matrix

$$F_n'(\xi, \eta) = \begin{pmatrix} \frac{a_n(1+b_n\eta)}{(1+\xi+b_n\eta)^2} & -\frac{a_n b_n \xi}{(1+\xi+b_n\eta)^2} \\ -d_n \eta e^{c_n - d_n\xi - \eta} & e^{c_n - d_n\xi - \eta}(1-\eta) \end{pmatrix} \quad \text{for all } n \in \mathbb{I}, \xi, \eta \geq 0.$$

Since we are interested in a robust analysis it is preferable to deal with uniform stability properties of solutions $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ to (Δ). They are determined by the dichotomy spectrum $\Sigma(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ (for this, see Appendix B.2) of (V), which indicates uniform asymptotic stability of a solution $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$, i.e. the fact that $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ is *uniformly attractive*

$$\exists \rho > 0 : \forall \varepsilon > 0 : \exists N \in \mathbb{N}_0 : (\bar{x}, \bar{y}) \in B_\rho(\xi_{n_0}, \eta_{n_0}) \Rightarrow \varphi(n; n_0, \bar{x}, \bar{y}) \in B_\varepsilon(\xi_n, \eta_n)$$

for all $n_0 \in \mathbb{I}$, $n \geq n_0 + N$, as well as *uniformly stable*

$$\forall \varepsilon > 0 : \exists \delta > 0 : \forall n_0 \in \mathbb{I} : \ (\bar{x}, \bar{y}) \in B_\delta(\xi_{n_0}, \eta_{n_0}) \Rightarrow \varphi(n; n_0, \bar{x}, \bar{y}) \in B_\varepsilon(\xi_n, \eta_n)$$

for all $n \geq n_0$.

THEOREM 2.1.
(a) If $\Sigma(\xi_{\mathbb{I}}, \eta_{\mathbb{I}}) \subseteq [0, 1)$, then $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ is uniformly asymptotically stable.
(b) If there exists a spectral interval $\sigma^+ \subseteq \Sigma(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ with $\min \sigma^+ > 1$, then $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ is unstable. Moreover, in case $\mathbb{I} = \mathbb{Z}$ there exists an unstable fiber bundle.

Proof. See [26, Props. 3.9 and 3.10]. ∎

For analogy reasons, a solution $(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$ to (Δ) is called a *(nonautonomous)*
- *sink*, if $\max \Sigma(\xi_{\mathbb{I}}, \eta_{\mathbb{I}}) < 1$,
- *saddle*, if $\mathbb{I} = \mathbb{Z}$ and there exist two nonempty disjoint $\sigma_1, \sigma_2 \subseteq (0, \infty)$ with

$$\Sigma(\xi_{\mathbb{I}}, \eta_{\mathbb{I}}) = \sigma_1 \cup \sigma_2, \qquad \max \sigma_1 < 1 < \min \sigma_2,$$

- *source*, if $1 < \min \Sigma(\xi_{\mathbb{I}}, \eta_{\mathbb{I}})$.

Returning to the concrete planar system (Δ), it has the trivial equilibrium $(0, 0)$ and both coordinate axes are forward invariant. Hence, in order to understand (Δ) we initially investigate its behavior restricted to the x- and y-axes.

Restricted to the x-axis, (Δ) is a scalar nonautonomous Beverton-Holt equation

$$x_{n+1} = f_n(x_n), \qquad f_n(x) := \frac{a_n x}{1 + x}, \tag{BH}$$

while the restriction to the y-axis results in a nonautonomous Ricker equation

$$y_{n+1} = g_n(y_n), \qquad g_n(y) := y e^{c_n - y}. \tag{R}$$

Given these scalar systems, we illustrate basic nonautonomous tools for analyzing their dynamics. First we note that their right-hand sides have at 0 the Taylor expansions

$$f_n(x) = a_n \sum_{i=0}^{\infty} (-1)^i x^{i+1}, \qquad g_n(y) = e^{c_n} \sum_{i=0}^{\infty} \frac{(-1)^i y^{i+1}}{i!}$$

with convergence radii 1 resp. ∞; see Fig. 2.1.

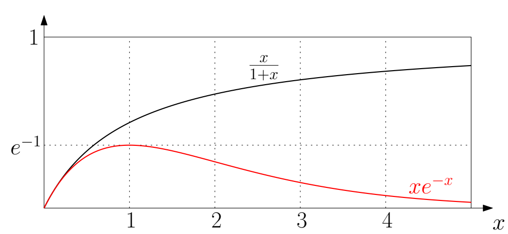

FIG. 2.1. *Right-hand sides of the Beverton-Holt model* (BH) *for $a_n = 1$ (black), and the Ricker model* (R) *for $c_n = 0$ (red).*

The function $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing with limit $a_n > 0$, whereas $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing for $y < 1$ with maximum $e^{c_n - 1}$ at $y = 1$ and monotone decreasing to 0 for $y > 1$. Using their common properties, we immediately find invariant and pullback absorbing sets (see Appendix A for the required terminology):

THEOREM 2.2 (pullback attractor). *Let $\mathbb{I} = \mathbb{Z}$ and $h_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{Z}$, be continuous. If $r_{\mathbb{Z}}$ is a bounded sequence such that $0 \leq h_n(z) \leq r_n$ for all $n \in \mathbb{Z}$, $z \geq 0$, then the nonautonomous set $\mathcal{Z} := \{(n, z) \in \mathbb{Z} \times \mathbb{R}_+ : 0 \leq z \leq r_{n-1}\}$ is forward invariant and pullback absorbing w.r.t. the scalar difference equation*

$$z_{n+1} = h_n(z_n).$$

Furthermore, the so-called pullback attractor

$$\mathcal{Z}^* := \{z_{\mathbb{Z}} \in \ell^\infty : z_{n+1} = h_n(z_n) \text{ on } \mathbb{Z}\} \subseteq \mathcal{Z}$$

is compact, invariant, connected and pullback attracts all bounded subsets of $\mathbb{Z} \times \mathbb{R}_+$.

Proof. Since $h_n(\mathbb{R}_+) \subseteq [0, r_n]$ holds true for all $n \in \mathbb{Z}$ it follows that the nonautonomous set $\mathcal{Z}$ is forward invariant and pullback absorbing.

The ω-limit set $\omega_{\mathcal{Z}}$ of the absorbing set $\mathcal{Z}$ is invariant, pullback attracting and satisfies the inclusion $\omega_{\mathcal{Z}} \subseteq \mathcal{Z}$ by [25, p. 19, Thm. 1.3.9]. Furthermore, [25, p. 14, Thm. 1.2.25] guarantees compactness and [25, p. 20, Cor. 1.3.11] its connectedness. Finally, the dynamical characterization [25, p. 17, Thm. 1.3.4] implies $\mathcal{Z}^* = \omega_{\mathcal{Z}}$. $\square$

3. Nonautonomous Beverton-Holt model. Let us first consider the dynamics of the nonautonomous Beverton-Holt equation (BH) in detail (cf. Fig. 3.1). Given an initial pair $(n_0, \bar{x}) \in \mathbb{I} \times \mathbb{R}_+$, we denote its solution satisfying the initial condition $x_{n_0} = \bar{x}$ by $x(\cdot; n_0, \bar{x})$ and obtain the following properties:

- The nonautonomous set $\mathbb{I} \times \{0\}$ is invariant, while $\{(n, x) \in \mathbb{I} \times \mathbb{R}_+ : x \leq a_{n-1}\}$ and $\{(n, x) \in \mathbb{I} \times \mathbb{R} : 0 < x \leq a_{n-1}\}$ are forward invariant w.r.t. (BH).
- Due to the monotonicity of each f_n, one has the comparison principle

  $$0 \leq \bar{x}_1 < \bar{x}_2 \quad \Rightarrow \quad x(n; \bar{n}, \bar{x}_1) < x(n; \bar{n}, \bar{x}_2) \quad \text{for all } \bar{n} \leq n.$$

  In particular, the Beverton-Holt equation (BH) is order-preserving.
- If $\mathbb{I} = \mathbb{Z}$, then Thm. 2.2 applies with $r_n = a_n$ and the pullback absorbing set

  $$\mathcal{X} := \{(n, x) \in \mathbb{Z} \times \mathbb{R}_+ : 0 \leq x \leq a_{n-1}\}$$

  yielding the pullback attractor $\mathcal{X}_a^* := \left\{ x_{\mathbb{Z}} \in \ell^\infty : x_{n+1} = \frac{a_n x_n}{1+x_n} \quad \text{on } \mathbb{Z} \right\}$.

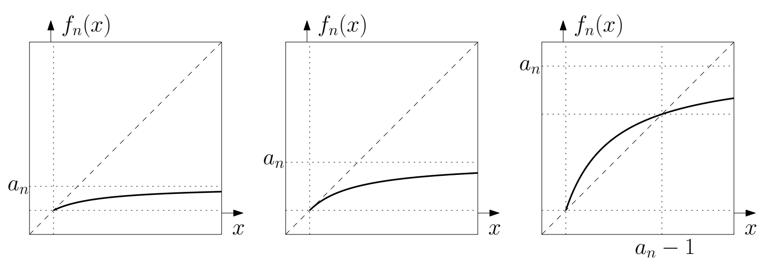

FIG. 3.1. *Right-hand side $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ of the Beverton-Holt equation* (BH) *for $a_n \in (0, 1)$ (left), $a_n = 1$ (center) and $a_n > 1$ (right).*

3.1. Pullback attractor of (BH)**.** In the autonomous Beverton-Holt equation

$$x_{n+1} = f(x_n), \qquad f(x) := \frac{\alpha x}{1 + x} \tag{BH'}$$

with the right-hand side $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a parameter $\alpha > 0$, the trivial solution transcritically bifurcates into the equilibrium $\alpha-1$, as α increases through the critical value 1. For $\alpha = 1$ it follows from [5, p. 478, Thm. A.3] that 0 is still asymptotically stable. Thus, its pullback attractor becomes

$$\mathcal{X}_\alpha^* = \begin{cases} \mathbb{Z} \times \{0\}, & \alpha \in (0,1], \\ \mathbb{Z} \times [0, \alpha-1], & \alpha > 1 \end{cases} \tag{3.1}$$

and for $\alpha > 1$ every solution starting in the open interval $(0, \alpha - 1)$ is a strictly increasing heteroclinic connection between the fixed points 0 and $\alpha - 1$.

Our next goal are similar information on the structure of the above nonautonomous set $\mathcal{X}_a^*$ for arbitrary time-varying sequences $a_{\mathbb{I}}$. Here, it is helpful to introduce

$$\Phi_a(n,m) := \begin{cases} a_{n-1} \cdots a_m, & m < n, \\ 1, & n = m, \\ a_n^{-1} \cdots a_{m-1}^{-1}, & n < m \end{cases} \tag{3.2}$$

and that the solutions of the Beverton-Holt equation (BH) are explicit:

Lemma 3.1. *The Beverton-Holt equation* (BH) *has the explicit solution*

$$x(n; n_0, \bar{x}) = \begin{cases} \dfrac{\bar{x}\Phi_a(n,n_0)}{1 + \bar{x}\sum_{i=n_0}^{n-1} \Phi_a(i,n_0)}, & n_0 \le n,\ \bar{x} \ge 0, \\ \dfrac{\bar{x}\Phi_a(n,n_0)}{1 - \bar{x}\sum_{i=n}^{n_0-1} \Phi_a(i,n_0)}, & n < n_0 \end{cases} \tag{3.3}$$

and $\bar{x} \ge 0$ satisfying $\bar{x}\sum_{i=n}^{n_0-1} \Phi_a(i,n_0) < 1$.

Proof. Let $\bar{x} \ge 0$. It holds that $x(n_0; n_0, \bar{x}) = \bar{x}$ and moreover we get

$$\begin{aligned} x(n+1; n_0, \bar{x}) &\overset{(3.3)}{=} \frac{\bar{x}\prod_{i=n_0}^{n} a_i}{1 + \bar{x}\sum_{i=n_0}^{n}\prod_{j=n_0}^{i-1} a_j} \overset{(3.2)}{=} \frac{a_n\bar{x}\Phi_a(n,n_0)}{1 + \bar{x}\sum_{i=n_0}^{n-1}\prod_{j=n_0}^{i-1} a_j + \bar{x}\Phi_a(n,n_0)} \\ &\overset{(3.3)}{=} \frac{a_n x(n; n_0, \bar{x})}{1 + x(n; n_0, \bar{x})} \quad \text{for all } n_0 \le n. \end{aligned}$$

In order to obtain the relation for times $n < n_0$ one has to solve

$$\bar{x} = x(n; n_0, x(n_0; n, \bar{x})) = x(n; n_0, \xi) \quad \text{for all } n_0 < n$$

w.r.t. the variable ξ and arrives at the second formula given in (3.3). ☐

A particularly important solution of (BH) results in the pullback limit $n_0 \to -\infty$.

Lemma 3.2. *For $\mathbb{I} = \mathbb{Z}$ the Beverton-Holt equation* (BH) *has the entire solution*

$$\xi_n^* = \frac{1}{\sum_{i=-\infty}^{n-1} \Phi_a(i,n)} \quad \textit{for all } n \in \mathbb{Z}$$

and the following holds: If there exists an $n \in \mathbb{Z}$ such that

(a) $\prod_{i=-\infty}^{n-1} a_i = \infty$, then the solution $\xi_{\mathbb{Z}}^*$ is pullback attracting in $\mathbb{Z} \times (0,\infty)$.

(b) $\prod_{i=-\infty}^{n-1} a_i = 0$ or

$$0 < \inf_{i<n} \Phi_a(n,i) \le \sup_{i<m\le n} \Phi_a(m,i) < \infty, \tag{3.4}$$

then the zero solution is pullback attracting in $\mathbb{Z} \times \mathbb{R}_+$.

Proof. (a) For $\bar{x} > 0$ we have the limit relation

$$\lim_{n_0 \to -\infty} \frac{\bar{x}\Phi_a(n, n_0)}{1 + \bar{x}\sum_{i=n_0}^{n-1} \Phi_a(i, n_0)} = \xi_n^* \quad \text{for all } n \in \mathbb{Z}$$

which, together with Lemma 3.1 and $\prod_{i=-\infty}^{n-1} a_i = \infty$, follows for $n_0 \le n$ from

$$x(n; n_0, \bar{x}) \overset{(3.3)}{=} \frac{\bar{x}\Phi_a(n, n_0)}{1 + \bar{x}\sum_{i=n_0}^{n-1} \Phi_a(i, n_0)} = \frac{1}{\frac{1}{\bar{x}}\Phi_a(n_0, n) + \sum_{i=n_0}^{n-1} \Phi_a(i, n)}$$

by passing over to the limit $n_0 \to -\infty$.

(b) Similarly to (a), the condition $\prod_{i=-\infty}^{n-1} a_i = 0$ yields $\lim_{n_0 \to -\infty} x(n; n_0, \bar{x}) = 0$ for all $\bar{x} \ge 0$. Under the assumption (3.4) we abbreviate

$$a_- := \inf_{i<n} \Phi_a(n, i), \qquad a_+ := \sup_{i<m\le n} \Phi_a(m, i)$$

and deduce

$$0 \le x(n; n_0, \bar{x}) \overset{(3.3)}{=} \frac{\bar{x}\Phi_a(n, n_0)}{1 + \bar{x}\sum_{i=n_0}^{n-1} \Phi_a(i, n_0)} \le \frac{a_+\bar{x}}{1 + \bar{x}\sum_{i=n_0}^{n-1} a_-} \xrightarrow[n_0 \to -\infty]{} 0.$$

Being a pullback solution, Thm. A.1 ensures that $\xi_{\mathbb{Z}}^*$ solves (BH). □

Remark 3.3 (sufficient conditions). *(1) Lemma 3.2(a) occurs for sequences $a_{\mathbb{Z}}$ satisfying $\underline{\beta}(a_{\mathbb{Z}}) > 1$, where $\underline{\beta}$ denotes the lower Bohl exponent, cf. Appendix B.1.*

(2) Otherwise, for coefficient sequences $a_{\mathbb{Z}}$ with upper Bohl exponent $\overline{\beta}(a_{\mathbb{Z}}) < 1$, Lemma 3.2(b) applies. Subexponential pullback convergence to 0 *occurs under* (3.4).

After these preparations we are in a position to characterize the set $\mathcal{X}_a^*$:

Theorem 3.4 (pullback attractor of (BH)). *For $\mathbb{I} = \mathbb{Z}$ the pullback attractor of the Beverton-Holt equation* (BH) *is given by*

$$\mathcal{X}_a^* = \begin{cases} \mathbb{Z} \times \{0\}, & \prod_{i=-\infty}^{n-1} a_i = 0 \text{ or } (3.4), \\ \{(n, x) : 0 \le x \le \xi_n^*\}, & \prod_{i=-\infty}^{n-1} a_i = \infty. \end{cases}$$

Remark 3.5. *If one restricts to initial values $\bar{x} > 0$ the entire solution $\xi_{\mathbb{Z}}^*$ becomes pullback attracting in the sense that $\omega_{\mathcal{B}} = \xi_{\mathbb{Z}}^*$ holds for all bounded $\mathcal{B} \subseteq \mathbb{Z} \times (0, \infty)$.*

Proof. Under $\prod_{i=-\infty}^{n-1} a_i = 0$ or (3.4) the assertion follows immediately from Lemma 3.2(b). In case $\prod_{i=-\infty}^{n-1} a_i = \infty$ we can apply Thm. A.3 and its Cor. A.4 with the bounded sequences $x_n^- := 0$, $x_n^+ := a_n$ and the nonautonomous set $\mathcal{X}$. It obviously holds that $\xi_n^- = x_n^- = 0$ and as in the proof of Lemma 3.2 one shows that

$$\xi_n^+ = \lim_{k \to -\infty} x(n; k, a_k) = \xi_n^* \quad \text{for all } n \in \mathbb{Z}.$$

□

We next illustrate that Thm. 3.4 captures the autonomous situation as well.

Example 3.6 (attractor change in (BH')). *For $a_n \equiv \alpha$ on $\mathbb{Z}$ one has $\Phi_a(n, n_0) = \alpha^{n-n_0}$ and therefore the pullback attractor $\mathcal{X}_\alpha^*$ precisely changes at $\alpha = 1$ (cf.* (3.1)*). In particular,*

$$\xi_n^* = \frac{1}{\sum_{i=-\infty}^{n-1} \prod_{j=i}^{n-1} \frac{1}{\alpha}} = \alpha - 1 \quad \text{for all } n \in \mathbb{Z},\ \alpha > 1$$

and accordingly the entire solution $\xi^*_{\mathbb{Z}}$ degenerates to the fixed point $\alpha - 1$.

In addition, the explicit representation (3.3) shows that structural assumptions (periodicity, almost periodicity, asymptotic constancy) on the coefficient sequence $a_{\mathbb{Z}}$ carry over to the entire solution $\xi^*_{\mathbb{Z}}$ and hence the pullback attractor $\mathcal{X}^*_a$.

While the nonhyperbolic case $\alpha = 1$ was easily settled in the autonomous situation, the following example shows that for corresponding critical Bohl exponents

$$\underline{\beta}(a_{\mathbb{Z}}) = \overline{\beta}(a_{\mathbb{Z}}) = 1 \tag{3.5}$$

a distinction between the attractors $\mathcal{X}^*_a$ is more subtle in a time-variant setting:

Example 3.7. *Let $p \in \mathbb{R}$ and consider coefficient sequences*

$$a_j := \begin{cases} \left|\frac{j-1}{j}\right|^p, & j < 0, \\ 1, & j \geq 0 \end{cases}$$

with the limit behavior

$$\prod_{j=i}^{n-1} a_j = \prod_{j=i}^{n-1} \left|\frac{j-1}{j}\right|^p = \left|\frac{i-1}{\min\{n,-1\}}\right|^p \xrightarrow[i \to -\infty]{} \begin{cases} 0, & p < 0, \\ 1, & p = 0, \\ \infty, & p > 0 \end{cases}$$

and Bohl exponents fulfilling (3.5). *In case $p < 0$ we derive from Lemma 3.2(b) that the trivial solution is pullback attracting. Nonetheless, from the explicit expression*

$$\xi^*_n = \frac{1}{\sum_{i=-\infty}^{n-1} \left|\frac{n}{i-1}\right|^p} \quad \text{for all } n < 0$$

we observe that $\xi^*_{\mathbb{Z}}$ degenerates to the trivial solution for all $p \leq 1$. However, for exponents $p > 1$ the nontrivial entire solution $\xi^*_{\mathbb{Z}}$ becomes pullback attracting.

The change of the pullback attractors from Thm. 3.4 is understood as bifurcation in (BH). Indeed, one can illustrate the different approaches to a nonautonomous transcritical bifurcation from [22, 30] in the parameter-dependent Beverton-Holt model

$$x_{n+1} = \frac{a_n(\lambda) x_n}{1 + x_n}. \tag{BH$_\lambda$}$$

3.2. Transcritical bifurcation.

3.2.1. The case $a_n(\lambda) = \lambda \frac{p_{n+1}}{p_n}$. We choose the parameter sequence

$$a_n(\lambda) = \lambda \frac{p_{n+1}}{p_n}, \quad \text{where } 0 < p_- \leq p_n \leq p_+ < \infty \text{ for all } n \in \mathbb{Z}. \tag{3.6}$$

In this case, the entire solution introduced in Lemma 3.2 simplifies to

$$\xi^*_n(\lambda) = \frac{\lambda^n p_n}{\sum_{i=-\infty}^{n-1} \lambda^i p_i},$$

and a plot of this solution for different parameter sequences $p_{\mathbb{Z}}$ is given in Fig. 3.2. It clearly illustrates that the attractivity of the trivial solution gets transferred to a nontrivial constant, periodic, asymptotically constant resp. entire solution of (BH$_\lambda$) as λ

passes through the critical value 1, if the parameter sequence $(p_n)_{n\in\mathbb{Z}}$ has the corresponding time-dependence. Note that the heteroclinic choice of $(p_n)_{n\in\mathbb{Z}}$ in Fig. 3.2(c) leads to $a_n(\lambda) = \lambda$ for $n \neq 19$ and $a_{19} = 3\lambda$, which explains the homoclinic behavior of the solution $\xi^*(\lambda)$.

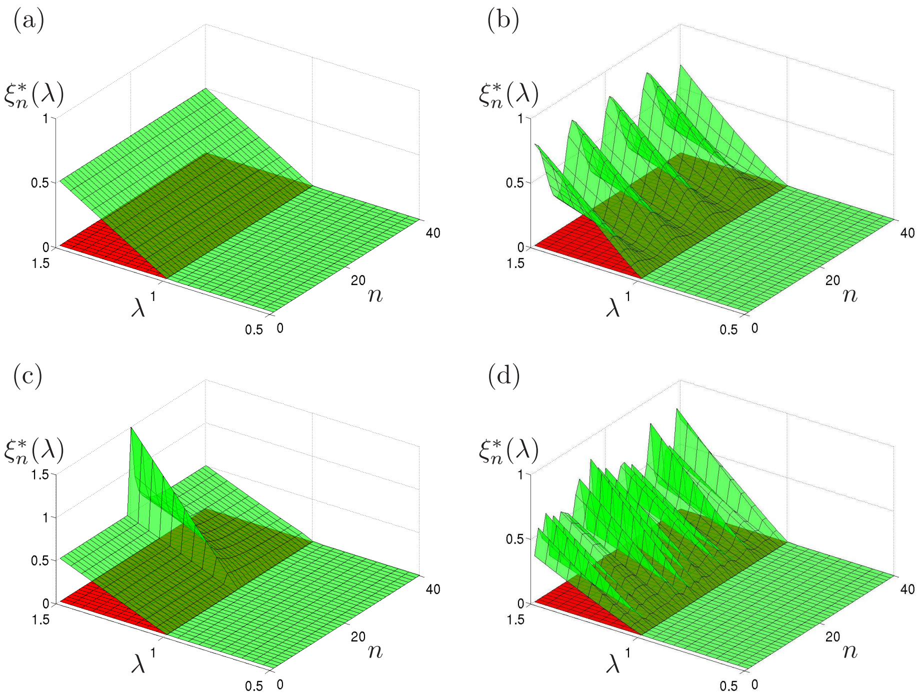

FIG. 3.2. *Nonautonomous transcritical bifurcation for the Beverton-Holt equation* (BH_λ) *with (a)* $p_n \equiv 1$*, (b)* $p_n = 1 + \frac{1}{2}\sin\frac{\pi n}{5}$*, (c)* $p_n = \frac{1}{2}$ $(n < 20)$ *and* $p_n = \frac{3}{2}$ $(n \geq 20)$*, (d)* $p_n = 1 + \frac{1}{2}r_n$ *with randomly chosen* $r_n \in [-1, 1]$.

In [11], the model function

$$x_{n+1} = h_n(x_n, \lambda), \quad \text{where } h_n(x, \lambda) = \frac{\lambda x}{1 + \frac{p_n}{\lambda}x}, \quad 0 < p_- \leq p_n \leq p_+ < \infty$$

is introduced that undergoes a nonautonomous transcritical bifurcation at $\lambda = 1$, as defined in [22] and [30]. Via the kinematic transformation $T_n(x) = \frac{p_n}{\lambda}x$ we get

$$T_{n+1} \circ h_n \circ T_n^{-1}(x) = \frac{\lambda\frac{p_{n+1}}{p_n}x}{1 + x},$$

which is our Beverton-Holt model (BH_λ) with $a_n(\lambda) = \lambda\frac{p_{n+1}}{p_n}$. Due to this topological equivalence, (BH_λ) also exhibits a transcritical bifurcation at $\lambda = 1$.

In the nonhyperbolic case $\lambda = 1$ one obtains

$$\prod_{i=-\infty}^{0} a_i(1) = \prod_{i=-\infty}^{0} \frac{p_{i+1}}{p_i} \in (0, \infty),$$

and by Lemma 3.2(b), all solutions of (BH_λ) pullback converge to 0. Thus, the trivial solution is pullback attractive for all parameters $0 < \lambda \leq 1$.

3.2.2. The case $a_n(\lambda) = 1 + \lambda\alpha_n$. We deal with sequences $a_n := 1 + \lambda\alpha_n$ and a bounded $\alpha_{\mathbb{I}}$. For $\lambda = 0$ equation (BH_λ) becomes autonomous and the unique equilibrium 0 is globally asymptotically stable. Provided the sequence is uniformly positive (or negative) it is possible to verify a transcritical bifurcation as understood in [30]. This means that attraction and repulsion radii of the trivial solution to (BH_λ) decay to 0 as $\lambda \to 0$.

3.3. Forward behavior. Since pullback attractors essentially capture the dynamical behavior in the past of a difference equation (cf. [17] and Ex. 3.12 below), we separately address the forward dynamics of (BH). First, one recaptures the feature that all forward solutions to the autonomous problem (BH') converge to the trivial solution for parameters $\alpha \in (0, 1]$. Second, it is worth to point out that already the stability of the linearization $x_{n+1} = a_n x_n$ implies an attractive zero solution to (BH):

PROPOSITION 3.8. *If $\Phi_a(\cdot, n_0) \in \ell^\infty$ for a $n_0 \in \mathbb{I}$, then*

$$\lim_{n\to\infty} x(n; \bar{n}, \bar{x}) = 0 \quad \textit{for all } \bar{n} \in \mathbb{I},\, \bar{x} \geq 0.$$

Note that the above assertion holds for a Bohl exponent $\overline{\beta}(a_{\mathbb{I}}) < 1$, whereas $\underline{\beta}(a_{\mathbb{I}}) > 1$ enforces the trivial solution of (BH) to be unstable (cf. Thm. 2.1(b)).

Proof. Let $\bar{n} \in \mathbb{I}$. Due to the inclusion $\ell^1 \subseteq \ell_0$ one has the decomposition

$$\ell^\infty = \ell_0 \cup (\ell^\infty \setminus \ell_0) = \ell_0 \cup (\ell^\infty \setminus \ell^1)$$

and therefore it suffices to consider the following two cases:

(I) $\Phi_a(\cdot, n_0) \in \ell_0$: Thanks to the elementary estimate $f_n(x) = \frac{a_n x}{1+x} \leq a_n x$ for all $n \in \mathbb{I}$ and $x \geq 0$, mathematical induction implies the inequalities

$$0 \leq x(n; \bar{n}, \bar{x}) \leq \bar{x}\Phi_a(n, \bar{n}) = \bar{x}\Phi_a(n_0, \bar{n})\Phi_a(n, n_0) \quad \text{for all } \bar{n} \leq n,\, \bar{x} \geq 0 \tag{3.7}$$

and the claim follows in the limit $n \to \infty$.

(II) $\Phi_a(\cdot, n_0) \in \ell^\infty \setminus \ell^1$: Here we can deduce

$$0 \leq x(n; \bar{n}, \bar{x}) \overset{(3.3)}{=} \frac{\bar{x}\Phi_a(n, \bar{n})}{1 + \bar{x}\sum_{i=\bar{n}}^{n-1} \Phi_a(i, \bar{n})} \leq \frac{\bar{x}\Phi_a(n_0, \bar{n}) \sup_{n_0 \leq k} \Phi_a(k, n_0)}{1 + \bar{x}\Phi_a(n_0, \bar{n}) \sum_{i=\bar{n}}^{n-1} \Phi_a(i, n_0)} \xrightarrow[n\to\infty]{} 0$$

and consequently obtain the assertion. ∎

The assumption of Prop. 3.8 ensures that all forward solutions to (BH) converge towards the trivial solution. An alternative summability condition — always fulfilled in the hyperbolic autonomous case $\alpha \neq 1$ (see Ex. 3.11) — that forward solutions approach each other is given in

PROPOSITION 3.9. *If $\lim_{n\to\infty} \left(\sum_{j=n_0}^{n-1} \Phi_a(j, n)\right) \sum_{i=n_0}^{n-1} \Phi_a(i, n_0) = \infty$ holds for a $n_0 \in \mathbb{I}$, then*

$$\lim_{n\to\infty} (x(n; n_0, \bar{x}_1) - x(n; n_0, \bar{x}_2)) = 0 \quad \textit{for all } 0 < \bar{x}_1, \bar{x}_2. \tag{3.8}$$

Proof. For $\bar{x}_1 = \bar{x}_2$ there is nothing to prove. Given reals $0 < \bar{x}_2 < \bar{x}_1$ the explicit representation from Lemma 3.1 yields

$$0 \leq x(n; n_0, \bar{x}_1) - x(n; n_0, \bar{x}_2) \overset{(3.3)}{=} \frac{\bar{x}_1\Phi_a(n, n_0)}{1 + \bar{x}_1 \sum_{i=n_0}^{n-1} \Phi_a(i, n_0)} - \frac{\bar{x}_2\Phi_a(n, n_0)}{1 + \bar{x}_2 \sum_{i=n_0}^{n-1} \Phi_a(i, n_0)}$$

$$= \frac{(\bar{x}_1 - \bar{x}_2)\Phi_a(n, n_0)}{\left(1 + \bar{x}_1 \sum_{i=n_0}^{n-1} \Phi_a(i, n_0)\right)\left(1 + \bar{x}_2 \sum_{i=n_0}^{n-1} \Phi_a(i, n_0)\right)}$$

$$\leq \frac{\bar{x}_1 - \bar{x}_2}{\bar{x}_1\bar{x}_2 \left(\sum_{j=n_0}^{n-1} \Phi_a(j, n)\right) \sum_{i=n_0}^{n-1} \Phi_a(i, n_0)} \xrightarrow[n\to\infty]{} 0,$$

by the assumed divergence condition. Interchanging $\bar{x}_1$ and $\bar{x}_2$ in the above estimate finally yields the claim for $0 < \bar{x}_1 \leq \bar{x}_2$, as well. □

In the autonomous case (BH') and for $\alpha > 1$ all nontrivial solutions converge to the fixed point $\alpha - 1$ in forward time. This behavior persists for the nonautonomous Beverton-Holt equation (BH) with $\alpha - 1$ replaced by the entire solution $\xi^*_{\mathbb{Z}}$:

Corollary 3.10. *If $\mathbb{I} = \mathbb{Z}$ and the limit relations*

$$\lim_{j\to-\infty} \Phi_a(n_0, j) = \infty, \qquad \lim_{n\to\infty} \left(\sum_{j=n_0}^{n-1} \Phi_a(j, n)\right) \sum_{i=n_0}^{n-1} \Phi_a(i, n_0) = \infty$$

hold for a $n_0 \in \mathbb{Z}$, then $\lim_{n\to\infty} (x(n; n_0, \bar{x}) - \xi^*_n) = 0$ for all $\bar{x} > 0$.

Proof. Set $\bar{x}_1 := \bar{x}$ and $\bar{x}_2 := \xi^*_{n_0}$ in (3.8) and use Lemma 3.2. □

For autonomous equations forward and pullback asymptotics coincide:

Example 3.11 (autonomous situation). *Suppose that $a_n \equiv \alpha$ on $\mathbb{Z}$ holds with some $\alpha > 0$. In case $\alpha = 1$ the asymptotic equivalence (and convergence to 0) follows from Prop. 3.8. On the other hand, for $\alpha \neq 1$, one gets $\Phi_a(n, j) = \alpha^{n-j}$ and*

$$\left(\sum_{j=n_0}^{n-1} \Phi_a(j, n)\right) \sum_{i=n_0}^{n-1} \Phi_a(i, n_0) = \frac{1 - \alpha^{n_0-n}}{\alpha - 1} \frac{\alpha^{n-n_0} - 1}{\alpha - 1}$$

$$= \frac{\alpha^{n-n_0} + \alpha^{n_0-n} - 2}{(\alpha - 1)^2} \xrightarrow[n\to\infty]{} \infty.$$

Hence, in case $\alpha \in [0, 1)$ all solutions converge to 0 in forward time. However, for $\alpha > 1$ the above Cor. 3.10 applies and yields forward convergence to the nontrivial fixed point $\alpha - 1$. In conclusion, the behavior from (3.1) occurs.

To illustrate that forward and pullback behavior of a nonautonomous equation (BH) can be different in general, we consider

Example 3.12. *Take a bounded sequence $p_{\mathbb{Z}}$ as in (3.6), choose a fixed $\alpha_- \in (0, 1)$ and consider $\lambda > 0$ as bifurcation parameter. We define the sequence*

$$a_n(\lambda) := \frac{p_{n+1}}{p_n} \begin{cases} \lambda, & n \geq 0, \\ \alpha_-, & n < 0 \end{cases} \quad \text{for all } n \in \mathbb{Z}$$

in (BH_λ) *and an easy computation implies*

$$\Phi_a(n, m) = \frac{p_n}{p_m} \begin{cases} \lambda^{n-m}, & 0 \leq m \leq n, \\ \lambda^n \alpha_-^{-m}, & m < 0 \leq n, \\ \alpha_-^{n-m}, & m \leq n < 0. \end{cases}$$

The boundedness of $p_{\mathbb{Z}}$ and the choice of $\alpha_- \in (0, 1)$ yields $\lim_{m\to-\infty} \Phi_a(n, m) = 0$ and by Thm. 3.4 the Beverton-Holt equation (BH_λ) has the pullback attractor $\mathbb{Z} \times \{0\}$ for all $\lambda > 0$. Yet, solutions starting in $\bar{x} > 0$ converge towards the entire solution $\xi^*_n(\lambda)$ in forward time, which needs not to be the trivial one.

4. Nonautonomous Ricker model. We focus on a nonautonomous Ricker equation (R) with right-hand side $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ (cf. Fig. 4.1) for bounded sequences $c_{\mathbb{I}}$ in $\mathbb{R}_+$. Given pairs $(n_0, \bar{y}) \in \mathbb{I} \times \mathbb{R}_+$, let us write $y(\cdot; n_0, \bar{y})$ for the general solution to (R) and obtain

- The nonautonomous set $\mathbb{I} \times \{0\}$ is invariant, while $\mathbb{I} \times \mathbb{R}_+$ and $\mathbb{I} \times (0, \infty)$ are forward invariant w.r.t. (R).
- All forward solutions are bounded, i.e. it is

$$0 < y(n; n_0, \bar{y}) \leq \frac{e^{c_{n-1}}}{e} \quad \text{for all } n > n_0,\ \bar{y} > 0, \tag{4.1}$$

  while backward solutions to initial values $\bar{y} > \frac{e^{c_n}}{e}$ do not exist.
- Comparing the Beverton-Holt and Ricker model for $a_n = e^{c_n}$, we immediately see that (cf. Fig. 2.1)

$$0 < y(n; n_0, \bar{y}) \leq x(n; n_0, \bar{y}) \overset{(3.3)}{=} \frac{\bar{y} \prod_{i=n_0}^{n-1} e^{c_i}}{1 + \bar{y} \sum_{i=n_0}^{n-1} \prod_{j=n_0}^{i-1} e^{c_j}},$$

  and one particularly has the estimate

$$0 < y(n; n_0, \bar{y}) \leq \bar{y} \prod_{j=n_0}^{n-1} e^{c_j} \quad \text{for all } n \geq n_0,\ \bar{y} > 0. \tag{4.2}$$

- Due to the assumption $c_n \geq 0$ one deduces $1 \leq \underline{\beta}(e^{c_{\mathbb{I}}}) \leq \overline{\beta}(e^{c_{\mathbb{I}}})$.

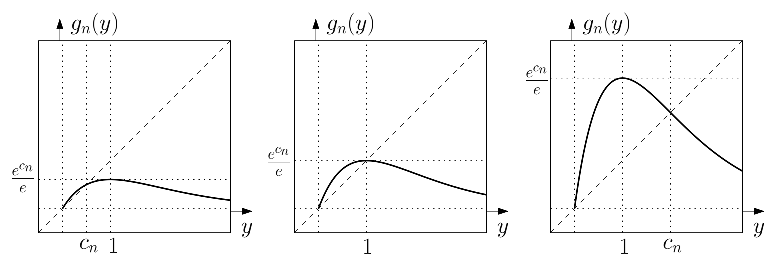

FIG. 4.1. *Right-hand side $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ of the Ricker equation* (R) *for $c_n \in (0, 1)$ (left), $c_n = 1$ (center) and $c_n > 1$ (right).*

As for the Beverton-Holt equation (BH) we obtain for $\mathbb{I} = \mathbb{Z}$ from Thm. 2.2:

- The nonautonomous set

$$\mathcal{Y} := \left\{(n, y) \in \mathbb{Z} \times \mathbb{R} : 0 \leq y \leq \tfrac{e^{c_{n-1}}}{e}\right\}$$

  is forward invariant and pullback absorbing w.r.t. (R).
- The set $\mathcal{Y}_c^*$ of all bounded entire solutions to (R) is invariant, compact, connected, it pullback attracts bounded subsets of $\mathbb{Z} \times \mathbb{R}_+$ and satisfies $\mathcal{Y}_c^* \subseteq \mathcal{Y}$.

Nevertheless, since (R) fails to be order-preserving in general (see Fig. 4.1) the dynamics inside of $\mathcal{Y}_c^*$ might be chaotic (cf. [24]).

4.1. Pullback attractor of (R)**.** For the autonomous Ricker equation

$$y_{n+1} = g(y_n), \qquad g(y) := ye^{\gamma - y} \tag{R'}$$

with the right-hand side $g : \mathbb{R}_+ \to \left[0, e^{\gamma-1}\right]$ the nontrivial equilibrium γ is asymptotically stable for $\gamma \in (0,2)$. Then one observes a flip bifurcation into a 2-periodic orbit, as γ crosses the critical value 2. For $\gamma = 2$ the right-hand side g of (R') has a negative Schwarzian $Sg(2) = -1$ and [5, p. 479, Thm. A.4] implies that $\gamma = 2$ is still an asymptotically stable fixed point of (R'). Moreover, as γ increases through 1, monotonicity of the solutions is lost. We observe that the pullback attractor is

$$\mathcal{Y}_\gamma^* = \begin{cases} \mathbb{Z} \times [0,\gamma], & \gamma \in (0,1], \\ \mathbb{Z} \times \left[0, \frac{e^\gamma}{e}\right], & \gamma > 1 \end{cases}$$

and at least for $\gamma \leq 1$ every solution starting in $(0,\gamma)$ is a strictly increasing heteroclinic connection between the fixed points 0 and γ.

We now approach the corresponding nonautonomous problem:

LEMMA 4.1. *If* $\sup_{n\in\mathbb{I}} c_n \leq 1$, *then the Ricker equation* (R) *is order preserving on the forward invariant set* $\mathbb{I} \times [0,1]$ *and its forward solutions satisfy*

$$\bar{y}_1 < \bar{y}_2 \quad \Rightarrow \quad y(n; n_0, \bar{y}_1) < y(n; n_0, \bar{y}_2) \quad \textit{for all } n_0 \leq n,\ \bar{y}_1, \bar{y}_2 \in [0,1].$$

Proof. Due to $c_n \leq 1$ one has $e^{c_n - 1} \leq 1$ for all $n \in \mathbb{I}$ and (4.1) implies that the set $\mathbb{I} \times [0,1]$ is forward invariant w.r.t. (R). Moreover, $g_n|_{[0,1]}$ is strictly increasing and consequently the forward solutions are order-preserving. □

THEOREM 4.2. *Let* $\mathbb{I} = \mathbb{Z}$. *The pullback attractor of the Ricker equation* (R) *is*

$$\mathcal{Y}_c^* = \begin{cases} \{(n,y) :\, 0 \leq y \leq \eta_n^*\}, & \sup_{n\in\mathbb{Z}} c_n \leq 1, \\ \left\{(n,y) :\, 0 \leq y \leq \frac{e^{c_n - 1}}{e}\right\}, & 1 \leq \inf_{n\in\mathbb{Z}} c_n \end{cases}$$

with the pullback solution $\eta_\mathbb{Z}^*$ *given by*

$$\eta_n^* := \lim_{k\to-\infty} y(n; k, 1) \quad \textit{for all } n \in \mathbb{Z}. \tag{4.3}$$

Proof. (I) First, we note that $c_n \leq 1$ implies the inclusion $\mathcal{Y} \subseteq \mathbb{Z} \times [0,1]$ and Lemma 4.1 guarantees that also $\mathbb{Z} \times [0,1]$ is forward invariant and pullback absorbing. Moreover, the dynamical characterization of the pullback attractor as the set consisting of all bounded entire solutions implies $\mathcal{Y}_c^* = \omega_\mathcal{Y} = \omega_{\mathbb{Z}\times[0,1]}$. Finally, because (R) is order preserving in $\mathbb{Z} \times [0,1]$, the claim results from Cor. A.4.

(II) In the complementary situation $c_n \geq 1$ for all $n \in \mathbb{Z}$, the pullback absorbing set $\mathcal{Y}$ turns out to be invariant. Hence, Cor. A.2 implies the claim. □

Although for time-varying coefficients c_n no nontrivial equilibrium of (R) exists, Thm. 4.2 shows that $\mathcal{Y}_c^*$ has a similar structure as in the autonomous situation, namely

$$\mathcal{Y}_c^* = \{(n,y) :\, 0 \leq y \leq \eta_n^*\}$$

with an entire bounded solution $\eta_\mathbb{Z}^*$ to (R) — at least in case $c_n \leq 1$. Let us next show that the pullback solution $\eta_\mathbb{Z}^*$ from (4.3) remains well-defined for a larger parameter regime, without essentially weakening its attraction properties.

This requires to introduce the *Lambert-W-function* $W : [-e^{-1}, \infty) \to [-1, \infty)$, i.e. the inverse of $x \mapsto xe^x$ restricted to $[-1,\infty)$.

LEMMA 4.3. *If the sequence* $c_\mathbb{I}$ *satisfies (cf. Fig. 4.2 (left))*

$$\tilde{y} := \underbrace{1 - W(e^{-1})}_{\approx 0.72} \leq \min\left\{c_n, g_n\left(\frac{e^{c_n-1}}{e}\right)\right\} \quad \textit{for all } n \in \mathbb{I}, \tag{4.4}$$

then the following holds true:

(a) (R) *possesses a forward invariant nonautonomous set*

$$\tilde{\mathcal{Y}}_c := \left\{(n,y) : \tilde{y} \leq y \leq \max\left\{1, \tfrac{e^{c_{n-1}}}{e}\right\}\right\}.$$

(b) For all $\bar{n} \in \mathbb{I}$ *and* $\bar{y}_1, \bar{y}_2 \in \tilde{\mathcal{Y}}_c(\bar{n})$ *one has*

$$|y(n;\bar{n},\bar{y}_1) - y(n;\bar{n},\bar{y}_2)| \leq \left(\prod_{j=\bar{n}}^{n-1} e^{c_j-2}\right)|\bar{y}_1 - \bar{y}_2| \quad \textit{for all } \bar{n} \leq n. \tag{4.5}$$

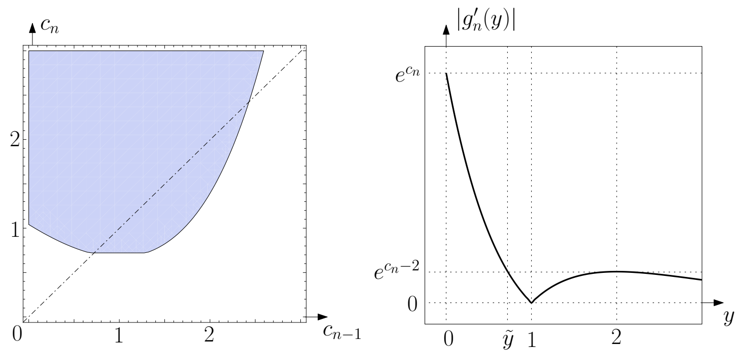

FIG. 4.2. *Parameter pairs* (c_{n-1}, c_n) *satisfying assumption* (4.4) *(left) and a plot of* $|g_n'|$ *to illustrate the value* $\tilde{y} \approx 0.72$ *from the proof of Lemma 4.3 (right).*

Proof. The derivative of g_n is $g_n'(y) = (1-y)e^{c_n-y}$ and consequently $|g_n'(\cdot)|$ has the global maximum $e^{c_n} \geq 1$ (for $y=0$) and the local maximum e^{c_n-2} (for $y=2$, cf. Fig. 4.2 (right)). Moreover, one easily sees that the equation $|g_n'(y)| = e^{c_n-2}$ possesses the two solutions $\tilde{y} = 1 - W(e^{-1})$ and 2. This particularly implies

$$\max_{\tilde{y}\leq y} |g_n'(y)| = e^{c_n-2} \quad \text{for all } n \in \mathbb{I}. \tag{4.6}$$

(a) We show the forward invariance of $\tilde{\mathcal{Y}}_c$. Due to (4.4) one has $\tilde{y} \leq c_n$ and thus

$$\tilde{y} \leq g_n(\tilde{y}) \quad \text{for all } n \in \mathbb{I} \tag{4.7}$$

(cf. Fig. 4.1), yielding the inclusions

$$\begin{aligned}
g_n(\tilde{\mathcal{Y}}_c(n)) &= g_n\left(\left[\tilde{y}, \max\left\{1, \tfrac{e^{c_{n-1}}}{e}\right\}\right]\right) \\
&\subseteq \left[\min\left\{g_n(\tilde{y}), g_n\left(\max\left\{1, \tfrac{e^{c_{n-1}}}{e}\right\}\right)\right\}, \max\left\{1, \tfrac{e^{c_n}}{e}\right\}\right] \\
&\subseteq \left[\min\left\{g_n(\tilde{y}), g_n\left(\tfrac{e^{c_{n-1}}}{e}\right)\right\}, \max\left\{1, \tfrac{e^{c_n}}{e}\right\}\right] \\
&\overset{(4.7)}{\subseteq} \left[\min\left\{\tilde{y}, g_n\left(\tfrac{e^{c_{n-1}}}{e}\right)\right\}, \max\left\{1, \tfrac{e^{c_n}}{e}\right\}\right] \\
&\overset{(4.4)}{\subseteq} \left[\tilde{y}, \max\left\{1, \tfrac{e^{c_n}}{e}\right\}\right] = \tilde{\mathcal{Y}}_c(n+1) \quad \text{for all } n \in \mathbb{I}.
\end{aligned}$$

(b) In order to show (4.5) we proceed by induction. The claim obviously holds for $n = \bar{n}$. In the induction step $n \to n+1$ we obtain from the mean value estimate

$$|y(n+1;\bar{n},\bar{y}_1) - y(n+1;\bar{n},\bar{y}_2)| \overset{(R)}{=} |g_n(y(n;\bar{n},\bar{y}_1)) - g_n(y(n;\bar{n},\bar{y}_2))|$$

$$\stackrel{(4.6)}{\leq} e^{c_n-2}\left|y(n;\bar n,\bar y_1)-y(n;\bar n,\bar y_2)\right| \leq \left(\prod_{j=\bar n}^{n} e^{c_j-2}\right)\left|\bar y_1-\bar y_2\right|$$

for all $\bar y_1, \bar y_2 \in \tilde{\mathcal{Y}}_c(\bar n)$. ∎

Having Lemma 4.3 at hand, the uniform bounds on $c_{\mathbb{Z}}$ in Thm. 4.2 can be weakened for the price of a smaller domain of attraction.

Proposition 4.4. *Let $\mathbb{I} = \mathbb{Z}$. If $c_{\mathbb{Z}}$ satisfies* (4.4) *and $\prod_{j=-\infty}^{n_0-1} e^{c_j-2} = 0$ holds for one $n_0 \in \mathbb{Z}$, then the pullback solution $\eta^*_{\mathbb{Z}}$ given in* (4.3) *solves* (R) *and pullback attracts the nonautonomous set $\tilde{\mathcal{Y}}_c$.*

Proof. For each fixed $n \in \mathbb{Z}$ our assumptions combined with

$$\begin{aligned}\left|y(n;k_1,1)-y(n;k_2,1)\right| &= \left|y(n;k_1,1)-y(n;k_1,y(k_1;k_2,1))\right| \\ &\stackrel{(4.5)}{=} \left(\prod_{j=k_1}^{n} e^{c_j-2}\right)\left|1-y(k_1;k_2,1)\right|\end{aligned}$$

for all $k_2 \leq k_1 \leq n$ (resp. an analogous estimate for $k_1 \leq k_2 \leq n$) readily yield that $(y(n;k,1))_{k\leq n}$ is a Cauchy sequence in the complete space $\tilde{\mathcal{Y}}_c(n)$ and thus convergent. Hence, by Thm. A.1 the sequence $\eta^*_{\mathbb{Z}}$ solves (R). Moreover, (4.5) also implies that $\tilde{\mathcal{Y}}_c$ is pullback attracted by $\eta^*_{\mathbb{Z}}$. ∎

In Fig. 4.3 we depicted the absorbing set $\tilde{\mathcal{Y}}_c$ and the entire bounded solution $\eta^*_{\mathbb{Z}}$ for different sequences $c_{\mathbb{Z}}$ fulfilling the assumptions of Prop. 4.4. Again it is indicated that structural properties of $c_{\mathbb{Z}}$ (constancy, periodicity, etc.) get transferred to the absorbing set and the pullback solution $\eta^*_{\mathbb{Z}}$:

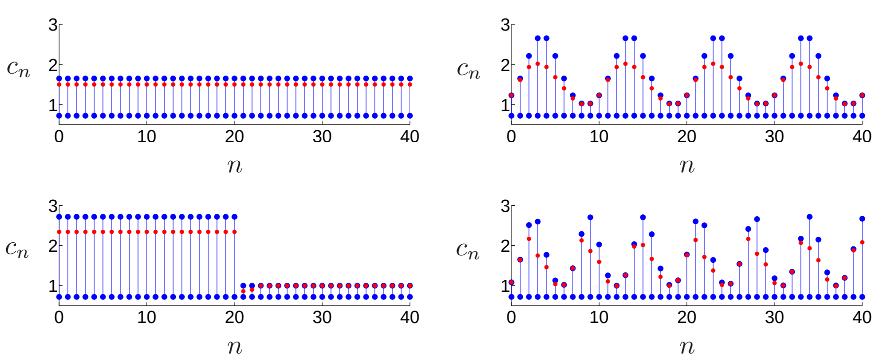

Fig. 4.3. *The pullback absorbing sets $\tilde{\mathcal{Y}}_c$ (blue shaded) containing the entire solutions $\eta^*_{\mathbb{Z}}$ (red dots) for sequences $c_n \equiv \frac{3}{2}$ (top left); $c_n = \frac{3}{2} + \frac{1}{2}\sin\frac{\pi n}{5}$ (10-periodic, top right); $c_n = 2$ ($n < 20$) and $c_n = 1$ ($n \geq 20$, bottom left); $c_n = \frac{3}{2} + \frac{1}{2}r_n$ ($r_n \in [-1,1]$ randomly chosen, bottom right).*

For 200 starting points y_0^i, $i = 1,\dots,200$, we computed the maximal distance between corresponding forward orbits $e_n := \max_{i,j=1,\dots,200}\left|y(n,0,y_0^i)-y(n,0,y_0^j)\right|$ in Fig. 4.4. In addition, parameter sequences are considered that (on purpose) do not satisfy the assumptions of Lemma 4.3. The convergence of these solutions towards each other in case (b) and (c) indicates that (4.4) might be of technical nature.

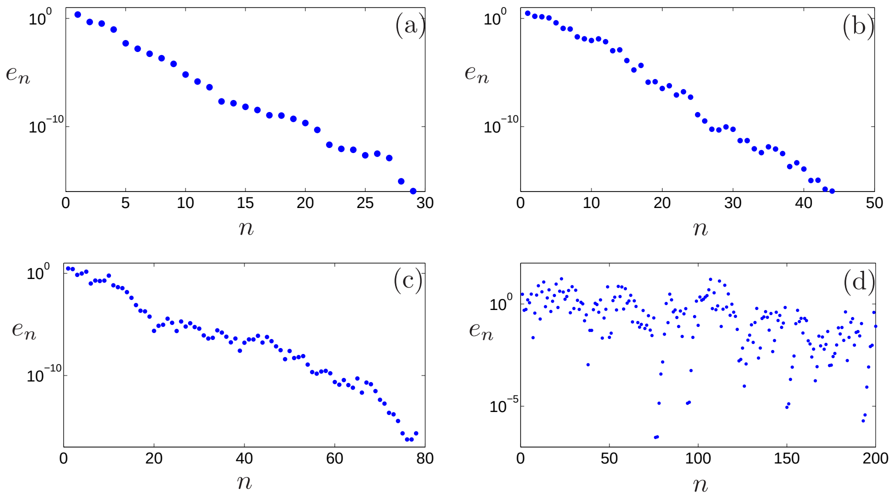

FIG. 4.4. *Maximal distance e_n between 200 solutions of* (R) *for (a): $y_0 \in (\tilde{y}, 3]$, $c_{\mathbb{I}} \in [\tilde{y}, 2]^{\mathbb{I}}$, (b): $y_0 \in (0, 3]$, $c_{\mathbb{I}} \in [0, 2]^{\mathbb{I}}$, (c): $y_0 \in (0, 3]$, $c_{\mathbb{I}} \in [0, 3]^{\mathbb{I}}$ and (d): $y_0 \in (0, 3]$, $c_{\mathbb{I}} \in [0, 4]^{\mathbb{I}}$, where the parameter sequences are uniformly distributed.*

Another possibility to obtain entire solutions to (R) in the pullback attractor $\mathcal{Y}_c^*$ follows from a perturbation argument: For $\gamma \neq 2$ the nontrivial equilibrium γ of the autonomous equation (R') persists as an entire bounded solution to (R) locally. Quantitatively one has

THEOREM 4.5. *Suppose that $\mathbb{I} = \mathbb{Z}$, $\gamma \neq 2$ and choose $\nu \in (0, 1)$. If $\varepsilon, \rho > 0$ fulfill*

$$\frac{2e^{\gamma}}{|\gamma - 2|} \frac{e^{2\rho} - 1}{\rho} \left(\rho + \frac{\varepsilon}{2}\right) \leq \nu, \qquad \frac{2\rho}{|\gamma - 2|} \left(1 + e^{\gamma+\rho}\right) \leq \varepsilon(1 - \nu),$$

then for every parameter sequence $c_{\mathbb{Z}}$ satisfying $\sup_{n\in\mathbb{Z}} |c_n - \gamma| < \rho$ there exists a unique bounded solution $\eta_{\mathbb{Z}}$ of (R) *such that $\sup_{n\in\mathbb{Z}} |\eta_n - \gamma| < \varepsilon$.*

Proof. We subdivide the proof into two steps:

(I) First, we introduce the functions $g, G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$$g(y, \gamma) := ye^{\gamma - y}, \qquad G(y, \gamma) := g_y(y, \gamma) = (1 - y)e^{\gamma - y}.$$

The elementary estimate

$$\left|(n - t)e^{\gamma - t}\right| \leq ne^{\gamma} \quad \text{for all } t \geq 0,\ n \in \mathbb{N},\ \gamma \in \mathbb{R} \tag{4.8}$$

yields $|G(y, \gamma)| \leq e^{\gamma}$ and therefore the partial derivative $g_y(\cdot, \gamma)$ is globally bounded. For all $y, \bar{y} \geq 0$ and $\gamma > 0$ the mean value theorem implies

$$|g(y, \gamma) - g(\bar{y}, \gamma)| \leq \int_0^1 |g_y(y + t(\bar{y} - y), \gamma)|\, dt\, |y - \bar{y}| \leq e^{\gamma} |y - \bar{y}|. \tag{4.9}$$

In addition, the derivative of G reads as $G'(y, \gamma) = (y - 2, 1 - y)e^{\gamma - y}$. Choosing an arbitrary $\rho > 0$, again the mean value theorem implies

$$|g_y(y, \gamma) - g_y(\bar{y}, \bar{\gamma})| \;\leq\; \int_0^1 \left| G'(y + t(\bar{y} - y), \gamma + t(\bar{\gamma} - \gamma)) \begin{pmatrix} y - \bar{y} \\ \gamma - \bar{\gamma} \end{pmatrix} \right| dt$$

$$\begin{aligned}
&\leq \int_0^1 |y + t(\bar{y} - y) - 2|\, e^{\gamma + t(\bar{\gamma}-\gamma) - y - t(\bar{y}-y)}\, dt\, |y - \bar{y}| \\
&\quad + \int_0^1 |y + t(\bar{y} - y) - 1|\, e^{\gamma + t(\bar{\gamma}-\gamma) - y - t(\bar{y}-y)}\, dt\, |\gamma - \bar{\gamma}| \\
&\overset{(4.8)}{\leq} 2e^{\gamma} \int_0^1 e^{t(\bar{\gamma}-\gamma)}\, dt\, |y - \bar{y}| + e^{\gamma} \int_0^1 e^{t(\bar{\gamma}-\gamma)}\, dt\, |\gamma - \bar{\gamma}| \\
&= e^{\gamma} \frac{e^{\bar{\gamma}-\gamma} - 1}{\bar{\gamma} - \gamma} \left(2\, |y - \bar{y}| + |\gamma - \bar{\gamma}|\right) \\
&\leq e^{\gamma} \frac{e^{2\rho} - 1}{2\rho} \left(2\, |y - \bar{y}| + |\gamma - \bar{\gamma}|\right) \quad \text{for all } y, \bar{y} \geq 0 \qquad (4.10)
\end{aligned}$$

and $\gamma, \bar{\gamma} > 0$ with $|\bar{\gamma} - \gamma| < 2\rho$, since $t \mapsto \frac{e^t - 1}{t}$ is strictly increasing.

(II) For fixed $\gamma > 0$ let $c_{\mathbb{Z}}, \bar{c}_{\mathbb{Z}}$ be sequences in $(0, \infty)$ satisfying $\sup_{n\in\mathbb{Z}} |c_n - \gamma| < \rho$ and $\sup_{n\in\mathbb{Z}} |\bar{c}_n - \gamma| < \rho$. This implies $c_n < \gamma + \rho$ and $|c_n - \bar{c}_n| < 2\rho$ for all $n \in \mathbb{Z}$, which yields the estimates

$$\begin{aligned}
|g(y, c_n) - g(\bar{y}, c_n)| &\overset{(4.9)}{\leq} \omega_0(|y - \bar{y}|), \\
|g_y(y, c_n) - g_y(\bar{y}, \bar{c}_n)| &\overset{(4.10)}{\leq} \omega_1\left(\sup_{n\in\mathbb{Z}} |c_n - \bar{c}_2|\,, |y - \bar{y}|\right) \quad \text{for all } y, \bar{y} \geq 0,\, n \in \mathbb{Z},
\end{aligned}$$

with the real functions $\omega_0(t) := e^{\gamma+\rho} t$ and $\omega_1(t, s) := e^{\gamma} \frac{e^{2\rho}-1}{\rho} \left(t + \frac{s}{2}\right)$. Then the claim follows from the quantitative perturbation result [27, Cor. 2.12]. □

4.2. Flip bifurcation. Standard results (cf. [31, p. 244, Thm. 3.1]) guarantee a flip bifurcation for the autonomous equation (R'), i.e. the nontrivial equilibrium γ bifurcates into an asymptotically stable 2-periodic solution for parameters $\gamma > 2$.

To obtain an impression of the behavior in the critical and nonhyperbolic situation $\gamma = 2$, we now investigate the autonomous (R') under time-varying perturbations

$$y_{n+1} = y_n e^{c_n(\gamma) - y_n} \qquad (\mathrm{R}_\gamma)$$

and suppose the parameter sequence is given as

$$c_n(\gamma) := \gamma + \varepsilon \gamma_n \quad \text{for all } n \in \mathbb{I},$$

with a bounded sequence $\gamma_{\mathbb{I}}$ and a real $\varepsilon \geq 0$. Even in this perturbed autonomous situation the basic problem arises that established nonautonomous bifurcation results (cf., e.g., [22, 30]) require a whole family (parametrized by γ) of bounded entire solutions $\eta(\gamma)_{\mathbb{I}}$ along which the bifurcation occurs with $\varepsilon = 0$ and $\eta(0) = \gamma$.

Numerically, we can approximate such bounded solutions $\eta(\gamma)$ by solving

$$y_{n+1} = y_n e^{c_n(\gamma) - y_n}, \quad n = n_-, \ldots, n_+ - 1$$

with projection or periodic boundary conditions $y_{n_-} = y_{n_+}$.

Assuming hyperbolicity (i.e. an exponential dichotomy of the variational equation) it turns out that approximation errors decay exponentially fast towards the midpoint of the finite interval, see [14].

With $\gamma_n \in [-1, 1]$ chosen randomly, we illustrate the accordingly computed bounded trajectories $\eta(\gamma)_{\mathbb{I}}$ in Fig. 4.5 for $\varepsilon = 0$ (left, autonomous case) and $\varepsilon = 0.02$ (right). The dichotomy spectrum of the corresponding linearization

$$y_{n+1} = (1 - \eta_n(\gamma)) e^{c_n(\gamma) - \eta_n(\gamma)} y_n$$

is illustrated in Fig. 4.6.

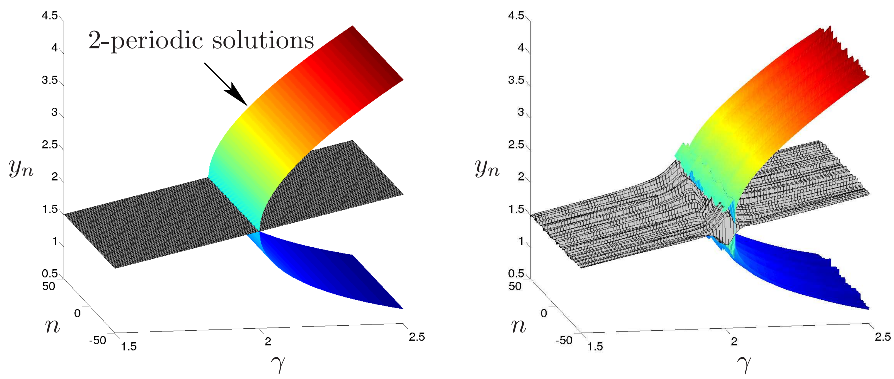

FIG. 4.5. *Bounded trajectories of the Ricker model* (R_γ) *for* $\varepsilon = 0$ *(left) and* $\varepsilon = 0.02$ *(right).*

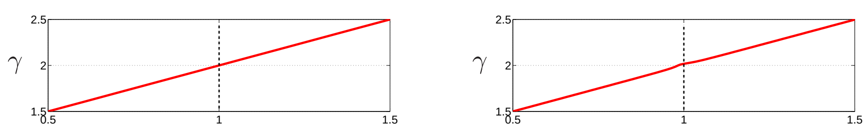

FIG. 4.6. *Dichotomy spectrum for the gray trajectories from Fig. 4.5 for parameters* $\varepsilon = 0$ *(left) and* $\varepsilon = 0.02$ *(right).*

Thus, we observe that the autonomous flip bifurcation in (R') at $\gamma = 2$ turns into a "nonautonomous flip bifurcation" in (R_γ) perturbation strengths $\varepsilon > 0$.

REMARK 4.6 (nonautonomous flip bifurcation). *Once a reference solution branch* $\eta(\gamma)$ *is known in a neighborhood of* $\gamma = 2$, *a nonautonomous flip bifurcation can be approached analytically as follows: As in the autonomous situation, where a period doubling means a pitchfork bifurcation in the second iterate of* (R'), *one introduces the equation of perturbed motion*

$$y_{n+1} = (y_n + \eta_n(\gamma))e^{c_n(\gamma)-y_n-\eta_n(\gamma)} - \eta_{n+1}(\gamma) =: \hat{g}_n(y_n, \gamma)$$

and applies nonautonomous bifurcation criteria from [22, 30] to the difference equation

$$y_{n+1} = G_n(y_n, \gamma), \qquad G_n(y, \gamma) := \hat{g}_{n+1}(\hat{g}_n(y, \gamma), \gamma)$$

near $\gamma = 2$. *Due to the tedious computations we skip the details here.*

Before addressing forward convergence of solutions, let us illustrate that the trivial solution to (R) can be both pullback attracting and unstable simultaneously:

EXAMPLE 4.7. *Let* $\mathbb{I} = \mathbb{Z}$. *For the sequence*

$$c_j := \begin{cases} \ln \frac{j+1}{j}, & j < -1, \\ \ln 2, & j \geq -1 \end{cases}$$

of positive coefficients, we obtain from (4.2) *the limit relation*

$$0 < y(n; n_0, \bar{y}) \leq \bar{y} \prod_{j=n_0}^{n-1} e^{c_j} = \frac{n}{n_0} \xrightarrow[n_0 \to -\infty]{} 0 \quad \textit{for all } n < -1, \, \bar{y} > 0$$

and therefore it is $\mathcal{Y}_c^* = \mathbb{Z}\times\{0\}$*. However, due to* $\underline{\beta}(e^{c_\mathbb{N}}) = \overline{\beta}(e^{c_\mathbb{N}}) = 2$ *our Thm. 2.1(b) implies that* 0 *is unstable.*

4.3. Forward behavior. Concerning the forward dynamics of the Ricker equation (R), Thm. 2.1(b) implies that the trivial solution is unstable, provided $\underline{\beta}(e^{c_\mathbb{I}}) > 1$. One readily deduces a criterion for asymptotic equivalence of all forward solutions.

PROPOSITION 4.8. *Under the assumptions*

$$\liminf_{n\to\infty} c_n > 0, \qquad\qquad \limsup_{n\to\infty} c_n < \ln 2 + 1 \approx 1.30$$

all nontrivial forward solutions to (R) *fulfill*

$$\lim_{n\to\infty}\left(y(n;\bar{n},\bar{y}_2) - y(n;\bar{n},\bar{y}_1)\right) = 0 \quad \text{for all } \bar{n}\in\mathbb{I},\ \bar{y}_1,\bar{y}_2 > 0.$$

Proof. The right-hand side g_n of (R) is strictly increasing in $[0,1]$ and strictly decreasing on $[1,\infty)$. At $y=1$ it achieves its global maximum e^{c_n-1}.

(I) Thanks to $c_* := \liminf_{n\to\infty} c_n > 0$ we can choose

$$\eta \in \left(0, \min\left\{c_*, 2e^{-2}\right\}\right)$$

and get an $N_1\in\mathbb{I}$ such that $c_n \geq \eta$ for all $n \geq N_1$. This implies $g_n(\eta) = \eta e^{c_n-\eta} \geq \eta$, as well as $\eta \leq \eta e^{c_n} \leq 2e^{c_n-2} = g_n(2) \leq g_n(2-\eta)$, since g_n decreases on the closed interval $[1,2]$. Due to $c^* := \limsup_{n\to\infty} c_n < \ln 2 + 1$ we can furthermore choose $\eta > 0$ so small that $c_n < \ln(2-\eta)+1$ holds for almost all $n\in\mathbb{I}$, say for every $n \geq N_2$. Consequently, it is $g_n(y) \leq e^{c_n-1} < 2-\eta$ for all $y \geq 0$ and we arrive at the inclusion

$$g_n([\eta, 2-\eta]) \subseteq [\eta, 2-\eta] \quad \text{for all } n \geq \max\{N_1, N_2\}.$$

(II) For every $y\in[\eta,2-\eta]$ we obtain

$$\left|\frac{g_n'(y)}{g_n(y)}\right| = \left|\frac{(1-y)}{y}\right| = \frac{|1-y|}{y} \leq \frac{|1-\eta|}{y} \quad \text{for all } y\in[\eta,2-\eta]$$

and according to [21, Lemma 2.1] the mappings g_n are $|1-\eta|$-cave functions. Now because of the estimate $|1-\eta| < 1$ we obtain from [21, Thm. 3.2(ii)] that

$$\lim_{n\to\infty}\left|y(n;\bar{n},\bar{y}_1) - y(n;\bar{n},\bar{y}_2)\right| = 0 \quad \text{for all } \bar{n} \geq \max\{N_1,N_2\}$$

and $\bar{y}_1,\bar{y}_2\in[\eta,2-\eta]$. Since the Ricker equation (R) is permanent (cf. [33]), every forward solution to (R) eventually enters an interval $[\eta, 2-\eta]$ with sufficiently small $\eta > 0$. Hence, the above limit relation even holds for all $\bar{n}\in\mathbb{I}$ and $\bar{y}_1,\bar{y}_2 > 0$. ∎

An alternative condition on $c_\mathbb{I}$ for asymptotic equivalence of all solutions is

PROPOSITION 4.9. *If* $c_\mathbb{I}$ *satisfies* (4.4) *and* $\prod_{j=n_0}^{\infty} e^{c_j-2} = 0$ *for a* $n_0\in\mathbb{I}$*, then*

$$\lim_{n\to\infty}\left(y(n;\bar{n},\bar{y}_1) - y(n;\bar{n},\bar{y}_2)\right) = 0 \quad \text{for all } \bar{n}\in\mathbb{I},\ \bar{y}_1,\bar{y}_2 > 0.$$

REMARK 4.10. *The limit relation* $\prod_{j=n_0}^{\infty} e^{c_j-2} = 0$ *always holds in the setting of Prop. 4.8. Indeed, even if* $c^* := \limsup_{n\to\infty} c_n < 2$*, then there exists a* $N\in\mathbb{I}$ *such that* $c_n < \frac{c^*+2}{2} < 2$ *for all* $n \geq N$ *and consequently* $\prod_{j=n_0}^{\infty} e^{c_j-2} = 0$.

Proof. Since $\prod_{j=n_0}^{\infty} e^{c_j-2} = 0$ implies $\prod_{j=\bar{n}}^{\infty} e^{c_j-2} = 0$ for all $\bar{n} \in \mathbb{I}$, it suffices to restrict to initial times $\bar{n} = n_0$. Then, in case $\bar{y}_1, \bar{y}_2 \in \tilde{\mathcal{Y}}_c(n_0)$ the proof follows immediately from Lemma 4.3(b).

Let $\bar{y}_1, \bar{y}_2 > 0$. If the parameter sequence $c_{\mathbb{I}}$ satisfies $\lim_{n\to\infty} c_n = \tilde{y}$, then both solutions converge towards the asymptotic fixed point $\tilde{y}$. Otherwise, there is an $N(\bar{y}_1, \bar{y}_2) \in \mathbb{I}$ such that $y(n; n_0, \bar{y}_1)$, $y(n; n_0, \bar{y}_2) \in \tilde{\mathcal{Y}}_c(n_0)$ for all $n \geq N(\bar{y}_1, \bar{y}_2)$ and the claim follows from Lemma 4.3(b). ∎

COROLLARY 4.11. *Let $\mathbb{I} = \mathbb{Z}$. If the sequence $c_{\mathbb{Z}}$ satisfies* (4.4) *and*

$$\prod_{j=-\infty}^{n_0-1} e^{c_j-2} = \prod_{j=n_0}^{\infty} e^{c_j-2} = 0 \quad \textit{for some } n_0 \in \mathbb{Z},$$

then one has the limit relation

$$\lim_{n\to\infty} (y(n; n_0, \bar{y}) - \eta_n^*) = 0 \quad \textit{for all } \bar{y} \in \tilde{\mathcal{Y}}_c(n_0).$$

Proof. Set $\bar{y}_1 := \bar{y}$ and $\bar{y}_2 := \eta_{n_0}^*$ in (4.5). ∎

5. Global Dynamics. In this section, we tackle the planar system (Δ) to obtain information on its global dynamics. Above all, the first (and biologically relevant) quadrant $\mathbb{R}_+^2$ is forward invariant w.r.t. (Δ), i.e. each mapping $F_n : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, $n \in \mathbb{I}$, is well-defined. As above, $\varphi = (\varphi_1, \varphi_2)$ is the general solution of (Δ).

5.1. Pullback attractor. LEMMA 5.1. *For $\mathbb{I} = \mathbb{Z}$ the nonautonomous set*

$$\mathcal{A} := \left\{(n, x, y) \in \mathbb{R}^2 : 0 \leq x \leq a_{n-1},\, 0 \leq y \leq \tfrac{e^{c_{n-1}}}{e}\right\}$$

is forward invariant and pullback absorbing w.r.t. (Δ).

Proof. For arbitrary reals $x, y \geq 0$ one has

$$0 \leq \frac{a_n x}{1 + x + b_n y} \leq \frac{a_n x}{1 + x} \leq a_n, \qquad 0 \leq y e^{c_n - d_n x - y} \leq y e^{c_n - y} \leq \frac{e^{c_n}}{e} \tag{5.1}$$

and thus the right-hand side of (Δ) satisfies $F_n(\mathbb{R}_+^2) \subseteq \mathcal{A}(n+1)$ for $n \in \mathbb{Z}$. By means of (5.1), induction yields $\varphi(n; n_0, \mathbb{R}_+^2) \subseteq \mathcal{A}(n)$ for $n_0 < n$ and $\mathcal{A}$ is pullback absorbing. In particular, $F_n(\mathcal{A}(n)) \subseteq \mathcal{A}(n+1)$ and consequently $\mathcal{A}$ is also forward invariant. ∎

THEOREM 5.2 (pullback attractor for (Δ)). *If $\mathbb{I} = \mathbb{Z}$, then the set $\mathcal{A}^*$ of all bounded entire solutions to* (Δ) *is invariant, compact, connected, it pullback attracts every bounded nonautonomous subset of $\mathbb{Z} \times \mathbb{R}_+^2$ and satisfies $\mathcal{A}^* \subseteq \mathcal{A}$.*

Proof. On the basis of Lemma 5.1 the proof follows along the lines of Thm. 2.2 applying the corresponding results from [25]. ∎

Since both coordinate axes are forward invariant w.r.t. (Δ), the nonautonomous set $\mathcal{A}^*$ contains the pullback attractors $\mathcal{X}_a^*$ of (BH) and $\mathcal{Y}_c^*$ of (R) in the sense that

$$\mathcal{X}_a^* \times \{0\} \subseteq \mathcal{A}^*, \qquad \{0\} \times \mathcal{Y}_c^* \subseteq \mathcal{A}^* \tag{5.2}$$

and also $\mathbb{Z} \times \{(0, 0)\} \subseteq \mathcal{A}^*$.

LEMMA 5.3. *For all $\xi, \eta \geq 0$ one has the estimates*

$$\begin{aligned} 0 \leq \varphi_1(n; n_0, \xi, \eta) &\leq x(n; n_0, \xi) \quad \textit{for all } n_0 \leq n, \\ 0 \leq \varphi_2(n; n_0, \xi, \eta) &\leq \tfrac{e^{c_n}}{e} \quad \textit{for all } n_0 < n. \end{aligned} \tag{5.3}$$

Proof. The second estimate is essentially a consequence of the forward invariance of $\mathcal{A}$ shown in Lemma 5.1. The first estimate follows by induction: It obviously holds for $n = n_0$. Concerning the induction step $n \to n+1$ one has

$$\begin{aligned} 0 \leq \varphi_1(n+1; n_0, \xi, \eta) &\stackrel{(\Delta)}{=} \frac{a_n \varphi_1(n; n_0, \xi, \eta)}{1 + \varphi_1(n; n_0, \xi, \eta) + b_n \varphi_2(n; n_0, \xi, \eta)} \\ &\leq f_n(\varphi_1(n; n_0, \xi, \eta)) \leq f_n(x(n; n_0, \xi)) = x(n+1; n_0, \xi), \end{aligned}$$

since f_n is strictly increasing, and this completes the proof. □

Our following result is a criterion ensuring that the pullback behavior of the planar difference equation (Δ) is dominated by the Ricker dynamics of (R):

THEOREM 5.4. *Let $\mathbb{I} = \mathbb{Z}$. If $\prod_{j=-\infty}^{n_0-1} a_j = 0$ for a $n_0 \in \mathbb{Z}$, then $\mathcal{A}^* = \{0\} \times \mathcal{Y}_c^*$.*

Proof. The set $\{0\} \times \mathcal{Y}_c^*$ is closed and a combination of Lemma 5.3 with (3.7) shows that $\{0\} \times \mathcal{Y}_c^*$ is pullback attracting. Then [25, p. 19, Thm. 1.3.9(b)] implies $\mathcal{A}^* \subseteq \{0\} \times \mathcal{Y}_c^*$ and the claim follows together with (5.2). □

LEMMA 5.5. *Let $\mathbb{I} = \mathbb{Z}$. If $\sup_{n\in\mathbb{Z}} c_n \leq 1$, then (Δ) is order-preserving (w.r.t. the south-east cone $C := \mathbb{R}_+ \times (-\infty, 0]$) on the forward invariant and pullback absorbing set*

$$\mathcal{A}_0 := \left\{(n, x, y) \in \mathbb{Z} \times \mathbb{R}_+^2 : x \leq a_{n-1},\, y \leq 1\right\}.$$

REMARK 5.6. *In the terminology of [34], Lemma 5.5 shows that (Δ) is* competitive *on $\mathcal{A}_0$.*

Proof. The estimates (5.1) imply that $\mathcal{A}_0$ is forward invariant and as a superset of $\mathcal{A}$, by Lemma 5.1 also pullback absorbing. Thus, (Δ) defines a difference equation in the nonautonomous set $\mathcal{A}_0$. With arbitrary pairs $(x, y) \in C$ we derive

$$F_n'(\xi, \eta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a_n(1+b_n\eta)}{(1+\xi+b_n\eta)^2} & -\frac{a_n b_n \xi}{1+\xi+b_n\eta} \\ -d_n\eta e^{c_n - d_n\xi - \eta} & e^{c_n - d_n\xi-\eta}(1-\eta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \in C \quad \text{for all } n \in \mathbb{I}$$

and $\xi \geq 0$, $\eta \in [0, 1]$. Therefore, the derivative $F_n'(\xi, \eta)$ is positive (w.r.t. the cone C) and the criterion [9, Lemma 2.2] ensures that $F_n|_{\mathcal{A}_0(n)}$ is order-preserving. □

THEOREM 5.7. *Let $\mathbb{I} = \mathbb{Z}$. If $\sup_{n\in\mathbb{Z}} c_n \leq 1$, then the pullback attractor of (Δ) fulfills*

$$\mathcal{A}^* \subseteq \left\{(n, \xi, \eta) \in \mathbb{Z} \times \mathbb{R}_+^2 : \xi \leq \lim_{k\to-\infty} x(n; k, a_{k-1}),\, \eta \leq \lim_{k\to-\infty} y\big(n; k, \tfrac{e^{c_{k-1}}}{e}\big)\right\}.$$

Proof. We use the terminology from the above proof of Lemma 5.5, where the south-east cone C induces a partial ordering

$$(x_1, y_1) \preceq (x_2, y_2) \quad :\Leftrightarrow \quad x_1 \leq y_1 \text{ and } y_2 \leq y_1$$

on $\mathbb{R}^2$. Let us apply Thm. A.3 to the sequences $(x_n^-, y_n^-) := (0, \frac{e^{c_{n-1}}}{e})$ and $(x_n^+, y_n^+) := (a_{n-1}, 0)$ satisfying $(x_n^-, y_n^-) \preceq (x_n^+, y_n^+)$ for all $n \in \mathbb{Z}$. The resulting set $\mathcal{A}_0$ is forward invariant and consequently Thm. A.3 yields the entire solutions

$$\zeta_n^- := \lim_{k\to-\infty} \varphi(n; k, (0, \tfrac{e^{c_{n-1}}}{e})) = \begin{pmatrix} 0 \\ \lim_{k\to-\infty} y(n; k, \frac{e^{c_{k-1}}}{e}) \end{pmatrix},$$

$$\zeta_n^+ := \lim_{k\to-\infty} \varphi(n;k,(a_{k-1},0)) = \begin{pmatrix} \lim_{k\to-\infty} x(n;k,a_{k-1}) \\ 0 \end{pmatrix}$$

for all $n \in \mathbb{Z}$. Hence, thanks to Cor. A.4 we deduce the claimed inclusion

$$\begin{aligned} \mathcal{A}^* &\subseteq \left\{(n,x,y) \in \mathbb{Z}\times\mathbb{R}_+^2 : \zeta_n^- \preceq (x,y) \preceq \zeta_n^+\right\} \\ &= \left\{(n,\xi,\eta) \in \mathbb{Z}\times\mathbb{R}_+^2 : \xi \le \lim_{k\to-\infty} x(n;k,a_{k-1}),\, \eta \le \lim_{k\to-\infty} y(n;k,\tfrac{e^{c_{n-1}}}{e})\right\} \end{aligned}$$

and the proof is completed. □

Corollary 5.8. *If $\prod_{j=-\infty}^{n_0-1} a_j = \infty$ for a $n_0 \in \mathbb{Z}$ and $c_{\mathbb{Z}}$ satisfies* (4.4)*, then*

$$\mathcal{A}^* \subseteq \left\{(n,x,y) \in \mathbb{Z}\times\mathbb{R}_+^2 : x \le \xi_n^*,\, y \le \eta_n^*\right\}.$$

Proof. On the one hand, by Lemma 3.2(a) we see that $\xi_{\mathbb{Z}}^*$ is pullback attracting w.r.t. (BH) and hence $\lim_{k\to-\infty} x(n;k,a_{k-1}) = \xi_n^*$ holds true for every $n \in \mathbb{Z}$. On the other hand, because of $c_n \le 1$ one has $\prod_{j=-\infty}^{n-1} e^{c_j-2} = 0$ and the set $\tilde{\mathcal{Y}}_c$ from Lemma 4.3 allows the inclusion $\left(n, \frac{e^{c_{n-1}}}{e}\right) \in \tilde{\mathcal{Y}}_c$. By Prop. 4.4, in particular the sequence $\left(n, \frac{e^{c_{n-1}}}{e}\right)$ is pullback attracted to $\eta_{\mathbb{Z}}^*$ and thus $\lim_{k\to-\infty} y(n;k,\frac{e^{c_{k-1}}}{e}) = \eta_n^*$ is satisfied for all $n \in \mathbb{Z}$. □

5.2. Forward behavior. Since the right-hand side of (Δ) is globally bounded, we obtain that all its forward solutions are bounded and immediately contained in the invariant rectangle (cf. (5.1))

$$\left[0, \sup_{n\in\mathbb{I}} a_n\right] \times \left[0, e^{\sup_{n\in\mathbb{I}} c_n - 1}\right].$$

The subsequent results illustrate the role of the parameter $a_{\mathbb{I}}$ in the extinction of one species x or y. Indeed, a Beverton-Holt equation (BH) with an attractive trivial solution (see Prop. 3.8) guarantees an asymptotically vanishing population x for (Δ):

Theorem 5.9 (extinction of x). *If $\Phi_a(\cdot,n_0) \in \ell^\infty$ for a $n_0 \in \mathbb{I}$, then*

$$\lim_{n\to\infty} \operatorname{dist}\left(\varphi(n;n_0,\bar{x},\bar{y}), \{0\}\times\mathbb{R}_+\right) = 0 \quad \textit{for all } \bar{x},\bar{y} \ge 0.$$

Proof. Thanks to Prop. 3.8 and Lemma 5.3 one has

$$0 \le \operatorname{dist}(\varphi(n;n_0,\bar{x},\bar{y}), \{0\}\times\mathbb{R}_+) = \varphi_1(n;n_0,\bar{x},\bar{y}) \overset{(5.3)}{\le} x(n;n_0,\bar{x}) \xrightarrow[n\to\infty]{} 0$$

and therefore the claim. □

On the other hand, sufficiently large values of $a_{\mathbb{I}}$ in comparison to $c_{\mathbb{I}}$ (cf. (5.4) for a concretization) yield to extinction of the y-species in the full equation (Δ):

Theorem 5.10 (extinction of y). *If there exists an $s > 0$ such that*

$$\limsup_{n\to\infty} \frac{e^{c_n}}{a_n^s} \max\left\{(sb_n)^s e^{\frac{1}{b_n}-s}, \left(\tfrac{s}{d_n}\right)^s e^{d_n-s}\right\} < 1, \tag{5.4}$$

then $\lim_{n\to\infty} \operatorname{dist}\left(\varphi(n;n_0,\bar{x},\bar{y}), \mathbb{R}_+\times\{0\}\right) = 0$ *for all* $\bar{x},\bar{y} > 0$.

Remark 5.11. *If the sequence $b_{\mathbb{I}}$ is eventually constant to a value $\beta > 0$, one can choose $s := \beta^{-1}$ and the condition* (5.4) *becomes*

$$\limsup_{n\to\infty} \tfrac{e^{c_n}}{a_n^s}\left(\tfrac{1}{\beta d_n}\right)^{\frac{1}{\beta}} e^{d_n-\beta} < 1.$$

For $d_{\mathbb{I}}$ being eventually equal to $\delta > 0$, the choice $s := \delta$ allows to simplify (5.4) *to*

$$\limsup_{n\to\infty} \tfrac{e^{c_n}}{a_n^s}(\delta b_n)^{\delta} e^{\frac{1}{b_n}-\delta} < 1;$$

this is based on the fact that the functions $s \mapsto (sb_n)^s e^{\frac{1}{b_n}-s}$ and $s \mapsto \left(\frac{s}{d_n}\right)^s e^{d_n-s}$ attain their global minimum 1 *on* $(0,\infty)$ *for $s = b_n^{-1}$ resp. $s = d_n$.*

Proof. Following [6, Thm. 5.2] we define $V(x,y) := \frac{y}{x^s}$. In order to show that V is a Lyapunov function for the nonautonomous equation (Δ) on $(0,\infty)^2$ one has

$$\frac{V(F_n(x,y))}{V(x,y)} = \frac{e^{c_n}}{a_n^s} v_n(x,y) \quad \text{for all } n \in \mathbb{I}$$

with $v_n(x,y) := (1+x+b_n y)^s e^{-d_n x - y}$ defined on $\mathbb{R}_+^2$. As in [6, Thm. 5.2] one derives

- $b_n d_n \geq 1$ implies that $v_n(x,y) \leq v_n\left(0, \frac{sb_n-1}{b_n}\right) = (sb_n)^s e^{\frac{1}{b_n}-s}$
- $b_n d_n \leq 1$ implies that $v_n(x,y) \leq v_n\left(\frac{s-d_n}{d_n}, 0\right) = \left(\frac{s}{d_n}\right)^s e^{d_n-s}$

for all $n \in \mathbb{I}$, $x, y \geq 0$. Therefore, our assumptions imply

$$\frac{V(F_n(x,y))}{V(x,y)} \leq \frac{e^{c_n}}{a_n^s} \max\left\{(sb_n)^s e^{\frac{1}{b_n}-s}, \left(\tfrac{s}{d_n}\right)^s e^{d_n-s}\right\},$$

thanks to (5.4) there exist $\mu \in (0,1)$, $N \in \mathbb{I}$ with $\frac{V(F_n(x,y))}{V(x,y)} \leq 1-\mu$ for all $n \geq N$ and $x, y > 0$. Because of $V(F_n(x,y)) - V(x,y) \leq -\mu V(x,y) \leq 0$ for all $n \geq N$, $x, y > 0$ this establishes V as Lyapunov function for (Δ) in the sense of [23, p. 49, Def. 7.1]. Since all solutions to (Δ) are bounded, [23, p. 50, Thm. 7.2] yields

$$\varphi(n; n_0, \bar{x}, \bar{y}) \xrightarrow[n\to\infty]{} \left\{(x,y) \in \mathbb{R}_+^2 : V(x,y) = 0\right\} = \mathbb{R}_+ \times \{0\} \quad \text{for all } \bar{x}, \bar{y} > 0$$

and the proof is complete. ◻

6. Nonautonomous equilibria and stability. Our next objective is a more detailed understanding of the dynamics in the pullback attractor $\mathcal{A}^*$ for (Δ). The starting point for our investigations are the equilibria of the autonomous system (Δ') and their stability, especially with regard to their persistence when passing over to the full nonautonomous problem (Δ). Since the corresponding linearizations are time-variant, it is well-known (cf., e.g., [5, p. 190, Ex. 4.17]) that eigenvalues in general do not yield stability information. Hence, the appropriate tools for stability investigations are the dichotomy spectrum and Bohl exponents (cf. [1, 13]); for readers unfamiliar with this concept, we have summarized some essential aspects in Appendix B. When dealing with Bohl exponents let us implicitly assume that the associated sequences have bounded inverses (cf. (B.1)).

We start our analysis with the trivial equilibrium and continue with increasingly more involved cases.

6.1. The trivial equilibrium. The trivial equilibrium $(0,0)$ of (Δ) exists for all parameter constellations and yields the linearization

$$F_n'(0,0) = \begin{pmatrix} a_n & 0 \\ 0 & e^{c_n} \end{pmatrix}$$

in the variational equation (V). Since all $F_n'(0,0)$ are diagonal, the dichotomy spectrum is the union of the spectra for the diagonal elements (cf. Prop. B.5), i.e.

$$\Sigma(0,0) = \left[\underline{\beta}(a_{\mathbb{I}}), \overline{\beta}(a_{\mathbb{I}})\right] \cup \left[\underline{\beta}(e^{c_{\mathbb{I}}}), \overline{\beta}(e^{c_{\mathbb{I}}})\right].$$

Due to $1 \leq \underline{\beta}(e^{c_{\mathbb{I}}})$ this offers the following possibilities for the local asymptotics of (Δ) near the origin, i.e. the stability properties of the trivial solution (cf. Thm. 2.1):

- In the Beverton-Holt-stable case $\overline{\beta}(a_{\mathbb{I}}) < 1 < \underline{\beta}(e^{c_{\mathbb{I}}})$ it is unstable, despite the x-axis as stable direction. For $\mathbb{I} = \mathbb{Z}$ the y-axis becomes the unstable fiber bundle and the origin is a *saddle*.
- Finally, for $\min\left\{\underline{\beta}(a_{\mathbb{I}}), \underline{\beta}(e^{c_{\mathbb{I}}})\right\} > 1$ it is unstable in form of a *source*.

6.2. The Beverton-Holt equilibrium. For a constant sequence $a_n \equiv \alpha > 0$ the difference equation (Δ) has the so-called *Beverton-Holt equilibrium* $(\alpha - 1, 0)$, which is only present for values $\alpha \geq 1$. The linearization

$$F_n'(\alpha - 1, 0) = \begin{pmatrix} \frac{1}{\alpha} & \frac{1-\alpha}{\alpha} b_n \\ 0 & e^{c_n - (\alpha-1)d_n} \end{pmatrix} \quad \text{for all } n \in \mathbb{I}$$

is upper-triangular and yields the dichotomy spectrum

$$\Sigma(\alpha - 1, 0) = \begin{cases} \left\{\frac{1}{\alpha}\right\} \cup \left[\underline{\beta}(e^{c_{\mathbb{I}} - (\alpha-1)d_{\mathbb{I}}}), \overline{\beta}(e^{c_{\mathbb{I}} - (\alpha-1)d_{\mathbb{I}}})\right], & \alpha > 1, \\ \{1\} \cup \left[\underline{\beta}(e^{c_{\mathbb{I}}}), \overline{\beta}(e^{c_{\mathbb{I}}})\right], & \alpha = 1. \end{cases}$$

On a semiaxis $\mathbb{I}$ this holds due to Prop. B.5(a), because here the dichotomy spectrum is determined by the corresponding diagonal system, as well as for $\mathbb{I} = \mathbb{Z}$, since one spectral interval is a singleton and therefore Prop. B.6(b) applies.

See Fig. 6.1 (left) for a visualization of $\Sigma(\alpha - 1, 0)$ for different α: An intersection of the green and red shapes with a horizontal line indicates the spectrum $\Sigma(\alpha - 1, 0)$ for a particular value of α. The red curve illustrates the singleton spectral interval $\left\{\frac{1}{\alpha}\right\}$, while the green shape represents the interval $\left[\underline{\beta}(e^{c_{\mathbb{I}} - (\alpha-1)d_{\mathbb{I}}}), \overline{\beta}(e^{c_{\mathbb{I}} - (\alpha-1)d_{\mathbb{I}}})\right]$.

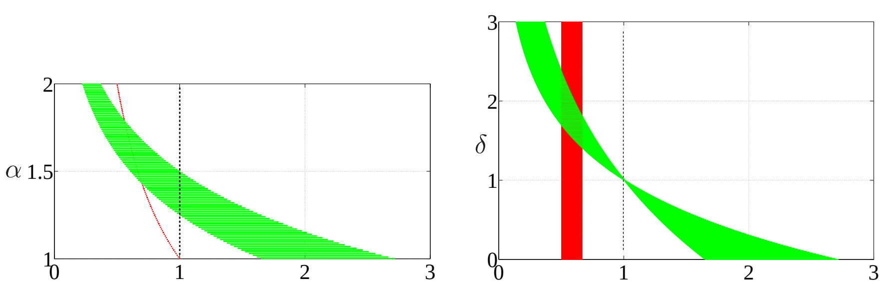

FIG. 6.1. *The dashed vertical line indicates the stability boundary:*
Left: Dichotomy spectrum $\Sigma(\alpha - 1, 0)$ for $a_n \equiv \alpha \in [1, 2]$ with $c_n := 1$ $(n \geq 0)$, $c_n := 0.5$ $(n < 0)$ and $d_n := 2 + \sin n$. It indicates uniform asymptotic stability for $\alpha > 1.5$.
Right: Dichotomy spectrum $\Sigma(\xi_{\mathbb{Z}}^*, 0)$ for $d_n \equiv \delta \in [0, 3]$ with $a_n := 2$ $(n \geq 0)$, $a_n := 1.5$ $(n < 0)$, $c_n := 1$ $(n \geq 0)$, $c_n := 0.5$ $(n < 0)$. It indicates uniform asymptotic stability for $\delta > 1$.

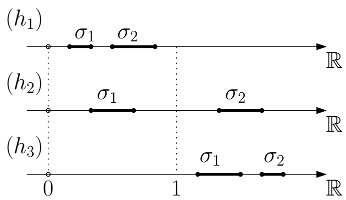

FIG. 6.2. *Hyperbolic behavior in* (6.1)*:* (h_1) *sink,* (h_2) *saddle-point and* (h_3) *source, or with switched spectral intervals* σ_1 *and* σ_2

This allows the following choices determining the asymptotic behavior near the equilibrium $(\alpha-1,0)$ for varying $\alpha>1$ (note $\frac{1}{\alpha}\in(0,1)$):

- If $\overline{\beta}(e^{c_{\mathbb{I}}-(\alpha-1)d_{\mathbb{I}}})<1$, then $(\alpha-1,0)$ is a uniformly asymptotically stable sink and the nonautonomous set $\mathbb{Z}\times\{(\alpha-1,0)\}$ is a local pullback attractor for (Δ); a sufficient condition for this behavior is $\overline{\beta}(e^{c_{\mathbb{I}}})<(\alpha-1)\underline{\beta}(e^{d_{\mathbb{I}}})$. Hence, an unstable behavior is enforced by choosing the coupling sequence $c_{\mathbb{I}}$ larger.
- if $1<\underline{\beta}(e^{c_{\mathbb{I}}-(\alpha-1)d_{\mathbb{I}}})$, then $(\alpha-1,0)$ becomes unstable with the y-axis as unstable direction, and for $\mathbb{I}=\mathbb{Z}$ even a saddle. Due to Prop. B.1 a sufficient condition for this behavior is $(\alpha-1)\overline{\beta}(e^{d_{\mathbb{I}}})<\underline{\beta}(e^{c_{\mathbb{I}}})$.

For time-varying coefficients $a_{\mathbb{I}}$, the planar system (Δ) does not have an equilibrium on the x-axis anymore. Nonetheless, on the whole integer axis $\mathbb{I}=\mathbb{Z}$ we know from Sect. 3 that the Beverton-Holt equilibrium $(\alpha-1,0)$ persists as the entire bounded solution $(\xi^*_{\mathbb{Z}},0)$ given in Lemma 3.2. Stability of the corresponding entire solution $(\xi^*_{\mathbb{Z}},0)$ can be obtained from the variational equation (cf. (V))

$$\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix}=F_n'(\xi_n^*,0)\begin{pmatrix}x_n\\ y_n\end{pmatrix},\qquad F_n'(\xi,0)=\begin{pmatrix}\frac{a_n}{(1+\xi)^2} & -\frac{a_nb_n\xi}{(1+\xi)^2}\\ 0 & e^{c_n-d_n\xi}\end{pmatrix},\tag{6.1}$$

where the dichotomy spectrum fulfills (cf. Prop. B.5 and Prop. B.6(b))

$$\Sigma(\xi^*_{\mathbb{Z}},0)\begin{cases}=\sigma_1\cup\sigma_2, & \sigma_1\cap\sigma_2 \text{ has no interior points,}\\ \subseteq\sigma_1\cup\sigma_2, & \text{otherwise}\end{cases}$$

with the spectral intervals

$$\sigma_1:=\left[\underline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}})^2}\big),\overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}})^2}\big)\right],\qquad \sigma_2:=\left[\underline{\beta}\big(e^{c_{\mathbb{Z}}-d_{\mathbb{Z}}\xi_{\mathbb{Z}}}\big),\overline{\beta}\big(e^{c_{\mathbb{Z}}-d_{\mathbb{Z}}\xi_{\mathbb{Z}}}\big)\right].$$

Keeping the parameter sequence $d_{\mathbb{Z}}$ constant to the value $\delta>0$, Fig. 6.1 (right) visualizes $\Sigma(\xi^*_{\mathbb{Z}},0)$ for different values of δ. The spectral intervals σ_1 are in red, while σ_2 are marked in green for varying δ.

Depending on the location of $\sigma_1,\sigma_2\subseteq(0,\infty)$ we also illustrated the stability properties of the Beverton-Holt solution $(\xi^*_{\mathbb{Z}},0)$ in Fig. 6.2. In particular, under the condition $\max\Sigma(\xi^*_{\mathbb{Z}},0)<1$ (this corresponds to Fig. 6.2(h_1)) the nonautonomous set

$$\left\{(n,\xi_n^*,0)\in\mathbb{Z}\times\mathbb{R}^2_+ : n\in\mathbb{Z}\right\}$$

is a local pullback attractor of the planar equation (Δ), whereas $(\xi^*_{\mathbb{Z}},0)$ becomes unstable for $\min\sigma_1>1$ or $\min\sigma_2>1$ (see Fig. 6.2(h_2) or (h_3)).

More subtle are the nonhyperbolic situations described in Fig. 6.3. Provided there exists a gap between the spectral intervals σ_1 and σ_2 such that

$$\max\sigma_1<\min\sigma_2\le 1,\qquad 1\in\sigma_2\tag{6.2}$$

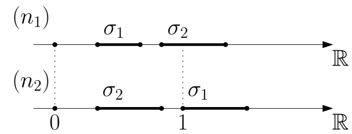

FIG. 6.3. *Nonhyperbolic behavior in* (6.1)*: One spectral interval is a subset of* $(0,1) \subseteq \mathbb{R}$*, while* 1 *is contained in another spectral interval* (n_1) *or touches it* (n_2)

(see Fig. 6.3(n_1), or with σ_1 and σ_2 exchanged), one can determine stability properties for $(\xi^*_{\mathbb{Z}}, 0)$ by means of a reduction to a center fiber bundle.

To exemplify this nonautonomous center manifold reduction we rely on

LEMMA 6.1. *Let* $\mathbb{I} = \mathbb{Z}$*. Under one of the assumptions*

(i) $\overline{\beta}(\frac{a_{\mathbb{Z}} e^{d_{\mathbb{Z}} \xi^*_{\mathbb{Z}} - c_{\mathbb{Z}}}}{(1+\xi^*_{\mathbb{Z}})^2}) < 1$ *or*

(ii) $1 < \underline{\beta}(\frac{a_{\mathbb{Z}} e^{d_{\mathbb{Z}} \xi^*_{\mathbb{Z}} - c_{\mathbb{Z}}}}{(1+\xi^*_{\mathbb{Z}})^2})$

the variational equation (6.1) *is kinematically similar to the diagonal system*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{a_n}{(1+\xi^*_n)^2} & 0 \\ 0 & e^{c_n - d_n \xi^*_n} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{6.3}$$

by means of the kinematic transformation $T_{\mathbb{Z}}$ *with* $T_n := \begin{pmatrix} 1 & -t^*_n \\ 0 & 1 \end{pmatrix}$ *and*

$$t^*_n := \begin{cases} -\sum_{j=-\infty}^{n-1} \frac{a_j b_j \xi^*_j e^{d_j \xi^*_j - c_j}}{(1+\xi^*_j)^2} \prod_{i=j+1}^{n-1} \frac{a_i e^{d_i \xi^*_i - c_i}}{(1+\xi^*_i)^2}, & \textit{if (i) holds}, \\ \sum_{j=n}^{\infty} \frac{a_j b_j \xi^*_j e^{d_j \xi^*_j - c_j}}{(1+\xi^*_j)^2} \prod_{i=n}^{j} \frac{(1+\xi^*_i)^2 e^{c_i - d_i \xi^*_i}}{a_i}, & \textit{if (ii) holds} \end{cases} \tag{6.4}$$

for all $n \in \mathbb{Z}$*.*

Proof. During the present proof we abbreviate

$$\alpha_n := \frac{a_n}{(1+\xi^*_n)^2}, \qquad \beta_n := -\frac{a_n b_n \xi^*_n}{(1+\xi^*_n)^2}, \qquad \gamma_n := e^{c_n - d_n \xi^*_n};$$

note that theses sequences, as well as $\frac{a_{\mathbb{Z}} b_{\mathbb{Z}} \xi^*_{\mathbb{Z}} e^{d_{\mathbb{Z}} \xi^*_{\mathbb{Z}} - c_{\mathbb{Z}}}}{(1+\xi^*_{\mathbb{Z}})^2} = \frac{\beta_{\mathbb{Z}}}{\gamma_{\mathbb{Z}}}$ are bounded. In order to determine the bounded sequence $t^*_{\mathbb{Z}}$, one applies the kinematic transformation $T_{\mathbb{Z}}$ to the variational equation (6.1) and obtains a linear difference equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \alpha_n & \alpha_n t^*_n + \beta_n - \gamma_n t^*_{n+1} \\ 0 & \gamma_n \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

which has the diagonal form (6.3), provided $t^*_{\mathbb{Z}}$ fulfills the scalar linear inhomogeneous equation $t^*_{n+1} = \frac{\alpha_n}{\gamma_n} t^*_n + \frac{\beta_n}{\gamma_n}$; here the inhomogeneity is assumed to be bounded. Due to [25, p. 153, Thm. 3.5.4], under (i) (meaning $\overline{\beta}(\frac{\alpha_{\mathbb{Z}}}{\gamma_{\mathbb{Z}}}) < 1$) or (ii) (corresponding to $1 < \underline{\beta}(\frac{\alpha_{\mathbb{Z}}}{\gamma_{\mathbb{Z}}})$) this problem possesses a unique bounded solution $t^*_{\mathbb{Z}}$ given by (6.4). ∎

From now on we consider the nonhyperbolic case by assuming that the spectral intervals σ_1, σ_2 fulfill (6.2).

PROPOSITION 6.2 (reduced equation). *Let* $\mathbb{I} = \mathbb{Z}$*. If*

$$\overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}})^2}\big) < \underline{\beta}\big(e^{c_{\mathbb{Z}} - d_{\mathbb{Z}} \xi_{\mathbb{Z}}}\big) \le 1, \qquad \overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}})^2}\big) < \underline{\beta}\big(e^{c_{\mathbb{Z}} - d_{\mathbb{Z}} \xi_{\mathbb{Z}}}\big)^2, \tag{6.5}$$

then the stability properties of the solution $(\xi^*_{\mathbb{Z}}, 0)$ *to the planar system* (Δ) *correspond to the stability of the trivial solution to the scalar equation (the* reduced equation*)*

$$y_{n+1} = e^{c_n - d_n \xi^*_n} \left(y_n - (1 + d_n t^*_n) y_n^2 + \frac{(1 + d_n t^*_n)^2 - d_n \omega_n}{2} y_n^3 + O(y_n^4) \right) \tag{6.6}$$

uniformly in $n \in \mathbb{Z}$ *with*

$$t_n^* = -\sum_{j=-\infty}^{n-1} \frac{a_j b_j \xi_j^* e^{d_j \xi_j^* - c_j}}{(1+\xi_j^*)^2} \prod_{i=j+1}^{n-1} \frac{a_i e^{d_i \xi_i^* - c_i}}{(1+\xi_i^*)^2},$$

$$\omega_n = \sum_{j=-\infty}^{n-1} h_j \prod_{i=j+1}^{n-1} \frac{a_i e^{2d_i \xi_i^* - 2c_i}}{(1+\xi_i^*)^2}$$

and $h_j := \frac{2e^{d_j \xi_j^* - 2c_j}\left(a_j(b_j+t_j^*)e^{d_j\xi_j^*}(b_j\xi_j^* - t_j^*) + e^{c_j} t_{j+1}^*(1+\xi_j^*)^3(d_j t_j^*+1)\right)}{(1+\xi_j^*)^3}$.

Proof. We proceed in three steps:

(I) First of all, our assumptions imply the estimate

$$\overline{\beta}\left(\frac{a_{\mathbb{Z}} e^{d_{\mathbb{Z}}\xi_{\mathbb{Z}}^* - c_{\mathbb{Z}}}}{(1+\xi_{\mathbb{Z}}^*)^2}\right) \stackrel{\text{(B.5)}}{\leq} \frac{\overline{\beta}(\frac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}}^*)^2})}{\underline{\beta}(e^{c_{\mathbb{Z}} - d_{\mathbb{Z}}\xi_{\mathbb{Z}}^*})} < 1$$

and therefore we are in the situation of Lemma 6.1(i). Then the difference equation of perturbed motion for (Δ) w.r.t. the solution $(\xi_{\mathbb{Z}}^*, 0)$ becomes

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F_n(x_n + \xi_n^*, y_n) - F_n(\xi_n^*, 0)$$

and applying the kinematic transformation $T_{\mathbb{Z}}$ from Lemma 6.1 yields

$$\begin{cases} x_{n+1} = \frac{a_n[x_n + (t_n^* - b_n\xi_n^*)y_n]}{(1+\xi_n^*)(1+x_n+b_n y_n + \xi_n^* + y_n t_n^*)} - y_n t_{n+1}^* e^{c_n - d_n(x_n+\xi_n^*+t_n^* y_n) - y_n}, \\ y_{n+1} = y_n e^{c_n - d_n(x_n + \xi_n^* + t_n^* y_n) - y_n}. \end{cases}$$

(II) This planar system has the trivial solution and our assumptions guarantee the existence of a center-unstable fiber bundle $\mathcal{W} \subseteq \mathbb{Z} \times U$, $U \subseteq \mathbb{R}^2$ being an open neighborhood of $(0,0)$, whose fibers $\mathcal{W}(n)$ are graphs of functions $w_n : U_0 \to \mathbb{R}$ defined on a neighborhood $U_0 \subset \mathbb{R}$ of 0 uniformly in $n \in \mathbb{Z}$ (see [25, p. 259, Thm. 4.6.4(b) and p. 260, Rem. 4.6.5(2)]). Due to the reduction principle [25, p. 267, Thm. 4.6.15] stability properties of $(\xi_{\mathbb{Z}}^*, 0)$ are determined by those of the trivial solution to the reduced equation

$$y_{n+1} = y_n e^{c_n - d_n(w_n(y_n) + \xi_n^* + t_n^* y_n) - y_n}. \tag{6.7}$$

Now the right inequality in (6.5) guarantees that each such function w_n is of differentiability class C^2. Hence, Taylor's theorem in connection with $w_n(0) = 0$ and the tangentiality property $w_n'(0) = 0$ yields the representation $w_n(y) = \frac{w_n''(0)}{2!} y^2 + O(y^3)$ uniformly in $n \in \mathbb{Z}$. Given this, a Taylor expansion in the right-hand side of (6.7) implies the claimed representation (6.6), where we have abbreviated $\omega_n := w_n''(0)$.

(III) It remains to establish the expression for the Taylor coefficient ω_n. The following argument is based on the fact that the mappings w_n defining the center-unstable fiber bundle $\mathcal{W}$ fulfill the invariance equation (cf. [29])

$$\begin{aligned} &w_{n+1}\left(y e^{c_n - d_n(w_n(y) + \xi_n^* + y t_n^*) - y}\right) \\ &\quad = \frac{a_n[w_n(y) + (t_n^* - b_n\xi_n^*)y]}{(1+\xi_n^*)(1 + w_n(y) + b_n y + \xi_n^* + y t_n^*)} - y t_{n+1}^* e^{c_n - d_n(w_n(y) + \xi_n^* + t_n^* y) - y} \end{aligned}$$

for every $y \in U_0$ and $n \in \mathbb{Z}$. The above Taylor expansion of w_n as ansatz in the invariance equation shows that the coefficients ω_n indeed satisfy the linear difference equation (see [29, Thm. 4.2(b)])

$$\omega_{n+1} = a_n \frac{e^{2d_n\xi_n^* - 2c_n}}{(1+\xi_n^*)^2}\omega_n + 2a_n \frac{e^{d_n\xi_n^* - 2c_n}(b_n + t_n^*)(b_n\xi_n^* - t_n^*)}{(1+\xi_n^*)^3} + 2e^{d_n\xi_n^* - c_n} t_{n+1}^*(1 + d_n t_n^*),$$

whose unique bounded solution is given by the expression stated above. ∎

A further analysis of the reduced equation (6.6) requires appropriate stability criteria for nonautonomous scalar difference equations with nonhyperbolic linear part.

6.3. The Ricker equilibrium. In the situation of a constant parameter sequence $c_n \equiv \gamma$ the so-called *Ricker equilibrium* $(0,\gamma)$ to (Δ) is present for $\gamma \geq 0$, yielding a variational equation (V) with lower-triangular coefficient matrix

$$F_n'(0,\gamma) = \begin{pmatrix} \frac{a_n}{1+\gamma b_n} & 0 \\ -\gamma d_n & 1-\gamma \end{pmatrix}.$$

Reasoning as above, due to a singleton as spectral interval, the dichotomy spectrum is determined by the diagonal elements (cf. Prop. B.6 and B.1)

$$\Sigma(0,\gamma) = \{|1-\gamma|\} \cup \left[\underline{\beta}\left(\tfrac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}}\right), \overline{\beta}\left(\tfrac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}}\right)\right].$$

Consequently, $(0,\gamma)$ is unstable for $\gamma > 2$ because of Thm. 2.1(b). A more detailed analysis of the Ricker equilibrium $(0,\gamma)$ reads as follows:

- For $\gamma \in (0,2)$ and $\overline{\beta}(\frac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}}) < 1$ it is a uniformly asymptotically stable sink.
- For $\gamma \in (0,2)$ and $1 < \underline{\beta}(\frac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}})$ it is unstable and for $\mathbb{I} = \mathbb{Z}$ even a saddle.
- If $\gamma > 2$ and $\overline{\beta}(\frac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}}) < 1$, then $(0,\gamma)$ is unstable and for $\mathbb{I} = \mathbb{Z}$ a saddle. Moreover, due to a flip bifurcation in (R') at $\gamma = 2$, there exists an asymptotically stable 2-periodic solution to (Δ) on the y-axis for small $\gamma - 2 > 0$.
- If $\gamma > 2$ and $1 < \underline{\beta}(\frac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}})$ it is an unstable source. For $\mathbb{I} = \mathbb{Z}$ the 2-periodic solution to (Δ) existing for small $\gamma - 2 > 0$ becomes a saddle.

This analysis shows that a strong coupling, i.e. large values of the sequence $b_{\mathbb{I}}$ in (Δ), yield a stabilization of the Ricker equilibrium $(\gamma, 0)$. See Fig. 6.4 for an illustration of $\Sigma(\gamma,0)$ as a function of the parameter γ. Here, the singleton spectral interval $\{|1-\gamma|\}$ is in red, while $\left[\underline{\beta}(\frac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}}), \overline{\beta}(\frac{a_{\mathbb{I}}}{1+\gamma b_{\mathbb{I}}})\right]$ is in green.

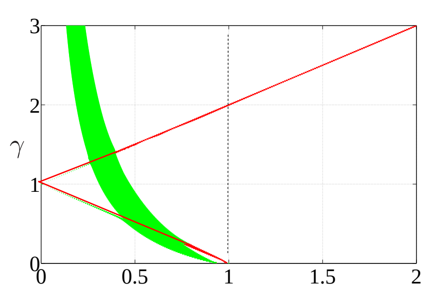

FIG. 6.4. *Dichotomy spectrum $\Sigma(0,\gamma)$ for $c_n \equiv \gamma \in [0, 2.5]$ with $a_n := 1 + 0.5\cos n$ and $b_n := 2$ $(n \geq 0)$, $b_n := 1$ $(n < 0)$. It indicates uniform asymptotic stability for $\gamma \in (0, 2)$.*

For nonconstant sequences $c_{\mathbb{Z}}$, Sect. 4 provides conditions that the equilibrium γ of (R') persists as the pullback solution $\eta^*_{\mathbb{Z}}$ for (R) given in (4.3). Hence, a similar stability analysis is possible for the solution $(0, \eta^*_{\mathbb{Z}})$ to (Δ), as we have done it for the entire Beverton-Holt solution $(\xi^*_{\mathbb{Z}}, 0)$. In the hyperbolic case this is possible on basis of the dichotomy spectrum alone, while nonlinear effects (and thus a center-unstable fiber bundle) come into play, if a dominant spectral interval contains 1. We do not give the details.

Nonetheless, the basic difference to the above situation is that there exists no explicit expression for $\eta^*_{\mathbb{Z}}$ anymore. During the whole analysis one rather has to work with a numerical approximation for $\eta^*_{\mathbb{Z}}$.

6.4. The coexistence equilibrium. We restrict to the autonomous situation (Δ') first, where there exists a coexistence equilibrium $(\xi^\star, \eta^\star) := \left(\frac{\alpha-\beta\gamma-1}{1-\beta\delta}, \frac{\delta(1-\alpha)+\gamma}{1-\beta\delta}\right)$ for $\beta\delta \neq 1$. Thanks to [6, Lemma 5.1] we know that $\gamma < (\alpha-1)\delta$ (yielding asymptotic stability of Beverton-Holt equilibrium) and $1 + \beta\gamma < \alpha$ (implying the Ricker equilibrium to be unstable) ensure that $(\xi^\star, \eta^\star)$ is not contained in $(0, \infty)^2$.

Due to the linearization

$$F'(\xi^\star, \eta^\star) = \frac{1}{1-\beta\delta} \begin{pmatrix} \frac{\beta\gamma-\alpha\beta\delta+1}{\alpha} & \frac{\beta(1-\alpha+\beta\gamma)}{\alpha} \\ \delta(\alpha\delta-\gamma-\delta) & 1-\gamma-(1-\alpha+\beta)\delta \end{pmatrix}$$

nonhyperbolicity occurs in one of the three cases:

- 1 is an eigenvalue of $F'(\xi^\star, \eta^\star)$. Since the remaining eigenvalue λ fulfills
$$\det F'(\xi^\star, \eta^\star) = \lambda, \qquad \operatorname{tr} F'(\xi^\star, \eta^\star) = 1 + \lambda,$$
this form of nonhyperbolicity is given if and only if $\beta\gamma = \alpha - 1$ or $\alpha\delta = \gamma + \delta$, i.e. $(\xi^\star, \eta^\star)$ is contained in one of the coordinate axes.
- -1 is an eigenvalue of $F'(\xi^\star, \eta^\star)$. For the remaining eigenvalue λ this means
$$\det F'(\xi^\star, \eta^\star) = -\lambda, \qquad \operatorname{tr} F'(\xi^\star, \eta^\star) = -1 + \lambda$$
and such a form of nonhyperbolicity occurs if and only if
$$\alpha^2\delta + \alpha\beta\gamma\delta - 4\alpha\beta\delta - \alpha\gamma + 2\alpha - \beta\gamma^2 - \beta\gamma\delta + 2\beta\gamma - \gamma - \delta + 2 = 0.$$
- $F'(\xi^\star, \eta^\star)$ has a pair of complex-conjugated eigenvalues $\lambda, \bar{\lambda}$ of modulus 1. Consequently, the parameters have to satisfy
$$1 = \det F'(\xi^\star, \eta^\star), \qquad \operatorname{tr} F'(\xi^\star, \eta^\star) = 2\Re\lambda.$$

In the complimentary hyperbolic situation, the coexistence equilibrium $(\xi^\star, \eta^\star)$ of (Δ') will persist as an entire bounded solution $(\xi, \eta)(a, b, c, d)_{\mathbb{Z}}$ to (Δ) satisfying

$$\sup_{n\in\mathbb{Z}} \|(\xi, \eta)(a, b, c, d)_n - (\xi^*, \eta^*)(\alpha, \beta, \gamma, \delta)\| < \varepsilon,$$

when the constant parameters $\alpha, \beta, \gamma, \delta > 0$ are replaced by time varying sequences $a_{\mathbb{Z}}, b_{\mathbb{Z}}, c_{\mathbb{Z}}$ resp. $d_{\mathbb{Z}}$ (see [14, Lemma 2] or [27, Thm. 2.17] for details), if

$$\sup_{n\in\mathbb{Z}} \max\{|a_n - \alpha|, |b_n - \beta|, |c_n - \gamma|, |d_n - \delta|\} < \rho$$

for small $\rho > 0$. In fact, as in Thm. 4.5 one can give an estimate for the size of $\varepsilon, \rho > 0$. However, since this is essentially a technical extension of the proof for Thm. 4.5, we omit the details.

Finally, Fig. 6.5 illustrates the discussed equilibria in the autonomous setup (left) as well as corresponding bounded solutions for a nonautonomous choice of parameters.

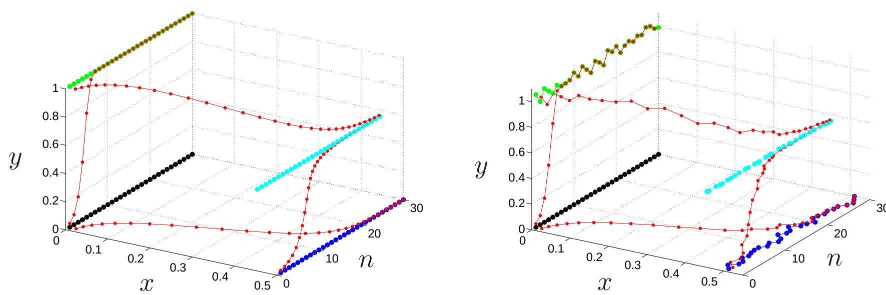

FIG. 6.5. *Left: Equilibria and trajectories starting nearby (in red) for* $\alpha = 1.5$, $\beta = 0.1$, $\gamma = \delta = 1$. *Right: The corresponding nonautonomous diagram with randomly perturbed parameters.*

The coexistence equilibrium $(\xi^\star, \eta^\star)$ of (Δ') is created via bifurcations from the two extinction equilibria $(\alpha - 1, 0)$ and $(0, \gamma)$. In the next section, we investigate to what extend this bifurcation scenario persists in the nonautonomous situation of (Δ).

7. Bifurcation. The above Sects. 3 and 4 provided a quite detailed insight on the dynamics of the nonautonomous Beverton-Holt equation (BH) resp. the Ricker equation (R) in terms of their attractors $\mathcal{X}_a^*$ and $\mathcal{Y}_c^*$. Thus, the uncoupled equations

$$\begin{cases} x_{n+1} = \frac{a_n x_n}{1+x_n}, \\ y_{n+1} = y_n e^{c_n - y_n} \end{cases}$$

simply form a product system and therefore its pullback attractor is $\mathcal{A}^* = \mathcal{X}_a^* \times \mathcal{Y}_c^*$. In particular, bifurcation phenomena in (BH) or (R) carry over to $\mathcal{A}^*$.

Another source yielding changes in the dynamics of (Δ) are the coupling sequences $b_{\mathbb{I}}$ and $d_{\mathbb{I}}$. For the sake of a bifurcation analysis, we restrict to the respective time-constant case $b_n \equiv \beta$ and $\delta_n \equiv \delta$ and interpret β or γ as bifurcation parameters.

7.1. Beverton-Holt entire solution. First, we investigate the behavior near the entire solution $(\xi_{\mathbb{Z}}^*, 0)$ to the planar difference equation

$$\begin{cases} x_{n+1} = \frac{a_n x_n}{1+x_n+b_n y_n}, \\ y_{n+1} = y_n e^{c_n - \delta x_n - y_n} \end{cases} \tag{Δ_δ}$$

being a special case of (Δ) with constant coupling $d_n \equiv \delta$.

Let us interpret δ as bifurcation parameter to study the loss of stability in the Beverton-Holt solution $(\xi_{\mathbb{Z}}^*, 0)$ as the coupling strength δ changes. If we assume the estimate $\overline{\beta}\big(\frac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}}^*)^2}\big) \leq \underline{\beta}\big(e^{c_{\mathbb{Z}} - \delta\xi_{\mathbb{Z}}^*}\big)$, then according to Prop. B.6 and following Sect. 6.2 the associate dichotomy spectrum reads as

$$\Sigma(\xi_{\mathbb{Z}}^*, 0) = \Big[\underline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}}^*)^2}\big), \overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi_{\mathbb{Z}}^*)^2}\big)\Big] \cup \Big[\underline{\beta}\big(e^{c_{\mathbb{Z}} - \delta\xi_{\mathbb{Z}}^*}\big), \overline{\beta}\big(e^{c_{\mathbb{Z}} - \delta\xi_{\mathbb{Z}}^*}\big)\Big].$$

From now on, we suppose that

$$\overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi^*_{\mathbb{Z}})^2}\big) < \underline{\beta}\big(e^{c_{\mathbb{Z}}-\delta^*\xi^*_{\mathbb{Z}}}\big), \qquad \overline{\beta}\big(e^{c_{\mathbb{Z}}-\delta^*\xi^*_{\mathbb{Z}}}\big) = 1 \tag{7.1}$$

holds for some critical value $\delta^* > 0$. A reduction to the center-unstable fiber bundle as in Sect. 6.2 yields that the bifurcation in (Δ_δ) is determined by the scalar equation

$$y_{n+1} = e^{c_n-\delta^*\xi^*_n}\left[(1-(\delta-\delta^*)\xi^*_n)y_n - (1+\delta^*\xi^*_n)y_n^2\right] + O((\delta-\delta^*)^2 y_n, (\delta-\delta^*)y_n^2, y_n^3)$$

uniformly in $n \in \mathbb{Z}$. This indicates a transcritical bifurcation as δ decreases through the critical value δ^*.

One can verify the spectral assumption (7.1) numerically with the algorithm from Appendix B.1. Let $r_{\mathbb{Z}} \in [-0.05, 0.05]^{\mathbb{Z}}$ be a uniformly distributed random sequence. For $n \in \mathbb{Z}$ we choose the parameters $a_n = r_n + 1.5$, $b_n = r_n + 0.1$, $c_n = r_n + 1$, $\delta = 2.5$ and obtain from our numerical experiments:

$$\overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi^*_{\mathbb{Z}})^2}\big) = 0.66771 < \underline{\beta}\big(e^{c_{\mathbb{Z}}-\delta\xi^*_{\mathbb{Z}}}\big) = 0.77618, \quad \overline{\beta}\big(e^{c_{\mathbb{Z}}-\delta\xi^*_{\mathbb{Z}}}\big) = 0.78163.$$

For $\delta^* = 2$ the assumptions (7.1) are approximately satisfied:

$$\overline{\beta}\big(\tfrac{a_{\mathbb{Z}}}{(1+\xi^*_{\mathbb{Z}})^2}\big) = 0.66771 < \underline{\beta}\big(e^{c_{\mathbb{Z}}-\delta^*\xi^*_{\mathbb{Z}}}\big) = 0.99781, \quad \overline{\beta}\big(e^{c_{\mathbb{Z}}-\delta^*\xi^*_{\mathbb{Z}}}\big) = 1.00250.$$

Note that neither the bounded Beverton-Holt trajectory $\xi^*_{\mathbb{Z}}$, nor the Ricker trajectory depend on δ. For $\delta < 2$ a bounded coexistence trajectory exists, that bifurcates from the Beverton-Holt and Ricker solution, respectively, see Fig. 7.1. The coexistence trajectory is attracting and can be computed via forward iteration, instead of solving boundary value problems as described in Section 4.2.

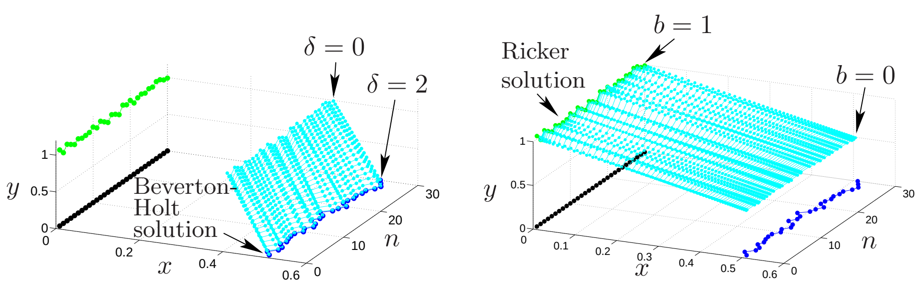

FIG. 7.1. *Coexistence trajectories (cyan) for fixed $b_n = r_n + 0.1$ and various values of $\delta \in [0, 2]$ (left) and for fixed $\delta = 1$ and various values of $b_n = r_n + b$, $b \in [0, 1]$ (right).*

7.2. Ricker equilibrium. Let us eventually investigate also the behavior of the Ricker equilibrium $(0, \gamma)$ to the planar difference equation

$$\begin{cases} x_{n+1} = \frac{a_n x_n}{1+x_n+\beta y_n}, \\ y_{n+1} = y_n e^{\gamma - d_n x_n - y_n}, \end{cases} \tag{Δ_β}$$

where $\gamma \in (0, 2)$ is assumed to be fixed from now on, while $\beta > 0$ will be considered as bifurcation parameter. Referring to Sect. 6.3 this yields the dichotomy spectrum

$$\Sigma(0,\gamma) \overset{\text{(B.3)}}{=} \{|1-\gamma|\} \cup \frac{1}{1+\gamma\beta}\left[\underline{\beta}(a_{\mathbb{Z}}), \overline{\beta}(a_{\mathbb{Z}})\right]$$

and the critical stability situation $\max\Sigma(0,\gamma)=1$ precisely holds for the parameter $\beta^*=\frac{\overline{\beta}(a_{\mathbb{Z}})-1}{\gamma}$. As β passes through β^* one can observe a *subcritical shovel bifurcation* (see [26, Thm. 3.15(a)]); this quite coarse bifurcation scenario precisely means

- For $\beta<\beta^*$ the additional assumption $\overline{\beta}(a_{\mathbb{N}})<1+\gamma\beta$ implies that $(0,\gamma)$ is asymptotically stable and, provided $\underline{\beta}(a_{\mathbb{Z}^-})=1+\gamma\beta^*$, embedded into a 1-parameter family of bounded entire solutions to (Δ_β).
- For $\beta=\beta^*$ the additional assumption $\overline{\beta}(a_{\mathbb{N}})<1+\gamma\beta^*=\overline{\beta}(a_{\mathbb{Z}})$ ensures $(0,\gamma)$ to be asymptotically stable.
- For $\beta>\beta^*$ the Ricker equilibrium is (locally) the unique bounded entire solution to (Δ_β) in a neighborhood of $(0,\gamma)$.

See Fig. 7.2 for the dichotomy spectra yielding a subcritical shovel bifurcation.

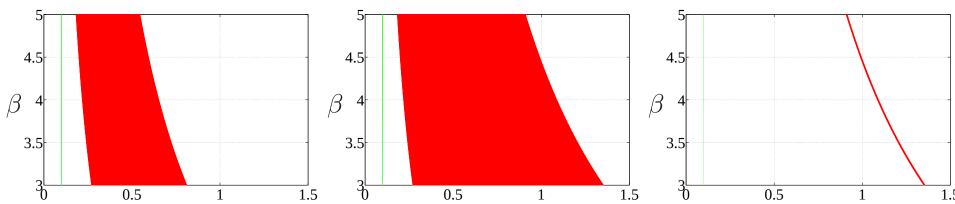

FIG. 7.2. *Dichotomy spectra $\Sigma(0,\gamma)$ for $\mathbb{I}=\mathbb{N}$ (left), $\mathbb{I}=\mathbb{Z}$ (center) and $\mathbb{I}=\mathbb{Z}^-$ (right) for various values of $\beta\in[3,5]$ with coefficients $a_n:=2+\sin\ln(1+n)$, $n>0$ and $a_n:=5$ $(n\le 0)$, $\gamma=0.9$, $d_n:=1+0.5\sin n$ and consequently $\beta^*=\frac{40}{9}$.*

Appendix A. Nonautonomous discrete dynamics. Let $\mathbb{J}$ be a discrete interval unbounded below, $\mathbb{J}':=\{k\in\mathbb{J}:\,k+1\in\mathbb{J}\}$ and $\Omega\subseteq\mathbb{R}^d$. Suppose that the mapping $F_n:\Omega\to\Omega$, $n\in\mathbb{J}'$, is continuous. Given a pair $n_0\in\mathbb{J}$, $\bar{x}\in\Omega$, we write $\varphi(\cdot;n_0,\bar{x})$ for the forward solution to the nonautonomous difference equation

$$x_{n+1}=F_n(x_n) \tag{A.1}$$

satisfying the initial condition $x_{k_0}=\bar{x}$ and denote it as *general solution* to (A.1). A solution to (A.1) existing on the whole discrete integer axis is called an *entire solution*.

A.1. Pullback attraction. First of all, one has

THEOREM A.1 (pullback solution). *Let $\bar{x}_{\mathbb{J}}$ denote a sequence in Ω. If the limits*

$$\xi_n^*:=\lim_{k\to-\infty}\varphi(n;k,\bar{x}_k)$$

exist for all $n\in\mathbb{J}$, then also this so-called pullback solution *$\xi_{\mathbb{J}}^*$ solves* (A.1).

Proof. Due to the continuity of F_n one has

$$\begin{aligned}\xi_{n+1}^*&=\lim_{k\to-\infty}\varphi(n+1;k,\bar{x}_k)\stackrel{(A.1)}{=}\lim_{k\to-\infty}F_n(\varphi(n;k,\bar{x}_k))\\&=F_n\left(\lim_{k\to-\infty}\varphi(n;k,\bar{x}_k)\right)=F_n(\xi_n^*)\quad\text{for all }n\in\mathbb{J}'\end{aligned}$$

and this completes the proof. ◻

A subset $\mathcal{A}\subseteq\mathbb{J}\times\Omega$ is denoted as *nonautonomous set* with the *n-fibers*

$$\mathcal{A}(n):=\{x\in\Omega:\,(n,x)\in\mathcal{A}\}\quad\text{for all }n\in\mathbb{J}.$$

For the *cartesian product* of two nonautonomous sets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{J} \times \Omega$ we define

$$\mathcal{A} \times \mathcal{B} := \left\{(n, x, y) \in \mathbb{J} \times \Omega^2 : (x, y) \in \mathcal{A}(n) \times \mathcal{B}(n)\right\}.$$

One says a nonautonomous set $\mathcal{A}$ (cf. [25]) is

- *bounded*, if there exists a $\rho > 0$ such that $\mathcal{A}(n) \subseteq B_\rho(0) \subseteq \mathbb{R}^d$ for all $n \in \mathbb{J}$,
- *compact* (resp. *connected*), if each fiber $\mathcal{A}(n) \subseteq \Omega$, $n \in \mathbb{J}$, is compact (resp. connected),
- *forward invariant*, if $F_n(\mathcal{A}(n)) \subseteq \mathcal{A}(n+1)$ for all $n \in \mathbb{J}'$,
- *invariant*, if $F_n(\mathcal{A}(n)) = \mathcal{A}(n+1)$ for all $n \in \mathbb{J}'$,
- *pullback absorbing*, if for every bounded nonautonomous set $\mathcal{B}$ there is a $K \in \mathbb{N}_0$ such that $\varphi(n; n-k, \mathcal{B}(n-k)) \subseteq \mathcal{A}(n)$ holds for all $n \in \mathbb{J}$ and $k \geq K$.

For given $\mathcal{A}$, the nonautonomous set $\omega_\mathcal{A}$ defined by the fibers

$$\omega_\mathcal{A}(n) := \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} \varphi(n; n-k, \mathcal{A}(k-n))} \quad \text{for all } n \in \mathbb{J}$$

is called the *ω-limit set* of $\mathcal{A}$. When $\mathcal{A}$ is pullback absorbing, one denotes the nonautonomous set $\mathcal{P} := \omega_\mathcal{A}$ as *pullback attractor* of (A.1).

COROLLARY A.2. *If $\mathcal{A}$ is invariant, then $\omega_\mathcal{A} = \overline{\mathcal{A}}$.*

Proof. The invariance of $\mathcal{A}$ implies

$$\omega_\mathcal{A}(n) = \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} \varphi(n; n-k, \mathcal{A}(k-n))} = \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} \mathcal{A}(n)} = \overline{\mathcal{A}(n)} \quad \text{for all } n \in \mathbb{J}$$

and this finishes the proof. □

A.2. Order-preserving difference equations. A *cone* $C \subseteq \mathbb{R}^d$ is a closed subset satisfying $\lambda C \subseteq C$ for $\lambda \geq 0$, $C + C \subseteq C$ and $C \cap (-C) = \{0\}$; we define the partial order

$$x \preceq y \quad :\Leftrightarrow \quad y - x \in C \quad \text{for all } x, y \in \mathbb{R}^d.$$

A difference equation (A.1) is called *order-preserving*, if one has the implication

$$x \preceq y \quad \Rightarrow \quad F_n(x) \preceq F_n(y) \quad \text{for all } n \in \mathbb{J}',\ x, y \in \Omega$$

and consequently mathematical induction yields the implication

$$\bar{x} \preceq \bar{y} \quad \Rightarrow \quad \varphi(n; n_0, \bar{x}) \preceq \varphi(n; n_0, \bar{y}) \quad \text{for all } n_0 \leq n,\ x, y \in \Omega. \tag{A.2}$$

The following result is a discrete-time counterpart to [3, pp. 257–258, Lemma 9.3]:

THEOREM A.3. *Assume that* (A.1) *is order-preserving. If $x_\mathbb{J}^-, x_\mathbb{J}^+$ are sequences in Ω such that the nonautonomous set*

$$\mathcal{A} := \left\{(n, x) \in \mathbb{J} \times \Omega : x_n^- \preceq x \preceq x_n^+\right\}$$

is forward invariant, then there exist entire solutions $\xi_\mathbb{J}^-, \xi_\mathbb{J}^+$ to (A.1) *being minimal resp. maximal in the following sense: Every entire solution $x_\mathbb{J}$ of* (A.1) *in $\mathcal{A}$ fulfills*

$$x_n^- \preceq \xi_n^- \preceq x_n \preceq \xi_n^+ \preceq x_n^+ \quad \textit{for all } n \in \mathbb{J}$$

and $\mathcal{A}$ contains in particular at least one entire solution.

Proof. Let $n \in \mathbb{J}$ be fixed. Since the set $\mathcal{A}$ is forward invariant one obtains

$$x_n^- \preceq \varphi(n;k,x_k^+) = \varphi(n;l,\varphi(l;k,x_k^+)) \overset{\text{(A.2)}}{\preceq} \varphi(n;l,x_l^+) \quad \text{for all } k \le l \le n$$

and thus the sequence $y_k := \varphi(n;k,x_k^+)$, $k \le n$, in Ω is decreasing and bounded below by x_n^-. Therefore (cones in finite-dimensional spaces are regular), it is convergent and thanks to Thm. A.1 the pullback solution $\xi_n^+ := \lim_{k\to-\infty} \varphi(n;k,x_k^+)$ solves equation (A.1) on $\mathbb{J}$. Similarly, one verifies that also the limit $\xi_n^- := \lim_{k\to-\infty} \varphi(n;k,x_k^-)$ exists and defines an entire solution to (A.1). □

Corollary A.4. *If* (A.1) *is order-preserving and has a bounded and pullback absorbing set $\mathcal{A}$, then the pullback attractor $\mathcal{A}^*$ fulfills*

$$\mathcal{A}^* \subseteq \left\{(n,x) \in \mathbb{J} \times \Omega : \xi_n^- \preceq x \preceq \xi_n^+\right\}.$$

Moreover, equality holds in case $\{(n,x) \in \mathbb{J} \times \Omega : \xi_n^- \preceq x \preceq \xi_n^+\}$ is invariant.

Proof. (I) First, [25, p. 19, Thm. 1.3.9] shows that the pullback attractor $\mathcal{A}^* = \omega_{\mathcal{A}}$ exists with $\mathcal{A}^* \subseteq \mathcal{A}$. Hence, it is bounded, [25, p. 17, Cor. 1.3.4] implies that $\mathcal{A}^*$ consists of all bounded entire solutions to (A.1) and thus

$$\mathcal{A}^* \subseteq \{(n,x) \in \mathbb{J} \times \Omega : \xi_n^- \preceq x \preceq \xi_n^+\} =: \mathcal{A}_0.$$

(II) If $\mathcal{A}_0$ is invariant, then Cor. A.2 implies $\omega_{\mathcal{A}_0} = \overline{\mathcal{A}_0} = \mathcal{A}_0$ and with (I) this yields $\mathcal{A}^* \subseteq \mathcal{A}_0 = \omega_{\mathcal{A}_0} \subseteq \omega_{\mathcal{A}} = \mathcal{A}^*$ (cf. [25, p. 9, Prop. 1.2.13(a)]), i.e. $\mathcal{A}^* = \mathcal{A}_0$. □

Appendix B. Nonautonomous hyperbolicity.

B.1. Bohl exponents. Let $a_{\mathbb{I}} = (a_n)_{n\in\mathbb{I}}$ be a real or complex sequence and suppose $\mathbb{I}$ is a discrete interval unbounded above. For simplicity we assume

$$0 < \inf_{n\in\mathbb{I}} |a_n| \le \sup_{n\in\mathbb{I}} |a_n| < \infty \tag{B.1}$$

throughout. Let us define the *lower* resp. *upper Bohl* exponent of $a_{\mathbb{I}}$ by

$$\underline{\beta}(a_{\mathbb{I}}) := \lim_{n\to\infty} \inf_{k\in\mathbb{I}} \sqrt[n]{\prod_{i=k}^{k+n-1} |a_i|}, \qquad \overline{\beta}(a_{\mathbb{I}}) := \lim_{n\to\infty} \sup_{k\in\mathbb{I}} \sqrt[n]{\prod_{i=k}^{k+n-1} |a_i|}. \tag{B.2}$$

Thanks to (B.1) we know that both Bohl exponents are finite and the limits in (B.2) exist due to [2, Lemma 3.1]. Moreover, they fulfill the following elementary

Proposition B.1. *For unbounded subintervals $\mathbb{J} \subseteq \mathbb{I}$ it is $\underline{\beta}(a_{\mathbb{I}}) \le \underline{\beta}(a_{\mathbb{J}}) \le \overline{\beta}(a_{\mathbb{J}}) \le \overline{\beta}(a_{\mathbb{I}})$ and*

$$\overline{\beta}(\lambda a_{\mathbb{I}}) = |\lambda|\, \overline{\beta}(a_{\mathbb{I}}), \qquad \underline{\beta}(\lambda a_{\mathbb{I}}) = |\lambda|\, \underline{\beta}(a_{\mathbb{I}}) \quad \text{for all } \lambda \in \mathbb{C}. \tag{B.3}$$

If the sequences $a_{\mathbb{I}}$, $b_{\mathbb{I}}$ additionally both satisfy (B.1)*, then the following relations hold*

$$\overline{\beta}(a_{\mathbb{I}} b_{\mathbb{I}}) \le \overline{\beta}(a_{\mathbb{I}})\overline{\beta}(b_{\mathbb{I}}), \qquad \underline{\beta}(a_{\mathbb{I}} b_{\mathbb{I}}) \ge \underline{\beta}(a_{\mathbb{I}})\underline{\beta}(b_{\mathbb{I}}), \tag{B.4}$$

$$\overline{\beta}\left(\frac{a_{\mathbb{I}}}{b_{\mathbb{I}}}\right) \le \frac{\overline{\beta}(a_{\mathbb{I}})}{\underline{\beta}(b_{\mathbb{I}})}, \qquad \underline{\beta}\left(\frac{a_{\mathbb{I}}}{b_{\mathbb{I}}}\right) \ge \frac{\underline{\beta}(a_{\mathbb{I}})}{\overline{\beta}(b_{\mathbb{I}})}. \tag{B.5}$$

Proof. The claimed inequality for the Bohl exponents on different intervals $\mathbb{I}$ and $\mathbb{J}$ immediately follows from elementary properties of sup and inf; so does (B.3). A proof of the remaining inequalities is left to the interested reader. □

EXAMPLE B.2. *Let $\mathbb{J} := [\kappa, \infty) \cap \mathbb{Z}$ with some $\kappa \in \mathbb{Z}$.*

(1) For a constant sequence $a_n :\equiv \alpha$, $\alpha \in \mathbb{C}$, all Bohl exponents coincide, i.e.,

$$\underline{\beta}(a_{\mathbb{J}}) = \overline{\beta}(a_{\mathbb{J}}) = \underline{\beta}(a_{\mathbb{Z}}) = \overline{\beta}(a_{\mathbb{Z}}) = |\alpha|.$$

(2) Also in case of an ω-periodic, $\omega \in \mathbb{N}$, sequence $a_{\mathbb{Z}}$ one deduces

$$\underline{\beta}(a_{\mathbb{J}}) = \overline{\beta}(a_{\mathbb{J}}) = \underline{\beta}(a_{\mathbb{Z}}) = \overline{\beta}(a_{\mathbb{Z}}) = \sqrt[\omega]{|a_\kappa \cdots a_{\kappa+\omega-1}|}.$$

(3) For a sequence a_n satisfying $\lim_{n\to\pm\infty} a_n = \alpha_\pm$ it is

$$\underline{\beta}(a_{\mathbb{J}}) = \overline{\beta}(a_{\mathbb{J}}) = |\alpha_+|, \quad \underline{\beta}(a_{\mathbb{Z}}) = \min\{|\alpha_-|, |\alpha_+|\}, \quad \overline{\beta}(a_{\mathbb{Z}}) = \max\{|\alpha_-|, |\alpha_+|\}.$$

(4) In the situation $a_n := 3 - \operatorname{sgn} n + \sin\ln(1+|n|)$ the sequence $(\sin\ln(1+|n|))_{n\in\mathbb{I}}$ comes arbitrarily close to the values ± 1 on increasingly larger intervals. Hence, it is

$$\underline{\beta}(a_{\mathbb{J}}) = 1, \qquad \overline{\beta}(a_{\mathbb{J}}) = 3,$$
$$\underline{\beta}(a_{\mathbb{Z}}) = 1, \qquad \overline{\beta}(a_{\mathbb{Z}}) = 5.$$

(5) For a bounded sequence $c_{\mathbb{I}}$ one has

$$\underline{\beta}(e^{c_{\mathbb{I}}}) = \exp\left(\lim_{n\to\infty} \frac{1}{n} \inf_{k\in\mathbb{I}} \sum_{i=k}^{k+n-1} c_i\right), \qquad \overline{\beta}(e^{c_{\mathbb{I}}}) = \exp\left(\lim_{n\to\infty} \frac{1}{n} \sup_{k\in\mathbb{I}} \sum_{i=k}^{k+n-1} c_i\right).$$

REMARK B.3 (computation of Bohl exponents). *For the geometric means*

$$\beta_{n,k}(a_{\mathbb{I}}) := \sqrt[n]{\prod_{i=k}^{k+n-1} |a_i|},$$

one has

$$\ln \beta_{n,k}(a_{\mathbb{I}}) = \frac{1}{n} \sum_{i=k}^{k+n-1} \ln |a_i|.$$

We consequently find approximations of $\ln \overline{\beta}(a_{\mathbb{I}})$ and $\ln \underline{\beta}(a_{\mathbb{I}})$ by computing

$$\sup_{k\in\mathbb{I}} \ln \beta_{n,k}(a_{\mathbb{I}}) \quad \text{and} \quad \inf_{k\in\mathbb{I}} \ln \beta_{n,k}(a_{\mathbb{I}}) \quad \text{for sufficiently large } n \in \mathbb{N},$$

respectively. Note the recursion

$$\ln \beta_{n,k+1}(a_{\mathbb{I}}) = \ln \beta_{n,k}(a_{\mathbb{I}}) + \frac{1}{n} \ln \left|\frac{a_{k+n}}{a_k}\right|,$$

which allows to compute the sequence $(\beta_{n,k})_{k\in\mathbb{I}}$ efficiently.

For the sequence $a_{\mathbb{I}}$ defined in Ex. B.2(4), we obtained the results from Tab. B.1 demonstrating that larger values of n for fixed $\mathbb{I}$ do not yield convergence (caused by the exponential increase in the "period" of the $\sin \circ \ln$-function). Indeed, for infinite $\mathbb{I}$ theoretical considerations guarantee that the approximations $\inf_{k\in\mathbb{I}} \beta_{n,k}(a_{\mathbb{I}})$ to $\underline{\beta}(a_{\mathbb{I}})$ are increasing, while the approximations $\sup_{k\in\mathbb{I}} \beta_{n,k}(a_{\mathbb{I}})$ to $\overline{\beta}(a_{\mathbb{I}})$ are decreasing as $n \to \infty$. These monotonicity properties can serve as an indicator for convergence and suggest to increase both n, as well as the length of $\mathbb{I}$ during the computation of Bohl exponents.

On the other hand, for the almost periodic sequence $a_n = \cos n$, Tab. B.2 indicates that the approximations to $\underline{\beta}(a_{\mathbb{I}})$ actually increase, while those to $\overline{\beta}(a_{\mathbb{I}})$ decrease to $\frac{1}{2}$.

n	$\underline{\beta}(a_{\mathbb{I}})$	$\overline{\beta}(a_{\mathbb{I}})$
10	1.00000000116	2.99999936133
10^2	1.00000011725	2.99993721913
10^3	1.00001172586	2.99377387672
10^4	1.00117000882	2.95027314483

n	$\underline{\beta}(a_{\mathbb{I}})$	$\overline{\beta}(a_{\mathbb{I}})$
10	1.00000000116	4.99999936133
10^2	1.00000011725	4.99993721934
10^3	1.00001172586	4.99377593377
10^4	1.00117000882	4.94711156401

Table B.1
Ex. B.2(4) with $\mathbb{I} = [1, 10^6]$ *(left) and* $\mathbb{I} = [-10^6, 10^6]$ *(right).*

n	$\underline{\beta}(a_{\mathbb{I}})$	$\overline{\beta}(a_{\mathbb{I}})$
10^2	0.42625376980	0.52837200045
10^3	0.48989566883	0.50335789629
10^4	0.49640107150	0.50116706302
10^5	0.49992964104	0.50021543720

n	$\underline{\beta}(a_{\mathbb{I}})$	$\overline{\beta}(a_{\mathbb{I}})$
10^2	0.42625376980	0.52837200045
10^3	0.48989566883	0.50335789629
10^4	0.49640107150	0.50116706302
10^5	0.49992964104	0.50021544123

Table B.2
$a_n = \cos n$ *with* $\mathbb{I} = [0, 10^6]$ *(left) and* $\mathbb{I} = [-10^6, 10^6]$ *(right).*

B.2. Exponential dichotomies. Let $A_{\mathbb{I}} = (A_n)_{n\in\mathbb{I}}$ denote a bounded sequence of square matrices $A_n \in \mathbb{R}^{d\times d}$. The associated linear difference equation

$$x_{n+1} = A_n x_n \tag{L}$$

induces an *evolution operator*

$$\Phi(n,m) := \begin{cases} A_{n-1}\cdots A_m, & m < n,\\ I, & n = m. \end{cases}$$

A difference equation (L) is said to possess an *exponential dichotomy* (ED for short, cf. [8]) on $\mathbb{I}$, if there exists a sequence $(P_n)_{n\in\mathbb{I}}$ of projections in $\mathbb{R}^{d\times d}$ (i.e. $P_n = P_n^2$) and real numbers $\alpha \in (0,1)$, $K \geq 1$, such that

(a) $P_{n+1}A_n = A_nP_n$ and $A_n|_{N(P_n)} : N(P_n) \to N(P_{n+1})$ is invertible,
(b) $\|\Phi(n,m)P_m\| \leq K\alpha^{n-m}$ for all $m \leq n$,
(c) $\left\|\bar\Phi(n,m)[I - P_m]\right\| \leq K\alpha^{m-n}$ for all $n \leq m$, and $\bar\Phi(n,m) : N(P_m) \to N(P_n)$, $n \leq m$, being the inverse of $\Phi(m,n)|_{N(P_n)}$.

Geometrically, an ED is a hyperbolic splitting of the extended state space $\mathbb{I}\times\mathbb{R}^d$ for (L) into a stable vector bundle $\mathcal{V}_s := \left\{(n,x) \in \mathbb{I}\times\mathbb{R}^d : x \in R(P_n)\right\}$ and a complementary unstable vector bundle $\mathcal{V}_u := \left\{(n,x) \in \mathbb{I}\times\mathbb{R}^d : x \in N(P_n)\right\}$. When restricted to the invariant set $\mathcal{V}_s$, a dichotomous equation (L) becomes uniformly asymptotically stable. On this basis, we define the *dichotomy spectrum*

$$\Sigma(A_{\mathbb{I}}) = \left\{\gamma > 0 : x_{n+1} = \gamma^{-1}A_nx_n \text{ does not have an ED on } \mathbb{I}\right\}$$

and the *dichotomy resolvent* $\rho(A) := (0,\infty)\setminus\Sigma(A)$. Referring to [1, Spectral theorem], the dichotomy spectrum consists of up to d disjoint *spectral intervals* and reads as

$$\Sigma(A_{\mathbb{I}}) = \begin{cases} (0,b_1] \\ [a_1,b_1] \end{cases} \cup \bigcup_{i=2}^{k} [a_i,b_i]$$

with real numbers $0 < a_1 \leq b_1 < a_2 \leq \ldots \leq b_k$ for some integer $k \leq d$. A spectral interval $(0,b_1]$ can occur, if some inverse A_n^{-1} does not exist or $\sup_{n\in\mathbb{I}}\left\|A_n^{-1}\right\| = \infty$.

There is a close relation between the dichotomy spectrum and Bohl exponents, which we are going to explore next:

PROPOSITION B.4. *Given a real sequence $a_{\mathbb{I}}$ satisfying* (B.1)*, the dichotomy spectrum of a scalar equation*

$$x_{n+1} = a_n x_n \tag{B.6}$$

is $\Sigma(a_{\mathbb{I}}) = [\underline{\beta}(a_{\mathbb{I}}), \overline{\beta}(a_{\mathbb{I}})]$ with the Bohl exponents $\underline{\beta}(a_{\mathbb{I}})$ and $\overline{\beta}(a_{\mathbb{I}})$.

Proof. Above all, the evolution operator $\Phi_a(n,m)$ of (B.6) is given by (3.2). Hence, the scaled difference equation $x_{n+1} = \gamma^{-1} a_n x_n$ has an ED on $\mathbb{I}$ if and only if, there exist reals $K \geq 1$, $\alpha \in (0,1)$ such that

$$\begin{aligned}
& \gamma^{m-n} |\Phi_a(n,m)| \leq K\alpha^{n-m} \quad \text{or} \quad \gamma^{n-m} |\Phi_a(m,n)| \leq K\alpha^{n-m} \quad \text{for all } m \leq n \\
\Leftrightarrow \quad & |\Phi_a(n,m)| \leq K(\alpha\gamma)^{n-m} \quad \text{or} \quad \frac{1}{K}\left(\frac{\alpha}{\gamma}\right)^{m-n} \leq |\Phi_a(n,m)| \quad \text{for all } m \leq n \\
\Leftrightarrow \quad & \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|} \leq K^{\frac{1}{n}}\alpha\gamma \quad \text{or} \quad \frac{1}{K^{\frac{1}{n}}}\frac{\gamma}{\alpha} \leq \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|} \quad \text{for all } k \in \mathbb{I},\ n \geq 0 \\
\Leftrightarrow \quad & \sup_{k\in\mathbb{I}} \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|} \leq K^{\frac{1}{n}}\alpha\gamma \quad \text{or} \quad \frac{1}{K^{\frac{1}{n}}}\frac{\gamma}{\alpha} \leq \inf_{k\in\mathbb{I}} \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|} \quad \text{for all } n \geq 0.
\end{aligned}$$

(I) In the limit $n \to \infty$ this immediately implies

$$\frac{\overline{\beta}(a_{\mathbb{I}})}{\alpha} \leq \gamma \quad \text{or} \quad \gamma \leq \alpha\underline{\beta}(a_{\mathbb{I}}). \tag{B.7}$$

(II) Conversely, if we choose an arbitrary $q > 1$, then (B.7) implies that there exists an $N_q \in \mathbb{N}_0$ such that

$$\sup_{k\in\mathbb{I}} \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|} \leq q^{\frac{1}{n}}\alpha\gamma \quad \text{or} \quad \frac{1}{q^{\frac{1}{n}}}\frac{\gamma}{\alpha} \leq \inf_{k\in\mathbb{I}} \sqrt[n]{\prod_{j=k}^{k+n-1} |a_j|} \quad \text{for all } n \geq N_q$$

and proceeding as in the chain of equivalences above, we obtain

$$\gamma^{m-n} |\Phi_a(n,m)| \leq q\alpha^{n-m} \quad \text{or} \quad \gamma^{n-m} |\Phi_a(m,n)| \leq q\alpha^{n-m} \quad \text{for all } N_q \leq n-m.$$

By choosing an appropriate constant $Q \geq q$, using the boundedness assumption on $a_{\mathbb{I}}$ and (B.1), it is easy to see that these inequalities extend to

$$\gamma^{m-n} |\Phi_a(n,m)| \leq Q\alpha^{n-m} \quad \text{or} \quad \gamma^{n-m} |\Phi_a(m,n)| \leq Q\alpha^{n-m} \quad \text{for all } m \leq n.$$

(III) If we combine the steps (I) and (II), the scaled equation $x_{n+1} = \gamma^{-1} a_n x_n$ has an ED on $\mathbb{I}$, if and only if (B.7) holds. The logical contraposition to this statement is that $\gamma \in \Sigma(a_{\mathbb{I}})$ is equivalent to $\gamma \in \left(\alpha\underline{\beta}(a_{\mathbb{I}}), \frac{\overline{\beta}(a_{\mathbb{I}})}{\alpha}\right)$ for all $\alpha \in (0,1)$, i.e.

$$\gamma \in \bigcap_{\alpha\in(0,1)} \left(\alpha\underline{\beta}(a_{\mathbb{I}}), \tfrac{\overline{\beta}(a_{\mathbb{I}})}{\alpha}\right) = [\underline{\beta}(a_{\mathbb{I}}), \overline{\beta}(a_{\mathbb{I}})]$$

and consequently the claim $\Sigma(a_{\mathbb{I}}) = [\underline{\beta}(a_{\mathbb{I}}), \overline{\beta}(a_{\mathbb{I}})]$. ∎

In order to meet the purposes of this paper we now restrict to two dimensions:

PROPOSITION B.5. *If* (L) *has a diagonal coefficient sequence*

$$A_n = \begin{pmatrix} a_n & \\ & b_n \end{pmatrix} \quad \text{for all } n \in \mathbb{I}$$

with bounded diagonal sequences $a_{\mathbb{I}}, b_{\mathbb{I}}$*, then*

(a) $\Sigma(A_{\mathbb{I}}) = \Sigma(a_{\mathbb{I}}) \cup \Sigma(b_{\mathbb{I}})$,

(b) $\Sigma(A_{\mathbb{I}}) = \left[\underline{\beta}(a_{\mathbb{I}}), \overline{\beta}(a_{\mathbb{I}})\right] \cup \left[\underline{\beta}(b_{\mathbb{I}}), \overline{\beta}(b_{\mathbb{I}})\right]$*, provided* $a_{\mathbb{I}}, b_{\mathbb{I}}$ *satisfy* (B.1).

Proof. We define the linear operators

$$\begin{aligned} &L_\gamma \in L(\ell^\infty(\mathbb{R}^2)), && (L_\gamma \phi)_n := \phi_{n+1} - \gamma^{-1} A_n \phi_n, \\ &L_\gamma^a \in L(\ell^\infty(\mathbb{R})), && (L_\gamma^a \phi)_n := \phi_{n+1} - \gamma^{-1} a_n \phi_n, \\ &L_\gamma^b \in L(\ell^\infty(\mathbb{R})), && (L_\gamma^b \phi)_n := \phi_{n+1} - \gamma^{-1} b_n \phi_n. \end{aligned} \tag{B.8}$$

(I) Suppose that $\mathbb{I}$ is bounded below. Due to [15, Thm. 3.2] we obtain

$$\begin{aligned} \gamma \notin \Sigma(A) &\Leftrightarrow L_\gamma \text{ is onto} \Leftrightarrow \forall \psi \in \ell^\infty(\mathbb{R}^2) : \exists \phi \in \ell^\infty(\mathbb{R}^d) : \; L_\gamma \phi = \psi \\ &\Leftrightarrow \forall \psi \in \ell^\infty(\mathbb{R}^2) : \; x_{n+1} = \gamma^{-1} A_n x_n + \psi_n \text{ has a bounded solution } \phi \\ &\Leftrightarrow \forall \psi^a, \psi^b \in \ell^\infty(\mathbb{R}) : \; x_{n+1} = \tfrac{a_n}{\gamma} x_n + \psi_n^a \text{ and } x_{n+1} = \tfrac{b_n}{\gamma} x_n + \psi_n^b \text{ have} \\ &\qquad \text{bounded solutions } \phi^a, \phi^b \\ &\Leftrightarrow \forall \psi^a, \psi^b \in \ell^\infty(\mathbb{R}) : \exists \phi^a, \phi^b \in \ell^\infty(\mathbb{R}) : \; L_\gamma^a \phi^a = \psi^a \text{ and } L_\gamma^b \phi^b = \psi^b \\ &\Leftrightarrow L_\gamma^a, L_\gamma^b \text{ are onto} \\ &\Leftrightarrow x_{n+1} = \tfrac{a_n}{\gamma} x_n \text{ and } x_{n+1} = \tfrac{b_n}{\gamma} x_n \text{ have an ED} \Leftrightarrow \gamma \notin \Sigma(a_{\mathbb{I}}) \cup \Sigma(b_{\mathbb{I}}), \end{aligned}$$

yielding claim (a) in the logical contraposition. Claim (b) results using Prop. B.4.

(II) The situation $\mathbb{I} = \mathbb{Z}$ can be shown similarly with the characterization [8, p. 230, Thm. 7.6.5] applied to the operators L_γ and L_γ^a, L_γ^b. ☐

We remind the reader of the symmetric difference of two sets M_1, M_2 defined as

$$M_1 \triangle M_2 := (M_1 \cup M_2) \setminus (M_1 \cap M_2),$$

containing all elements which are either in M_1 or in M_2. We note that the intersection of sets distributes over the symmetric difference, i.e. for arbitrary sets M one has

$$(M_1 \triangle M_2) \cap M = (M_1 \cap M) \triangle (M_2 \cap M). \tag{B.9}$$

PROPOSITION B.6. *Let* $\mathbb{I} = \mathbb{Z}$*. If* (L) *has an upper triangular coefficient sequence*

$$A_n = \begin{pmatrix} a_n & c_n \\ 0 & b_n \end{pmatrix} \quad \text{for all } n \in \mathbb{Z}$$

and $a_{\mathbb{Z}}, b_{\mathbb{Z}}, c_{\mathbb{Z}}$ *are bounded sequences, then the following holds true:*

(a) $\overline{\Sigma(a_{\mathbb{Z}}) \triangle \Sigma(b_{\mathbb{Z}})} \subseteq \Sigma(A_{\mathbb{Z}}) \subseteq \Sigma(a_{\mathbb{Z}}) \cup \Sigma(b_{\mathbb{Z}})$.

(b) If $\Sigma(a_{\mathbb{Z}}) \cap \Sigma(b_{\mathbb{Z}})$ *has no interior points, then* $\Sigma(A_{\mathbb{Z}}) = \Sigma(a_{\mathbb{Z}}) \cup \Sigma(b_{\mathbb{Z}})$.

REMARK B.7. *(1) The same assertion holds for difference equations* (L) *with lower triangular coefficient matrices.*

(2) An alternative condition guaranteeing $\Sigma(A_{\mathbb{Z}}) = \Sigma(a_{\mathbb{Z}}) \cup \Sigma(b_{\mathbb{Z}})$ *is* $1 \notin \Sigma(\frac{a_{\mathbb{Z}}}{b_{\mathbb{Z}}})$ *(in the upper triangular case) resp.* $1 \notin \Sigma(\frac{b_{\mathbb{Z}}}{a_{\mathbb{Z}}})$ *(in the lower triangular case).*

Proof. Let us introduce a weighted shift operator $T_A \in L(\ell^\infty(\mathbb{R}^2))$ as

$$(T_A\phi)_n := A_{n-1}\phi_{n-1} = \begin{pmatrix} a_{n-1}\phi^1_{n-1} + c_{n-1}\phi^2_{n-1} \\ b_{n-1}\phi^2_{n-1} \end{pmatrix} \quad \text{for all } n \in \mathbb{Z}.$$

If we define the bounded projection $P \in L(\ell^\infty(\mathbb{R}^2))$, $(P\phi)_n := \binom{\phi^1_n}{0}$ and the closed subspaces $X := R(P)$, $Y := N(P)$ of $\ell^\infty(\mathbb{R}^2)$, then the following holds true

$$(T_A P\phi)_n = \begin{pmatrix} a_{n-1}\phi^1_{n-1} \\ 0 \end{pmatrix}, \qquad (T_A(I-P)\phi)_n = \begin{pmatrix} c_{n-1}\phi^2_{n-1} \\ b_{n-1}\phi^2_{n-1} \end{pmatrix} \quad \text{for all } n \in \mathbb{Z},$$

as well as $\ell^\infty(\mathbb{R}^2) = X \oplus Y$. Furthermore, T_A can be represented as block-diagonal operator $T_A = \begin{pmatrix} T_a & C \\ 0 & T_b \end{pmatrix} \in L(X \oplus Y)$ with

$$\begin{array}{ccc} T_a \in L(X), & T_b \in L(Y), & C \in L(Y, X), \\ (T_a\phi)_n := a_{n-1}\phi_{n-1}, & (T_b\phi)_n := b_{n-1}\phi_{n-1}, & (C\phi)_n := c_{n-1}\phi_{n-1}. \end{array}$$

(a) Due to [7, Cor. 4] one has $\sigma(T_a) \triangle \sigma(T_b) \subseteq \sigma(T_A) \subseteq \sigma(T_a) \cup \sigma(T_b)$ and thus the first claimed inclusion follows from (B.9), if we set $M_a := \sigma(T_a)$, $M_b := \sigma(T_b)$, $M := \mathbb{R}^+$ and make use of $\Sigma(A_\mathbb{Z}) = \sigma(T_A) \cap \mathbb{R}^+$ and corresponding relations for T_a, T_b (cf. [28, (1.1)]). The second claimed inclusion follows directly using [28, (1.1)].

(b) Thanks to the relation $\sigma(T_a) \cap \sigma(T_b) = \{\lambda \in \mathbb{C} : |\lambda| \in \Sigma(a_\mathbb{Z}) \cap \Sigma(b_\mathbb{Z})\}$ and our assumption on interior points, the intersection $\sigma(T_a) \cap \sigma(T_b) \subseteq \mathbb{C}$ is a finite union of circles centered around 0 (or empty) and has thus no interior points. Hence, [7, Cor. 8] ensures $\sigma(T_A) = \sigma(T_a) \cup \sigma(T_b)$ and again [28, (1.1)] yields the claim. □

The following example ultimately illustrates that Prop. B.6(b) fails without the additional assumption on interior points:

EXAMPLE B.8. *We investigate* (*L*) *on* $\mathbb{Z}$ *with upper-triangular coefficients.*

(1) In the situation

$$A_n := \begin{cases} A^+, & n \geq 0, \\ A^-, & n < 0, \end{cases} \qquad A^+ := \begin{pmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad A^- := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

the diagonal sequences of A_n *fulfill* $\Sigma(a_\mathbb{Z}) = \Sigma(b_\mathbb{Z}) = [\frac{1}{2}, 2]$ *(see Prop. B.4), whereas* $\Sigma(A_\mathbb{Z}) = \left\{\frac{1}{2}, 1\right\}$ *(cf. [28, Exam. 5.5]).*

(2) Prop. B.6(b) can be used to detect gaps in the dichotomy spectrum. In case

$$A_n := \begin{cases} A^+, & n \geq 0, \\ A^-, & n < 0, \end{cases} \qquad A^+ := \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A^- := \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

the diagonal sequences of A_n *imply* $\Sigma(a_\mathbb{Z}) = [1, 4]$, $\Sigma(b_\mathbb{Z}) = [2, 3]$ *(see Prop. B.4). Hence, it is* $\overline{[1, 2) \cup (3, 4]} \subseteq \Sigma(A_\mathbb{Z}) \subseteq [1, 4]$*, which holds for* $\Sigma(A_\mathbb{Z}) = [1, 2] \cup [3, 4]$.

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